On the $n$-dimensional Dirichlet problem for isometric maps

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Abstract

We exhibit explicit Lipschitz maps from $\mathbb{R}^n$ to $\mathbb{R}^n$ which have almost everywhere orthogonal gradient and are equal to zero on the boundary of a cube. We solve the problem by induction on the dimension $n$.

1 Introduction

We consider in the general $n$—dimensional case ($n > 1$) the nonlinear system of pde’s

$$ Du^t Du = I, $$

where $Du^t$ denotes the transpose matrix of the gradient $Du$ of a map $u : \mathbb{R}^n \to \mathbb{R}^n$, while $I$ is the identity matrix. A map $u$ satisfying (1) is said to be an isometric map or rigid map and its gradient is an orthogonal matrix; briefly as usual we write $Du \in O(n)$.

To the system (1) we associate the homogeneous boundary condition $u = 0$ on the boundary of a bounded open set of $\mathbb{R}^n$. The Dirichlet problem that we obtain is critical; i.e., it is incompatible with classical solutions. In fact any isometric map $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ of class $C^1$ on an open connected set $\Omega$ of $\mathbb{R}^n$ is affine by the classical Liouville theorem, and it therefore cannot be equal to zero on its boundary $\partial \Omega$. Even more: since its invertibility, it cannot be equal to zero in more than a single point. We can then consider Lipschitz continuous maps $u : \mathbb{R}^n \to \mathbb{R}^n$, satisfying the system (1) almost everywhere; then, if $u$ is equal to zero at the boundary $\partial \Omega$ it must be not differentiable at any neighbourhood of any boundary point, thus presenting a fractal behaviour at the boundary.

In this paper we find an explicit Lipschitz solutions to the differential problem

$$ \begin{cases} 
Du(x) \in O(n) & \text{a.e. } x \in Q \\
u(x) = 0 & x \in \partial Q, 
\end{cases} $$

where $Q = (0,1)^n$ is the unit cube and $O(n)$ stands, as said above, for the set of orthogonal matrices in $\mathbb{R}^{n \times n}$. 

The study of \textit{differential inclusions} of the form
\[
\begin{cases}
    Du(x) \in E & \text{a.e. } x \in \Omega \\
    u(x) = u_0(x) & x \in \partial \Omega,
\end{cases}
\]
where \( E \subset \mathbb{R}^{N \times n} \), \( u: \Omega \subset \mathbb{R}^n \to \mathbb{R}^N \) and \( u_0 \) is a given map, has received considerable attention. In the vectorial case \( n, N \geq 2 \), general theories of existence have been developed either via the Baire category method (see Dacorogna–Marcellini \cite{3}, \cite{4}, \cite{5}) or via the convex integration method by Gromov (see Müller–Sverak \cite{9}). These methods are purely existential and do not give a way of constructing explicit solutions. In parallel, for some special problems mostly related to the case when \( E \) is the set of orthogonal matrices, some solutions were provided in a constructive way. This started with the work of Cellina–Perrotta \cite{1} when \( n = N = 3 \) and \( u_0 = 0 \), Dacorogna–Marcellini–Paolini \cite{6}, \cite{7} when \( n = N = 2 \) or \( n = N = 3 \) and Iwaniec–Verchota–Vogel \cite{8} for \( n = N = 2 \). In this context there are also some related unpublished arguments by R. D. James for \( n = N = 2 \). In \cite{7} the connection between this problem with \textit{isometric immersions} and \textit{origami} has been made. Moreover in \cite{7} we also dealt with inhomogeneous linear boundary data.

In the present article we give a self contained and purely analytical construction in any dimension. Despite its generality our proof is shorter than the existing ones which were, however, restricted to the cases \( n = 2, 3 \). We first solve the problem by induction on the dimension in the half space \( (0, \infty) \times \mathbb{R}^{n-1} \). We then get the solution to our problem by composing the solution in the half space with a map that sends the whole boundary of the unit cube in \( \mathbb{R}^n \) to one of its faces. We should point out that our construction in fact solves the problem in a more precise way: instead of considering matrices in the whole of \( O(n) \), we use only a finite number of them, namely \textit{permutation matrices} whose non zero entries are \( \pm 1 \).

\section{The fundamental brick}

Define \( f: \mathbb{R} \to \mathbb{R} \) by
\[
f(t) = \min\{t, 1-t\}.
\]
Then define \( h: \mathbb{R}^2 \to \mathbb{R}^2 \) by
\[
h(x, y) = (h^1(x, y), h^2(x, y)) = \begin{cases}
(x, f(y)) & \text{if } x \leq y, \\
(y, f(x)) & \text{if } x \geq y.
\end{cases}
\]
Finally we define a map \( \phi_n: \mathbb{R}^n \to \mathbb{R}^n \), for \( n = 2, 3, \ldots \), by induction on \( n \)
\[
\begin{align*}
\phi_2(x_1, x_2) &= h(x_1, x_2) \\
\phi_{n+1}(x_1, x_2, \ldots, x_{n+1}) &= (\phi_n(h^1(x_1 - n + 1, x_{n+1}) + n - 1, x_2, \ldots, x_n), h^2(x_1 - n + 1, x_{n+1})).
\end{align*}
\]
More in details, \( \phi_{n+1} \) can be written as a composition of the following maps

\[
\begin{align*}
(x_1, \cdots, x_{n+1}) &\mapsto (x_1 - n + 1, x_2, \cdots, x_{n+1}) \\
(y_1, \cdots, y_{n+1}) &\mapsto (h^1(y_1, y_{n+1}), y_2, \cdots, y_n, h^2(y_1, y_{n+1})) \\
(z_1, \cdots, z_{n+1}) &\mapsto (z_1 + n - 1, z_2, \cdots, z_{n+1}) \\
(w_1, \cdots, w_{n+1}) &\mapsto (\phi_n(w_1, \cdots, w_n), w_{n+1}).
\end{align*}
\]

We recall that \( u: \mathbb{R}^n \to \mathbb{R}^n \) is called a rigid map, or equivalently an isometric map, if it is Lipschitz continuous and \( Du(x) \in O(n) \) for almost every \( x \in \mathbb{R}^n \); i.e., if \( u \) satisfies (1) for almost every \( x \in \mathbb{R}^n \).

**Theorem 1 (properties of \( \phi_n \))** The map \( \phi_n: \mathbb{R}^n \to \mathbb{R}^n \), for every \( n = 2, 3, \ldots \), satisfies the following properties

1. \( \phi_n \) is a piecewise affine rigid map;
2. if \( x_2, \cdots, x_{n+1} \in [0, 1] \) then
   \[
   \phi_n(0, x_2, \cdots, x_n) = (0, f(x_2), \cdots, f(x_n));
   \]
3. on the cube \( [n - 1, n] \times [0, 1]^{n-1} \) the map \( \phi_n \) is affine.

**Proof.** We will use the following properties of the map \( h \) defined in (2):

1. \( h \) is a piecewise affine rigid map;
2. if \( y \geq 0 \) and \( x \leq 0 \) then \( h(x, y) = (x, f(y)) \);
3. if \( x \geq 1 \) and \( y \in [0, 1] \) then \( h(x, y) = (y, 1 - x) \).

We prove the theorem by induction on \( n \). In the case \( n = 2 \) the claims are direct consequences of the properties of \( h \). We assume now that the theorem holds true for \( n \), and we prove it for \( n + 1 \).

Claim (i) is a consequence of the fact that the composition of piecewise affine rigid maps is again a piecewise affine rigid map.

To prove (ii) we compute, for any \( x_2, \cdots, x_{n+1} \in [0, 1] \),

\[
\begin{align*}
\phi_{n+1}(0, x_2, \cdots, x_{n+1}) &= \phi_n(h^1(1 - n, x_{n+1}) + n - 1, x_2, \cdots, x_n, h^2(1 - n, x_{n+1})) \\
&= (\phi_n(0, x_2, \cdots, x_n), f(x_{n+1}))
\end{align*}
\]

since \( 1 - n \leq 0 \leq x_{n+1} \), by property 2 of the function \( h \) we have \( h(1 - n, x_{n+1}) = (1 - n, f(x_{n+1})) \), hence we continue

\[
= (\phi_n(0, x_2, \cdots, x_n), f(x_{n+1}))
\]

and by the induction hypothesis

\[
(0, f(x_2), \cdots, f(x_n), f(x_{n+1}))
\]

which is the claim.

Let us conclude by proving (iii). Let \( x_1 \in [n, n + 1] \) and \( x_2, \cdots, x_{n+1} \in [0, 1] \). We have

\[
\begin{align*}
\phi_{n+1}(x_1, x_2, \cdots, x_{n+1}) &= \phi_n(h^1(x_1 - n + 1, x_{n+1}) + n - 1, x_2, \cdots, x_n, h^2(x_1 - n + 1, x_{n+1})) \\
&= (\phi_n(0, x_2, \cdots, x_n), f(x_{n+1}))
\end{align*}
\]
since \( x_1 - n + 1 \geq 1 \) and \( x_{n+1} \in [0,1] \), by the property 3 of \( h \) we find that \( h(x_1 - n + 1, x_{n+1}) = (x_{n+1}, n - x_1) \), hence
\[
= (\phi_n(x_{n+1} + n - 1, x_2, \cdots, x_n), n - x_1);
\]
since now \( x_{n+1} + n - 1 \in [n-1, n] \), by the induction hypothesis we conclude that \( \phi_{n+1} \) is affine on this region. ■

3 The pyramid construction

Let us start with some notations. Let \( f \) be as in the previous section. For every \( x = (x_1, \cdots, x_n) \) we order the real numbers \( f(x_1), \cdots, f(x_n) \) so that
\[
f(x_i) \leq f(x_{i_2}) \leq \cdots \leq f(x_{i_n}).
\]
We then define \( v: [0,1]^n \to \mathbb{R}^n \) as
\[
v(x) = (f(x_1), f(x_2), \cdots, f(x_n)).
\]
Note that for \( v(x) = (v^1(x), \cdots, v^n(x)) \) we have
\[
v^1(x) = \min_{i=1,\cdots,n} \{f(x_i)\}, \quad v^n(x) = \max_{i=1,\cdots,n} \{f(x_i)\}
\]
\[
v^k(x) = \max_{i_1,\cdots,i_{k-1}} \left[ \min_{\ell\neq i_1,\cdots,i_{k-1}} \{f(x_i)\} \right], \quad k = 2, \cdots, n - 1,
\]
in particular, when \( n = 3 \),
\[
v^2(x) = \max \{\min \{f(x_1), f(x_2)\}, \min \{f(x_1), f(x_3)\}, \min \{f(x_2), f(x_3)\}\}.
\]

Theorem 2 (pyramid construction) Let \( Q = (0,1)^n \subset \mathbb{R}^n \). The map \( v: Q \to \mathbb{R}^n \) defined above, has the following properties

(i) \( v \) is a piecewise affine rigid map;
(ii) \( v(Q) \subset (0,1/2]^n \subset \{x \in \mathbb{R}^n : x_1 > 0\} \);
(iii) \( v(\partial Q) \subset \{x \in \mathbb{R}^n : x_1 = 0\} \), meaning that \( v^1 = 0 \) on \( \partial Q \).

Proof. The map \( v \) is constructed as the composition of piecewise affine rigid maps, so it is piecewise affine rigid. The second property is a consequence of the fact that if \( x_1, \cdots, x_n \in (0,1) \) then \( f(x_1), \cdots, f(x_n) \in (0,1/2] \). If we take \( x \in \partial Q \) we know that at least one component \( x_k \) of \( x \) is equal to either 0 or 1. So \( f(x_k) = 0 \). Since \( f(x_j) \geq 0 \) for every \( x_j \in [0,1] \) we conclude that \( f(x_k) = 0 \) is the first component of \( v(x) \). ■
4 The solutions to the Dirichlet problem

Now we are going to construct a locally piecewise rigid map $w: [0, +\infty) \times \mathbb{R}^{n-1} \to \mathbb{R}^n$ with zero boundary condition.

First we consider the zigzag function $F: \mathbb{R} \to \mathbb{R}$ which is defined by the conditions
\[
\begin{align*}
F(t) &= 2f(t/2) = \min\{t, 2 - t\}, \quad \text{when } t \in [0, 2], \\
F(t) &= F(t + 2), \quad \text{for every } t \in \mathbb{R}.
\end{align*}
\]
We also consider the affine map $x \mapsto Jx + a$, with $J \in O(n)$, $a \in \mathbb{R}^n$, such that (as mentioned in Theorem 1)
\[
\phi_n(x) = Jx + a \quad \text{when } x_1 \in [n - 1, n] \text{ and } x_2, \ldots, x_n \in [0, 1].
\]
Define, for $k \in \mathbb{Z}$, the vector $b_k \in \mathbb{R}^n$ as
\[
b_k = \sum_{j=k}^{+\infty} \frac{J^{-j}}{2^{j+1}} a',
\]
where $a' = (n - 1, 0, \ldots, 0) + J^{-1} a$.

Let $H = (0, +\infty) \times \mathbb{R}^{n-1}$. Given $x \in H$ there exists $k \in \mathbb{Z}$ such that
\[
(n - 1)2^{-k} \leq x_1 < (n - 1)2^{1-k}.
\]
Then, for such a point $x$, we define
\[
w(x_1, \ldots, x_n) = 2^{-k} J^{-k} \phi_n(2^k x_1 - n + 1, F(2^k x_2), \ldots, F(2^k x_n)) + b_k,
\]
where $\phi_n$ is the map considered in Theorem 1, while for $x_1 = 0$ we define
\[
w(0, x_2, \ldots, x_n) = 0 \quad \text{for all } x_2, \ldots, x_n \in \mathbb{R}.
\]

**Theorem 3 (solution in the half space)** Let $H = (0, +\infty) \times \mathbb{R}^{n-1}$. The map $w: \overline{H} \to \mathbb{R}^n$ is locally piecewise affine in $H$ and it is rigid on $\overline{H}$. Moreover $w(\partial H) = 0$.

**Proof.** We first want to check the continuity of $w$ on the planes $x_1 = (n-1)2^{-k}$, for every $k \in \mathbb{Z}$. So let $x$ be a point on such a plane and let us check that
\[
w(x_1, x_2, \ldots, x_n)
= 2^{-k-1} J^{-k-1} \phi_n(2^{k+1} x_1 - n + 1, F(2^{k+1} x_2), \ldots, F(2^{k+1} x_n)) + b_{k+1}. \quad (3)
\]
With the substitution $x_1 = (n - 1)2^{-k}$ in the definition of $w$, the left hand side of (3) becomes
\[
2^{-k} J^{-k} \phi_n(0, F(2^k x_2), \ldots, F(2^k x_n)) + b_k;
\]
by Theorem 1, since $F(t) \in [0,1]$ for all $t$,

$$= 2^{-k} J^{-k}(0, f(F(2^k x_2)), \ldots, f(F(2^k x_n))) + b_k$$

and by the identity $f(F(t)) = F(2t)/2$

$$= 2^{-k-1} J^{-k}(0, F(2^{k+1} x_2), \ldots, F(2^{k+1} x_n)) + b_k.$$ 

While the right hand side of (3) is, for $x_1 = (n-1)2^{-k}$, equal to

$$2^{-k-1} J^{-k-1}\phi_n(n-1, F(2^{k+1} x_2), \ldots, F(2^{k+1} x_n)) + b_{k+1};$$

since $F(t) \in [0,1]$ for every $t \in \mathbb{R}$, by Theorem 1 we can replace $\phi_n$ with the affine map $Jx + a$, and get

$$= 2^{-k-1} J^{-k-1}[J(n-1, F(2^{k+1} x_2), \ldots, F(2^{k+1} x_n)) + a] + b_{k+1}$$

$$= 2^{-k-1} J^{-k}(n-1, F(2^{k+1} x_2), \ldots, F(2^{k+1} x_n)) + 2^{-k-1} J^{-k-1}a + b_{k+1}$$

$$= 2^{-k-1} J^{-k}(0, F(2^{k+1} x_2), \ldots, F(2^{k+1} x_n)) + 2^{-k-1} J^{-k-1}a + b_{k+1}$$

$$= 2^{-k-1} J^{-k}(0, F(2^{k+1} x_2), \ldots, F(2^{k+1} x_n)) + 2^{-k-1} J^{-k}a + b_{k+1};$$

by recalling the definition of $b_k$, we obtain, as desired

$$= 2^{-k-1} J^{-k}(0, F(2^{k+1} x_2), \ldots, F(2^{k+1} x_n)) + b_k.$$ 

So the map $w$ on $H = (0, +\infty) \times \mathbb{R}^{n-1}$ is locally piecewise affine and rigid. We now inspect the boundary values of $w$. Take any $k \in \mathbb{Z}$, and $i_2, \ldots, i_n \in \mathbb{Z}$. We have

$$w(2^{-k}(n-1), 2^{-k} i_1, \ldots, 2^{-k} i_n) = 2^{-k} J^{-k} \phi_n(0, F(i_1), \ldots, F(i_n)) + b_k;$$

by Theorem 1 we get

$$= 2^{-k} J^{-k}(0, f(F(i_1)), \ldots, f(F(i_n))) + b_k;$$

now notice that $F(i_k)$ is either 0 or 1 hence $f(F(i_k)) = 0$, so we find

$$= b_k.$$ 

Now since $b_k \to 0$ as $k \to +\infty$ and $w$ is Lipschitz continuous, we conclude that $w \to 0$ at every point of $\partial H$ and hence is continuous on the whole set $\overline{H}$. 

**Theorem 4 (solution in the cube)** Let $Q = (0,1)^n$, $w$ be as above and $v$ as in Section 3. The map $u = w \circ v : \overline{Q} \to \mathbb{R}$ is locally piecewise affine in $Q$ and it is rigid on $\overline{Q}$. Moreover $u(\partial Q) = 0$. 

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Proof. The map \( w \) of Theorem 3 is a Lipschitz solution to the Dirichlet problem

\[
\begin{aligned}
Dw & \in O(n) \quad \text{a.e. in } H \\
 w &= 0 \quad \text{on } \partial H
\end{aligned}
\]

where \( H \) is the half space of \( \mathbb{R}^n \). Since \( u = w \circ v \), we clearly have that \( u \) is rigid and so \( Du \in O(n) \) a.e. Moreover since \( v(\partial Q) \subset \partial H \) and \( w(\partial H) = 0 \), we get the condition \( u(\partial Q) = 0 \).

Notice that we have solved a more precise problem, namely

\[
Du(x) \in \Pi(n) \subset O(n)
\]

where \( \Pi(n) \) is the set of permutation matrices whose non zero entries are \( \pm 1 \). In particular we have used at most \( n!2^n \) different matrices in the construction of \( w \) and \( v \).

References


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