The problem addressed is that of finding the harvesting policy that will maximize the yield and maintain a given population in a steady state. It is assumed that the equations describing the evolution of the population and the yield are linear but that the equation describing reproduction is nonlinear (density dependence). The optimal strategy consists in harvesting two age classes only: a younger age class is partially harvested, and an older age class is completely harvested as in the linear case. This problem is a particular case of a general abstract maximization problem in which, for example, nonlinear mortality or nonlinear value of the individuals can also be introduced. The appropriate compactness for this problem is obtained from the fact that maximizing sequences are decreasing.

1. INTRODUCTION

Harvesting theory is concerned with the rational exploitation of biological resources such as populations of game, fish, and trees. In this paper we present such an application by considering a relatively simple matter: the determination of the harvesting policy that ensures the maximum sustainable yield (MSY) from a population in a steady state. Clark [4, 5] has developed the theory in the context of models where the population is represented by a simple scalar variable. Getz and Haight [6] have developed structured models as generalizations of the classical Leslie matrix approach. This discrete-time model applies to populations exhibiting distinct reproductive seasons, like many biological resources in temperate environments. Reed [9] considers such an example by
determining the age-specific harvesting policy for maximizing equilibrium yield in a discrete-time, self-generating, and age-structured population model. However, many species of animals and plants reproduce continuously throughout the year, especially in the tropics. In such cases, the discrete approach is not appropriate. Continuous models lead naturally to differential equations rather than difference equations.

Arditi and Dacorogna [1] have already treated the problem for linear fecundity, but they had to introduce a constraint on the population size to avoid an infinite solution. Rorres and Fair [10] also studied the linear problem with another constraint on the population: they consider that the harvester is constrained to spend one economic unit (money or work) to maintain the total equilibrium population. In both cases, the authors showed that the optimal strategy is to harvest two age classes only: a younger age class is partially harvested, and an older age class is completely harvested (implying that no individual will exist beyond this age). This strategy will be called the two-age policy.

In our work the population is characterized by continuous age structure and therefore leads to a differential equation. Here we assume that the equations describing the evolution of the population and the yield are linear but that the equation describing reproduction is nonlinear. That is, we assume that fecundity (per capita) decreases with density. This density dependence is due to competition among all individuals, among reproducing individuals, or among newly hatched eggs (or offspring). This nonlinearity avoids the introduction of a constraint on the population size and is thus more consistent.

To develop our formalism, we use the notations of Arditi and Dacorogna [1]. Let \( a \) denote the age of an individual. The state variable \( x(a) \) is the equilibrium population density, and \( \int_0^a x(\alpha) d\alpha \) represents the number of individuals whose age is between 0 and \( a \). Therefore \( x(a) \) represents the age distribution and will be called for simplicity the number of individuals of age \( a \). The life history of this population is characterized by two schedules: The fecundity \( b(a) \) is the rate at which individuals of age \( a \) reproduce, and the mortality \( \mu(a) \) is the rate at which individuals of age \( a \) die.

The action of harvesting is represented by the function \( u(a) \), the number of individuals of age \( a \) that are removed. We also define the function \( h(a) = u(a) / x(a) \) as the removal rate. It is the additional mortality rate due to harvesting. Thereafter, it will also be called the control. Let \( w(a) \) denote the value (e.g., in dollars) of an individual of age \( a \). Then the performance of harvesting strategy \( h \) can be written as

\[
J(h) = \int_0^A w(a) h(a) x(a) \, da,
\]
where $A$ denotes the maximal age that any individual of the given population can reach (implying that no individual will exist beyond this age).

The number of newborns is given by

$$x(0) = G(x),$$

where $G$ is a given functional. It represents (see below for examples) the density dependence due to the number of eggs or to competition between individuals themselves. To give a better interpretation of the functional $G$, we can suppose that there is a linear production of eggs (which can be fictitious in the case of viviparous animals) and then that this virtual number is reduced by density dependence to obtain finally the real number of newborns $G(x)$. To be more precise, we consider three different examples of $G$, for which we can give some qualitative results.

In the first example, the number of eggs that survive depends nonlinearly (less than linearly) on the total number of eggs produced by all individuals. This corresponds to competition among eggs or among newly hatched individuals. In this case $G$ is of the form

$$G(x) = g\left(\int_0^A b(a) x(a) \, da\right).$$

We will see that the solution is qualitatively the same as in the linear case (i.e., with no density dependence). This optimal strategy consists in harvesting two age classes only. A younger age class is partially harvested, and an older age class is completely harvested.

The second example will introduce nonlinearity in a different way. The number of newborns depends on the total number of individuals in the population. The number of potential offspring is reduced by a multiplying factor $g(P)$, where $P = \int_0^A x(a) \, da$ denotes the total number of individuals. In this case $G$ is of the form

$$G(x) = g\left(\int_0^A x(a) \, da\right) \int_0^A b(a) x(a) \, da.$$

The optimal strategy will also be a two-age policy as above. Note that in this case this result has already been obtained by Murphy and Smith [8]. They have shown that if the nonlinearity depends only on the population size $P$, the optimal policy is then bimodal.

Finally, in the third example, nonlinearity is due to competition within age classes, that is, between individuals of the same age. In this
case $G$ is of the form

$$G(x) = \int_0^A b(a) g(x(a)) \, da.$$  

We will see here that results can be quite different. The optimal strategy will not, in general, be a two-age policy. This can be seen as a generalization of the result obtained by Getz and Haight [6, p. 911], who show for the discrete model that each additional aggregation variable that characterizes the population structure implies that an additional age class may have to be harvested. Here we say that the number of newborns depends on $g(x(.))$, which means that this number depends on each age class. Since we are in the continuous case, $g(x(.))$ reflects an infinite number of aggregation variables. Clearly the two-age policy is therefore not optimal.

All these problems are particular cases of a general abstract problem in which, for example, nonlinear mortality or nonlinear value of the individuals can also be introduced. The existence of a solution for this general problem is the essential part of our work. Related results can be found in Gurtin and MacCamy [7].

In Section 2 we consider this general abstract problem. We prove that, under mild hypotheses on the functionals $J$ and $G$, the problem

$$\max_{x \in X} J(x), \quad (P)$$

where

$$X = \{ x \in L^\infty(0, A); \ x \geq 0, \ x \text{ decreasing and } G(x) \geq x(a) \ \forall a \in [0, A] \},$$

has a solution. Note that since $h$ and $x$ are related (see Section 3) we indifferently use $J(h)$ or $J(x)$ to mean the quantity

$$\int_0^A w(a) h(a) x(a) \, da,$$

the choice depending on the context.

A similar problem has been studied by Ball and Knowles [2]. They consider the minimization of integrals of the form

$$I(u) = \int_\omega f(x, u(x)) \, dx$$

in the set of integrable maps $u: \omega \subset \mathbb{R}^n \to \mathbb{R}^k$ satisfying the linear constraint

$$\int_\omega u(x) \, dx = a.$$
To prove existence, they use the tool of Young measures. However, our method for solving the problem is very different from that of Ball and Knowles [2]. We obtain the appropriate compactness from the fact that maximizing sequences are decreasing.

In Section 3, we prove, with the help of Section 2, the qualitative results on the ecological examples we have presented above.

In Section 4 we discuss the consequences of our results.

2. A GENERAL ABSTRACT THEOREM

The problem we consider is the following. Find $M$ defined by

$$M = \sup_{x \in X} \{ J(x) \text{ with } G(x) \geq x(a) \forall a \in [0, A] \},$$

where $J, G: L^\infty(0, A) \to \mathbb{R}$ are continuous (with respect to $L^1$ convergence) and $X = \{ x \in L^\infty(0, A) : x \geq 0 \text{ a.e. and } x \text{ is decreasing a.e.} \}$.

We see later that many problems of mathematical as well as biological interest can be put into this formalism.

Before proceeding further we make some conventions.

CONVENTIONS

Since functions in $X$ are defined only almost everywhere, we agree to replace any function $x \in X$ by $\bar{x} \in X$ where $\bar{x}$ is defined as follows.

Let $E \subset [0, A]$ be the set where $[x(a) - x(b)](a - b) \leq 0 \forall a, b \in E$. By definition of $X$, $\text{meas}(E) = A$. Let $a \in [0, A]$ and $\{a_k\}$ be a decreasing sequence in $E$ such that $\lim_{k \to \infty} a_k = a$. We can then define $\bar{x}$ by

$$\bar{x}(a) = \begin{cases} \inf_{a \in E} x(a) & \text{if } a = A, \\ x(a) & \text{if } a \in E, \\ \sup_{\{a_k\} \in E} \limsup_{k \to \infty} x(a_k) & \text{if } a \in E^c. \end{cases}$$

Note that $\bar{x} = x$ almost everywhere and that $\bar{x}$ is everywhere decreasing.

Remark. We could also replace $x$ by $\tilde{x}$, where

$$\tilde{x}(a) = \begin{cases} \sup_{a \in E} x(a) & \text{if } a = 0, \\ x(a) & \text{if } a \in E, \\ \inf_{\{b_k\} \in E} \liminf_{k \to \infty} x(b_k) & \text{if } a \in E^c. \end{cases}$$
where \( \{b_k\} \) is an increasing sequence in \( E \) such that \( \lim_{k \to \infty} b_k = a \). \( \tilde{x} \) is also everywhere decreasing. Observe that in general \( \tilde{x} \) and \( \bar{x} \) are not equal.

We now give a simple criterion to determine whether or not a given function \( x \) is in \( X \).

**LEMMA 2.1**

Let \( x: [0, A] \to \mathbb{R}_+ \) be a measurable and bounded function.

(i) If \( x \) is decreasing on \( [0, A] \), then \( \int_0^Ax(a)\phi'(a)\,da \geq 0 \ \forall \phi \in C_0^\infty(0, A), \phi \geq 0 \).

(ii) Conversely, let \( E \subset [0, A] \) be the Lebesgue set of \( x \) [as is well known, \( \text{meas}(E) = A \)]. If \( \int_0^Ax(a)\phi'(a)\,da \geq 0 \ \forall \phi \in C_0^\infty(0, A), \phi \geq 0 \), then \( x \) is decreasing on \( E \) (i.e., \( x \) is decreasing almost everywhere on \( [0, A] \)).

**Remarks.**

(1) \( C_0^\infty(0, A) \) stands for the set of \( C^\infty(0, A) \) functions with compact support.

(2) With the convention we have given above, we can say that \( x \) is decreasing everywhere.

**Proof.** The proof being elementary, we only outline it.

(i) This is obviously true if \( x \in C^1 \). Noting that if \( x \in L^\infty \), the regularization (by convolution) \( x_n \in C^1 \) of \( x \) is also decreasing, the result then follows at once.

(ii) By density it is sufficient to show that if \( E \) is the Lebesgue set of \( x \) and if \( \int_0^Ax(a)\phi'(a)\,da \geq 0 \ \forall \phi \in W_{0, \infty}^1(0, A), \phi \geq 0 \), then \( x \) is decreasing on \( E \). Here \( \phi \in W_{0, \infty}^1(0, A) \) means that \( \phi \in C(0, A), \phi(0) = \phi(A) = 0 \) and \( \phi' \in L^\infty(0, A) \).

Let \( a_1 \) and \( a_2 \) be with \( 0 < a_1 < a_2 < A \). Then we define a sequence \( \phi_n \) (see Figure 1) by

\[
\phi_n(a) = \begin{cases} 
0 & \text{if } a \in [0, a_1 - 1/n), \\
na + 1 - na_1 & \text{if } a \in [a_1 - 1/n, a_1), \\
1 & \text{if } a \in [a_1, a_2), \\
-na + 1 + na_2 & \text{if } a \in [a_2, a_2 + 1/n), \\
0 & \text{if } a \in [a_2 + 1/n, A]. 
\end{cases}
\]

Note that \( \phi_n \) is in \( W_{0, \infty}^1(0, A) \). Since \( \int_0^Ax(a)\phi_n'(a)\,da \geq 0 \), we have

\[
\int_0^Ax(a)\phi_n'(a)\,da = n\int_{a_1 - 1/n}^{a_1}x(a)\,da - n\int_{a_2}^{a_2 + 1/n}x(a)\,da \geq 0.
\]
Since $a_1$ and $a_2$ are Lebesgue points of $x$,

$$\lim_{n \to \infty} n \int_{a_1 - 1/n}^{a_1} x(a) \, da = \lim_{n \to \infty} \frac{1}{1/n} \int_{a_1 - 1/n}^{a_1} x(a) \, da = x(a_1).$$

Similarly for $a_2$. We therefore deduce that

$$\lim_{n \to \infty} \int_{0}^{A} x(a) \phi_n'(a) \, da = x(a_1) - x(a_2) \geq 0,$$

$$\forall a_1, a_2 \in E \text{ with } a_1 < a_2$$

(thus $x$ is decreasing almost everywhere). With the adopted convention, we have that actually $x$ is decreasing everywhere. \[\Box\]

We now give a result that will be essential to proving existence of solutions of (P). It asserts that any weakly convergent sequence in $X$ is in fact strongly convergent. Before doing that, we recall the following.

**Notations.** By $x_n \rightharpoonup x$ in $L^\infty$ we mean that $x_n, x \in L^\infty(0, A)$ and

$$\lim_{n \to \infty} \int_{0}^{A} [x_n(a) - x(a)] f(a) \, da = 0 \quad \forall f \in L^1(0, A).$$
LEMMA 2.2

Let \( \{x_n\} \) be a sequence in \( X \). If \( x_n \to x \) in \( L^\infty(0, A) \) then

(i) \( x \in X \), that is, \( x \) is decreasing.
(ii) \( x_n \to x \) in \( L^1 \).

Proof. (i) We know that \( \lim_{n \to \infty} \int_0^A [x_n(a) - x(a)] f(a) \, da = 0 \) \( \forall f \in L^1(0, A) \). In particular, it is true for all \( \phi' \in C_0^\infty(0, A) \) with \( \phi \geq 0 \). Therefore, we have

\[
\lim_{n \to \infty} \int_0^A x_n(a) \phi'(a) \, da = \int_0^A x(a) \phi'(a) \, da \geq 0.
\]

since \( x_n \) are decreasing and Lemma 2.1 holds. Thus, from Lemma 2.1 we conclude that \( x \) is decreasing.

(ii) Without loss of generality we assume that \( A = 1 \). Let \( N > 0 \) be an integer. We write (see Figure 2)

\[
[0, A] = [0, 1] = \bigcup_{k=0}^{N-1} \left( \frac{k}{N}, \frac{k+1}{N} \right).
\]

Since \( x_n \to x \) we have for every \( \epsilon > 0 \) that there exists an integer \( n_0 = n_0(\epsilon, N) \) such that

\[
\left| \int_{k/N}^{(k+1)/N} [x_n(a) - x(a)] \, da \right| \leq \frac{\epsilon}{N} \quad \forall k \in \{0, 1, \ldots, N-1\} \text{ and } \forall n \geq n_0.
\]

We assume from now on that \( n \geq n_0 \). For \( a \in (k/N, (k+1)/N) \), we must have

\[
x_n(a) \leq x \left( \frac{k-1}{N} \right) + 2\epsilon.
\]

To show (2), we proceed by contradiction. Assume that there exists an \( a \in (k/N, (k+1)/N) \) such that \( x_n(a) > x((k-1)/N) + 2\epsilon \). Therefore for every \( b \in ((k-1)/N, k/N) \), we have \( x_n(b) \geq x_n(a) > x((k-1)/N) + 2\epsilon \). From (1) we deduce

\[
\frac{\epsilon}{N} \geq \int_{(k-1)/N}^{k/N} [x_n(b) - x(b)] \, db \geq \int_{(k-1)/N}^{k/N} \left[ x \left( \frac{k-1}{N} \right) + 2\epsilon - x \left( \frac{k-1}{N} \right) \right] \, db \geq \frac{2\epsilon}{N},
\]

which is absurd.
In the same way, we can show if \( a \in (k/N, (k + 1)N) \), then

\[
x_n(a) \geq x\left(\frac{k + 2}{N}\right) - 2\varepsilon. \tag{3}
\]

Therefore for every \( a \in (k/N, (k + 1)N) \), we have

\[
x\left(\frac{k + 2}{N}\right) - x(a) - 2\varepsilon \leq x_n(a) - x(a) \leq x\left(\frac{k - 1}{N}\right) - x(a) + 2\varepsilon. \tag{4}
\]

Since \( x(a) \leq x((k - 1)/N) \) and \( x(a) \geq x((k + 2)/N) \), we deduce from (4) that

\[
x\left(\frac{k + 2}{N}\right) - x\left(\frac{k - 1}{N}\right) - 2\varepsilon \leq x_n(a) - x(a) \leq x\left(\frac{k - 1}{N}\right) - x\left(\frac{k + 2}{N}\right) + 2\varepsilon,
\]

which implies that

\[
|x_n(a) - x(a)| \leq x\left(\frac{k - 1}{N}\right) - x\left(\frac{k + 2}{N}\right) + 2\varepsilon \quad \forall k \in \{2, \ldots, N - 2\}. \tag{5}
\]
Since $x_n \rightharpoonup x$, there exists $K \geq 0$ such that $|x(a)| \leq K$ and $|x_n(a)| \leq K$ almost everywhere. Finally, we have from (5) that for every $n \geq n_0$

$$
\int_0^1 |x_n(a) - x(a)| \, da
$$

\begin{align*}
= & \int_0^{1/N} |x_n(a) - x(a)| \, da + \sum_{k=1}^{N-2} \int_{k/N}^{(k+1)/N} |x_n(a) - x(a)| \, da \\
+ & \int_{(N-1)/N}^1 |x_n(a) - x(a)| \, da \\
\leq & \frac{4K}{N} + 2\varepsilon \left(1 - \frac{2}{N}\right) + \sum_{k=1}^{N-2} \frac{1}{N} \left[ x\left(\frac{k-1}{N}\right) - x\left(\frac{k+2}{N}\right) \right] \\
= & \frac{4K}{N} + 2\varepsilon \left(1 - \frac{2}{N}\right) \\
+ & \frac{1}{N} \left[ x(0) + x\left(\frac{1}{N}\right) + x\left(\frac{2}{N}\right) - x\left(\frac{N-2}{N}\right) - x\left(\frac{N-1}{N}\right) - x(1) \right] \\
\leq & \frac{1}{N} \left[ 4K + 2\varepsilon (N-2) + 6x(0) \right] \\
\leq & \frac{1}{N} \left( 10K + 2\varepsilon N \right) \leq \frac{10K}{N} + 2\varepsilon .
\end{align*}

We have therefore proved that

$$
\lim_{n \to \infty} \int_0^A |x_n(a) - x(a)| \, da = 0,
$$

which is the claimed result (i.e., strong convergence in $L^1$).

We now turn our attention to problem (P). Recall that

$$
M = \sup_{x \in X} \{ J(x) \text{ with } G(x) \geq x(a) \ \forall a \in [0, A] \}. \quad (P)
$$

We make the following hypotheses on $J$ and $G$:

(H1)

$$
\exists x \in X \text{ such that } G(x) \geq x(a) \quad \forall a \in [0, A].
$$
[Note that if \( G(x = 0) = 0 \), then (H1) is automatically satisfied for \( x = 0 \).]  

(H2a)  
\[ G(x) \leq \alpha \|x\|_{L^\infty} + \beta, \quad \forall x \in X \text{ and for some } 0 \leq \alpha < 1, \beta \geq 0. \]

or  

(H2b)  
\[ J(x) \leq -\gamma \|x\|_{L^\infty} + \delta, \quad \forall x \in X \text{ and for some } \gamma > 0, \delta \geq 0. \]

**THEOREM 2.3**  

*Under the above hypotheses, problem (P) attains its maximum.*

**Proof.** The proof is divided into three steps.

**Step 1.** (H1) implies that the problem (P) is well-posed, that is, there is at least one admissible \( x \in X \) with \( G(x) \geq x(a), \forall a \in [0, A] \).

**Step 2.** Let \( \{x_n\} \) be a maximizing sequence of (P), that is, \( x_n \in L^\infty \) is decreasing, positive, \( G(x_n) \geq x_n(a) \forall a \in [0, A] \) and \( J(x_n) \to M \) as \( n \to \infty \). We now show that with either of the hypotheses (H2), \( \{x_n\} \) is compact in \( X \).

**Step 2a.** Assume that (H2a) holds. Then by hypothesis we have  
\[ x_n(a) \leq G(x_n) \leq \alpha \|x_n\|_{L^\infty} + \beta \quad \forall a \in [0, A]. \]

Thus  
\[ \|x_n\|_{L^\infty}(1 - \alpha) \leq \beta. \]

Since \( \alpha < 1 \), we deduce that  
\[ \|x_n\|_{L^\infty} \leq \frac{\beta}{1 - \alpha} = K. \quad (2a) \]

**Step 2b.** If instead of (H2a) we have (H2b), the proof is almost identical. Indeed, since \( J(x) \leq \delta < + \infty \), we have \( \sup J(x) - M \leq \delta < + \infty \). We have from (H2b) that  
\[ -|M| \leq J(x_n) \leq -\gamma \|x_n\|_{L^\infty} + \delta. \]
Hence

\[ \|x_n\|_{L^\infty} \leq \frac{\delta + |M|}{\gamma} = K \]  

(2b)

and thus is bounded.

**Step 3.** Using step 2 [(2a) or (2b)] we can therefore extract a subsequence \( \{x_{n_k}\} \) and find \( \bar{x} \in L^\infty \) such that \( x_{n_k} \to \bar{x} \) in \( L^\infty \). Since \( x_{n_k} \) are decreasing functions, we deduce from Lemma 2.2 that \( \bar{x} \) is decreasing and \( x_{n_k} \to \bar{x} \) in \( L^1 \). Since \( J \) is continuous with respect to the \( L^1 \) convergence, we obtain the claimed result:

\[ \lim_{k \to \infty} J(x_{n_k}) = J(\bar{x}) = M. \]

3. ECOLOGICAL EXAMPLES

Several examples enter into the general framework of the preceding section. As already mentioned, the problem of finding the harvesting strategy that will maximize the yield and maintain the population in a steady state can be put into the formalism examined in the preceding section.

Let us first introduce the notations. We follow mostly those of Getz and Haight [6], as modified by Arditi and Dacorogna [1] to the continuous setting of the model. We handle only the steady-state problem. We recall that \( A \) denotes the maximal age that any individual of the given population can reach. We also recall that \( a \in [0, A] \) denotes the age of an individual. The state variable \( x(a) \) is the number of individuals of age \( a \), \( b(a) \) is the rate at which individuals of age \( a \) reproduce, and \( \mu(a) \), is the rate at which individuals of age \( a \) die.

We first present the linear evolution model. We assume that the equation describing the evolution of the population and the yield are linear. For fecundity, we examine three cases motivated by three different ways of introducing competition. The action of harvesting is represented by the function \( u(a) \), the number of individuals of age \( a \) that are removed at time \( t \). We also define the function \( h(a) = u(a)/x(a) \) as the removal rate. It is the additional mortality rate due to harvesting. Thereafter, it will also be called the control. Then the population dynamics is described by the linear differential equation

\[ \frac{dx}{da}(a) = -[\mu(a) + h(a)]x(a), \]

and the yield is defined by

\[ J(h) = \int_0^A w(a)h(a)x(a) \, da, \]
where \( w(a) \) is a differentiable function that represents the value of an individual of age \( a \).

Since \( x(a)h(a) = -\mu(a)x(a) - x'(a) \), we have

\[
J(h) = \int_0^A \left[ -w(a)\mu(a)x(a) - w(a)x'(a) \right] da.
\]

Assume that \( w(0) = 0 \). Integrating by parts and using the fact that \( x(A) = 0 \), we can replace the yield by

\[
J(x) = \int_0^A \left[ w'(a) - \mu(a)w(a) \right] x(a) da.
\]

Remarks.

(1) In order to avoid distribution theory, it is preferable (as in Arditi and Dacorogna [1]) to replace the problem max\{\( J(h) = \int_0^A w(a)h(a)x(a) da \)\}, where \( h \) is a sum of “Dirac masses,” by max\{\( J(x) = \int_0^A [w'(a) - \mu(a)w(a)]x(a) da \)\}, since \( J(x) \) is defined for piecewise continuous \( x \). Note that the two problems are equivalent if the functions belong to appropriate spaces.

(2) The number of newborns \( x(0) \) is given, for instance, by

\[
x(0) = G(x),
\]

where the functional \( G \) represents the density dependence. It will be defined more precisely in the examples.

(3) Let \( H(a) = \int_0^a h(\alpha) d\alpha \) and \( l(a) = \exp\left[-\int_0^a \mu(\alpha) d\alpha\right] \); then the solution of the linear differential equation is given by

\[
\frac{x(a)}{l(a)} - \frac{x(0)}{l(0)} \exp\left[-H(a)\right].
\]

Since \( h \geq 0 \) (i.e., no stocking) the function \( x/l \) is decreasing. To use results of Section 2 about decreasing functions, we replace the function \( x/l \) by \( x \). This means that the solution \( x \) we give below has to be interpreted as \( x/l \).

Let \( f(a) = [w'(a) - \mu(a)w(a)]l(a) \). Then the problem can be written as follows:

\[
sup \left\{ J(x) = \int_0^A f(a)x(a) da \text{ with } x(0) = G(x), \right. \]

\[
x \text{ decreasing and } x \geq 0 \right\}. \tag{P}
\]
For the regularity of this problem, we suppose that $f$ is piecewise continuous and $x$ is a function in $X = \{ x \in L^2(0, A) \text{ with } x \text{ positive and decreasing} \}$.

(4) Of course, our theorem will apply to much more general problems, for example, to nonlinear population dynamics such as

$$x'(a) = -\left[ k(a, \mu(a), x(a)) + h(a) \right] x(a)$$

and yield such as

$$J(h) = \int_0^A w(a) h(a) x(a) \, da$$

$$= \int_0^A w(a) \left[ -x'(a) - k(a, \mu(a), x(a)) x(a) \right] \, da$$

$$= \int_0^A \left[ w'(a) - k(a, \mu(a), x(a)) w(a) \right] x(a) \, da = J(x),$$

since we supposed that $w(0) = 0$ and $x(A) = 0$.

Theorem 2.3 (with appropriate conditions) applies to all these cases and gives the existence of an optimal solution of (P). However, in the case of linear population dynamics, we can give qualitative properties of optimal solutions. In particular, if the competition appears in a way such as in Cases 1 and 2, the optimal solution is the so-called two-age policy, which consists in harvesting two age classes only: a younger age class is partially harvested, and an older age class is completely harvested.

We now discuss three cases in detail.

CASE 1. DENSITY DEPENDENCE DUE TO COMPETITION AMONG OFFSPRING

If we suppose that fecundity of individuals is linear and that competition occurs only among eggs and among the newborn (or number of "potential individuals"), we can write

$$x(0) = G(x) = g \left( \int_0^A b(a) x(a) \, da \right).$$

With this assumption, the problem we consider is the following:

$$\sup_{x \in X} \left\{ J(x) = \int_0^A f(a) x(a) \, da \text{ with } G(x) \right\}$$

$$= g \left( \int_0^A b(a) x(a) \, da \right) \geq x(a) \, \forall a \in [0, A], \quad (P1)$$
where \( f \) and \( b \) are piecewise continuous functions such that \( b \geq 0 \), \( g \) is continuous, and \( g(t) \leq \alpha t + \beta \) with \( \alpha \int_0^1 b(a)\,da < 1 \) and \( \beta \geq 0 \).

Remarks.

(1) Observe that this restriction on \( g \) is not very strong. It excludes, however, the linear case (see below). All the classical nonlinear functions that characterize this sort of density dependence satisfy this condition. In Getz and Haight [6], competition for recruitment is defined by a function \( \psi \), which in our notation is related to \( g \) by \( g(t) = t\psi(t) \). To be more precise, the recruitment function \( \psi \) is, for example:

- Beverton and Holt:
  \[
  \psi(t) = \frac{1}{1 + \lambda t},
  \]
  Ricker:
  \[
  \psi(t) = \exp(-\lambda t),
  \]
- Schaefer:
  \[
  \psi(t) = (1 - \lambda t),
  \]
- Power function:
  \[
  \psi(t) = t^{-\lambda},
  \]
- Depensation:
  \[
  \psi(t) = \frac{t}{1 + \lambda t^2},
  \]

where the parameter \( \lambda > 0 \).

Note that all these nonlinear models satisfy the condition

\[
\exists \beta > 0 \text{ such that } t\psi(t) \leq \beta \quad \forall t \geq 0
\]

and therefore satisfy our condition \( g(t) \leq \beta \) (i.e., \( \alpha = 0 \)).

(2) In the linear case \([g(t) = t]\), the optimal solution has two jumps (see Arditi and Dacorogna [1] or Rorres and Fair [10]). Note that the linear case cannot be put into our framework since \( g(t) = t \) does not satisfy in general the condition \( g(t) \leq \alpha t + \beta \) with \( \alpha \int_0^1 b(a)\,da < 1 \) since \( \int_0^1 b(a)\,da \geq 1 \) and \( \alpha = 1 \) in the linear case.

We now show that (P1) admits an optimal solution with two jumps.
THEOREM 3.1

The maximum of \((P1)\) is attained by a piecewise constant function that has at most two jumps.

Remark. If we compare these results with that of Getz and Haight [6], we find that they are qualitatively the same. The theorems given by Getz and Haight [6] or Beddington and Taylor [3] gave only necessary conditions and were restricted to the discrete case. Our theorem not only includes continuous as well as discrete cases but also shows that the necessary conditions are sufficient.

Before proving this theorem, we present a lemma that shows that the result of the theorem is true if we restrict \((P1)\) to piecewise constant functions.

Let \(Y = \{y: \mathbb{R} \to \mathbb{R} \mid y \text{ is piecewise constant, positive and decreasing} \} \subseteq X\). We then consider the problem

\[
\sup_{y \in Y} \{ J(y) : G(y) \geq y(0) \}. \tag{P1'}
\]

LEMMA 3.2

Under the hypotheses of Theorem 3.1, problem \((P1')\) attains its maximum, and this maximum has at most two jumps.

Remark. It is clear that \((P1')\) is equivalent to \((P1)\) if we restrict our study to functions that belong to \(Y\).

Proof. We decompose the proof into two steps.

Step 1. We first show that if \(y \in Y\) has \(N\) jumps with \(N \geq 3\), then we can find \(\tilde{y} \in Y\) with \(N - 1\) jumps and \(G(\tilde{y}) \geq \tilde{y}(0)\) such that

\[
J(y) \leq J(\tilde{y}).
\]

Let \(0 = a_0 < a_1 < \cdots < a_N = A\) and \(y_0 > y_1 > \cdots > y_{N-1} > y_N = 0\) be such that

\[
y(a) = \begin{cases} y_k & \text{if } a \in [a_k, a_{k+1}], \quad k = 0, \ldots, N - 1, \\ 0 & \text{if } a = A. \end{cases}
\]

Denote

\[
f_k = \int_{a_k}^{a_{k+1}} f(a) \, da \quad \text{and} \quad b_k = \int_{a_k}^{a_{k+1}} b(a) \, da, \quad k = 0, 1, \ldots, N - 1.
\]
MSY IN THE CONTINUOUS CASE

Suppose we can find $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$ not all equal to zero such that

(a) $f_1 \varepsilon_1 + f_2 \varepsilon_2 \geq 0,$
(b) $b_1 \varepsilon_1 + b_2 \varepsilon_2 = 0,$
(c) $y_0 \geq y_1 + \varepsilon_1 \geq y_2 + \varepsilon_2 \geq y_3$ with at least one of these inequalities an equality.

Then, setting $\tilde{y} = (y_0, y_1 + \varepsilon_1, y_2 + \varepsilon_2, y_3, \ldots, y_{N-1}, y_N = 0)$, we have immediately that $\tilde{y} \in Y$, $\tilde{y}$ has at most $N - 1$ jumps [from (c)], $G(\tilde{y}) \geq \tilde{y}(0)$ [from (b)], and $J(\tilde{y}) \geq J(y)$ [from (a)], thereby establishing step 1.

Let us now find $\varepsilon_1, \varepsilon_2$ with the claimed properties.

Assume that $b_1 > 0$ and $b_2 > 0$. If this is not true, for example, if $b_1 = 0$ (and similarly for $b_2 = 0$), choose $\varepsilon_2 = 0$ and $\varepsilon_1 = y_0 - y_1$ if $f_1 > 0$ or $\varepsilon_1 = y_2 - y_1$ if $f_1 < 0$. For the nontrivial case (i.e., $b_1 > 0$ and $b_2 > 0$), we divide the discussion of the choice of $\varepsilon_1$ and $\varepsilon_2$ into two cases.

(1) If $b_1 f_2 - b_2 f_1 \geq 0$, then choose

$$\varepsilon_1 = -\frac{b_2}{b_1 + b_2}(y_1 - y_2) \quad \text{and} \quad \varepsilon_2 = \frac{b_1}{b_1 + b_2}(y_1 - y_2).$$

We have clearly that

(a) $f_1 \varepsilon_1 + f_2 \varepsilon_2 = \frac{y_1 - y_2}{b_1 + b_2}(b_1 f_2 - b_2 f_1) \geq 0,$
(b) $b_1 \varepsilon_1 + b_2 \varepsilon_2 = 0,$
(c) $y_0 \geq y_1 + \varepsilon_1 = y_2 + \varepsilon_2 = \frac{b_1}{b_1 + b_2}y_1 + \frac{b_2}{b_1 + b_2}y_2 \geq y_3.$

(2) If $b_1 f_2 - b_2 f_1 < 0$, and if $(b_1 / b_2)(y_0 - y_1) \leq y_2 - y_3$, then choose

$$\varepsilon_1 = y_0 - y_1 \quad \text{and} \quad \varepsilon_2 = -\frac{b_1}{b_2}(y_0 - y_1).$$

if $(b_1 / b_2)(y_0 - y_1) > y_2 - y_3$, then choose

$$\varepsilon_1 = -\frac{b_2}{b_1}(y_3 - y_2) \quad \text{and} \quad \varepsilon_2 = (y_3 - y_2).$$

In these two cases, we can easily verify that (a), (b), and (c) are satisfied.

Step 2. In step 1 we have just shown that for every admissible $y \in Y$ with more than two jumps we can find (by iteration) $\tilde{y}$ admissible with two jumps so that

$$J(y) \leq J(\tilde{y}).$$
Therefore, if we now prove that among all admissible functions with two jumps there is a maximal one $\tilde{y}$, we shall have

$$J(y) \leq J(\tilde{y}) \leq J(\bar{y}) \quad \forall y \in Y,$$

thus establishing the lemma.

Therefore we consider a function $y \in Y$ with two jumps: $y(a) = y_0$ if $a \in [0, a_1]$, $y(a) = y_1$ if $a \in [a_1, a_2]$ and $y(a) = 0$ if $a \in [a_2, A]$ with $0 < a_1 \leq a_2 < A$ and $y_0 \geq y_1 \geq 0$. Problem (P1') can then be written as

$$\max J(y_0, y_1, a_1, a_2) = y_0 \int_0^{a_1} f(a) \, da + y_1 \int_{a_1}^{a_2} f(a) \, da$$

with

$$G(y_0, y_1, a_1, a_2) = g \left( y_0 \int_0^{a_1} b(a) \, da + y_1 \int_{a_1}^{a_2} b(a) \, da \right) \geq y_0.$$

Therefore from the hypotheses we obtain

$$y_0 \leq g \left( y_0 \int_0^{a_1} b(a) \, da + y_1 \int_{a_1}^{a_2} b(a) \, da \right) \leq \alpha \left( y_0 \int_0^{a_1} b(a) \, da + y_1 \int_{a_1}^{a_2} b(a) \, da \right) + \beta$$

$$\leq \alpha y_0 \int_0^{a_2} b(a) \, da + \beta \leq \alpha y_0 \int_0^{A} b(a) \, da + \beta,$$

which implies

$$y_0 \leq \frac{\beta}{1 - \alpha \int_0^{A} b(a) \, da}. \quad (6)$$

If we consider a maximizing sequence $\{(y_0)_n, (y_1)_n, (a_1)_n, (a_2)_n\}$ of (P1'), we deduce from (6) that up to the extraction of a subsequence we have $\{(y_0)_n, (y_1)_n, (a_1)_n, (a_2)_n\} \rightarrow \{(y_0, y_1, a_1, a_2)\}$. The hypotheses on $b, f,$ and $g$ imply then that $\{(y_0, y_1, a_1, a_2)\}$ realizes the maximum and thus the result.

We are now in a position to prove Theorem 3.1.

**Proof of the Theorem.** Let $x \in X$ be an optimal solution of (P1). The existence of this solution $x$ has been shown in the first section. Let
$n > 0$ be an integer. We can then choose a piecewise constant function $x_n$ such that $\|x_n - x\|_{L^1} \leq 1/n$ with $J(x) \leq J(x_n) + 1/n$ and $x_n$ is decreasing on $[0, A]$. Since $x_n \in Y$, there exists an integer $N > 0$ and $0 = a_{n,0} < a_{n,1} < \cdots < a_{n,N} = A$ such that

$$x_n(a) = \begin{cases} x_{n,k} & \text{for each } a \in [a_{n,k}, a_{n,k+1}] \\ 0 & \text{if } a = A. \end{cases}$$

For such $x_n$, we have

$$J(x_n) = \sum_{k=0}^{N-1} f_k x_{n,k}, \quad G(x_n) = g \left( \sum_{k=0}^{N-1} b_k x_{n,k} \right) \geq x_{n,0},$$

$$x_{n,k} - x_{n,k+1} \geq 0, \quad k \in \{0, \ldots, N-1\},$$

with

$$f_k = \int_{a_{n,k}}^{a_{n,k+1}} f(a) \, da \quad \text{and} \quad b_k = \int_{a_{n,k}}^{a_{n,k+1}} b(a) \, da.$$

Using Lemma 3.2, we deduce immediately that there exists $\bar{y} \in Y$ with two jumps so that $G(\bar{y}) \geq \bar{y}(0)$ and

$$J(\bar{y}) \geq J(x_n).$$

We therefore get

$$J(x) \leq J(x_n) + \frac{1}{n} \leq J(\bar{y}) + \frac{1}{n}.$$  

Letting $n \to \infty$, we have the conclusion. 

\textbf{CASE 2. COMPETITION AMONG THE WHOLE POPULATION}

If we suppose that fecundity depends on the density of the whole population, we can write

$$G(x) = g \left( \int_0^Ax(a) \, da \right) \int_0^Ab(a)x(a) \, da,$$

and the problem we consider is the following:

$$\sup_{x \in X} \left\{ J(x) = \int_0^Af(a)x(a) \, da \right\},$$

with

$$G(x) = g \left( \int_0^Ax(a) \, da \right) \int_0^Ab(a)x(a) \, da \geq x(a) \quad \forall a \in [0, A].$$

(P2)
where \( f \) and \( b \) are piecewise continuous functions such that \( b \geq 0 \) and \( g: [0, \infty[ \to \mathbb{R} \) is assumed to be continuous, \( \lim_{t \to \infty} g(t) = 0 \), and

\[
\exists \beta \geq 0 \text{ such that } 0 < g(t) \leq \frac{\beta}{t} \quad \forall t > 0.
\]

**Remark.** Note that this restriction on \( g \) is not very strong. For example, if we use the nonlinearity as in the preceding example (recruitment function \( \psi \) given by Beverton and Holt, Ricker, Schaefer, power function, or depensation), we can easily observe that all of them satisfy the constraint given above if we let \( g(t) = \psi(t) \).

**THEOREM 3.3**

The maximum of (P2) is attained by a piecewise constant function that has at most two jumps.

Note that the result of Theorem 3.3 has already been shown by Murphy and Smith [8]. However, it is interesting to see that our approach, although fundamentally different from theirs, leads to the same conclusion.

To establish Theorem 3.3 we first prove it for a discrete case.

Let \( Y = \{ y: [0, A] \to \mathbb{R} | y \text{ is piecewise constant, positive and decreasing} \} \subset X \). We then consider the problem

\[
\text{Problem (P2')} \quad \sup_{y \in Y} \{ J(y) : G(y) \geq y(0) \}.
\]

**LEMMA 3.4**

Under the above hypotheses on \( f, b, \) and \( g \), problem (P2') attains its maximum with a function \( y \in Y \) that has at most two jumps.

**Proof.** The proof is divided into two steps.

**Step 1.** We first show that if \( y \in Y \) has \( N \) jumps, with \( N \geq 3 \), then we can find \( \tilde{y} \in Y \) with \( N - 1 \) jumps and \( G(\tilde{y}) \geq \tilde{y}(0) \) such that

\[
J(y) < J(\tilde{y}).
\]

Let \( 0 = a_0 < a_1 < \cdots < a_N = A \) and \( y_0 > y_1 > \cdots > y_{N-1} > y_N = 0 \) be such that

\[
y(a) = \begin{cases} 
y_k & \text{if } a \in [a_k, a_{k+1}], \quad k = 0, \ldots, N-1, \\
o & \text{if } a = A.
\end{cases}
\]

Denote

\[
f_k = \int_{a_k}^{a_{k+1}} f(a) \, da \quad \text{and} \quad b_k = \int_{a_k}^{a_{k+1}} b(a) \, da, \quad k = 0, 1, \ldots, N - 1.
\]
Suppose we can find $\varepsilon_0$, $\varepsilon_1$, and $\varepsilon_2$ not all equal to zero such that

(a) $f_0\varepsilon_0 + f_1\varepsilon_1 + f_2\varepsilon_2 \geq 0$,

(b) $g(p)(b_0\varepsilon_0 + b_1\varepsilon_1 + b_2\varepsilon_2) - \varepsilon_0 \geq 0$, where $p = \sum_{k=0}^{N-1}(a_{k+1} - a_k)y_k$,

(c) $y_0 + \varepsilon_0 \geq y_1 + \varepsilon_1 \geq y_2 + \varepsilon_2 \geq y_3$ with at least one of these inequalities an equality,

(d) $\varepsilon_0(a_1 - a_0) + \varepsilon_1(a_2 - a_1) + \varepsilon_2(a_3 - a_2) = 0$.

Then setting $\tilde{y} = (y_0 + \varepsilon_0, y_1 + \varepsilon_1, y_2 + \varepsilon_2, y_3, \ldots, y_{N-1}, y_N = 0)$, we have immediately $\tilde{y} \in Y$, $\tilde{y}$ has at most $N - 1$ jumps [from (c)], $G(\tilde{y}) - \tilde{y}(0) \geq G(y) - y(0) \geq 0$ [from (b) and since from (d) we have $p = \sum_{k=0}^{N-1}(a_{k+1} - a_k)y_k = \sum_{k=0}^{N-1}(a_{k+1} - a_k)\tilde{y}_k$, and $J(\tilde{y}) \geq J(y)$ [from (a)], therefore establishing step 1.

Let us now find $\varepsilon_0$, $\varepsilon_1$, and $\varepsilon_2$ with the claimed properties. Let

$$\gamma_1 = \frac{a_2 - a_1}{a_1 - a_0} > 0, \quad \gamma_2 = \frac{a_3 - a_2}{a_1 - a_0} > 0,$$

$$\alpha_1 = f_1 - f_0\gamma_1, \quad \alpha_2 = f_2 - f_0\gamma_2,$$

$$\beta_1 = g(p)(b_1 - b_0\gamma_1) + \gamma_1, \quad \beta_2 = g(p)(b_2 - b_0\gamma_2) + \gamma_2.$$

Then, since $\varepsilon_0 = -(\gamma_1\varepsilon_1 + \gamma_2\varepsilon_2)$, we have to find $\varepsilon_1$ and $\varepsilon_2$ not all equal to zero such that

(a') $\alpha_1\varepsilon_1 + \alpha_2\varepsilon_2 \geq 0$,

(b') $\beta_1\varepsilon_1 + \beta_2\varepsilon_2 \geq 0$,

(c') $y_0 - (\gamma_1\varepsilon_1 + \gamma_2\varepsilon_2) \geq y_1 + \varepsilon_1 \geq y_2 + \varepsilon_2 \geq y_3$, with at least one of these inequalities an equality.

To find such $\varepsilon_1$ and $\varepsilon_2$, we consider five cases:

(i) $\alpha_1 \geq 0$ and $\beta_1 \geq 0$, then choose $\varepsilon_1 = (y_0 - y_1)/(1 + \gamma_1) > 0$ and $\varepsilon_2 = 0$.

(ii) $\alpha_1 \leq 0$ and $\beta_1 \leq 0$, then choose $\varepsilon_1 = y_2 - y_1 < 0$ and $\varepsilon_2 = 0$.

(iii) $\alpha_2 \geq 0$ and $\beta_2 \geq 0$, then choose $\varepsilon_1 = 0$ and $\varepsilon_2 = \min[(y_0 - y_1)/\gamma_2, y_1 - y_2] > 0$.

(iv) $\alpha_2 \leq 0$ and $\beta_2 \leq 0$, then choose $\varepsilon_1 = 0$ and $\varepsilon_2 = y_3 - y_2 < 0$.

(v) $\alpha_1 < 0$, $\beta_1 > 0$, $\alpha_2 > 0$, and $\beta_2 < 0$.

We represent conditions (a'), (b'), and (c') in Figure 3. Note that any $(\varepsilon_1, \varepsilon_2)$ in the shaded region is admissible.

The other cases—$(\alpha_1 < 0$, $\beta_1 > 0$, $\alpha_2 < 0$, and $\beta_2 > 0)$ or $(\alpha_1 > 0$, $\beta_1 < 0$, $\alpha_2 < 0$, and $\beta_2 > 0)$ or $(\alpha_1 > 0$, $\beta_1 < 0$, $\alpha_2 > 0$, and $\beta_2 < 0)$—are handled in the same way.

With the above discussion we conclude that it is always possible to find such $\varepsilon_1$ and $\varepsilon_2$ and consequently $\varepsilon_0$. 
Case where \( \alpha_1 \beta_2 - \alpha_2 \beta_1 \geq 0 \)

Case where \( \alpha_1 \beta_2 - \alpha_2 \beta_1 \leq 0 \)

FIG. 3. (a) Case where \( \alpha_1 \beta_2 - \alpha_2 \beta_1 > 0 \); (b) case where \( \alpha_1 \beta_2 - \alpha_2 \beta_1 \leq 0 \). (1) \( \alpha_1 \varepsilon_1 + \alpha_2 \varepsilon_2 = 0 \); (2) \( \beta_1 \varepsilon_1 + \beta_2 \varepsilon_2 = 0 \); (3) \( \varepsilon_2 \gamma_2 = -(1 + \gamma_1) \varepsilon_1 + y_0 - y_1 \); (4) \( \varepsilon_2 = \varepsilon_1 + y_1 - y_2 \); (5) \( \varepsilon_2 = y_3 - y_2 \).
**Step 2.** If we show that among all \( y \in Y \) with two jumps such that \( G(y) \geq y(0) \), we can find a maximal one \( \bar{y} \in Y \) with two jumps such that \( G(\bar{y}) \geq \bar{y}(0) \) and

\[
J(\bar{y}) \geq J(y) \quad \forall y \in Y,
\]

we will have, with the help of step 1, established the lemma. Therefore we consider a function \( y \in Y \) with two jumps: \( y(a) = y_0 \) if \( a \in [0, a_1] \), \( y(a) = y_1 \) if \( a \in [a_1, a_2] \), and \( y(a) = 0 \) if \( a \in [a_2, A] \) with \( 0 \leq a_1 \leq a_2 \leq A \) and \( y_0 \geq y_1 \geq 0 \). Problem \( (P2') \) can then be written as

\[
\sup \left\{ J(y_0, y_1, a_1, a_2) = y_0 \int_0^{a_1} f(a) \, da + y_1 \int_{a_1}^{a_2} f(a) \, da \right\}
\]

with \( G(y_0, y_1, a_1, a_2) = g(a_1 y_0 + (a_2 - a_1) y_1) \times \left( y_0 \int_0^{a_1} b(a) \, da + y_1 \int_{a_1}^{a_2} b(a) \, da \right) \geq y_0 \).

Therefore, from the hypotheses we obtain

\[
y_0 \leq g(a_1 y_0 + (a_2 - a_1) y_1) \left( y_0 \int_0^{a_1} b(a) \, da + y_1 \int_{a_1}^{a_2} b(a) \, da \right)
\]

\[
\leq \beta \frac{y_0 \int_0^{a_1} b(a) \, da + y_1 \int_{a_1}^{a_2} b(a) \, da}{a_1 y_0 + (a_2 - a_1) y_1},
\]

which implies

\[
y_0 \left[ a_1 y_0 + (a_2 - a_1) y_1 \right] \leq \beta \left[ y_0 \int_0^{a_1} b(a) \, da + y_1 \int_{a_1}^{a_2} b(a) \, da \right].
\]

Let \( B = \max_{0 \leq a \leq A} b(a) \). We therefore have

\[
y_0 \left[ a_1 y_0 + (a_2 - a_1) y_1 \right] \leq \beta B \left[ a_1 y_0 + (a_2 - a_1) y_1 \right],
\]

which implies

\[
y_0 \leq \beta B. \quad (7)
\]

Now letting \( \{ (y_0)_n, (y_1)_n, (a_0)_n, (a_1)_n \} \) be a maximizing sequence of \( (P2') \), we deduce from (7) that up to the extraction of a subsequence we have \( \{ (y_0)_n, (y_1)_n, (a_0)_n, (a_1)_n \} \rightarrow \{ (y_0, y_1, a_0, a_1) \} \). The hypotheses on \( b, f, \) and \( g \) imply then that \( \{ (y_0, y_1, a_0, a_1) \} \) realizes the maximum and thus the result.
The proof of Theorem 3.3 follows easily from Lemma 3.4 exactly as for Theorem 3.1 above, and we therefore omit it.

CASE 3. COMPETITION AMONG INDIVIDUALS OF IDENTICAL AGES

We now suppose that competition occurs only among individuals belonging to the same age class. The functional $G$ is then defined by

$$G(x) = \int_0^A b(a) g(x(a)) \, da.$$  

This sort of density dependence leads us to the problem

$$\sup_{x \in X} \left\{ J(x) = \int_0^A f(a) x(a) \, da \right\}$$

with $G(x) = \int_0^A b(a) g(x(a)) \, da \geq x(a) \, \forall a \in [0, A]$, (P3)

where $f$ and $b$ are piecewise continuous functions such that $b > 0$, $g$ is continuous and increasing and $g(t) \leq \alpha t + \beta$ with $\alpha \int_0^A b(a) \, da < 1$ and $\beta \geq 0$.

Since $f$ is piecewise continuous, there exists an integer $M > 0$ and $c_0 = 0 < c_1 < \cdots < c_M = A$ such that $f(a) \geq 0$ or $f(a) \leq 0$, $\forall a \in [c_l, c_{l+1}]$, $l = 0, 1, \ldots, M$ (see Figure 4).

![Figure 4](image-url)
THEOREM 3.5

The maximum of (P3) is attained. In addition, if \( f(a) \geq 0 \ \forall a \in [c_i, c_{i+1}] \), then the solution is constant on the interval \([c_i, c_{i+1}]\).

Remarks.

(1) In particular Theorem 3.5 implies that if \( f(a) \geq 0 \ \forall a \in [0, A] \), then the maximum of (P3) is attained by a constant function on \([0, A]\).

(2) In general, the solution of (P3) can have as many discontinuities as desired (this depends on the sign of \( f \)). In the case where \( f(a) \leq 0 \ \forall a \in [0, A] \), we can find an example in which the solution is attained only by a function that has a number of jumps as big as we want. We give such an example below.

Example. Let \( N \geq 4 \) and \( a_0 = 0 < a_1 = 1/N < \cdots < a_k = k/N < \cdots < a_N = A = 1 \). Let \( g(t) = t/(1 + t) \), and let \( f \) be such that

\[
\begin{align*}
&f_0 = 1, \quad \text{and} \quad f_k = \int_{k/N}^{(k+1)/N} f(a) \, da - \frac{k^2}{(k+1)^2}, \\
&k = 1, \ldots, N-1, \\
&b_k = \int_{k/N}^{(k+1)/N} b(a) \, da \quad \text{with} \quad b_0 = 0; \quad b_1 = \cdots = b_{N-1} = 1.
\end{align*}
\]

The discrete version of (P3) is then equivalent to

\[
\max_{x_0 \geq x_1 \geq \cdots \geq x_{N-1} \geq 0} \left\{ \sum_{k=0}^{N-1} f_k x_k \right\} \text{ with } \sum_{k=1}^{N-1} \frac{x_k}{x_k+1} \geq x_0. \quad (P3')
\]

It is clear that since \( f_0 = 1 \geq 0 \), we can still transform it to

\[
\max_{\sum_{k=1}^{N-1} \frac{x_k}{x_k+1} \geq x_1 \geq \cdots \geq x_{N-1} \geq 0} \left\{ \sum_{k=1}^{N-1} \left( \frac{x_k}{x_k+1} + f_k x_k \right) \right\} = F(x).
\]

The stationary points of \( F \) are given by

\[
\frac{1}{(x_k+1)^2} = -f_k = \frac{k^2}{(k+1)^2} \iff x_k = \frac{1}{k}.
\]

Since \( N \geq 4 \), we deduce that \( \sum_{k=1}^{N-1} (x_k/(x_k+1)) \geq x_1 \geq \cdots \geq x_{N-1} \geq 0. \) Combining this fact with the strict concavity of \( F \), we conclude that \( x_k = 1/k, \ k \geq 1, \) and \( x_0 = \sum_{k=1}^{N-1} 1/(1+k) \) is the unique solution of (P3'). It obviously has \( N \) jumps.
To prove the above theorem, as we did in Sections 3.1 and 3.2, we first consider problem (P3) in the discrete case. Let $N > 0$ be an integer. We then consider the set

$$Y_N = \{ y : [0, A] \rightarrow \mathbb{R} | y \text{ is a positive decreasing piecewise constant function and has at most } N \text{ jumps} \}$$

and the problem

$$\sup_{y \in Y_N} \{ J(y) : G(y) \geq y(0) \}. \quad (P3_N)$$

**LEMMA 3.6**

Under the hypotheses of Theorem 3.5, the maximum of $(P3_N)$ is attained, and this maximum is constant on the interval $[c_l, c_{l+1}]$ where $f(a) > 0$ for all $a \in [c_l, c_{l+1}]$.

**Proof.** The proof is divided into two steps.

**Step 1.** Let $I_l = [c_l, c_{l+1}]$ with $f(a) \geq 0$ for all $a \in I_l$. We first show that if $y \in Y_N$ with $y$ not constant on $I_l$, then we can find $\tilde{y}$ constant on $I_l$ and $G(\tilde{y}) \geq \tilde{y}(0)$ such that

$$J(y) \leq J(\tilde{y}).$$

Let $0 = a_0 < a_1 < \cdots < a_N = A$ and $y_0 > y_1 > \cdots > y_{N-1} > y_N = 0$ be such that

$$y(a) = \begin{cases} y_k & \text{if } a \in [a_k, a_{k+1}] ; \quad k = 0, \ldots, N-1, \\ 0 & \text{if } a = A. \end{cases}$$

Since $y$ is not constant on $I_l$, there exists $k_0 \in \{1, \ldots, N-1\}$ such that

$$c_l \leq a_{k_0-1} < a_{k_0} < a_{k_0+1} \leq c_{l+1} \quad \text{and} \quad y_{k_0-1} > y_{k_0}.$$

Let $\tilde{y} = (y_0, \ldots, y_{k_0-2}, y_{k_0}, y_{k_0}, y_{k_0+1}, \ldots, y_{N-1}, y_N = 0)$. Since $g$ is increasing, we deduce that $G(\tilde{y}) \geq G(y) \geq y(0) = \tilde{y}(0) = y_0$. Denote

$$f_k = \int_{a_k}^{a_{k+1}} f(a) \, da \quad \text{and} \quad b_k = \int_{a_k}^{a_{k+1}} b(a) \, da, \quad k = 0, 1, \ldots, N-1.$$

Clearly, $f_{k_0-1} \geq 0$ and $f_{k_0} > 0$, and finally we have the claimed property $J(y) \leq J(\tilde{y})$ since

$$f_{k_0-1} y_{k_0-1} + f_{k_0} y_{k_0} < f_{k_0-1} y_{k_0} + f_{k_0} y_{k_0}.$$
Step 2. To show that the maximum of $(P_{3,N})$ is attained, we have to show (since $y_0 \geq y_1 \geq \cdots \geq y_{N-1} \geq 0$) that $y_0$ is bounded independently of the subdivision $0 = a_0 < a_1 < \cdots < a_N = A$ of $[0, A]$.

Under the hypotheses of Theorem 3.5, we obtain

$$0 \leq \sum_{k=0}^{N-1} b_k g(y_k) - y_0 \leq \sum_{k=0}^{N-1} b_k g(y_0) - y_0$$

$$\leq g(y_0) \sum_{k=0}^{N-1} b_k - y_0 \leq (\alpha y_0 + \beta) \int_0^A b(a) \, da - y_0$$

$$= y_0 \left( \alpha \int_0^A b(a) \, da - 1 \right) + \beta \int_0^A b(a) \, da,$$

which implies that

$$0 \leq y_0 \leq \frac{\beta \int_0^A b(a) \, da}{1 - \alpha \int_0^A b(a) \, da}.$$

Combining this fact with step 1, we have indeed established the lemma.

The proof of the theorem is then almost identical to that of Theorem 3.1, and we therefore omit it.

4. DISCUSSION

In the first two examples, the optimal strategy is identical to the linear case studied by Arditi and Dacorogna [1]. It consists in harvesting individuals at exactly two ages: partial harvest at a younger age $a$ and total harvest at an older age $b$. The advantage of the nonlinearity due to competition is that it avoids the constraint on the size of the population. This constraint had to be introduced in the linear case; otherwise the optimal solution $x$ takes an infinite value. With density dependence, the evolution is in some sense more natural since it controls or regularizes the size of the population. In mathematical terms, we have shown that nonlinearity due to competition implies that $x$ is bounded.

On the other hand, some particularities appear in the third example if $f(a)$ is not always greater than zero. We recall that $f(a) = [w'(a) - \mu(a) w(a)] l(a)$ [i.e., $f$ is a function depending on the value $w(a)$ of an individual of age $a$ and the function $\mu(a)$, which is the rate of mortality]. As we have seen, the optimal solution can have an infinite number of jumps. When dealing with fisheries, the solution in this last case is hardly applicable, contrary to the two-age policy found in the two
preceding examples. However, for some other harvesting problems, it might be applicable if we decide to harvest a maximal fixed number of age classes that correspond to the jumps of the optimal solution. Note that the greater this number of possible harvested classes, the better the solution. Finally, observe that the hypothesis that says that competition between individuals occurs only among individuals of the same age is generally unrealistic. However, this situation can appear in a population in which the individuals live in a specific space determined by their age class.

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REFERENCES

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