I. INTRODUCTION:

In this article we are concerned with the following problem, find a diffeomorphism \( u: \tilde{\Omega} \rightarrow \tilde{\Omega} \), where \( \Omega \subset \mathbb{R}^n \) is a bounded open set, satisfying

\[
\begin{align*}
& \left\{ \begin{array}{ll}
g(u(x)) \det \nabla u(x) = f(x) & x \in \Omega \\
u(x) = x & x \in \partial \Omega
\end{array} \right. \tag{1.1}
\end{align*}
\]

where \( f, g > 0 \) in \( \tilde{\Omega} \) and satisfy the compatibility condition

\[
f_\Omega g(x) dx = f_\Omega f(x) dx. \tag{1.2}
\]

In geometrical terms the problem is to show that two volume elements of the same total volume (i.e. (1.2)) are diffeomorphically equivalent.

Let us state precisely the result.

**Theorem 1:**

Let \( k \geq 0 \) be an integer, \( 0 < \alpha < 1 \) and \( \Omega \subset \mathbb{R}^n \) a bounded open set with \( C^{k+3, \alpha} \) boundary \( \partial \Omega \) (\( C^{k, \alpha} \) denoting the usual Hölder space). Let \( f, g \in C^{k, \alpha}(\tilde{\Omega}) \), \( f, g > 0 \) in \( \tilde{\Omega} \) satisfying (1.2). Then there exists \( u \in \text{Diff}^{k+1, \alpha}(\tilde{\Omega}) \) (the set of diffeomorphisms \( u: \tilde{\Omega} \rightarrow \tilde{\Omega} \) with \( u, u^{-1} \in C^{k+1, \alpha}(\tilde{\Omega}, \tilde{\Omega}) \)) satisfying (1.1).
Remark:

It is clear that (1.1) is underdetermined and uniqueness of solutions does not hold. Indeed let \( f = g = 1, \ N \) an integer then
\[
u(x) = r(\cos(\theta + 2N\pi r^2), \sin(\theta + 2N\pi r^2)) \]
satisfy (1.1) when \( \Omega \) is the unit disk of \( \mathbb{R}^2 \) and \((r, \theta)\) are the polar coordinates.

The above result has been obtained by Dacorogna-Moser [1]. The first result concerning (1.1) was established by Moser [1] for manifolds without boundary and for \( k = \infty \). It was then extended by Banyaga [1] for manifolds with boundary and for \( k = \infty \). Zehnder [1] obtained Theorem 1 for manifolds without boundary and with a smallness assumption on the \( C^{0,\alpha} \) norm of \((f-g)\). Independently Tartar [1] and Dacorogna [1] proved this result for \( k = \infty \) and \( \Omega \) diffeomorphic to the unit ball of \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \) respectively.

In Dacorogna-Moser [1] it is also shown how to weaken the regularity of the boundary \( \partial \Omega \), so as to include Lipschitz boundaries as well as isolated boundary points (as in a punctured disk)... In this article it is shown that if \( k > 0, f, g \in C^k(\tilde{\Omega}) \) (\( f, g > 0 \) in \( \tilde{\Omega} \), \( f + \frac{1}{f} \) and \( g + \frac{1}{g} \) bounded) and satisfy (1.2), then there exists a diffeomorphism \( u \in \text{Diff}^k(\tilde{\Omega}) \cap \text{Diff}^0(\tilde{\Omega}) \) (where \( \text{Diff}^0(\tilde{\Omega}) \) stands for the set of homeomorphisms \( u: \tilde{\Omega} \to \tilde{\Omega} \) with \( u, u^{-1} \in C^0(\tilde{\Omega};\tilde{\Omega}) \)). If \( k = 0 \) and \( f, g \) are as above, it is proved that there exists \( u \in \text{Diff}^0(\tilde{\Omega}) \) satisfying a weak form of (1.1) namely

\[
\int_E f(x) dx = \int_E g(x) dx \quad (1.3)
\]

for every open set \( E \subset \Omega \) and \( u(x) = x \) on \( \partial \Omega \). Note that this result is weaker, from the point of view of regularity, than Theorem 1, since the expected gain in regularity is not ensured.

Theorem 1 has several applications. The first one is that we can construct a volume preserving diffeomorphism with given boundary data. More precisely let \( \psi_0 \in \text{Diff}^{k,0}(\tilde{\Omega}) \) with \( k \geq 1 \), then there exists \( \psi \in \text{Diff}^{k,0}(\tilde{\Omega}) \) satisfying

\[
\begin{align*}
\det \nabla \psi & = 1 \quad \text{in} \ \Omega \\
\psi & = \psi_0 \quad \text{on} \ \partial \Omega.
\end{align*}
\]

(1.4)
With the above hypotheses (1.4) is equivalent to (1.1) and the existence is therefore deduced from Theorem 1.

The construction of volume preserving diffeomorphism plays a role in ergodic theory (c.f. Alpern [1], Anosov-Katok [1]) and is important in incompressible elasticity (c.f. Ball [1], Dacorogna [2]). It can also be used in the calculus of variations and in problems of equilibrium of gases (c.f. Dacorogna [1],[2]). For example if one looks for solutions of

$$\inf \{ \int_{\Omega} \phi(\text{det} V u(x)) dx : u = u_0 \text{ on } \partial \Omega, \, u \in \text{Diff}^{k,\alpha}(\tilde{\Omega}) \}$$

where $\phi: \mathbb{R} \to \mathbb{R}$ is convex and $u_0 \in \text{Diff}^{k,\alpha}(\tilde{\Omega})$ with $k \geq 1$. Indeed it follows from Jensen inequality that

$$\int_{\Omega} \phi(\text{det} V u(x)) dx \geq \text{meas} \Omega \, \phi(\frac{1}{\text{meas} \Omega} \int_{\Omega} \text{det} V u(x) dx) = \text{meas} \Omega \, \phi(\frac{1}{\text{meas} \Omega} \int_{\Omega} \text{det} V u_0(x) dx) \text{meas} \Omega \, \phi(1);$$

the last identity following from the fact that $u_0(\Omega) = \tilde{\Omega}$. Therefore using the fact that there exists $\tilde{u} \in \text{Diff}^{k,\alpha}(\tilde{\Omega})$ satisfying (1.4) we find, because of the above inequality, that $\tilde{u}$ is actually a minimum of (P).

Before concluding with some open problems, let us mention an interesting result for the corresponding linear problem, which will be used for obtaining Theorem 1.

**Theorem 2:**

Let $k \geq 0$ be an integer, $0 < \alpha < 1$ and $\Omega \subset \mathbb{R}^n$ a bounded open set with $C^{k+3,\alpha}$ boundary $\partial \Omega$. Let $f \in C^{k,\alpha}(\tilde{\Omega})$ with

$$\int_{\Omega} f(x) dx = 0. \quad (1.5)$$

Then there exists $u \in C^{k+4,\alpha}(\tilde{\Omega}; \mathbb{R}^n)$ satisfying
\[ \begin{cases} \text{div } u = f \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial \Omega. \end{cases} \quad (1.6) \]

Remark:

Several authors have proved Theorem 2 under some slightly different conditions, however as stated above it has been established by Dacorogna-Moser [1].

We now state three open problems

**Problem 1:**

Let \( \Omega \) be \( C^\infty \) smooth, \( k \geq 0 \), \( f \in C^k(\tilde{\Omega}) \), \( f > 0 \) in \( \tilde{\Omega} \) satisfying (1.2) with \( g \equiv 1 \). Does there exist \( u \in \text{Diff}^{k+1}(\tilde{\Omega}) \) satisfying (1.1)?

**Problem 2:**

Let \( \Omega \) be \( C^\infty \) smooth, \( k \geq 1 \), \( f \in C^k(\tilde{\Omega}) \) satisfying (1.5). Does there exist \( u \in C^{k+1}(\tilde{\Omega}; \mathbb{R}^n) \) satisfying (1.6)?

**Problem 3:**

Let \( \Omega \) be \( C^\infty \)-smooth, \( k \geq 0 \), \( p > 1 \), \( f \in W^{k,p}(\Omega) \) (the usual Sobolev space), \( f(x) \geq C_0 > 0 \) for almost all \( x \in \Omega \) and \( f \) satisfying (1.2) with \( g \equiv 1 \). Does there exist \( u \in W^{k+1,p}(\Omega) \) satisfying (1.1)?

**Remarks**

i) Note that \( C^k \) as well as \( C^{k,\alpha} \) spaces have the advantage over \( L^p \) spaces to form an algebra. In that sense one has, when working with Sobolev
spaces with low \( k \) and \( p \), to ask for solution in \( W^{k+1, np} \) instead of \( W^{k+1, p} \) (this last case can be treated by the method developed below).

ii) The third conjecture is false when \( k = 0 \) and \( p = 1 \). Indeed Müller [1] and Coifman-Lions-Meyer-Semmes [1] have shown that given any \( u \in W^{1,k}(\Omega) \) with \( \det Vu \geq 0 \) a.e. then \( \det Vu \log(2+\det Vu) \in L^1_{\text{loc}}(\Omega) \).

II. PROOF OF THEOREM 1 AND 2

We now sketch the proofs of the theorems and we refer for more details to Dacorogna-Moser [1]. We first assume that Theorem 2 has been proved.

Proof of Theorem 1:

The proof is divided in four steps.

Step 1: We first show that there is no loss of generality in assuming \( g \equiv 1 \) in (1.1). Indeed writing

\[
\begin{align*}
\det V \phi(x) &= \frac{\text{meas } \Omega}{\int_{\Omega} f(x) dx} f(x) \text{ in } \Omega \\
\det V \psi(x) &= \frac{\text{meas } \Omega}{\int_{\Omega} g(x) dx} g(x) \text{ in } \Omega \\
\phi(x) &= \psi(x) = x \quad \text{on } \partial \Omega
\end{align*}
\]

(2.1)

we have that \( u = \psi^{-1} \circ \phi \) satisfies (1.1).

Step 2: We then obtain Theorem 1 without the regularity result, i.e. we find \( u \in \text{Diff}^k, a(\bar{\Omega}) \) (instead of \( \text{Diff}^{k+1}, a(\bar{\Omega}) \)) satisfying (1.1) with \( g \equiv 1 \). This is obtained by a deformation argument explained below.
Step 3: Using Theorem 2 and a smallness assumption on the $C^{0,\beta}$ norm $0 < \beta < \alpha < 1$ of $f^{-1}$ we obtain by a fixed point argument a $\text{Diff}^{k+1,\alpha}$ solution of (1.1).

Step 4: We remove the smallness assumption in Step 3 by combining two diffeomorphisms one constructed via Step 2 the other one via Step 3.

We now prove Step 2.

Step 2: Let $v \in C^{k+1,\alpha}(\tilde{\Omega};\mathbb{R}^n)$ satisfying

\[
\begin{aligned}
\text{div } v &= f^{-1} \quad \text{in } \Omega \\
v &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

(2.2)

Such a $v$ exists by Theorem 2 and the fact that $g \equiv 1$ in (1.2) i.e.

\[
\int_{\tilde{\Omega}} (f^{-1}) \, dx = 0.
\]

Assume that $k \geq 1$. Define then a vector field

\[
v(t, z) = \frac{v(z)}{t+(1-t)f(z)}.
\]

(2.3)

Observe that $v(t, \cdot) \in C^{k,\alpha}$ (and not in $C^{k+1,\alpha}$). We then define the flow associated to this vector field as

\[
\begin{aligned}
\frac{\partial}{\partial t} \Phi(t, x) &= v(t, \Phi(t, x)) \quad t > 0 \\
\Phi(0, x) &= x.
\end{aligned}
\]

(2.4)

Note that because of the assumptions there exists a unique solution \( \Phi: [0,1] \times \tilde{\Omega} \rightarrow \mathbb{R}^n \) in $\text{Diff}^{k,\alpha}$ satisfying (2.4). We claim that $u(x) = \Phi(1, x)$ has all the desired properties, i.e., $u \in \text{Diff}^{k,\alpha}(\tilde{\Omega})$ and

\[
\begin{aligned}
\text{det} \nabla u(x) &= f(x), \quad x \in \Omega \\
u(x) &= x \quad \text{in } \partial \Omega.
\end{aligned}
\]

(2.5)
The aim of the next two steps will be to obtain $u \in \text{Diff}^{k+1,\alpha}$ instead of $\text{Diff}^{k,\alpha}$.

First observe that the fact that $u(x) = x$ on $\partial \Omega$ follows at once from the unicity of solutions of (2.4) and of the fact that $v(t,x) = 0$ if $x \in \partial \Omega$ (since $v = 0$ on $\partial \Omega$, c.f., (2.2)). It therefore remains to show that $\det \nabla u = f$ in $\Omega$. This can be deduced from the fact that the following quantity

$$h(t,x) \equiv \det \nabla \phi(t,x)[t + (1-t)f(\phi(t,x))]$$

(2.6)

is independent of $t$ (c.f. below for a proof). The result follows then from the fact that $h(1,x) = h(0,x)$. To prove that (2.6) is independent of $t$ is obtained from (2.2) and the fact that the solution of (2.4) satisfies (as well known)

$$\frac{\partial}{\partial t} [\det \nabla \phi(t,x)] = \det \nabla \phi(t,x) \text{div} v(t,\phi(t,x)).$$

We refer for more details to Dacorogna-Moser [1].

**Step 3:** A classical fixed point argument allows to deduce from Theorem 2 the following: let $0 < \beta \leq \alpha < 1$, then there exists $\varepsilon = \varepsilon(\alpha, \beta, k, \Omega) > 0$ such that if $\|f - 1\|_{C^0, \beta} \leq \varepsilon$ then there exists $u \in \text{Diff}^{k+1,\alpha}(\tilde{\Omega})$ satisfying (2.5).

**Step 4:** We now are in a position to conclude the proof of the theorem. By density of $C^\infty$ functions in $C^{k,\alpha}$ with the $C^{0,\beta}$ norm $0 < \beta < \alpha < 1$ (in general if $k = 0$ density does not occur in $C^{0,\alpha}$ norm) we can find $\tilde{f} \in C^\infty(\tilde{\Omega})$, $\tilde{f} > 0$ on $\tilde{\Omega}$ such that

$$\|\frac{f}{\tilde{f}} - 1\|_{C^{0,\beta}} \leq \varepsilon$$

(2.7)

$$\int_{\Omega} \frac{f(x)}{\tilde{f}(x)} \, dx = \text{meas} \, \Omega$$

(2.8)

where $\varepsilon > 0$ is as in Step 3.
We then define $b \in \text{Diff}^{k+1, \alpha}(	ilde{\Omega})$ to be a solution (which exists by Step 3, (2.7) and (2.8)) of
\[
\begin{cases}
\det \nabla b(x) = \frac{f(x)}{\tilde{f}(x)} & \text{in } \Omega \\
b(x) = x & \text{on } \partial \Omega.
\end{cases}
\quad (2.9)
\]

We then define $a \in \text{Diff}^{k+1, \alpha}(	ilde{\Omega})$ to be a solution (which exists by Step 2, since $f \circ b^{-1} \in C^{k+1, \alpha}$) of
\[
\begin{cases}
\det \nabla a(y) = \tilde{f}(b^{-1}(y)) & y \in \Omega \\
a(y) = y & y \in \partial \Omega.
\end{cases}
\quad (2.10)
\]

Note that the compatibility condition is satisfied since
\[
\int_{\Omega} \tilde{f}(b^{-1}(y)) dy = \int_{\Omega} \tilde{f}(x) \det \nabla b(x) dx = \int f(x) dx = \text{meas} \Omega.
\]

Finally we get the desired diffeomorphism by setting $u = a \circ b$. 

Before sketching the proof of Theorem 2, we introduce some notations.

**Notations:**

Let $w \in C^1(\mathbb{R}^n; \mathbb{R}^{n(n-1)/2})$ with $w = (w_{ij})_{1 \leq i < j \leq n}$. We also let $w_{ij} = -w_{ji}$ if $i \geq j$. We then define
\[
\text{curl}^w = ((\text{curl}^w)_{ij})_{1 \leq i < j \leq n} \in \mathbb{R}^n
\]

by
\[
(\text{curl}^w)_{ij} = \sum_{i=1}^{n} (-1)^{i+j} \frac{\partial w_{ij}}{\partial x_i}.
\]

**Remarks:**

i) If $n = 2$ then 
\[
\text{curl}^w = \left( \frac{\partial w}{\partial x_2}, -\frac{\partial w}{\partial x_1} \right)
\]
and if \( n = 3 \) then \( \text{curl}^*w = \text{curl}w \) (the usual curl).

ii) In terms of differential forms \( (\text{curl}^*w)_j \) are just the components of \( d\alpha \) where \( \alpha \) is an \( (n-2) \) form with components \( w_{ij} \).

iii) One always have

\[
\text{div}(\text{curl}^*w) = 0,
\]

which corresponds to \( \text{dd}\alpha = 0 \).

**Proof of Theorem 2:**

The idea for finding \( u \) satisfying

\[
\begin{aligned}
\text{div} \mathbf{u} &= f \text{ in } \Omega \\
u &= 0 \text{ on } \partial \Omega
\end{aligned}
\]

is to seek for solutions of the form

\[
u = \text{grad} v + \text{curl}^*w.
\]

We therefore solve (\( \nu \) being the outward unit normal)

\[
\begin{aligned}
\Delta \nu &= f \text{ in } \Omega \\
\frac{\partial \nu}{\partial n} &= 0 \text{ on } \partial \Omega
\end{aligned}
\]

which, provided \( \int f = 0 \), has a unique \( C^{k+2,\alpha} \) solution by the classical results. In view of (2.10) and (2.11) it therefore remains to find \( w \in C^{k+2,\alpha} \) satisfying

\[
\text{curl}^*w = -\text{grad} v \text{ on } \partial\Omega
\]

knowing that \( v \in C^{k+2,\alpha} \) and
Therefore Theorem 2 is reduced to finding $w \in C^{k+2,\alpha}$ satisfying

$$\text{curl}^*w = g \text{ on } \partial \Omega$$

(2.12)

knowing that $g \in C^{k+1,\alpha}$ with

$$\langle g; v \rangle = 0 \text{ on } \partial \Omega$$

(2.13)

where $\langle \cdot ; \cdot \rangle$ stands for scalar product in $\mathbb{R}^n$.

The equations (2.12) can be further reduced to solving

$$\text{grad} w_{ij} = (-1)^{i+j} (g_j v_i - g_i v_j) \text{ on } \partial \Omega.$$  

(2.14)

Indeed if the above holds on $\partial \Omega$ we have, recalling that $w_{ij} = -w_{ji}$,

$$(\text{curl}^*w)_j = \sum_{i=1}^{n} (-1)^{i+j} \frac{\partial w_{ij}}{\partial x_i} = \sum_{i=1}^{n} (g_j v_i - g_i v_j) v_i = g_j |v|^2 - \langle g; v \rangle v_j = g_j$$

where we have used (2.13) as well as the fact that $|v| = 1$.

We therefore have reduced the problem to finding $w_{ij} \in C^{k+2,\alpha}$ satisfying

$$\text{grad} w_{ij} = g_{ij} v \text{ on } \partial \Omega$$

(2.15)

given $g_{ij} \in C^{k+1,\alpha}$ (here $g_{ij} = (-1)^{i+j} (g_j v_i - g_i v_j)$). We show how to solve (2.15) and find $w_{ij} \in C^{k+1,\alpha}$ (to obtain a more regular solution in $C^{k+2,\alpha}$ we refer to Dacorogna-Moser [1]). Observing that the gradient of the distance $d(x, \partial \Omega)$ is $-v$ whenever $x \in \partial \Omega$ we obtain a solution by setting

$$w_{ij} = -g_{ij} \phi(d(x, \partial \Omega))$$

where $\phi(0) = 0$, $\phi'(0) = 1$, $\phi \equiv 0$ outside a small neighbourhood of 0 and $\phi \in C^\infty$. \qed
ACKNOWLEDGMENTS

I would like to thank M. Chipot and J. Saint Jean Paulin for organizing this nice conference in Metz.

REFERENCES

- S. Alpern [1]:
  New proofs that weak mixing is generic; Inventiones Math., Vol.32 (1976) 263-279.

- D.V. Anosov; A.B. Katok [1]:

- J.M. Ball [1]:
  Convexity conditions and existence theorems in nonlinear elasticity;

- A. Banyaga [1]:

- R. Coifman; P.L. Lions; Y. Meyer; S. Semmes [1]:

- B. Dacorogna [1]:
  A relaxation theorem and its application to the equilibrium of gases;

- B. Dacorogna [2]:

- B. Dacorogna; J. Moser [1]:
  On a partial differential equation involving the Jacobian determinant;

- J. Moser [1]:

- S. Müller [1]:

- L. Tartar [1]:
  Private communication (1979).

- E. Zehnder [1]: