1. - Introduction.

In this report we deal with the convexity of the integral

\[ I(u) = \int_0^1 f(x, u(x), u'(x)) \, dx \]  

where \( f : (0, 1) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is \( C^2 \). We seek conditions on \( f \) which ensure the convexity of the integral \( I \) over the space \( W^{1, \infty}(0, 1) \) (the space of Lipschitz functions vanishing at 0 and 1). We shall show the following.

Necessary condition.

If \( I \) is convex over \( W^{1, \infty}(0, 1) \), then \( f(x, u, \cdot) \) is convex (i.e., \( f \) is convex with respect to the last variable).

We then study three examples

i) Let \( f(x, u, \xi) = a(u) \xi^n \), with \( n \geq 1 \), an integer, then \( I \) is convex over \( W^{1, \infty}_0(0, 1) \) if \( a(u) = \text{constant} \).

ii) Let \( f(x, u, \xi) = a(u) + b(\xi) \), then \( I \) is convex over \( W^{1, \infty}_0(0, 1) \)

\[ \Leftrightarrow \begin{align*} 
1) & \quad b'(\xi) \geq 0, \text{ for every } \xi \in \mathbb{R} \\
2) & \quad \xi^2 b''(\xi) + a''(u) \geq 0 \text{ for every } (u, \xi) \in \mathbb{R}^2. 
\end{align*} \]

iii) Let \( f(x, u, \xi) = a(x) \xi^2 + 2b(x) u \xi + c(x) u^2 \), with \( a(x) > 0 \) then \( I \) is convex over \( W^{1, \infty}_0(0, 1) \) (or over \( H^1_0(0, 1) \)) if and only if there exists

\[ (*) \]  

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$g \in C^1(0,1)$ satisfying Jacobi equation, i.e.

$$a(x)[g'(x) + c(x)] - [b(x) + g(x)]^2 = 0.$$  

(This means that there is no conjugate point in $(0,1)$.)

Furthermore in the three cases (trivially in the first one) one has that the following conjecture is true.

**Conjecture.** $I$ is convex over $W^{1,\infty}_0(0,1)$ if and only if there exists $\tilde{f} : (0,1) \times R \times R \to R$ with $\tilde{f}(x, \cdot, \cdot)$ convex for every $x \in (0,1)$ (i.e. $\tilde{f}$ is convex with respect to the last two variables) satisfying

$$I(u) = \int_0^1 \tilde{f}(x,u(x),u'(x)) \, dx$$

for every $u \in W^{1,\infty}_0(0,1)$ with compact support in $(0,1)$.

In the last section we discuss the higher dimensional case, i.e.,

$$I(u) = \int_\Omega f(x,u(x),\nabla u(x)) \, dx$$

where $u : \Omega \subset R^n \to R^m$ and $f : \Omega \times R^m \times R^{nm} \to R$. We show that if $m = 1$, the necessary condition is still true while if $m, n > 1$, both the necessary condition and the conjecture are false.

Finally one should mention a closely related problem studied by Boccardo-Dacorogna [BD] where the monotonicity of operators of the form

$$A(u) = -(a(x,u(x),u'(x)))'$$

is investigated.

2. - Main results.

We begin with the necessary condition.

**Theorem 1.** Let $f : (0,1) \times R \times R \to R$ be continuous and such that

$$|f(x,u,\xi)| \leq a(x,|u|,|\xi|)$$

where $a$ is increasing with respect to $|u|$ and $|\xi|$ and locally integrable in $x$. If $I$ is convex over $W^{1,\infty}_0(0,1)$ then $f(x,u,\cdot)$ is convex.
**Proof.** Under the above hypotheses, a direct application of Mazur lemma gives that $I$ is in fact weak* lower semicontinuous in $W^{1,m}_0$ and thus that $f(x,u,\cdot)$ is convex. (For more details see Dacorogna [D2]).

We now turn to necessary and sufficient conditions.

**Theorem 2.** Let $a \in C^2(\mathbb{R})$ be such that
\[ a(u) \equiv a_0 > 0, \quad \text{for every } u \in \mathbb{R}. \]
Let
\[ f(x,u,\xi) = a(u)\xi^2 \]
with $n \geq 1$, an integer. Then $I$ is convex over $W^{1,m}_0(0,1)$ if and only if $a$ is constant.

**Theorem 3.** Let $a, b \in C^2(\mathbb{R})$ and
\[ f(x,u,\xi) = a(u) + b(\xi), \]
$I$ is convex over $W^{1,m}_0(0,1)$ if and only if
i) $b''(\xi) \geq 0$ for every $\xi \in \mathbb{R}$,
ii) $\pi^2 b''(\xi) + a''(u) \geq 0$ for every $(u,\xi) \in \mathbb{R}^2$.

Furthermore let
\[ a_0 = \inf \{ a''(u): u \in \mathbb{R} \}, \quad b_0 = \inf \{ b''(\xi): \xi \in \mathbb{R} \}, \]
\[ g(x) = \begin{cases} 0, & \text{if } a_0 \equiv 0, \\ \frac{1}{2} \sqrt{-a_0 b_0} \tan \left[ \sqrt{-a_0 b_0} \left( x - \frac{1}{2} \right) \right], & \text{if } a_0 < 0. \end{cases} \]
\[ \bar{f}(x,u,\xi) = f(x,u,\xi) + 2g(x)u\xi + g'(x)u^2. \]
If $\pi^2 b_0 + a_0 \geq 0$ (i.e. $I$ is convex) then $\bar{f}(x,\cdot,\cdot)$ is convex and satisfies
\[ I(u) = \int_0^1 \bar{f}(x,u(x),u'(x)) \, dx \]
for every $u \in W^{1,m}_0(0,1)$ with compact support in $(0,1)$ (this last condition can be dropped if $\pi^2 b_0 + a_0 > 0$).

**Theorem 4.** Let $a, b, c \in C([0,1])$ with $a(x) > 0$ for every $x \in [0,1]$ and
\[
(2.1) \quad f(x,u,\xi) = a(x)\xi^2 + 2b(x)u\xi + c(x)u^2.
\]
I is convex over $W^{1,\infty}_0(0,1)$ or $H^1(0,1)$ if and only if there exists $g \in C^1(0,1)$ satisfying Jacobi equation, i.e.,

\[(2.2)\quad a(x)[g'(x) + c(x)] - [g(x) + b(x)]^2 = 0.\]

Furthermore, set

\[(2.3)\quad \tilde{f}(x, u, \xi) = f(x, u, \xi) + 2g(x)u\xi + g'(x)u^2,
\]

with $g$ satisfying (2.2). Then $I$ convex implies that $\tilde{f}(x, \cdot, \cdot)$ is convex and that

\[(2.4)\quad I(u) = \int_0^1 \tilde{f}(x, u(x), u'(x)) \, dx
\]

for every $u \in W^{1,\infty}_0(0,1)$ with compact support in $(0,1)$.

**REMARKS.**

i) Theorems 2 and 3 have been proved in [D1]. Theorem 4 can be considered as classical in the calculus of variations c.f. for example Gelfand-Fomin [GF], however it is hard to find it, in the existing litterature, as stated and proved here.

ii) Note that the condition $\pi^2 b_0 + a_0 \geq 0$ in Theorem 3 or (2.2) in Theorem 4 ensure the existence of a global field, in the terminology of the field theories.

iii) In Theorem 4 note (c.f. the proof below) that if $I$ is strictly convex then $g$ satisfying (2.2) can be chosen $C^1([0,1])$, i.e. there is no conjugate point in the closed interval $[0,1]$. Similarly if $I$ is strictly convex then (2.4) holds without the condition on the support of $u$.

iv) Note that in Theorem 3 as well as Theorem 4 we have

$$\tilde{f}(x, u, u') - f(x, u, u') = \frac{d}{dx} [g(x)u^2].$$

However, in general, there is no reason to suppose that the right hand side is always of this form. For example if one takes $f(x, u, u') = u^4 + u^2 u'^2$ and $\Phi(x, u) = \frac{1}{24} \left( \tan x \right) u^4$ we have that if

$$\tilde{f}(x, u, u') = f(x, u, u') + \frac{d}{dx} [\Phi(x, u)]$$

then

$$\tilde{f}(x, \cdot, \cdot)$$

is convex.
EXAMPLE. Let $f(x, u, \xi) = \xi^2 - \alpha u^2$ with $\alpha \geq 0$.

i) By Theorem 3 one finds $\alpha \leq \pi^2$, which corresponds to the best constant in Poincaré inequality.

ii) We have that $f(x) = \sqrt{\alpha} \tan \sqrt{\alpha} (x - \frac{1}{2})$ is a solution of (2.2) i.e.,

$g' - \alpha - g^2 = 0$, which is globally defined in $(0, 1)$ only if $\alpha \leq \pi^2$; using Theorem 4 we also find that $I$ is convex provided $0 \leq \alpha \leq \pi^2$.

**Proof of Theorem 4.** Before proceeding with the proof, observe that the convexity of $I$ over $W^{1, \infty}_0(0, 1)$ is equivalent to $I(u) \geq 0$ for every $u \in W^{1, \infty}_0(0, 1)$. Furthermore convexity of $f(x, \cdot, \cdot)$ follows from its positivity which is ensured by (2.2).

**Sufficiency.** Let $u \in W^{1, \infty}_0(0, 1)$ with compact support in $(0, 1)$, then

$I(u) = \int_0^1 \left[ a(x) u'^2 + 2b(x) uu' + c(x) u^2 + \frac{d}{dx} [g(x) u^2] \right] dx = \int_0^1 f(x, u(x), u'(x)) dx = \int_0^1 \left[ a(x) u'^2 + 2b(x) uu' + c(x) u^2 + \frac{d}{dx} [g(x) u^2] \right] dx \geq 0,$

since $g$ satisfies (2.2).

**Necessity.** We assume without loss of generality, that the coefficients $a, b, c$ are in $C^1([0, 1])$ and that $f$ is strictly convex. Otherwise approximating $a, b, c$ with $a_\epsilon, b_\epsilon, c_\epsilon$, and applying the result to

$I_\epsilon(u) = \int_0^1 \left[ a_\epsilon u'^2 + 2b_\epsilon uu' + c_\epsilon u^2 \right] dx + \epsilon \int_0^1 u'^2 dx$

we shall have the result by letting $\epsilon$ tend to 0. We shall now prove that there exists $f \in C^1([0, 1])$ satisfying (2.2). We shall proceed by contradiction and assume the contrary. We shall then show that we can find $w \in W^{1, \infty}_0(0, 1), w \neq 0$ with

(2.5) $I(w) = 0$.

(2.5) will then immediately contradict the strict convexity of $I$, since letting $\lambda \in (0, 1)$ we should have

$0 \leq I(\lambda \cdot 0 + (1 - \lambda) w) < \lambda I(0) + (1 - \lambda) I(w) = 0.$
We therefore assume that there is no \( g \in C^1([0, 1]) \) satisfying (2.2). We then extend the functions \( a, b, c \) in a \( C^1 \) way from \([0, 1]\) to \( \mathbb{R} \) with \( a > 0 \) over \( \mathbb{R} \). Let \( 0 \leq \delta \leq 1 \) and consider

\[
\begin{aligned}
\begin{cases}
(a(x)[g'(x) + c(x)] - [g(x) + b(x)]^2 = 0, \\
g(\delta) = 0.
\end{cases}
\end{aligned}
\tag{2.6}
\]

By the classical existence theorem for ordinary differential equations with Cauchy data we have that there exists a maximal open interval of existence \((a_1(\delta), a_0(\delta))\) containing \( \delta \). The structure of the equation implies that \( g(a_1(\delta)) = -\infty \) and \( g(a_0(\delta)) = +\infty \). By continuity of the solutions on the initial point \( \delta \) we may choose \( \delta > 0 \) so that \( a_1(\delta) = 0 \) (this follows from the fact that, trivially, \( a_1(0) < 0 \) and that, by absurd assumption \( a_1(1) > 0 \)). Observe also that for this \( \delta \), \( a_0(\delta) \leq 1 \), otherwise, using again the continuity on the initial point \( \delta \), we could choose \( \delta > 0 \) so that \( a_1(\delta) < 0 \) and \( a_0(\delta) > 1 \), which contradicts the fact that there is no \( C^1([0, 1]) \) solution of (2.2). We may therefore assume that \( \delta > 0 \) has been chosen so that (2.6) admits a solution with \( a_1(\delta) = 0 \), \( a_0(\delta) = a_0 \leq 1 \) and with

\[
\begin{aligned}
g(0) = -\infty, \quad g(a_0) = +\infty.
\end{aligned}
\tag{2.7}
\]

Looking again at the structure of the equation (2.6) we can find \( \epsilon > 0 \) small and \( \beta = \beta(\epsilon) > 0, \gamma = \gamma(\epsilon) > 0 \) with the properties (c.f. the fig. 1).

\[
\begin{cases}
g(x + b(x) < -\beta \text{ and } g'(x) > \gamma, \quad \text{for every } x \in (0, \epsilon), \\
g(x) + b(x) > \beta \text{ and } g'(x) > \gamma, \quad \text{for every } x \in [a_0 - \epsilon, a_0].
\end{cases}
\tag{2.8}
\]

We then let

\[
w(x) = \begin{cases}
0, & \text{if } x = 0, \\
\exp \left\{ -\int_{[0, x]} \frac{b(t) + g(t)}{a(t)} \, dt \right\}, & \text{if } x \in (0, a_0), \\
0, & \text{if } x \geq a_0,
\end{cases}
\]

or equivalently

\[
w'(x) = \begin{cases}
-\frac{b(x) + g(x)}{a(x)} - w(x), & x \in (0, a_0), \\
0, & x \in (a_0, 1).
\end{cases}
\tag{2.9}
\]
We then claim that \( w \in W_0^{1,\infty}(0, 1) \) and satisfies (2.5) (i.e. \( I(w) = 0 \)). Observe that if we can show that

\[
\lim_{x \to 0} [g(x) w^2(x)] = \lim_{x \to x_0} [g(x) w^2(x)] = 0
\]

we shall have (2.5), since using (2.6), (2.9), (2.10)

\[
I(w) = \int_0^1 \{aw'^2 + 2bww' + cw^2\} \, dx = \\
= \int_0^{x_0} \{aw'^2 + 2bww' + cw^2 + \frac{d}{dx} [gw^2]\} \, dx = \int_0^{x_0} a \left[ w' + \frac{b + g}{a} \right]^2 \, dx = 0.
\]

We therefore only need to show that \( w \in W_0^{1,\infty}(0, 1) \) and (2.10). We consider three cases.
CASE 1. Let \( x \in (0, \varepsilon] \), then by (2.8) \( b(x) + g(x) \neq 0 \) and thus by (2.5) we have

\[
(2.11) \quad w(x) = \exp \left\{ - \int_{\varepsilon}^{x} \frac{b(t) + g(t)}{a(t)} \, dt \right\} = \exp \left\{ - \int_{\varepsilon}^{a(b + g)} \, dt \right\} =
\]

\[
\exp \left\{ - \int_{\varepsilon}^{g'} + \frac{b'}{g} \, dt + \int_{\varepsilon}^{b} - \frac{c}{g + b} \, dt \right\} = \exp \left\{ \frac{[g(x) + b(x)]}{[a(x) + b(x)]} \right\} \exp \left\{ \frac{b(x)}{g} \right\}.
\]

Therefore since \( g(0) = -\infty \) we have that \( w \) is continuous at \( 0 \) and that by (2.9) and (2.11), \( w' \in L^\infty (0, \varepsilon) \).

To show (2.10), observe that for \( x \in (0, \varepsilon) \), we have

\[
(2.12) \quad |g(x) w^2(x)| = \frac{a(x)}{a(x)} \left| \frac{g(x) + b(x)}{w(x)} \right| \leq a(x) \left| w(x) + \left| b(x) w^2(x) \right| \right| \leq a(x) \left| w'(x) \right| \left| w(x) \right| + \left| b(x) w^2(x) \right|,
\]

where we have used (2.9) in the last inequality. Since \( w(0) = 0 \) and \( w' \in L^\infty (0, \varepsilon) \), we have immediately \( \lim_{x \to 0} g(x) w^2(x) = 0 \).

CASE 2. Let \( x \in [\varepsilon, x_0 - \varepsilon] \), then \( w \) is well defined and even \( C^1 ([\varepsilon, x_0 - \varepsilon]) \).

CASE 3. Let \( x \in [x_0 - \varepsilon, x_0) \), the using (2.8) we have \( b(x) + g(x) \neq 0 \). Furthermore by (2.5) as in (2.11) we have

\[
(2.13) \quad w(x) = \exp \left\{ - \int_{x_0 - \varepsilon}^{x} \frac{b + g}{a} \, dt \right\} = \exp \left\{ - \int_{x_0 - \varepsilon}^{x} \frac{b + g}{a} \, dt \right\} =
\]

\[
\exp \left\{ - \int_{x_0 - \varepsilon}^{x} \frac{b + g}{a} \, dt \right\} \cdot \frac{b(x_0 - \varepsilon) + g(x_0 - \varepsilon)}{b(x) + g(x)} \exp \left\{ - \int_{x_0 - \varepsilon}^{x} \frac{b - c}{b + g} \, dt \right\}.
\]

Using (2.7) we have indeed that \( w \) is continuous at \( x_0 \). Combining (2.9) and (2.13) we have that \( w' \in L^\infty (x_0 - \varepsilon, x_0) \).

To conclude the proof we only need to show (2.10) at \( x = x_0 \) and this is done exactly in the same way as (2.12).
3. - Higher dimensional case.

We now consider a bounded open set \( \Omega \subset \mathbb{R}^n \), \( u: \Omega \to \mathbb{R}^m (\nabla u \in \mathbb{R}^{nm}) \) and a continuous function \( f: \Omega \times \mathbb{R}^m \times \mathbb{R}^{nm} \to \mathbb{R} \). We let

\[
I(u) = \int_{\Omega} f(x, u(x), \nabla u(x)) \, dx.
\]

As it is well known the result of Theorem 1 is still valid provided \( m = 1 \) or \( n = 1 \) (c.f. for example Dacorogna [D2]).

However when one considers the general vectorial case, i.e. \( m, n > 1 \), the necessary condition and the conjecture are both false and we give two examples below.

**Example 1.** We first show that the necessary condition is false. Let \( m = n = 2 \) and

\[
f(\xi) = \det \xi
\]
then \( f \) is not convex while \( I \) is convex over \( W^{1, \infty}_0(\Omega; \mathbb{R}^2) \) since

\[
I(u) = \int_{\Omega} \det \nabla u(x) \, dx = 0
\]
for every \( u \in W^{1, \infty}_0(\Omega; \mathbb{R}^2) \).

**Example 2** (c.f. Terpstra [T], Serre [S]). We now show that the conjecture is also false. The example is however more involved. We let \( m = n = 3 \) and denote the components of a \( 3 \times 3 \) matrix \( \xi \) by \( \xi_{ij}, 1 \leq i, j \leq 3 \). We let \( \varepsilon > 0 \) and

\[
f(\xi) = (\xi_{11} - \xi_{22} - \xi_{33})^2 + (\xi_{12} - \xi_{21} + \xi_{13})^2 + (\xi_{21} - \xi_{12} - \xi_{23})^2 + \xi_{31}^2 + \xi_{32}^2 - \varepsilon \sum_{i,j=1}^{3} \xi_{ij}^2.
\]

One can then show (c.f. [D2] for details) that for \( \varepsilon > 0 \) sufficiently small

i) The following holds

\[
I(u) = \int_{\Omega} f(\nabla u(x)) \, dx \geq 0
\]
for every \( u \in W^{1, \infty}_0(\Omega; \mathbb{R}^3) \). Since \( f \) is quadratic the positivity of \( I \) implies that \( I \) is convex over \( W^{1, \infty}_0(\Omega; \mathbb{R}^3) \);

ii) \( f \) cannot be written as

\[
f(\xi) = f(x, u, \xi) + \Phi(x, u, \xi)
\]
with $f(x,\cdot,\cdot)$ convex (even with $f(x,u,\cdot)$ convex) and $\Phi$ satisfying

\[(3.4) \int_\Omega \Phi(x,u(x),\nabla u(x)) \, dx = 0\]

for every $u \in W^{1,\infty}_0(\Omega; \mathbb{R}^3)$.

(In the terminology of the vectorial calculus of variations, a function $f$ satisfying (3.2) is said to be quasiconvex, (3.3) polyconvex, (3.4) quasiaffine or null Lagrangian.)

Therefore there is no $\tilde{f} : \Omega \times \mathbb{R}^m \times \mathbb{R}^{nm} \to \mathbb{R}$, $\tilde{f}(x,\cdot,\cdot)$ convex so that

$$I(u) = \int_\Omega f(\nabla u(x)) \, dx = \int_\Omega \tilde{f}(x,u(x),\nabla u(x)) \, dx$$

for every $u \in W^{1,\infty}_0(\Omega; \mathbb{R}^3)$, although $I$ is convex.

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RIASSUNTO

Vedi l'introduzione.

SUMMARY

See the introduction.

REFERENCES


