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A CHARACTERIZATION OF PSEUDO-MONOTONE DIFFERENTIAL OPERATORS IN DIVERGENCE FORM.

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I. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

In this article we will be concerned with pseudo-monotone operators, which have been studied intensively by Brezis [Br1], Browder [Bo1], Hartmann-Stampacchia [HS1], Hess [He1], Leray-Lions [LL1]...

We will give a characterization of such operators when they are differential operators, in divergence form. Our result will use the methods of the calculus of variations that are commonly used in order to obtain necessary conditions for weak lower semicontinuity; (see, among others [Tal], [Mol], [Gil], [Dal] and we will follow more closely [MS1] and [ET1]).

We now give the definition (see [Br1]) of pseudo-monotonicity.

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Definition: Let $V$ be a reflexive Banach space, $V'$ its dual and $\langle \cdot, \cdot \rangle$ the bilinear canonical form on $V' \times V$. Then $A: V \to V'$ is said to be pseudo-monotone if $A$ is bounded and if for every sequence $(u_m)$ such that
\[
\begin{cases}
  u_m \xrightarrow{\text{w}} u \\
  \lim_{m \to \infty} \langle A(u_m) ; u_m - u \rangle \in \mathcal{O}
\end{cases}
\] (1.1)
then
\[
\lim_{m \to \infty} \langle A(u_m) ; u_m - v \rangle \geq \langle A(u) ; u - v \rangle
\] (1.2)
for every $v \in V$ and where $\xrightarrow{\text{w}}$ denotes weak convergence in $V$.

In the following we will restrict ourselves to operators in divergence form over $V = W^{1,p}(\Omega)$, $p > 1$ ($V' = W^{-1,p'}$, with $p' = p/p-1$) where $\Omega \subset \mathbb{R}^n$ is a bounded open set and
\[
A(u) = - \operatorname{div} a(x,u(x), \nabla u(x))
\] (1.3)
where

(H) $a: \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory function (i.e., $a(x,\cdot,\cdot)$ is continuous on $\mathbb{R} \times \mathbb{R}^n$ for almost every $x \in \Omega$ and $a(\cdot,s,\xi)$ is measurable in $\Omega$ for every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$) satisfying
\[
|a(x,s,\xi)| \leq k(x) + \beta(|s|^{p-1} + |\xi|^{p-1})
\] (1.4)
for almost every $x \in \Omega$, for every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$ and for some $k \in L^{p'}(\Omega), \beta > 0, p > 1$.

Our main result is

Theorem 1: Let $A$ and $a$ be as above. If $A$ is pseudo monotone on $W^{1,p}(\Omega)$ then $a$ is monotone with respect to the last variable, i.e.,
\[ a(x,s,\xi) = a(x,s,\eta) ; \xi - \eta \to 0 \] (1.5)

for almost every \( x \in \mathbb{N} \), for every \( s \in \mathbb{R}, \xi, \eta \in \mathbb{N} \) and where \( \langle \cdot, \cdot \rangle \) denotes scalar product in \( \mathbb{N} \).

It is interesting to note that if one adds a coercivity condition on the function \( a \), then Leray-Lions [20] and Brezis [21] have shown that the strict monotonicity of \( a \) (i.e., (1.5) is a strict inequality whenever \( \xi \neq \eta \)) implies the surjectivity of the operator \( A \), in other words that \( A \) is pseudo-monotone and coercive. Note that the condition of strict monotonicity may be removed using an approximation argument. Therefore, roughly speaking, the above theorem can be considered as the converse of Leray-Lions and Brezis result.

II. PROOF OF THEOREM 1

In order to prove the above theorem we will need two lemmas

**Lemma 1**: Let \( I \subset \mathbb{R}^n \) be a bounded open set with Lipschitz boundary and let \( u_m, u \) be a such that

\[ u_m \overset{w}{\to} u \text{ in } W^{1,\infty}(I) \] (2.1)

\[ \| \text{grad } u_m \|_{L^\infty(I)} \leq \sigma \] (2.2)

where \( \sigma > 0 \) is fixed and \( \overset{w}{\to} \) denotes weak * convergence. Then there exist a sequence \( v_m \) such that

\[ v_m \overset{w}{\to} u \text{ in } W^{1,\infty}(I) \] (2.3)

\[ v_m = u \text{ on } \partial I \] (2.4)
\[ \| \nabla v_m \|_{L^\infty(I)} \leq 2 \sigma \]  
\[ \left\langle a(x,v_m(x),\nabla v_m(x));\nabla(v_m(x) - z(x)) \right\rangle \] 
\[ = a(x,u_m,\nabla u_m);\nabla(u_m - z) \right\rangle - \sigma (x) - \sigma \] 
for every \( x \in W^{1,\infty}(I) \).

Roughly speaking the lemma shows that one can modify slightly the original sequence \( \{u_m\} \) in order to fix its value on the boundary without altering too much the value of \( a(x,u_m,\nabla u_m) \).

**Proof:** The proof is rather standard in the calculus of variations and relies on a lemma of Mac Shanes (see [ET1] p.276). Let

\[ \delta_m = \frac{1}{\sigma} \| u_m - u \|_{L^\infty} \]  
\[ I_m = \{ x \in I : \text{dist}(x, \partial I) > \delta_m \}. \]

Define

\[ \tilde{v}_m(x) = \begin{cases} 
  u_m(x) & \text{if } x \in I_m \\
  u(x) & \text{if } x \in \partial I_m.
\end{cases} \]

Observe that \( \tilde{v}_m \) is locally Lipschitz with constant \( 2 \sigma \) on \( I_m \cup \partial I \), since if \( x \in I_m \) and \( y \in \partial I \), choose \( \{ \xi \} = \{ x,y \} \cap \partial I_m \), then

\[ |\tilde{v}_m(x) - \tilde{v}_m(y)| \leq |\tilde{v}_m(x) - \tilde{v}_m(\xi)| + |\tilde{v}_m(\xi) - \tilde{v}_m(y)| \]

\[ \leq \sigma |x - \xi| + \| u_m(\xi) - u(y) \|_{L^\infty} \]

\[ \leq \sigma |x - \xi| + \| u_m - u \|_{L^\infty} + \| u(\xi) - u(y) \|_{L^\infty} \leq 2 \sigma |x - y|. \]  

Using Mac Shanes's lemma we find \( v_m \) an extension of \( \tilde{v}_m \) to \( \bar{I} \) satisfying (2.3) - (2.5); it therefore remains to show (2.6). For simplicity we denote the expression on the right hand side of (2.6) as \( r_m \). Using the definition of \( v_m \) we get
Let $A$ be as in Theorem 1 and be pseudo-monotone. Then, for every bounded open set $0 \subset \Omega$ and for every sequence $\{u_m\}$ satisfying
\begin{equation}
\lim_{m \to \infty} \int_{\Omega} \langle a(x, u_m, \nabla u_m), \nabla(u_m - u) \rangle dx = 0 \tag{2.13}
\end{equation}
\begin{equation}
\lim_{m \to \infty} \int_{\Omega} \langle a(x, u_m, \nabla u_m), \nabla(u_m - u) \rangle dx \in \mathcal{A} \tag{2.14}
\end{equation}
one has
\begin{equation}
\lim_{m \to \infty} \int_{\Omega} \langle a(x, u_m, \nabla u_m), \nabla(u_m - u) \rangle dx \geq \int_{\Omega} \langle a(x, u, \nabla u), \nabla(u - \omega) \rangle dx \text{ for every } \omega \in W^{1,\infty}(\Omega). \tag{2.15}
\end{equation}

Proof: Let $\{u_m\}$ be as in (2.13) and (2.14). Then Lemma 1 implies that we can find $v_m$ such that
\begin{equation}
v_m \rightharpoonup u \text{ in } W^{1,\infty}(\Omega) \tag{2.16}
\end{equation}
Combining (2.18) and (2.14) we obtain (2.20). Therefore, using the fact that A is pseudo-monotone, we get

\[
\begin{align*}
\lim_{m \to \infty} \int_\Omega \langle a(x, \nabla \vec{v}_m); \nabla (\vec{v}_m - z) \rangle \, dx &= \int_\Omega \langle a(x, \nabla \vec{v}); \nabla (\vec{v} - z) \rangle \, dx \\
\end{align*}
\]

for every \( z \in W_0^1, P(\Omega) \).

We now choose \( z \) in such a way that \( z \in W_0^1, P(\Omega) \) and \( z = w \) in \( \Omega \) where \( w \in W_{\infty}^1(0) \). Returning to (2.22) and using the fact that \( \vec{v}_m = \hat{u} \) in
\( n = 0 \), we get
\[
\lim_{m \to \infty} \int_{\Omega} \langle a(x, \nabla u_m), \nabla (v_m - w) \rangle \, dx
\]
\[
\geq \int_{\Omega} \langle a(x, u), \nabla (u - w) \rangle \, dx
\]  \hspace{1cm} (2.23)
for every \( w = \psi \phi^\varepsilon(0) \).

Combining (2.23) and (2.18) we obtain (2.15) and thus the lemma.

We are now in a position of proving Theorem 1.

**Proof:** We will show (1.5) for every Lebesgue point \( x \in \Omega \). This will then imply (1.5) for almost all \( x \). We now proceed by contradiction. Let \( x_0 \in \Omega \) be a Lebesgue point of \( a \) and let \( s, \xi, \eta \in \mathbb{R}^n \) (\( \xi \neq \eta \)) be such that
\[
\langle a(x_0, s, \xi) - a(x_0, s, \eta); \xi - \eta \rangle < 0. \hspace{1cm} (2.24)
\]
We subdivide the proof into three steps.

**Step 1:** Since \( x_0 \) is a Lebesgue point of \( a \) we have
\[
\lim_{R \to 0} \frac{1}{\text{meas } H_R(x_0)} \int_{H_R(x_0)} \langle a(x, s, \xi) - a(x, s, \eta); \xi - \eta \rangle \, dx < 0 \hspace{1cm} (2.25)
\]
where \( H_R(x_0) \) is the hypercube of \( \mathbb{R}^n \) centered at \( x_0 \) and of side length \( R \). Therefore for \( R \) small enough, one has \( H_R(x_0) \subset \Omega \) and
\[
\int_{H_R(x_0)} \langle a(x, s, \xi) - a(x, s, \eta); \xi - \eta \rangle \, dx < 0. \hspace{1cm} (2.26)
\]
Choosing \( R \) smaller, if necessary, we have also that if
\[
u(x) := \frac{\xi + \eta}{2}; \hspace{1cm} x - x_0 > s; \hspace{1cm} x \in \Omega \hspace{1cm} (2.27)
\]
then
\[
\int_{H_R(x_0)} \langle a(x, \nu(x), \xi) - a(x, \nu(x), \eta); \xi - \eta \rangle \, dx < 0. \hspace{1cm} (2.28)
\]
For simplicity we will denote such an $H_R(x_0)$ by $0$.

**Step 2:** We now construct a sequence $(u_m)$ (where $u$ is as in (2.27)) such that

$$ u_m \rightharpoonup u \quad \text{in} \quad W^{1,\infty}(O) $$

$$ \text{grad} u_m(x) = \begin{cases} \xi & \text{if } x \in O^m_1 \\ \eta & \text{if } x \in O^m_2 \end{cases} $$

where $O^m_1, O^m_2 \subseteq O$, $O^m_1 \cap O^m_2 = \emptyset$ and

$$ \lim_{m \to \infty} \text{mes} O^m_1 = \lim_{m \to \infty} \text{mes} O^m_2 = \frac{1}{2} \text{mes} O. $$

Such a construction is classical in the calculus of variations and it can be done for example as follows (see [ET1] Chap.X,[MS1], or [MS1] Theorem 2.4). Up to a change of coordinates in $\mathbb{R}^n$, we may suppose that

$$ \langle \xi - \eta; x \rangle = \langle \xi_1 - \eta_1 \rangle x_1 $$

where $x = (x_1, \ldots, x_n)$ and $(\xi - \eta) = (\xi_1 - \eta_1, \ldots, \xi_n - \eta_n)$.

We then write

$$ O = H_R(x) = \prod_{i=1}^n (\alpha_i, \beta_i) = (\alpha_1, \beta_1) \times \prod_{i=2}^n (\alpha_i, \beta_i). $$

We divide $(\alpha_i, \beta_i)$ into intervals of length $(\beta_i - \alpha_i)2^{-m}$ where $m \in \mathbb{N}$. Each of these intervals is then subdivided into two parts of equal length $(\beta_i - \alpha_i)2^{-(m+1)}$. The union of the first (resp. the second) subintervals is denoted by $I_m$ (resp. $J_m$). We denote by

$$ O^m_1 = I_m \times \prod_{i=2}^n (\alpha_i, \beta_i). $$
From the definition of \( \omega \) and from (2.28) we get

\[
\lim_{m \to \infty} \int_0^1 \frac{1}{2} \langle a(x, u_m(x), \nabla u_m(x)) \rangle_{\mathbb{R}^n} < 0.
\]

Using Lemma 2 with \( a = u \) we deduce that

\[
\lim_{m \to \infty} \int_0^1 \frac{1}{2} \langle a(x, u_m, \nabla u_m) \rangle_{\mathbb{R}^n} \geq 0,
\]

contradicting (2.34) and hence \( A \) is not pseudo-monotone.

\[\square\]
Remarks:

(i) It is interesting to note the relationship between weak lower semicontinuity of functionals of the type

\[ F(u) = \int f(x, u(x), \nabla u(x)) \, dx \]

\((u: \mathbb{R}^n \to \mathbb{R})\) and the pseudo-monotonicity of \(A\) in \(W^{1, p}_0(\Omega)\) where

\[ A(u) = -\text{div} \, a(x, u(x), \nabla u(x)). \quad (1.3) \]

More precisely it is known in the calculus of variations ([Mo1], [ET1],[Ms1]...), that the weak lower semicontinuity of \(F\) is equivalent to the convexity of \(\varepsilon\) with respect to the last argument. Similarly we have just seen that, roughly speaking, the pseudo-monotonicity of \(A\) is equivalent to the monotonicity of \(\varepsilon\).

(ii) However it should be pointed out that the proofs and the conclusions given in the theorem are strictly restricted to operators \(A\) as above, i.e., for a single equation \((\text{since } u: \mathbb{R}^n \to \mathbb{R})\). It is an open problem to find necessary conditions for the pseudo-monotonicity of \(A\) when \((1.3)\) is a system, i.e., \(u: \mathbb{R}^n \to \mathbb{R}^m, m > 1\).

(iii) Finally, one can show by similar methods that if \(A\) is an operator of type \(M\) (i.e., if \(u_n \rightharpoonup u, A(u_n) \rightharpoonup g\) and \(\lim \langle A u_n; u_n \rightharpoonup \varepsilon, u \rangle \geq 0\), then \(g = A(u)\)) and is in divergence form then \(a(x, s, \cdot, \varepsilon)\) is either linear or monotone. Note, in particular, that a weakly continuous operator is of type \(M\) and it has been shown by Giachetti [Gl1] that if \(A\) is weakly continuous and in divergence form \((\text{c.f.} \quad (1.3))\) then \(a(x, s, \varepsilon)\) must be linear with respect to the variable \(\varepsilon\).
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