Convexity of certain integrals of the calculus of variations

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Synopsis

In this paper we study the convexity of the integral \( I(u) = \int_0^1 f(x, u(x), u'(x)) \, dx \) over the space \( W^{1,\infty}_0(0, 1) \). We isolate a necessary condition on \( f \) and we find necessary and sufficient conditions in the case where \( f(x, u, u') = a(u)u'^{2n} \) or \( g(u) + h(u') \).

1. Introduction

In this paper we are concerned with integrals of the calculus of variations of the type

\[
I(u) = \int_0^1 f(x, u(x), u'(x)) \, dx,
\]

where \( f : (0, 1) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is \( C^2 \). We study the conditions on \( f \) under which the integral \( I \) is convex over the space \( W^{1,\infty}_0(0, 1) \), which denotes the space of Lipschitz functions vanishing at 0 and 1.

We first give a necessary condition on \( f \), which is that \( f(x, u, \cdot) \) is convex. We then give examples showing that no implication can be inferred \textit{a priori} on the convexity of \( f \) with respect to the variable \( u \). We then study two examples

(i) \( f(x, u, \xi) = a(u)\xi^{2n} \)

with \( n \geq 1 \), \( n \) an integer, and we show in this case that

\[ I \text{ convex over } W^{1,\infty}_0(0, 1) \Leftrightarrow a(u) = \text{constant}. \]

(ii) \( f(x, u, \xi) = g(u) + h(\xi) \) and we show that if

\[
g_0 = \inf \{ g''(u) : u \in \mathbb{R} \} \]

\[
h_0 = \inf \{ h''(\xi) : \xi \in \mathbb{R} \},
\]

then

\[ I \text{ convex over } W^{1,\infty}_0(0, 1) \Leftrightarrow \pi^2h_0 + g_0 \geq 0 \quad \text{and} \quad h_0 \geq 0. \]

In this last example we show that even if \( f(x, u, \xi) \) is not convex in the variables \( (u, \xi) \), while \( I \) is convex over \( W^{1,\infty}_0(0, 1) \), there exists \( f : (0, 1) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) such that \( f(x, \cdot, \cdot) \) is convex and

\[
I(u) = \int_0^1 f(x, u(x), u'(x)) \, dx = \int_0^1 (g(u(x)) + h(u'(x))) \, dx
\]

for every \( u \in W^{1,\infty}_0(0, 1) \).
The question of the convexity of the integral $I$ is important in the sense that one can then apply the abstract results of convex analysis to $I$; in particular a solution of the Euler equation must then be a minimiser of $I$.

Usually in the direct methods of the calculus of variations one studies the weak lower semicontinuity of $I$ in a Sobolev space $W^{1,p}$ and we have the following result

(i) $I$ convex $\Rightarrow$ $I$ weakly lower semicontinuous;
(ii) $I$ weakly lower semicontinuous $\Leftrightarrow f(x, u, \cdot)$ is convex.

So, in particular, if

$$f(x, u, \xi) = \xi^4 + (u^2 - 1)^2,$$

then, in view of the above results, we have that the associated $I$ is weakly lower semicontinuous but not convex.

### 2. Main results

We start with a necessary condition.

**Theorem 1.** Let $f : (0, 1) \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be continuous and satisfy

$$|f(x, u, \xi)| \leq a(x, |u|, |\xi|),$$

where $a$ is increasing with respect to $|u|$ and $|\xi|$ and locally integrable in $x$. If $I$ is convex over $W^{1,p}_0(0, 1)$, then $f(x, u, \cdot)$ is convex.

**Proof.** Since $f$ is continuous and $I$ is convex over $W^{1,p}_0(0, 1)$ then $I$ is weak* lower semicontinuous in $W^{1,p}$ (this is a direct application of Mazur’s lemma, see for example [1]). However, it is well known that under the above hypotheses on $f$ and if $I$ is weak* lower semicontinuous in $W^{1,p}$, then $f(x, u, \cdot)$ is convex (see for example [3] and the references quoted therein). $\Box$

**Remark.** The above result is still true for multiple integrals of the type

$$I(u) = \int_{\Omega} f(x, u(x), \nabla u(x)) \, dx,$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set and $u : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$. However, it is false if $u : \mathbb{R}^n \rightarrow \mathbb{R}^m$ with $n, m > 1$; for example, if $m = n = 2$ and

$$f(x, u, \xi) = |\xi|,$$

then $f$ is obviously not convex, while $I(u) = 0$ for every $u \in W^{1,\infty}_0(\Omega)$ and hence $I$ is convex.

We now turn our attention to sufficient conditions in some particular cases. The most important and the simplest is, of course, the case with no dependence on $u$, i.e.

$$f(x, u, \xi) = f(x, \xi).$$

We then have, trivially, the following:

**Proposition 2.** $I$ is convex over $W^{1,\infty}_0(0, 1)$ if and only if $f(x, \cdot)$ is convex.
We now give a trivial example showing that no convexity on the variable $u$ can in general be inferred from the convexity of $I$.

**Proposition 3.** Let $g: \mathbb{R} \rightarrow \mathbb{R}$ be continuous and let

$$f(x, u, \xi) = g(u) \xi.$$

Then

$$I(u) = 0$$

for every $u \in W^{1,\infty}_0(0, 1)$.

**Remark.** Note, however, in the above example that there exists $f: (0, 1) \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, namely $f = 0$, convex in the last two variables such that

$$I(u) = \int_0^1 f(x, u(x), u'(x)) \, dx$$

for every $u \in W^{1,\infty}_0(0, 1)$.

We now turn our attention to the last two cases.

**Proposition 4.** Let $a \in C^\infty(\mathbb{R})$ be such that

$$a(u) \geq a_0 > 0$$

for every $u \in \mathbb{R}$ and for $n \geq 1$, $n$ an integer, let

$$f(x, u, \xi) = a(u) \xi^{2n}.$$

Then $I$ is convex over $W^{1,\infty}_0(0, 1)$ if and only if $a$ is constant.

**Proposition 5.** Let $g, h \in C^\infty(\mathbb{R})$ and

$$f(x, u, \xi) = g(u) + h(\xi)$$

and let

$$g_0 = \inf \{g''(u): u \in \mathbb{R}\}, \quad h_0 = \inf \{h''(\xi): \xi \in \mathbb{R}\}.$$

Then

(i) There exist nonconvex $g$ and convex $h$ such that $I$ is convex over $W^{1,\infty}_0(0, 1)$, for example

$$g(u) = \frac{1}{2}(u^2 - 1)^2, \quad h(\xi) = \xi^2.$$

(ii) $I$ is convex over $W^{1,\infty}_0(0, 1)$ if and only if $h_0 \geq 0$ and

$$\pi^2 h_0 + g_0 \geq 0.$$

Case 1. If $g_0 \geq 0$ and $h_0 \geq 0$, then $f(x, u, \xi) = g(u) + h(\xi)$ is convex in the variables $(u, \xi)$.

Case 2. If $g_0 < 0$ and $\pi^2 h_0 + g_0 > 0$, then let

$$\varphi(x, u, \xi) = \sqrt{-g_0 h_0} \tan \left[ \sqrt{-\frac{g_0}{h_0}} (x - \frac{1}{2}) \right] u_\xi - \frac{g_0}{2} \left( 1 + \tan^2 \left[ \sqrt{-\frac{g_0}{h_0}} (x - \frac{1}{2}) \right] \right) u^2.$$

(3)
If

$$\Phi(x, u) = \frac{1}{2} \sqrt{-g_0 h_0} \left( \tan \left[ \sqrt{-g_0 h_0} (x - \frac{1}{2}) \right] \right) u^2,$$

then

$$\frac{d}{dx} (\Phi(x, u(x))) = \varphi(x, u(x), u'(x)) \text{ almost everywhere}$$

for every $u \in W^{1, \infty}(0, 1)$ and

$$\tilde{f}(x, u, \xi) = g(u) + h(\xi) + \varphi(x, u, \xi)$$

is convex in the variables $(u, \xi)$ for every $x \in [0, 1]$ and satisfies

$$I(u) = \int_0^1 \tilde{f}(x, u(x), u'(x)) \, dx$$

for every $u \in W^{1, \infty}(0, 1)$.

**Case 3.** If $g_0 \equiv 0$ and $\pi^2 h_0 + g_0 = 0$ then $\tilde{f}$ defined by (6) is convex in $(u, \xi)$ for every $x \in (0, 1)$ and (7) holds if $u \in \mathcal{C}^0(0, 1) = \{ u \in C^0(0, 1) : \text{supp } u \subset (0, 1) \}$.

**Remarks.** (i) Note that the function $\varphi(x, u, \xi)$ in (3) is linear in $\xi$ and it is such that

$$\int_0^1 \varphi(x, u(x), u'(x)) \, dx = 0$$

for every $u \in W^{1, \infty}(0, 1)$ if $\pi^2 h_0 + g_0 > 0$; such an integral is called an invariant integral in the field theories in the calculus of variations.

(ii) Note also that if $\pi^2 h_0 + g_0 = 0$, then the function $\varphi$ is not defined at the boundary points $x = 0$ and 1.

Before proceeding with the proof, we quote a lemma whose proof is obvious.

**Lemma 6.** Let $f : (0, 1) \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be $C^2$, let

$$I(u) = \int_0^1 f(x, u(x), u'(x)) \, dx$$

and for $\lambda \in [0, 1]$, for $u, v \in W^{1, \infty}(0, 1)$ let

$$\psi(\lambda) = I(\lambda(u - v) + v) - \lambda I(u) - (1 - \lambda) I(v).$$

Then the three following assertions are equivalent:

(i) $I$ is convex over $W^{1, \infty}_0(0, 1)$.

(ii) $\psi$ is convex for every $u, v \in W^{1, \infty}_0(0, 1)$.

(iii) $\psi''(\lambda) \equiv 0$ for every $\lambda \in [0, 1]$, $u, v \in W^{1, \infty}_0(0, 1)$, where

$$\psi''(\lambda) = \int_0^1 \left[ (u - v)^2 f_{uv}(x, \lambda(u - v) + v, \lambda(u' - v') + v') + 2(u - v)(u' - v') f_u + (u' - v') f_v \right] \, dx,$$
Proof of Proposition 4. The fact that if \( a \) is constant then \( I \) is convex is trivial. We therefore prove the converse. We divide the proof into three steps.

Step 1. In the above lemma we let \( w = u - v \) and \( z = \lambda (u - v) + v \). We then have

\[
0 \leq \psi''(\lambda) = \int_0^1 \left[ w^2 a''(z) z^{2n} + 4nww' a'(z) z^{2n-1} + 2n(2n-1)w^2 a(z) z^{2n-2} \right] \, dx
\]

\[
= \int_0^1 2n(2n-1)a(z) z^{2n-2} \left[ w'^2 + \frac{2}{(2n-1)} \left( \frac{w a'(z)}{a(z)} \right) z' \right.
\]

\[
+ \left( \frac{w a'(z)}{(2n-1)a(z)} \right)^2 - \left( \frac{w a'(z)}{(2n-1)a(z)} \right)^2 + \frac{w^2 a''(z)}{2n(2n-1)a(z)} \right] \, dx
\]

\[
= \int_0^1 2n(2n-1)a(z) z^{2n-2} \left[ \left( \frac{w'}{2n-1} \right) \left( \frac{a'(z)}{a(z)} \right) \right]^2
\]

\[
- \frac{w^2 z'^2}{2n(2n-1)^2} \left( \frac{(a(z))^{1/(2n-1)} (a(z))^{1/(2n-1)} \right)^2
\]

\[
- \frac{w^2 z'^2 (a(z))^{1/(2n-1)} 2n(a'(z))^2 - (2n-1)a''(z) a(z)}{(a(z))^{2+1/(2n-1)}} \right] \, dx. \tag{8}
\]

On letting

\[
b(t) = (a(t))^{-1/(2n-1)},
\]

we have

\[
b''(t) = -\frac{1}{2n-1} \left( a^{-1-1/(2n-1)} a' \right)
\]

\[
= + \frac{1}{(2n-1)^2} \left( \frac{2n(a')^2 - (2n-1)a''}{a^{-2+1/(2n-1)}} \right).
\]

Therefore, returning to (8), we have

\[
0 \leq \psi''(\lambda) = \int_0^1 2n(2n-1) \left( \frac{z'^2}{b(z)} \right)^{2n-2} \left[ (b(z)) \left( \frac{w}{b(z)} \right) \right]^2 - \frac{w^2 z'^2}{2nb(z)} b''(z) \right] \, dx. \tag{10}
\]

Step 2. We now show that (10) implies that

\[
b''(t) \leq 0 \text{ for every } t \in \mathbb{R}. \tag{11}
\]

Assume, for the sake of contradiction, that there exists \( \alpha \in \mathbb{R} \) such that

\[
b''(\alpha) > 0. \tag{12}
\]
By continuity of \( b'' \), we can choose \( \alpha \neq 0 \). We then construct \( z \) and \( w \) in the following way.

**Construction of \( z \).** We define for \( N \) an integer

\[
\begin{cases}
N\alpha x & \text{if } x \in \left(0, \frac{1}{N}\right), \\
\alpha + N\alpha \left(x - \frac{k}{N} - \frac{m}{N^2}\right) & \text{if } x \in \left(\frac{k}{N}, \frac{k}{N} + \frac{m}{N^2}\right), 1 \leq k \leq N - 2, \\
\alpha - N\alpha \left(x - \frac{N-1}{N}\right) & \text{if } x \in \left(\frac{N-1}{N}, 1\right).
\end{cases}
\]

We then have that \( z \in W_{1,1}^1(0, 1) \) and

\[
\begin{align*}
|z(x) - \alpha| & \leq \frac{|\alpha|}{2N} & \text{if } x \in \left(\frac{1}{N}, \frac{N-1}{N}\right), \\
|z'(x)| & = N |\alpha| & \text{almost everywhere in } (0, 1).
\end{align*}
\]  

(13)

Therefore if \( \varepsilon > 0 \) is fixed, there exists \( N \) sufficiently large that

\[
|b''(z) - b''(\alpha)|, \ |b'(z) - b'(\alpha)|, \ |b(z) - b(\alpha)| \leq \varepsilon
\]  

(14)

for every \( x \in (1/N, N - 1/N) \).

**Construction of \( w \).** We choose \( w \) in such a way that

\[
\frac{1}{b(z(x))} w(x) =
\begin{cases}
0 & \text{if } x \in \left(0, \frac{1}{N}\right), \\
\sin \pi N \frac{x-1}{N-2} & \text{if } x \in \left(\frac{1}{N}, \frac{N-1}{N}\right), \\
0 & \text{if } x \in \left(\frac{N-1}{N}, 1\right).
\end{cases}
\]  

(15)

Returning to (10) we have

\[
0 \leq \psi''(\lambda) = \int_{1/N}^{1-(1/N)} 2n(2n-1) \left(\frac{N^{2n-2} |\alpha|^{2n-2}}{b(z(x))^{2n-1}}\right)
\times \left[\left(b(z) \frac{\pi N}{N-2} \cos \left(\frac{\pi N}{N-2} \left(x - \frac{1}{N}\right)\right)\right)^2
- \frac{b(z) N^2 \alpha^2}{2n} \sin^2 \left(\frac{\pi N}{N-2} \left(x - \frac{1}{N}\right)\right) b''(z)\right] dx.
\]

With \( K_1 \) and \( K_2 > 0 \) denoting constants depending on \( n, \alpha \) and \( b(\alpha) \), but not on \( N \), and using (14), we have

\[
0 \leq \psi''(\lambda) \leq K_1 \left(\varepsilon + \frac{1}{N}\right) + K_2(N^{2n-2} - b''(\alpha) N^{2n}).
\]  

(16)
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By letting $N$ tend to infinity and using (12), we have a contradiction with (16). Therefore (11) holds.

Step 3. The conclusion then follows immediately from (11), i.e. from the concavity of $b$. Recall that $a(t) \geq a_0 > 0$, therefore

$$0 < b(t) = \left( \frac{1}{a(t)} \right)^{(2n-1)/2} \leq \left( \frac{1}{a_0} \right)^{(2n-1)/2};$$

the fact that $b$ is concave, and bounded, implies that $b$, and therefore $a$, is constant. □

We now conclude with the following proof.

**Proof of Proposition 5.** Recall that

$$f(x, u, \xi) = g(u) + h(\xi).$$

Recall also the Poincaré-Wirtinger inequality, that is

$$\int_0^1 (w(x))^2 \, dx \leq \frac{1}{\pi^2} \int_0^1 (w'(x))^2 \, dx$$

for every $w \in W^{1,\infty}_a(0, 1)$ and that equality holds if $w(x) = \sin \pi x$ (see [2]).

(i) We now prove that if

$$h(\xi) = \xi^2 \quad \text{and} \quad g(u) = \frac{1}{2}(u^2 - 1)^2,$$

i.e. $g$ is not convex, then the associated $I$ is convex. We use Lemma 6 and we have

$$\psi'(\lambda) = \int_0^1 [(u' - v')^2 + 6(\lambda(u - v) + v)^2 - 2(u - v)^2] \, dx \geq 2 \int_0^1 [(u' - v')^2 - (u - v)^2] \, dx.$$

The Poincaré-Wirtinger inequality then immediately implies the positivity of $\psi''$ and therefore the convexity of $I$ over $W^{1,\infty}_a(0, 1)$.

(ii) We always have

$$\psi''(\lambda) = \int_0^1 [(u - v)^2 h''(\lambda(u - v) + v) + (u' - v')^2 h''(\lambda(u' - v') + v')] \, dx. \quad (17)$$

($\Leftarrow$) We now wish to show that if $h_0 = \inf \{h''(t) : t \in \mathbb{R} \} \geq 0$ and $\pi^2 h_0 + g_0 \geq 0$ where $g_0 = \inf \{g''(t) : t \in \mathbb{R} \}$ then $I$ is convex over $W^{1,\infty}_a(0, 1)$.

It is clear that

$$\psi''(\lambda) \geq \int_0^1 [(u' - v')^2 h_0 + (u - v)^2 g_0] \, dx.$$

By using the Poincaré-Wirtinger inequality, we have

$$\psi''(\lambda) \geq \int_0^1 (\pi^2 h_0 + g_0)(u - v)^2 \geq 0$$

and therefore from Lemma 6, $I$ is convex.
We now assume that $I$ is convex over $W^{1,\infty}(0, 1)$ and we wish to show that $h_0 \geq 0$ and $\pi h_0 + g_0 \geq 0$. First, as before, we let

$$w = u - v, \quad z = \lambda(u - v) + v.$$ 

Then $w, z \in W^{1,\infty}(0, 1)$ and (17) becomes

$$\psi''(\lambda) = \int_0^1 \left[ w^2 g''(z) + (w')^2 h''(z') \right] dx \geq 0.$$ \hspace{1cm} (18)

Since $I$ is convex, it then follows immediately from Theorem 1 that $h_0 \geq 0$. It therefore remains to show that $\pi^2 h_0 + g_0 \geq 0$. Observe that if $g_0 \geq 0$, then the result is trivial; we therefore assume that $g_0 < 0$.

We now fix $N$ an integer, then there exist $\xi_0, u_0 \in \mathbb{R}$ such that

$$0 \leq h''(\xi_0) - h_0 \leq \frac{1}{N},$$

$$0 \leq g''(u_0) - g_0 \leq \frac{1}{N}.$$ \hspace{1cm} (19)

The aim of the following construction is to choose $w, z \in W^{1,\infty}(0, 1)$ such that the left-hand side of (18) is up to a multiplicative constant equal to $\pi^2 h_0 + g_0$, the positivity of $\psi''(\lambda)$ then implying the result.

**Construction of $z$.** We let

$$z(x) = \begin{cases} Nu_0 x & \text{if } x \in \left(0, \frac{1}{N}\right) \\ u_0 + \xi_0 \left(x - \frac{k}{N}\right) & \text{if } x \in \left(\frac{k}{N}, \frac{k+1}{N} - \frac{1}{N^2}\right), \quad 1 \leq k \leq N - 3 \\ u_0 - \xi_0 (N-1) \left(x - \frac{k+1}{N}\right) & \text{if } x \in \left(\frac{k+1}{N} - \frac{1}{N^2}, \frac{k+1}{N}\right), \quad 1 \leq k \leq N - 3 \\ u_0 & \text{if } x \in \left(\frac{N-2}{N}, \frac{N-1}{N}\right), \\ -Nu_0 (x-1) & \text{if } x \in \left(\frac{N-1}{N}, 1\right). \end{cases}$$

We then obviously have that $z \in W^{1,\infty}(0, 1)$ and that

$$|z(x) - u_0| \leq \xi_0 \left(\frac{1}{N} - \frac{1}{N^2}\right) \quad \text{if } x \in \left(\frac{1}{N}, \frac{N-1}{N}\right),$$

$$z'(x) = \xi_0 \quad \text{if } x \in \bigcup_{k=1}^{N-3} \left(\frac{k}{N}, \frac{k+1}{N} - \frac{1}{N^2}\right).$$ \hspace{1cm} (20)

Hence for $\varepsilon > 0$ fixed we may choose $N$ sufficiently large so that for $x \in (1/N, N - 1/N)$

$$|g''(z) - g''(u_0)| \leq \varepsilon.$$ \hspace{1cm} (21)
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Construction of $w$. We let

$$
w(x) = \begin{cases} 
0 & \text{if } x \in \left(0, \frac{1}{N}\right), \\
\sin \frac{N\pi}{N-2} \left( x - \frac{1}{N} \right) + a_{2k-1} & \text{if } x \in \left(\frac{k}{N}, \frac{k+1}{N} - \frac{1}{N^2}\right), \quad 1 \leq k \leq N-3 \\
a_{2k} & \text{if } x \in \left(\frac{k+1}{N} - \frac{1}{N^2}, \frac{k+1}{N}\right), \quad 1 \leq k \leq N-3, \\
- Na_{2(N-3)} \left( x - \frac{N-1}{N} \right) & \text{if } x \in \left(\frac{N-2}{N}, \frac{N-1}{N}\right), \\
0 & \text{if } x \in \left(\frac{N-1}{N}, 1\right),
\end{cases}
$$

where for $1 \leq k \leq N-3$

$$\begin{align*}
a_1 &= 0 \\
a_{2k} &= a_{2k-1} + \sin \frac{Nk-1}{N(N-2)} \pi, \\
a_{2k+1} &= a_{2k} - \sin \frac{k\pi}{N-2}.
\end{align*}$$

Therefore $w \in W^1_0(0, 1)$ and

$$a_{2k} = \sum_{v=1}^{k} \sin \left[ \frac{(Nv-1)\pi}{N(N-2)} \right] - \sum_{v=1}^{k-1} \sin \left[ \frac{v\pi}{N-2} \right] = \sin \left[ \frac{Nk-1}{N(N-2)} \pi \right] - 2 \sin \frac{\pi}{2N(N-2)} \sum_{v=1}^{k-1} \cos \left[ \frac{2Nv-1}{2N(N-2)} \pi \right],$$

and similarly

$$a_{2k+1} = -2 \sin \frac{\pi}{2N(N-2)} \sum_{v=1}^{k} \cos \left[ \frac{2Nv-1}{2N(N-2)} \pi \right].$$

We then deduce that

$$\begin{align*}
|a_{2k+1}| &\to 0 \quad \text{as } N \to \infty, \\
|a_{2k} - \sin \left[ \frac{Nk-1}{N(N-2)} \pi \right]| &\to 0 \quad \text{as } N \to \infty,
\end{align*}$$

and therefore

$$|w(x) - \sin \left[ \frac{N\pi}{N-2} \left( x - \frac{1}{N} \right) \right]| \to 0 \quad \text{as } N \to \infty \quad x \in \left(\frac{1}{N}, \frac{N-2}{N}\right). \quad (22)$$

More precisely if $k = N-3$ from (22) we have

$$|a_{2(N-3)}| \leq \left| \sin \left[ \frac{N(N-3)-1}{N(N-2)} \pi \right] \right| + \frac{K' N}{N},$$

where $K$ and $K'$ are constant independent of $N$ and hence $Na_{2(N-3)}$ is uniformly bounded.
Summarising the results we have for $\varepsilon > 0$ fixed that there exists $N$ sufficiently large that
\[
\begin{cases}
|w'| \leq K & \text{almost everywhere in } (0, 1) \text{ uniformly in } N, \\
|w(x) - \sin \left[ \frac{N\pi}{N - 2} \left( x - \frac{1}{N} \right) \right]| \leq \varepsilon, x \in (0, 1), \\
w'(x) = \frac{N\pi}{N - 2} \cos \left[ \frac{N\pi}{N - 2} \left( x - \frac{1}{N} \right) \right] & \text{if } x \in \bigcup_{k=1}^{N-3} \left( \frac{kN}{N^2}, \frac{k+1}{N^2} - \frac{1}{N} \right), \\
w'(x) = 0 & \text{if } x \in \bigcup_{k=1}^{N-3} \left( \frac{k+1}{N^2}, \frac{k+1}{N^2} - \frac{1}{N} \right).
\end{cases}
\] (23)

Returning to (18), we have
\[
0 \leq \psi'(\lambda) = \int_0^1 \left[ w^2 h''(z) + w^2 g''(z) \right] dx \\
= \int_{\mathbb{S}^{N-1}/\mathbb{N}} \left[ w^2 h''(z') + w^2 g''(z) \right] dx \\
= \int_{\mathbb{S}^{N-1}/\mathbb{N}} \left( \sin^2 \left[ \frac{N\pi}{N - 2} \left( x - \frac{1}{N} \right) \right] g'(z) \right) dx \\
+ \int_{\mathbb{S}^{N-1}/\mathbb{N}} \left[ w^2(x) - \sin^2 \frac{N\pi}{N - 2} \left( x - \frac{1}{N} \right) \right] g'(z) dx \\
+ \sum_{k=1}^{N-3} \int_{\mathbb{S}^{N-1}/\mathbb{N}} \left( \frac{N\pi}{N - 2} \right)^2 \cos^2 \left[ \frac{N\pi}{N - 2} \left( x - \frac{1}{N} \right) \right] h''(\xi_k) dx \\
+ \int_{\mathbb{S}^{N-1}/\mathbb{N}} \left( Na^{(N-3)2} h''(0) \right) dx.
\]

Using (20) and (23) we have, with $K$ denoting a generic constant independent of $N$ and $\varepsilon$, that
\[
0 \leq -K \left( \varepsilon + \frac{1}{N} \right) + \int_{\mathbb{S}^{N-1}/\mathbb{N}} \sin^2 \left[ \frac{N\pi}{N - 2} \left( x - \frac{1}{N} \right) \right] g'(z) dx \\
+ h''(\xi_0) \left( \frac{N\pi}{N - 2} \right)^2 \sum_{k=1}^{N-3} \int_{\mathbb{S}^{N-1}/\mathbb{N}} \cos^2 \left[ \frac{N\pi}{N - 2} \left( x - \frac{1}{N} \right) \right] dx.
\]

Using (19) and (21), we have
\[
0 \leq -K \left( \varepsilon + \frac{1}{N} \right) + g_0 \int_{\mathbb{S}^{N-1}/\mathbb{N}} \sin^2 \left[ \frac{N\pi}{N - 2} \left( x - \frac{1}{N} \right) \right] dx \\
+ h_0 \left( \frac{N\pi}{N - 2} \right)^2 \int_{\mathbb{S}^{N-1}/\mathbb{N}} \cos^2 \left[ \frac{N\pi}{N - 2} \left( x - \frac{1}{N} \right) \right] dx \\
- h_0 \left( \frac{N\pi}{N - 2} \right)^2 \sum_{k=1}^{N-3} \int_{\mathbb{S}^{N-1}/\mathbb{N}} \cos^2 \left[ \frac{N\pi}{N - 2} \left( x - \frac{1}{N} \right) \right] dx \\
+ \int_{\mathbb{S}^{N-1}/\mathbb{N}} \cos^2 \left[ \frac{N\pi}{N - 2} \left( x - \frac{1}{N} \right) \right] dx.
\]
Finally, we have
\[ 0 \leq K \left( \epsilon + \frac{1}{N} \right) + (\pi^2 h_0 + g_0) \int_0^1 \sin^2 \pi y \, dy. \]

Letting \( N \to \infty \) and using the arbitrariness of \( \epsilon \) we have indeed obtained
\[ \pi^2 h_0 + g_0 \geq 0 \]
and thus the result.

(iii) Case 1 is trivial and we now show that if \( \tilde{f} \) is defined by
\[ \tilde{f}(x, u, \xi) = g(u) + h(\xi) + \varphi(x, u, \xi), \]
then
(a) \( \tilde{f}(x, ., .) \) is convex over \( \mathbb{R}^2 \) for every \( x \in (0, 1) \).
(b) For every \( u \in W^{1,0}_1(0, 1) \) we have
\[ I(u) = \int_0^1 \tilde{f}(x, u(x), u'(x)) \, dx. \]
In (a), since \( \pi^2 h_0 > -g_0 \) (Case 2) and \( g_0 < 0 \), then
\[ -\frac{\pi}{2} < \frac{-g_0}{\pi h_0} (x - \frac{1}{2}) < \frac{\pi}{2} \quad \text{if} \quad x \in [0, 1] \]
and if \( \pi^2 h_0 = -g_0 \) (Case 3), then the above inequality holds only if \( x \in (0, 1) \), so that \( \tilde{f} \) is well defined if \( x \in (0, 1) \). In order to show the convexity of \( \tilde{f} \) we show that, denoting by \( \gamma = \sqrt{-g_0/\pi h_0} (x - \frac{1}{2}) \),
\[ \nabla^2 \tilde{f} = \left( \begin{array}{c}
\tilde{f}_{xx} \\
\tilde{f}_{xu} \\
\tilde{f}_{uu}
\end{array} \right) = \left( \begin{array}{c}
h''(\xi) \\
h''(\xi) \tan \gamma \\
\sqrt{-g_0} \tan \gamma \\
\sqrt{-g_0} \tan \gamma \\
g''(u) - g_0 (1 + \tan^2 \gamma)
\end{array} \right) \]
the above matrix is positive definite for every \( (u, \xi) \in \mathbb{R}^2 \). Since \( h''(\xi) \geq h_0 \) and \( g''(u) \geq g_0 \), and \( g_0 < 0 \) it remains to show that
\[ \det \nabla^2 \tilde{f} = h''(\xi) (g''(u) - g_0 (1 + \tan^2 \gamma)) + h_0 g_0 \tan^2 \gamma \geq 0. \]
We have immediately that
\[ \det \nabla^2 \tilde{f} \geq h_0 (g_0 - g_0 (1 + \tan^2 \gamma)) + h_0 g_0 \tan^2 \gamma = 0, \]
and thus \( \tilde{f} \) is convex.

In (b), we observe that if \( u \in W^{1,0}_1(0, 1) \) then
\[ \tilde{f}(x, u, u') = g(u) + h(u') + \frac{d}{dx} \left[ \frac{-h_0 g_0}{2} \tan \left( \sqrt{\frac{-g_0}{h_0}} (x - \frac{1}{2}) \right) u^2 \right] \text{almost everywhere in } (0, 1), \]
and therefore
\[ I(u) = \int_0^1 [g(u(x)) + h(u'(x))] \, dx = \int_0^1 \tilde{f}(x, u(x), u'(x)) \, dx, \]
for every \( u \in W^{1,0}_1(0, 1) \) if \( \pi^2 h_0 + g_0 > 0 \) and only in \( \mathcal{O}(0, 1) \) if \( \pi^2 h_0 + g_0 = 0. \) \( \square \)
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References


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