SOME NEW RESULTS ON DIFFERENTIAL INCLUSIONS FOR DIFFERENTIAL FORMS

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Abstract. In this article we study some necessary and sufficient conditions for the existence of solutions in $W^{1,\infty}_0(\Omega; \Lambda^k)$ of the differential inclusion

$$d\omega \in E \quad \text{a.e. in } \Omega$$

where $E \subset \Lambda^{k+1}$ is a prescribed set.

In this article we discuss the existence of a $k$–form $\omega$, $0 \leq k \leq n-1$, verifying

$$\begin{cases}
  d\omega \in E & \text{in } \Omega \\
  \omega = 0 & \text{on } \partial \Omega
\end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set and $E \subset \Lambda^{k+1}$ is a given set of $(k+1)$–forms. For the precise notations we refer to the following section.

This problem has been mostly studied in the case $k = 0$ ($d\omega$ can then be identified with $\text{grad} \, \omega$), see [3], [4], [5], [9], [10], [11] and for an extensive bibliography on the subject see [7].

The case $k = 1$ ($d\omega$ is then identified with $\text{curl} \, \omega$) has also received some attention, see [1], [2], [8], [12].

The general case, $0 \leq k \leq n-1$, has been first considered in [1].

We here improve on [1] in two directions. The first result concerns the existence part (cf. Theorem 3.7).

**Theorem 0.1.** Let $0 \leq k \leq n-1$ be two integers, $\Omega \subset \mathbb{R}^n$ be a bounded open set, $b \in \Lambda^k \setminus \{0\}$ and $E \subset \Lambda^{k+1}$. Then the following statements are equivalent.

(i) There exists $\omega \in W^{1,\infty}_0(\Omega; \Lambda^k)$ of the form $\omega(x) = u(x) \, b$ where $u \in W^{1,\infty}_0(\Omega)$ such that

$$d\omega = (\text{grad} \, u) \wedge b \in E \quad \text{a.e. in } \Omega \quad \text{and} \quad \int_{\Omega} \omega \neq 0.$$

(ii) The following holds

$$0 \in \text{int}_{\mathbb{R}^n} b \co \left[ E \cap (\mathbb{R}^n \wedge b) \right].$$

This result was already obtained in [1] but only for $b$ of the form

$$b = b^1 \wedge \cdots \wedge b^k$$

2010 Mathematics Subject Classification. 35F60.

Part of the present work was done while the first and the third authors were visiting EPFL, whose hospitality is gratefully acknowledged.
where $b^1, \ldots, b^k \in \Lambda^1$. Our present theorem allows us (cf. Corollary 3.9) to get a complete picture when
\[ \dim \text{span } E = n - k. \]

Our second contribution concerns necessary conditions (cf. Theorem 2.5).

**Theorem 0.2.** Let $0 \leq k \leq n - 1$ be two integers, $\Omega \subset \mathbb{R}^n$ be a bounded open set, $E \subset \Lambda^{k+1}$ and $\omega \in W^{1,\infty}_0(\Omega; \Lambda^k)$ be such that
\[ d\omega \in E \text{ a.e. in } \Omega \text{ and } \int_{\Omega} \omega \neq 0. \]

Then
\[ \dim \text{span } E \geq n - k \]

and more precisely
\[ \mathbb{R}^n \wedge \left( \int_{\Omega} \omega \right) \subset \text{span } E. \]

Moreover if
\[ \dim \text{span } E = n - k \]

then
\[ \mathbb{R}^n \wedge \left( \int_{\Omega} \omega \right) = \text{span } E \text{ and } \int_{\Omega} \omega = b^1 \wedge \cdots \wedge b^k \]

for some $b^1, \ldots, b^k \in \Lambda^1$.

1. **Notations**

We gather here the notations which we will use throughout this article. For more details on exterior algebra and differential forms see [6] and for convex analysis see [7] or [13].

(1) Let $k, n$ be two integers.

- We write $\Lambda^k(\mathbb{R}^n)$ (or simply $\Lambda^k$) to denote the vector space of all alternating $k$-linear maps $f : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$. For $k = 0$, we set
  \[ \Lambda^0(\mathbb{R}^n) = \mathbb{R}. \]
  Note that $\Lambda^k(\mathbb{R}^n) = \{0\}$ for $k > n$ and, for $k \leq n$, \[ \dim \left( \Lambda^k(\mathbb{R}^n) \right) = \binom{n}{k}. \]
- $\wedge, \cdot, \langle \cdot, \cdot \rangle$ and, respectively, $\ast$ denote the exterior product, the interior product, the scalar product and, respectively, the Hodge star operator.
- For $b \in \Lambda^k$, \text{rank}[b]$ denotes the rank of the exterior $k$-form $b$.
- If $\{e^1, \cdots, e^n\}$ is a basis of $\mathbb{R}^n$, then, identifying $\Lambda^1$ with $\mathbb{R}^n$,
  \[ \{e^{i_1} \wedge \cdots \wedge e^{i_k} : 1 \leq i_1 < \cdots < i_k \leq n\} \]
  is a basis of $\Lambda^k$.
- For $E \subset \Lambda^k$, \text{span } E denotes the subspace spanned by $E$.
- Let $W$ be a subspace of $\Lambda^k$. We write $\dim W$ to denote the dimension of $W$ and $W^\perp$ to denote the orthogonal complement of $W$.
- For $b \in \Lambda^k$, we write, identifying again $\Lambda^1$ with $\mathbb{R}^n$,
  \[ \mathbb{R}^n \wedge b = \Lambda^1 \wedge b = \{x \wedge b : x \in \Lambda^1\} \subset \Lambda^{k+1}. \]

(2) Let $\Omega \subset \mathbb{R}^n$ be a bounded open set.

- The spaces $C^1(\Omega; \Lambda^k)$, $W^{1,p}(\Omega; \Lambda^k)$ and $W^{1,p}_0(\Omega; \Lambda^k)$, $1 \leq p \leq \infty$ are defined in the usual way.
• For $\omega \in W^{1,p}(\Omega; \Lambda^k)$, $\int_{\Omega} \omega$ denotes the exterior $k$-form obtained integrating componentwise the differential form $\omega$. Explicitly, for $1 \leq i_1 < \cdots < i_k \leq n$,

$$
\left( \int_{\Omega} \omega \right)_{i_1 \cdots i_k} = \int_{\Omega} \omega_{i_1 \cdots i_k}.
$$

• For $\omega \in W^{1,p}(\Omega; \Lambda^k)$, the exterior derivative $d\omega$ belongs to $L^p(\Omega; \Lambda^{k+1})$ and is defined by

$$(d\omega)_{i_1 \cdots i_{k+1}} = \sum_{j=1}^{k+1} (-1)^{j+1} \frac{\partial \omega_{i_1 \cdots \hat{i}_j \cdots i_{k+1}}}{\partial x_{i_j}} ,$$

for $1 \leq i_1 < \cdots < i_{k+1} \leq n$. If $k = 0$, then $d\omega \simeq \text{grad} \omega$. If $k = 1$, for $1 \leq i < j \leq n$,

$$(d\omega)_{ij} = \frac{\partial \omega_j}{\partial x_i} - \frac{\partial \omega_i}{\partial x_j} ,$$

i.e. $d\omega \simeq \text{curl} \omega$.

(3) For subsets $C, V \subset \Lambda^k$,

• $\text{co} C$ denotes the convex hull of $C$;

• $\text{int}_V C$ denotes the interior of $C$ with respect to the topology relative to $V$.

(4) For a convex set $C \subset \Lambda^k$,

• $\text{aff} C$ denotes the affine hull of $C$ which is the intersection of all affine subsets of $\Lambda^k$ containing $C$;

• $\text{ri} C$ denotes the relative interior of $C$ which is the interior of $C$ with respect to the topology relative to the affine hull of $C$. Equivalently $\text{ri} C = \text{int}_{\text{aff} C} C$;

• $\text{rbd} C$ denotes the relative boundary of $C$ which is $\overline{C} \setminus \text{ri} C$.

2. Necessary Conditions

2.1. Preliminaries.

Lemma 2.1. Let $0 \leq k \leq n - 1$ and let $b \in \Lambda^k \setminus \{0\}$. Then

$$\dim (\mathbb{R}^n \wedge b) \geq n - k .$$

Furthermore

$$\dim (\mathbb{R}^n \wedge b) = n - k \iff b = b^1 \wedge \cdots \wedge b^k \text{ for some } b^i \in \Lambda^1.$$

Proof. Step 1. We prove the first part. By definition of the interior product and the Hodge star operator (cf. Definition 2.11 in [6]), we have that

$$\mathbb{R}^n \lrcorner (\ast b) = (-1)^{k^2} \ast (\mathbb{R}^n \wedge b) .$$

Hence, since (cf. Proposition 2.32 (i) in [6])

$$\dim (\mathbb{R}^n \lrcorner (\ast b)) = \text{rank } [\ast b]$$

and since (cf. Proposition 2.37 (ii) of [6])

$$\text{rank } [\ast b] \geq n - k$$

we have proved the first part of the lemma.
Step 2. We prove the second part. First note that if \( b = b_1 \wedge \cdots \wedge b_k \) where \( b_i \in \Lambda^1 \), it is elementary to see that

\[
\dim (\mathbb{R}^n \wedge b) = n - k.
\]

We prove the converse. In this case

\[
n - k = \dim (\mathbb{R}^n \wedge b) = \dim (\mathbb{R}^n \wedge \ast b),
\]

and so, as in Step 1, \( \text{rank} \ast b = n - k \) and hence (cf. Proposition 2.43 (ii) in [6]) there exist \( c_1, \ldots, c^{n-k} \in \Lambda^1 \) such that

\[
\ast b = c_1 \wedge \cdots \wedge c^{n-k}.
\]

Using Proposition 2.19 in [6], it is not difficult to see that \( b = b_1 \wedge \cdots \wedge b_k \) for some \( b_i \in \Lambda^1 \). The lemma is therefore proved.

Proposition 2.2. Let \( 0 \leq k \leq n-1 \) be two integers and \( f : \mathbb{R}^n \to \Lambda^k \) be continuous at 0 and such that \( f(0) \neq 0 \). Then

\[
\dim \text{span} \{ x \wedge f(x) : x \in \mathbb{R}^n \} \geq n - k,
\]

and therefore

\[
\dim \text{span} \{ x \wedge f(x) : x \in \mathbb{R}^n \} = n - k,
\]

for some \( b_i \in \Lambda^1 \).

Proof. Step 1. From Lemma 2.1, since \( f(0) \in \Lambda^k \) and \( f(0) \neq 0 \), we have

\[
\dim \text{span} \{ x \wedge f(x) : x \in \mathbb{R}^n \} \geq n - k.
\]

Moreover if \( \dim \text{span} \{ x \wedge f(0) : x \in \mathbb{R}^n \} = n - k \), then

\[
f(0) = b_1 \wedge \cdots \wedge b_k
\]

for some \( b_i \in \Lambda^1 \).

Step 2. We prove the first assertion. Let \( x \in \mathbb{R}^n \setminus \{0\} \) be fixed. Note that, for every \( \lambda \in \mathbb{R} \setminus \{0\} \),

\[
x \wedge f(\lambda x) = \frac{1}{\lambda} [\lambda x \wedge f(\lambda x)] \in \text{span} \{ y \wedge f(y) : y \in \mathbb{R}^n \}.
\]

Since \( \text{span} \{ y \wedge f(y) : y \in \mathbb{R}^n \} \) is closed and \( f \) is continuous at 0, it follows, letting \( \lambda \to 0 \),

\[
x \wedge f(0) \in \text{span} \{ y \wedge f(y) : y \in \mathbb{R}^n \}.
\]

Therefore

\[
\mathbb{R}^n \wedge f(0) \subset \text{span} \{ y \wedge f(y) : y \in \mathbb{R}^n \}
\]

which directly shows that (cf. Step 1)

\[
\dim \text{span} \{ y \wedge f(y) : y \in \mathbb{R}^n \} \geq \dim (\mathbb{R}^n \wedge f(0)) \geq n - k.
\]

Step 3. We finally prove the extra assertion. We already know (cf. Step 2) that

\[
\mathbb{R}^n \wedge f(0) \subset \text{span} \{ x \wedge f(x) : x \in \mathbb{R}^n \}.
\]
Since, by hypothesis,
\[
\dim \text{span}\{x \wedge f(x) : x \in \mathbb{R}^n\} = n - k,
\]
we directly deduce from Step 1 that
\[
\mathbb{R}^n \wedge f(0) = \text{span}\{x \wedge f(x) : x \in \mathbb{R}^n\}
\]
and that
\[
f(0) = b_1 \wedge \ldots \wedge b_k
\]
for some \(b_i \in \Lambda^1\). The proof is therefore complete. \(\square\)

**Remark 2.3.** (i) With a very similar proof one can show that if \(f : \mathbb{R}^n \to \Lambda^k\) is differentiable at some point \(x_0 \in \mathbb{R}^n\),
\[
x_0 \wedge Df(x_0) + x \wedge f(x_0) \subset \text{span}\{y \wedge f(y) : y \in \mathbb{R}^n\}
\]
where \(Df(x_0)\) denotes the directional derivative of \(f\) at \(x_0\) in the direction of \(x\). In particular if \(Df(x_0) = 0\) then
\[
\mathbb{R}^n \wedge f(x_0) \subset \text{span}\{y \wedge f(y) : y \in \mathbb{R}^n\}.
\]
(ii) It can also be proved that if \(f : \mathbb{R}^n \to \Lambda^k\) is continuous and \(x_0 \in \mathbb{R}^n\) is such that
\[
x_0 \wedge f(x_0) \neq 0
\]
then, necessarily (even if \(f(0) = 0\)),
\[
\dim \text{span}\{x \wedge f(x) : x \in \mathbb{R}^n\} \geq n - k.
\]

The following lemma will be used in the proofs of Theorems 2.6 and 3.7 and Corollary 3.9.

**Lemma 2.4.** Let \(0 \leq k \leq n - 1\) be two integers, \(\Omega \subset \mathbb{R}^n\) be a bounded open set, \(E \subset \Lambda^{k+1}\) and \(\omega \in W^{1,\infty}_0(\Omega; \Lambda^k)\) be such that
\[
d\omega \in E \quad \text{a.e. in } \Omega.
\]
Then
\[
0 \in \overline{\co E}.
\]
Moreover, if \(0 \notin \ri \co E\), then there exists \(D \subset \Omega\) such that \(\text{meas}(\Omega \setminus D) = 0\), \(d\omega(D) \subset E\) and
\[
\dim \text{span}(d\omega(D)) < \dim \text{span} E.
\]

**Proof.** Since \(\omega \in W^{1,\infty}_0(\Omega; \Lambda^k)\) and hence
\[
\int_{\Omega} d\omega = 0,
\]
we deduce, using Proposition 2.36 in \cite{7} and Jensen inequality, that
\[
0 \in \overline{\co E}.
\]
Suppose that \(0 \notin \ri \co E\) and hence (using the previous observation)
\[
0 \in \ri \co E.
\]
Using Separation Theorem (cf. Theorem 2.10 in \cite{7}), we easily deduce from the previous equation that there exists \(b \in (\text{span} E) \setminus \{0\}\) such that
\[
\langle h; b \rangle \geq 0 \quad \text{for every } h \in \co E.
\]
In particular \( \langle d\omega; b \rangle \geq 0 \) a.e. in \( \Omega \). But, as \( \omega \in W^{1,\infty}_0(\Omega; \Lambda^k) \) and
\[
\int_{\Omega} \langle d\omega; b \rangle = 0
\]
we therefore obtain that
\[
\langle d\omega; b \rangle = 0 \text{ a.e. in } \Omega.
\]
Let \( D \subset \Omega \) be the set where the previous equation holds. Taking \( D \) smaller if necessary we can assume without loss of generality that \( d\omega(D) \subset E \) (and of course \( \text{meas}(\Omega \setminus D) = 0 \)) and hence
\[
\text{span } d\omega(D) \subset \text{span } E.
\]
As \( \langle d\omega(x); b \rangle = 0 \) for every \( x \in D \) and \( b \in (\text{span } E) \setminus \{0\} \), we deduce that
\[
\text{span } d\omega(D) \subset \neq \text{span } E
\]
and thus
\[
\dim \text{span } d\omega(D) < \dim \text{span } E
\]
as wished. \( \square \)

2.2. The main results. We first state the main results of this section.

Theorem 2.5. Let \( 0 \leq k \leq n-1 \) be two integers, \( \Omega \subset \mathbb{R}^n \) be a bounded open set, \( E \subset \Lambda^{k+1} \) and \( \omega \in W^{1,\infty}_0(\Omega; \Lambda^k) \) be such that
\[
d\omega \in E \text{ a.e. in } \Omega \quad \text{and} \quad \int_{\Omega} \omega \neq 0.
\]
Then
\[
\mathbb{R}^n \wedge \left( \int_{\Omega} \omega \right) \subset \text{span } E \quad \text{and} \quad \dim \text{span } E \geq n-k.
\]
Moreover if
\[
\dim \text{span } E = n-k
\]
then
\[
\mathbb{R}^n \wedge \left( \int_{\Omega} \omega \right) = \text{span } E \quad \text{and} \quad \int_{\Omega} \omega = b^1 \wedge \cdots \wedge b^k
\]
for some \( b^i \in \Lambda^1 \).

The following result improves Theorem 3.6 of [1], since \( E \) is not assumed to be finite and \( F \) is given explicitly.

Theorem 2.6. Let \( 0 \leq k \leq n-1 \) be integers, \( \Omega \subset \mathbb{R}^n \) be a bounded open set and \( E \subset \Lambda^{k+1} \setminus \{0\} \). Let \( \omega \in W^{1,\infty}_0(\Omega; \Lambda^k) \) be such that
\[
d\omega \in E \text{ a.e. in } \Omega.
\]
Then there exists \( F \subset E \) such that
\[
0 \in \text{rico } F.
\]
More precisely \( F \) can be taken as \( d\omega(D) \) for some \( D \subset \Omega \) with \( \text{meas}(\Omega \setminus D) = 0 \).

We start with the proof of Theorem 2.5.
Proof. Let $P : \Lambda^{k+1} \to \Lambda^{k+1}$ denote the projection onto the orthogonal complement of $\text{span } E$. Since $\omega \in W^{1,\infty}_0 (\Omega; \Lambda^k)$, extending $\omega$ by 0 to $\mathbb{R}^n$, it follows that $P(d\omega) = 0$ a.e. in $\mathbb{R}^n$.

Applying Fourier transform we obtain (recalling that the Fourier transform of $\omega$ is continuous)

$$P \left( x \wedge \left[ \int_{\mathbb{R}^n} \omega(y) \cos (2\pi \langle x; y \rangle) \, dy \right] \right) = 0 \quad \text{for every } x \in \mathbb{R}^n$$

which is equivalent to

$$x \wedge \left[ \int_{\mathbb{R}^n} \omega(y) \cos (2\pi \langle x; y \rangle) \, dy \right] \in \text{span } E \quad \text{for every } x \in \mathbb{R}^n.$$

Letting $f(x) = \int_{\mathbb{R}^n} \omega(y) \cos (2\pi \langle x; y \rangle) \, dy$

we get

$$f(0) = \int_{\Omega} \omega \neq 0.$$

Applying then Proposition 2.2 to the above $f$ we have indeed established the theorem. □

We next prove Theorem 2.6.

Proof. Let $D \subset \Omega$ (not necessarily unique) be such that

$$\text{meas}(\Omega \setminus D) = 0, \quad d\omega(D) \subset E$$

and, for every $D_1 \subset D$ with $\text{meas}(\Omega \setminus D_1) = 0$, then

$$(2.1) \quad \dim \text{span } d\omega(D_1) = \dim \text{span } d\omega(D).$$

If such a $D$ did not exist, we would find, after a finite induction on the dimension, that

$$d\omega = 0 \quad \text{a.e. in } \Omega$$

which contradicts the fact that $0 \notin E$. Letting $F = d\omega(D) \subset E$, it remains to show that

$$0 \in \text{rice } F.$$

For the sake of contradiction suppose that this is not the case. Hence using Lemma 2.4 (with $E$ replaced by $F$) there exists a set $D_1 \subset D$ (in the conclusion of Lemma 2.4 the set $D_1$ is only contained in $\Omega$ but taking it smaller, if necessary, we can assume that $D_1 \subset D$) such that $\text{meas}(\Omega \setminus D_1) = 0$ and

$$\dim \text{span } d\omega(D_1) < \dim \text{span } F.$$

This is the desired contradiction since, using (2.1),

$$\dim \text{span } d\omega(D_1) = \dim \text{span } F.$$

The proof is therefore complete. □
2.3. Further remarks. In this section we want to discuss the hypothesis
\[ \int_\Omega \omega \neq 0 \]
that has been made in Theorem 2.5. We will concentrate on the case \( k = 1 \). We start with an elementary lemma.

**Lemma 2.7.** Let \( E \subset \Lambda^2(\mathbb{R}^n) \) be such that there exists a non-degenerate \( g \in \Lambda^2 \) (i.e. \( h \cdot g \neq 0 \) for every \( h \in \Lambda^1 \setminus \{0\} \)) and
\[ g \in (\text{span } E)^\perp. \]

Then the two following statements hold true.
(i) There exists no \( b \in \Lambda^1(\mathbb{R}^n) \setminus \{0\} \) such that
\[ \mathbb{R}^n \wedge b \subset \text{span } E. \]
(ii) There exists no \( \omega \in W^{1,\infty}_0(\Omega; \Lambda^1) \) such that
\[ d\omega \in E \text{ a.e. in } \Omega \text{ and } \int_\Omega \omega \neq 0. \]

**Proof.** (i) We proceed by contradiction and assume that \( \mathbb{R}^n \wedge b \subset \text{span } E \) with \( b \neq 0 \). Since \( g \in (\text{span } E)^\perp \), we deduce that
\[ g \in (\mathbb{R}^n \wedge b)^\perp, \]
which is equivalent to
\[ \langle g; a \wedge b \rangle = 0 \quad \text{for every } a \in \Lambda^1. \]
This in turn is equivalent to (cf. Proposition 2.16 in [6])
\[ \langle b \cdot g; a \rangle = 0 \quad \text{for every } a \in \Lambda^1. \]
Since the previous equation is the same as
\[ b \cdot g = 0, \]
we deduce, appealing to the fact that \( g \) is non-degenerate, that \( b = 0 \). This is the desired contradiction.

(ii) Indeed, if such a solution exists, then (cf. Theorem 2.5)
\[ \mathbb{R}^n \wedge \left( \int_\Omega \omega \right) \subset \text{span } E. \]
Define \( b = \int_\Omega \omega \) and apply the previous point (i) to get the result. \( \square \)

The next proposition shows that the hypothesis \( \int_\Omega \omega \neq 0 \) cannot be removed, in general, in Theorem 2.5.

**Proposition 2.8.** Let \( B \subset \mathbb{R}^4 \) be the unit ball centered at 0. Then there exists \( E \subset \Lambda^2(\mathbb{R}^4) \setminus \{0\} \) with the following properties.
(i) There exists \( \omega_0 \in W^{1,\infty}_0(B; \Lambda^1) \) such that
\[ d\omega_0 \in E \text{ a.e. in } B. \]
(ii) There exists no \( b \in \Lambda^1(\mathbb{R}^4) \setminus \{0\} \) such that
\[ \mathbb{R}^4 \wedge b \subset \text{span } E. \]
(iii) For every \( \omega \in W^{1,\infty}_0(B; \Lambda^1) \) such that \( d\omega \in \text{span } E \) a.e. in \( B \), then, necessarily,
\[
\int_B \omega = 0.
\]

**Remark 2.9.** (i) Using Vitali covering theorem (see [10]), the same set \( E \) also works for any open set \( \Omega \) (instead of \( B \)).

(ii) This phenomenon only occurs where \( n \geq 4 \). Indeed if \( n \leq 3 \) we always can find (cf. Theorem 3.11) a solution with a non-zero average (as far as there exists a non-trivial solution).

(iii) The previous proposition has an interesting implication for differential inclusions where we seek solutions \( u \in W^{1,\infty}_0(\Omega; \mathbb{R}^N) \) and \( \Omega \subset \mathbb{R}^n \), verifying
\[
\nabla u 
\in
F \quad \text{a.e. in } \Omega.
\]

In the scalar case \( N = 1 \), we can always ensure (cf. Theorem 3.5) that if there is a non-trivial solution \( u \) of the differential inclusion, then there are some solutions with non-zero average. This is no longer the case in the vectorial context when \( n, N > 1 \). Indeed let \( n = N = 4 \) and define \( F \subset \mathbb{R}^{4 \times 4} \approx \mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R}^4 \) by
\[
F = \{(u_1, u_2, u_3, u_4) \in \mathbb{R}^{4 \times 4} : u_1 \wedge dx^1 + u_2 \wedge dx^2 + u_3 \wedge dx^3 + u_4 \wedge dx^4 \in E \}.
\]
where \( E \) is as in the proposition. Noticing that, for \( \omega = u_1 dx^1 + \cdots + u_4 dx^4 \),
\[
d\omega \in E \iff \nabla u = (\nabla u_1, \nabla u_2, \nabla u_3, \nabla u_4) \in F.
\]

we have the result.

(iv) If \( \dim \text{span } E = n - 1 \) then, as far as there exists a non-trivial solution, (cf. Theorem 4.13 [1]) we always have (without assuming the existence of a solution with non-zero average) \( \text{span } E = \mathbb{R}^n \ominus b \) for some \( b \in \Lambda^1 \setminus \{0\} \).

**Proof.** (i) Let
\[
\omega_0(x) = \left( |x|^2 - 1 \right) \left( x_1 dx^1 + x_2 dx^2 + 2x_3 dx^3 + 2x_4 dx^4 \right).
\]

Obviously \( \omega_0 \in W^{1,\infty}_0(B; \Lambda^1) \) and
\[
d\omega_0(x) = 2x_1 x_3 dx^1 \wedge dx^3 + 2x_1 x_4 dx^1 \wedge dx^4 + 2x_2 x_3 dx^2 \wedge dx^3 + 2x_2 x_4 dx^2 \wedge dx^4.
\]

Let
\[
\Sigma = \{x_1 = 0\} \cup \{x_2 = 0\} \cup \{x_3 = 0\} \cup \{x_4 = 0\}.
\]

Note in particular that
\[
d\omega_0(x) \neq 0 \quad \text{for every } x \in \mathbb{R}^4 \setminus \Sigma.
\]
Observe that \( \omega_0 \) and
\[
E = d\omega_0(B \setminus \Sigma)
\]
satisfy all the requirements of the first statement of the proposition, since, trivially, \( 0 \notin E, \omega_0 \in W^{1,\infty}_0(B; \Lambda^1) \) and \( d\omega_0 \in E \) a.e. in \( B \).

(ii) We now prove the second statement. First observe that for every \( h \in E \) we have \( h_{12} = h_{34} = 0 \). We thus deduce that for every \( h \in \text{span } E \) we also have \( h_{12} = h_{34} = 0 \). In other words \( dx^1 \wedge dx^2, dx^3 \wedge dx^4 \in (\text{span } E)^\perp \) and hence, in particular,
\[
g = dx^1 \wedge dx^2 + dx^3 \wedge dx^4 \in (\text{span } E)^\perp.
\]
Note that $g$ is non-degenerate. We therefore can invoke Lemma 2.7 (i) to get the claim.

(iii) The last statement of the proposition is a direct consequence of Theorem 2.5 and of the previous point (ii). \hfill $\square$

3. Necessary and sufficient conditions

3.1. Preliminaries. We start with an elementary lemma whose proof is straightforward.

**Lemma 3.1.** Let $X$ be a normed linear space, $Y \subset X$ be a subspace and $A \subset X$. Then $0 \in \text{int} \, \text{co} \,(A \cap Y)$ if and only if $\text{span} \,(A \cap Y) = Y$ and $0 \in \text{ri} \, \text{co} \,(A \cap Y)$.

We next recall Caratheodory theorem (see Corollary 2.16 in [7]).

**Lemma 3.2** (Caratheodory theorem). Let $X$ be a normed linear space and $E \subset X$. Then $0 \in \text{ri} \, \text{co} \, E$ if and only if there exist $m \in \mathbb{N}, m \geq \text{dim} \, \text{span} \, E + 1$, $z^i \in E$, $t^i > 0$ for every $1 \leq i \leq m$, such that

$$m \sum_{i=1}^m t^i z^i = 0, \quad m \sum_{i=1}^m t^i = 1 \quad \text{and} \quad \text{span} \{z^1, \ldots, z^m\} = \text{span} \, E.$$

In what follows we will constantly identify $\Lambda^1$ with $\mathbb{R}^n$.

**Lemma 3.3.** Let $0 \leq k \leq n-1$ be two integers, $b \in \Lambda^k \setminus \{0\}$ and $E \subset \Lambda^{k+1}$ be such that $0 \in \text{int}_{\mathbb{R}^n \wedge b} \text{co} \,[E \cap (\mathbb{R}^n \wedge b)]$.

Then there exists $F \subset \mathbb{R}^n$ such that $E \cap (\mathbb{R}^n \wedge b) = F \wedge b$ and $0 \in \text{int} \, F$.

**Proof.** Step 1. Let us define

$$F = \{x \in \mathbb{R}^n : x \wedge b \in E\}.$$  

Evidently, $E \cap (\mathbb{R}^n \wedge b) = F \wedge b$. Since $0 \in \text{int}_{\mathbb{R}^n \wedge b} \text{co} \,[E \cap (\mathbb{R}^n \wedge b)]$, it follows from Lemma 3.1 that

$$\text{span} \,[E \cap (\mathbb{R}^n \wedge b)] = \mathbb{R}^n \wedge b \quad \text{and} \quad 0 \in \text{ri} \, \text{co} \,[E \cap (\mathbb{R}^n \wedge b)].$$

Using Lemma 3.2 we find $m \in \mathbb{N}, m \geq \text{dim} \, \text{span} \,[E \cap (\mathbb{R}^n \wedge b)] + 1$, $z^i \in E \cap (\mathbb{R}^n \wedge b), t^i > 0$ for every $1 \leq i \leq m$, such that

$$m \sum_{i=1}^m t^i z^i = 0, \quad m \sum_{i=1}^m t^i = 1$$

and

$$\text{span} \{z^1, \ldots, z^m\} = \text{span} \,[E \cap (\mathbb{R}^n \wedge b)] = \mathbb{R}^n \wedge b.$$  

As $z^i \in E \cap (\mathbb{R}^n \wedge b)$, for every $1 \leq i \leq m$, there exists $a^i \in F$ such that

$$z^i = a^i \wedge b \quad \text{for every} \ 1 \leq i \leq m.$$  

It therefore follows from (3.2) that

$$\left(\sum_{i=1}^m t^i a^i\right) \wedge b = 0.$$
We next define
\[ W = \{ x \in \mathbb{R}^n : x \wedge b = 0 \}. \]

Note that it follows from Lemma 2.1 that \( 0 \leq \dim W \leq k \). We now consider two cases (cf. Steps 2 and 3).

**Step 2.** We prove the lemma when \( \dim W = 0 \). Note that in this case \( \dim (\mathbb{R}^n \wedge b) = n \) and hence \( m \geq n + 1 \). Since \( \dim W = 0 \), we deduce from (3.5) that
\[
\sum_{i=1}^{m} t^i a^i = 0.
\]
Observe that \( \text{span}\{a^1, \ldots, a^m\} = \mathbb{R}^n \). Indeed if \( \text{span}\{a^1, \ldots, a^m\} \) was a proper subspace of \( \mathbb{R}^n \), so would \( \text{span}\{z^1, \ldots, z^m\} \) be a proper subspace of \( \mathbb{R}^n \wedge b \) which contradicts (3.3). Therefore, using Lemma 3.2, we have that \( 0 \in \text{int co} F \). This proves the lemma when \( \dim W = 0 \) (or equivalently when \( b \) does not have any \( 1 \)-form as a factor).

**Step 3.** We prove the lemma when \( \dim W = r \), where \( 1 \leq r \leq k \).

**Step 3.1.** In this case, we have that
\[
\dim (\mathbb{R}^n \wedge b) = n - r
\]
and thus \( m \geq n - r + 1 \). Let \( \{b^1, \ldots, b^r\} \) be an orthonormal basis of \( W \). Without loss of generality, we can assume that
\[
b^j = e^j \quad \text{for every } 1 \leq j \leq r
\]
where \( \{e^1, \ldots, e^n\} \) is the standard basis of \( \mathbb{R}^n \). Since \( e^i \wedge b = 0 \) for every \( 1 \leq i \leq r \), we have, invoking Cartan Lemma (cf. Theorem 2.42 in [6]), that \( b \) can be written as
\[
b = e^1 \wedge \cdots \wedge e^r \wedge \bar{b}
\]
where \( \bar{b} \in \Lambda^{k-r} \) does not have any \( 1 \)-form as a factor i.e. we have that
\[
x \wedge \bar{b} \neq 0 \quad \text{for every } x \in \mathbb{R}^n \setminus \{0\}.
\]
Note that
\[
z^i = a^i \wedge b = a^i \wedge e^1 \wedge \cdots \wedge e^r \wedge \bar{b} = \left( \sum_{j=r+1}^{n} a^i_j e^j \right) \wedge e^1 \wedge \cdots \wedge e^r \wedge \bar{b}.
\]
Hence, without loss of generality, we assume that \( \{a^1, \ldots, a^n\} \) in (3.4) also satisfy
\[
a^i_j = 0 \quad \text{for every } 1 \leq i \leq m \text{ and } 1 \leq j \leq r.
\]
In addition, since \( x \wedge \bar{b} \neq 0 \) for every \( x \in \mathbb{R}^n \setminus \{0\} \) and using (3.6), we deduce that
\[
\dim (\text{span}\{a^1, \ldots, a^m\}) = \dim (\text{span}\{z^1, \ldots, z^m\}) = n - r.
\]
Step 3.2. To show that $0 \in \text{int co } F$, let us define $v^{i,j} \in \mathbb{R}^n$, for $1 \leq i \leq m$ and $1 \leq j \leq 2^r$, as

\begin{equation}
(3.8) \quad v^{i,1} = \begin{pmatrix}
1 \\
\vdots \\
1 \\
a_i^{r+1}
\end{pmatrix}, \quad v^{i,2} = \begin{pmatrix}
1 \\
\vdots \\
1 \\
a_i^{r+1}
\end{pmatrix}, \ldots, \quad v^{i,2^r} = \begin{pmatrix}
-1 \\
\vdots \\
-1 \\
a_i^{r+1}
\end{pmatrix}.
\end{equation}

Clearly $v^{i,j} \in F$, for every $1 \leq i \leq m$ and $1 \leq j \leq 2^r$, since $v^{i,j} \wedge b = a^i \wedge b \in E$ for every $1 \leq i \leq m$ and $1 \leq j \leq 2^r$.

Furthermore defining

$$t^{i,j} = \frac{t^i}{2^r} \quad \text{for every } 1 \leq i \leq m \text{ and } 1 \leq j \leq 2^r$$

it is easy check that $m2^r \geq n + 1$ for every $1 \leq r \leq n - 1$ and

$$\sum_{i=1}^{m} \sum_{j=1}^{2^r} t^{i,j} v^{i,j} = 0$$

and

$$\sum_{i=1}^{m} \sum_{j=1}^{2^r} t^{i,j} = 1.$$

If we show (cf. Step 3.3) that

\begin{equation}
(3.9) \quad \text{span}\{v^{i,j} : 1 \leq i \leq m; \ 1 \leq j \leq 2^r\} = \mathbb{R}^n
\end{equation}

then, using Lemma 3.2, we will have proved that $0 \in \text{int co } F$ and the proof will be finished.

Step 3.3. We show (3.9). Let $y \in \mathbb{R}^n$ be such that $\langle y; v^{i,j} \rangle = 0$, for every $1 \leq i \leq m$ and $1 \leq j \leq 2^r$. It is enough to prove that $y = 0$ to have the claim. Let $1 \leq i \leq m$ be fixed. For every $1 \leq p \leq r$, we choose $1 \leq j, l \leq 2^r$ such that

$$v^{i,j} - v^{i,l} = 2e^p.$$

Then

$$0 = \langle y; v^{i,j} - v^{i,l} \rangle = 2\langle y; e^p \rangle \quad \text{for every } p \in \{1, \ldots, r\}$$

which shows that $y_p = 0$, for every $1 \leq p \leq r$. The system

$$\langle y; v^{i,j} \rangle = 0 \quad \text{for every } 1 \leq i \leq m \text{ and } 1 \leq j \leq 2^r,$$

is therefore equivalent to the following system

\begin{equation}
(3.10) \quad \sum_{j=r+1}^{n} a_{ij} y_j = 0 \quad \text{for every } 1 \leq i \leq m.
\end{equation}

We now define a matrix $A \in \mathbb{R}^{m \times (n-r)}$ as

$$A^i_j = a_{ij}^{r+1} \quad \text{for every } 1 \leq i \leq m \text{ and } 1 \leq j \leq n - r.$$

Then (3.10) can be written as

$$A\hat{y} = 0$$
where \( \bar{y} = (y_{r+1}, \cdots, y_n) \).

To show that \( \bar{y} = 0 \), it is enough to prove that \( \text{rank } A = n - r \) (recall that \( m \geq n - r + 1 \)) which directly follows from (3.6) and (3.7). Therefore, \( \bar{y} = 0 \) and hence, \( y = 0 \). This proves the claim and concludes the proof. \( \square \)

**Definition 3.4.** A map \( \omega : \Omega \to \Lambda^k(\mathbb{R}^n) \) is said to be (locally finite) piecewise affine if there exist \( A_i \in \mathbb{R}^{|\mathbb{R}^n|^k} \times |\mathbb{R}^n|^k \), \( \alpha_i \in \mathbb{R}^{|\mathbb{R}^n|^k} \) and \( \Omega_i \subset \Omega \) open sets where \( i \) runs through an at most countable set \( I \) such that

1. \( \omega(x) = A_i x + \alpha_i \) in \( \Omega_i \) for every \( i \in I \),
2. \( \text{meas } (\Omega \setminus \bigcup_{i \in I} \Omega_i) = 0 \) and \( \Omega_i \cap \Omega_j = \emptyset \) for \( i, j \in I \) with \( i \neq j \),
3. for every compact \( K \subset \Omega \{ i \in I : K \cap \Omega_i \neq \emptyset \} \) is finite.

We recall a result concerning the scalar gradient case (see Lemma 2.11 in [10] and its proof).

**Theorem 3.5** (Gradient case). Let \( \Omega \subset \mathbb{R}^n \) be open and \( F \subset \mathbb{R}^n \). Then there exists \( u \in W^{1,\infty}_0(\Omega) \) such that \( \text{grad } u \in F \) a.e. in \( \Omega \)

if and only if

\[ 0 \in F \cup \text{int } \text{co } F. \]

If moreover \( 0 \in \text{int } \text{co } F \), then \( u \) can be taken piecewise affine, \( u \geq 0 \) and

\[ \int_{\Omega} u > 0. \]

Finally we give the following elementary lemma.

**Lemma 3.6.** Let \( 0 \leq k \leq n \) be two integers and \( \omega \in C^\infty(\mathbb{R}^n; \Lambda^k) \) be such that \( \omega(x) = A x + a \) for every \( x \in \mathbb{R}^n \)

where \( a \in \mathbb{R}^{|\mathbb{R}^n|^k} \) and \( A \in \mathbb{R}^{|\mathbb{R}^n|^k} \times |\mathbb{R}^n|^k \) is such that \( \text{rank } A \leq 1 \).

Then there exists \( b \in \Lambda^1 \setminus \{0\} \) such that

\[ b \wedge d\omega = 0. \]

**Proof.** Since we will use the lemma only when \( k = 1 \), we prove it in this context; the general case is similar. The result being trivial if \( \text{rank } A = 0 \) we assume that \( \text{rank } A = 1 \). Hence there exist \( \alpha \in \mathbb{R}^n \setminus \{0\} \) and \( b \in \mathbb{R}^n \setminus \{0\} \) such that

\[ A_{ij} = \alpha_i b_j \]

for every \( 1 \leq i, j \leq n \).

We claim that

\[ b \wedge d\omega = 0 \]

which will prove the lemma. Since, for \( 1 \leq j_1 < j_2 \leq n \),

\[ (d\omega)_{j_1j_2} = \alpha_{j_1} b_{j_2} - \alpha_{j_2} b_{j_1}, \]

we deduce that, for every \( 1 \leq r_1 < r_2 < r_3 \leq n \),

\[ (b \wedge d\omega)_{r_1r_2r_3} = (\alpha_{r_1} b_{r_2} - \alpha_{r_2} b_{r_1}) b_{r_3} - (\alpha_{r_1} b_{r_3} - \alpha_{r_3} b_{r_1}) b_{r_2} + (\alpha_{r_2} b_{r_3} - \alpha_{r_3} b_{r_2}) b_{r_1}, \]

\[ = 0. \]
This proves the claim and concludes the lemma.

3.2. Statement of the main results.

**Theorem 3.7** (Necessary and sufficient condition). Let $0 \leq k \leq n-1$ be two integers, $\Omega \subset \mathbb{R}^n$ be a bounded open set, $b \in \Lambda^k \setminus \{0\}$ and $E \subset \Lambda^{k+1}$. Then the following statements are equivalent.

(i) There exists $u \in W^{1,\infty}_0(\Omega)$ such that

$$(\text{grad } u) \wedge b \in E \text{ a.e. in } \Omega \text{ and } \int_{\Omega} u \neq 0.$$  

(ii) The following holds

$$0 \in \operatorname{int}_{\mathbb{R}^n \wedge b} \text{co}\{E \cap (\mathbb{R}^n \wedge b)\}.$$  

**Remark 3.8.**

(i) The $u$ will be constructed piecewise affine with $u \geq 0$.

(ii) Letting $\omega(x) = u(x) b$, we get that $\omega \in W^{1,\infty}_0(\Omega; \Lambda^k)$,

$$d\omega \in E \text{ a.e. in } \Omega \text{ and } \int_{\Omega} \omega \neq 0.$$

**Corollary 3.9.** Let $0 \leq k \leq n-1$ be two integers, $\Omega \subset \mathbb{R}^n$ be a bounded open set and $E \subset \Lambda^{k+1}$ be such that

$$\dim \text{span } E = n-k.$$  

Then the following statements are equivalent.

(i) There exists $\omega \in W^{1,\infty}_0(\Omega; \Lambda^k)$ such that

$$d\omega \in E \text{ a.e. in } \Omega \text{ and } \int_{\Omega} \omega \neq 0.$$  

(ii) There exist $b = b^1 \wedge \cdots \wedge b^k \neq 0$ where $b^i \in \Lambda^1$ such that

$$(3.11) \quad 0 \in \operatorname{ri co} E \quad \text{and} \quad \text{span } E = \mathbb{R}^n \wedge b.$$  

**Remark 3.10.** Using Lemma 3.1, note that (3.11) implies (since in this case $E = E \cap (\mathbb{R}^n \wedge b)$)

$$0 \in \operatorname{int}_{\mathbb{R}^n \wedge b} \text{co } E.$$  

In the case $n = 3$ the result, obtained in Theorem 4.15 of [1], takes the following optimal form. Note that in this case we can identify $d\omega$ with curl $\omega$ and $\Lambda^2(\mathbb{R}^3)$ with $\mathbb{R}^3$.

**Theorem 3.11.** Let $\Omega \subset \mathbb{R}^3$ be a bounded open set and $E \subset \mathbb{R}^3$. Then the three following assertions are equivalent.

(i) There exists $\omega \in W^{1,\infty}_0(\Omega; \mathbb{R}^3)$ such that

$$\text{curl } \omega \in E \text{ a.e. } \Omega.$$  

(ii) There exists a piecewise affine $\omega \in W^{1,\infty}_0(\Omega; \mathbb{R}^3)$ such that

$$\text{curl } \omega \in E \text{ a.e. } \Omega \text{ and } \int_{\Omega} \omega \neq 0.$$  

(iii) There exists $F \subset E$ such that

$$\dim \text{span } F \geq 2 \quad \text{and} \quad 0 \in \operatorname{ri co} F.$$
It is interesting to compare Theorems 3.7 and 3.11. Indeed one should not infer from the first theorem that any solution of $d\omega \in E$ is of the form $\omega = ub$, as the following proposition shows.

**Proposition 3.12.** Let $\Omega \subset \mathbb{R}^3$ be a bounded open set. Then there exists a set $E \subset \Lambda^2 \setminus \{0\}$ such that

$$0 \in \text{int co } E$$

with the following properties.

(i) There exists no $b \in \Lambda^1 \setminus \{0\}$ such that

$$0 \in \text{int co } \left[ E \cap (\mathbb{R}^3 \wedge b) \right]$$

and therefore there is no $u \in W^{1,\infty}_0(\Omega)$ such that

$$(\text{grad } u) \wedge b \in E \text{ a.e. in } \Omega.$$ 

(ii) There exists $\omega \in W^{1,\infty}_0(\Omega; \Lambda^2)$ such that

$$d\omega \in E \text{ a.e. in } \Omega.$$ 

The general situation when $n \geq 4$ is considerably harder in view of the following considerations. The situation in Theorem 3.11 is very specific to the dimension $n = 3$. In the next proposition we will exhibit a set $E \subset \Lambda^2(\mathbb{R}^4)$ with

$$0 \in \text{int co } \left[ E \cap (\mathbb{R}^3 \wedge e^1 + \mathbb{R}^3 \wedge e^2) \right]$$

for which there cannot exist a piecewise affine solution $\omega$ of $d\omega \in E$. However when $n = 3$, since $\mathbb{R}^3 \wedge e^1 + \mathbb{R}^3 \wedge e^2 = \Lambda^2(\mathbb{R}^3)$, we always have

$$\text{int co } \left[ E \cap (\mathbb{R}^3 \wedge e^1 + \mathbb{R}^3 \wedge e^2) \right] = \text{int co } E.$$ 

Therefore, appealing to Theorem 3.11, if

$$0 \in \text{int co } \left[ E \cap (\mathbb{R}^3 \wedge e^1 + \mathbb{R}^3 \wedge e^2) \right]$$

we can find a piecewise affine solution $\omega$ of $d\omega \in E$.

**Proposition 3.13.** There exists a set $E \subset \Lambda^2(\mathbb{R}^4)$ with

$$0 \in \text{int co } \left[ E \cap (\mathbb{R}^4 \wedge e^1 + \mathbb{R}^4 \wedge e^2) \right]$$

with the following property: for every bounded open set $\Omega \subset \mathbb{R}^4$ there exists no piecewise affine $\omega \in W^{1,\infty}_0(\Omega; \Lambda^1)$ such that

$$d\omega \in E \text{ a.e. in } \Omega.$$ 

3.3. **Proof of the main results.** We start by showing Theorem 3.7.

**Proof.** Part 1. $(i) \Rightarrow (ii)$. Using Lemma 3.1 we have to show that

$$(3.12) \quad \mathbb{R}^n \wedge b = \text{span } [E \cap (\mathbb{R}^n \wedge b)]$$

and

$$(3.13) \quad 0 \in \text{ri co } [E \cap (\mathbb{R}^n \wedge b)].$$

Let $\omega = ub \in W^{1,\infty}_0(\Omega; \Lambda^b)$. Obviously

$$d\omega \in E \cap (\mathbb{R}^n \wedge b) \text{ a.e. in } \Omega \quad \text{and} \quad \int_{\Omega} \omega = \left( \int_{\Omega} u \right) b \neq 0.$$
Hence using Theorem 2.5 (with $E$ replaced by $E \cap (\mathbb{R}^n \wedge b)$), we obtain that
\[ \mathbb{R}^n \wedge b \subset \text{span} \left[ E \cap (\mathbb{R}^n \wedge b) \right] \]
which obviously implies (3.12). We now show (3.13). For the sake of contradiction suppose that (3.13) does not hold. Then using Lemma 2.4 (with $E$ replaced by $E \cap (\mathbb{R}^n \wedge b)$) there exists a set $D \subset \Omega$ such that $\text{meas}(\Omega \setminus D) = 0$, $d\omega(D) \subset E \cap (\mathbb{R}^n \wedge b)$ and
\[ (3.14) \quad \dim \text{span} \ d\omega(D) < \dim \text{span} \left[ E \cap (\mathbb{R}^n \wedge b) \right]. \]
But, since $d\omega \in d\omega(D)$ a.e. in $\Omega$, we deduce, using again Theorem 2.5, that
\[ \mathbb{R}^n \wedge b \subset \text{span} \left( d\omega(D) \right). \]
This is the desired contradiction since, using (3.12) and (3.14), we have
\[ \dim (\mathbb{R}^n \wedge b) \leq \dim \text{span} \left( d\omega(D) \right) < \dim \text{span} \left[ E \cap (\mathbb{R}^n \wedge b) \right] = \dim (\mathbb{R}^n \wedge b). \]

Part 2. $(ii) \Rightarrow (i)$. Using Lemma 3.3, we find $F \subset \mathbb{R}^n$ such that
\[ (3.15) \quad E \cap (\mathbb{R}^n \wedge b) = F \wedge b \quad \text{and} \quad 0 \in \text{int co} F. \]
Appealing to Theorem 3.5, we find $u \in W^{1,\infty}_0(\Omega)$ such that
\[ (3.16) \quad \text{grad } u \in F \quad \text{a.e. in } \Omega. \]
Moreover $u$ can be chosen piecewise affine and such that $\int_{\Omega} u \neq 0$ (and also such that $u \geq 0$). Let us now define $\omega \in W^{1,\infty}_0(\Omega; \Lambda^k)$ by
\[ \omega(x) = u(x)b \quad \text{for every } x \in \Omega. \]
It is easy to check that
\[ d\omega = (\text{grad } u) \wedge b, \quad \text{a.e. in } \Omega. \]
It therefore follows from (3.15) and (3.16) that
\[ d\omega \in E \cap (\mathbb{R}^n \wedge b) \subset E \quad \text{a.e. in } \Omega. \]
This finishes Part (ii) and the proof.

We now prove Corollary 3.9.

Proof. Part 1: $(i) \Rightarrow (ii)$. Since $\dim \text{span} E = n - k$, using Theorem 2.5, there exist $b^1, \ldots, b^k \in \Lambda^1$ such that
\[ \mathbb{R}^n \wedge \left( \int_{\Omega} \omega \right) = \text{span} E \quad \text{and} \quad \int_{\Omega} \omega = b^1 \wedge \cdots \wedge b^k \]
which shows the second part of (3.11). Proceeding exactly as in Part 1 of the proof of Theorem 3.7, we can prove that
\[ 0 \in \text{ri co} E. \]
This concludes Part 1.

Part 2: $(ii) \Rightarrow (i)$. Using Lemma 3.1, we have that (3.11) implies (since $E = E \cap (\mathbb{R}^n \wedge b)$)
\[ 0 \in \text{int}_{\mathbb{R}^n \wedge b} \text{co} \left[ E \cap (\mathbb{R}^n \wedge b) \right]. \]
Hence, by Theorem 3.7, there exists $u \in W^{1,\infty}_0(\Omega)$ such that
\[ (\text{grad } u) \wedge b \in E \quad \text{a.e. in } \Omega \quad \text{and} \quad \int_{\Omega} u \neq 0. \]
Thus $\omega = ub$ has the desired properties. This concludes Part 2 and the proof. □

We next turn to the proof of Proposition 3.12.

Proof. Let

$$E = \{ e_1 \wedge e_2, -e_1 \wedge e_2 + e_1 \wedge e_3, -e_1 \wedge e_2 - e_1 \wedge e_3 + e_2 \wedge e_3, $$

$$-e_1 \wedge e_2 - e_1 \wedge e_3 - e_2 \wedge e_3 \}.$$

It is easy to see that

$$0 \notin \text{int}_{\mathbb{R}^3 \wedge b} \text{co} [E \cap (\mathbb{R}^3 \wedge b)] \quad \text{for every } b \in \Lambda^1$$

but that

$$0 \in \text{int} \text{co} E.$$

(i) The first statement combined with Theorem 3.7 shows that there is no $u \in W^{1, \infty}_0 (\Omega)$ such that

$$(\text{grad } u) \wedge b \in E \quad \text{a.e. in } \Omega.$$ 

(ii) The second statement implies (cf. Theorem 3.11) the existence of $\omega \in W^{1, \infty}_0 (\Omega; \Lambda^1)$ such that

$$d\omega \in E \quad \text{a.e. in } \Omega.$$

The proof of the proposition is therefore complete. □

Finally we establish Proposition 3.13.

Proof. Step 1. Let

$$a^1 = e_1 \wedge e_2, \quad a^2 = (e_1 + e_2) \wedge e_3, \quad a^3 = (e_1 + 2e_2) \wedge e_3, \quad a^4 = (e_1 + 3e_2) \wedge e_4, \quad a^5 = (e_1 + 4e_2) \wedge e_4,$$

$$a^6 = -\sum_{j=1}^5 a^j = -(e_1 \wedge e_2 + 2e_1 \wedge e_3 + 2e_1 \wedge e_4 + 3e_2 \wedge e_3 + 7e_2 \wedge e_4)$$

and finally let

$$E = \{ a^1, \ldots, a^6 \}.$$

We will show that $E$ has all the desired properties (cf. Steps 2 and 3).

Step 2. Note that

$$\text{span } E = \mathbb{R}^4 \wedge e_1 + \mathbb{R}^4 \wedge e_2.$$

Since obviously

$$\sum_{j=1}^6 \frac{1}{6} a^j = 0,$$

it follows directly from Lemmas 3.1 and 3.2 that

$$0 \in \text{int}_{\mathbb{R}^4 \wedge e_1 + \mathbb{R}^4 \wedge e_2} \text{co} [E \cap (\mathbb{R}^4 \wedge e_1 + \mathbb{R}^4 \wedge e_2)].$$

A simple calculation shows that, for every $1 \leq j \leq 5$,

$$(a^j - a^6) \wedge (a^j - a^6) \neq 0.$$

This says that

$$\text{rank } [a^j - a^6] = 4 \quad \text{for every } 1 \leq j \leq 5.$$
which is equivalent, in view of Cartan Lemma (cf. Theorem 2.42 in [6]), to saying that, for every $1 \leq j \leq 5$

\[(3.17) \quad b \wedge (a^j - a^6) \neq 0 \quad \text{for every } b \in \Lambda^1 \setminus \{0\}.
\]

**Step 3.** Let $\Omega \subset \mathbb{R}^4$ be a bounded open set. We claim that there does not exist $\omega \in W^{1,\infty}_0(\Omega; \Lambda^1)$ piecewise affine such that

$$d\omega \in E = \{a^1, \ldots, a^6\} \quad \text{a.e. in } \Omega.$$  

By contradiction assume that such a map exists. Therefore (cf. Definition 3.4) there exist $A_i \in \mathbb{R}^{4 \times 4}$, $\alpha_i \in \mathbb{R}^4$ and $\Omega_i \subset \Omega$ open sets where $i$ runs through an at most countable set $I$ such that

\[
\begin{align*}
(1) \quad & \omega(x) = A_i x + \alpha_i \text{ in } \Omega_i \quad \text{for every } i \in I \quad \\
(2) \quad & \text{meas}(\Omega \setminus \bigcup_{i \in I} \Omega_i) = 0 \quad \text{and } \Omega_i \cap \Omega_j = \emptyset \quad \text{for every } i, j \in I \text{ with } i \neq j \quad \\
(3) \quad & \text{for every compact } K \subset \Omega \setminus \bigcup_{i \in I} \Omega_i \text{ is finite.}
\end{align*}
\]

It is well known and easy to see that if $\partial \Omega_i \cap \partial \Omega_j$ contains a 3 dimensional subset of a hyperplane then necessarily

\[(3.18) \quad \text{rank} [A_i - A_j] \leq 1.
\]

From now on we assume that $\Omega$ is connected, otherwise we reason separately on every connected component of $\Omega$. Since the partition of $\Omega$ is locally finite (cf. Point 3 above) it is easy to see that for every $i, j \in I$ with $i \neq j$ there exist $l_1 = i$, $l_2, \ldots, l_N = j$ such that, for every $2 \leq m \leq N$, either $A_{l_{m-1}} - A_{l_m}$ or

\[(3.19) \quad \partial \Omega_{l_{m-1}} \cap \partial \Omega_{l_m} \text{ contains a 3 dimensional subset of a hyperplane.}
\]

Since (cf. Lemma 2.4) $0 \in \text{co}E = \text{co}E$ and since

$$\{a^1, \ldots, a^5\} \text{ are linearly independent,}$$

we deduce that there exists $\tilde{i}$ such that

$$d\omega = a^6 \text{ in } \Omega_{\tilde{i}}.$$  

Let $I_1 \subset I$ be defined by

$$\{i \in I : d\omega \in \{a^6\} \text{ in } \Omega_i\}.$$  

We claim that $I_1 = I$ which implies that

$$d\omega \in \{a^6\} \quad \text{a.e. in } \Omega$$

and this is the desired contradiction. Choose $i \in I$ with $i \neq \tilde{i}$. Combining (3.18) and (3.19), there exist $l_1 = i$, $l_2, \ldots, l_N = \tilde{i}$ such that, for every $2 \leq j \leq N$,

$$\text{rank} [A_{l_{j-1}} - A_{l_j}] \leq 1.$$  

For every $1 \leq j \leq N$, let $r_j \in \{1, \ldots, 6\}$ be such that

$$d\omega = a^{r_j} \text{ in } \Omega_{l_j}.$$  

Note that in particular, $r_N = 6$. Using Lemma 3.6 we find that, for every $2 \leq j \leq N$, there exists $b^j \in \Lambda^1 \setminus \{0\}$ such that

$$b^j \wedge (a^{r_{j-1}} - a^{r_j}) = 0.$$  

Combining the previous equation with (3.17) we immediately deduce that

$$r_1 = \cdots = r_N = 6.$$
and hence $i \in I_1$. This shows that $I_1 = I$ and proves the proposition.  

\section*{References}


