REGULARIZATION OF NON ELLIPTIC VARIATIONAL PROBLEMS

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INTRODUCTION

In this article we discuss the existence of solutions for

$$\text{(P)} \quad \inf\{\int_\Omega f(x,u(x),\nabla u(x))\,dx : u \in u_0 + W_0^{1,\alpha}(\Omega;\mathbb{R}^m)\}$$

where

(i) Ω is a bounded open set of $\mathbb{R}^n$; $u: \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ and thus $\nabla u \in \mathbb{R}^{nm}$.

(ii) $f: \Omega \times \mathbb{R}^m \times \mathbb{R}^{nm} \to \mathbb{R}$ is continuous and satisfies for every $(x,u,F) \in \Omega \times \mathbb{R}^m \times \mathbb{R}^{nm}$

$$a_1(x) + c_1|F|^\alpha \leq f(x,u,F) \leq a_2(x) + b|u|^\alpha + c_2|F|^\alpha \tag{0.1}$$

where $a_1, a_2 \in L^1(\Omega)$, $b \geq 0$, $c_2 \geq c_1 > 0$ and $\alpha > 1$.

(iii) $W^{1,\alpha}(\Omega;\mathbb{R}^m) = \{u = (u_1,\dots,u_m) : u_j \in L^\alpha(\Omega)$ and grad $u_j \in L_n^\alpha(\Omega)$, $j = 1,\dots,m\}$. By $u \in u_0 + W_0^{1,\alpha}(\Omega;\mathbb{R}^m)$ we just mean that $u, u_0 \in W^{1,\alpha}(\Omega;\mathbb{R}^m)$ and $u = u_0$ on $\partial\Omega$ in the sense of traces (see Adams [Ad1], for details).

In order to show that the problem (P) has a solution one usually uses the so called "direct methods of the calculus of variations", i.e., if

$$I(u,\Omega) = \int_\Omega f(x,u(x),\nabla u(x))\,dx \tag{0.2}$$

one tries to show that I is (sequentially) <u>weakly lower semicontinuous</u> i.e.,

J. M. Ball (ed.), Systems of Nonlinear Partial Differential Equations, 385–400.

$$I(u,\Omega) \leqslant \liminf_{u_\nu \xrightarrow{W^{1,\alpha}} u} I(u_\nu,\Omega). \tag{O.3}$$

(By $\rightharpoonup$, we denote weak convergence.)

Roughly speaking, the property (O.1) ensures (with $\alpha > 1$) that minimizing sequences are bounded in a reflexive space, and thus are (up to a subsequence) weakly convergent; property (O.3) then implies that the weak limit is the actual minimum of I.

A functional I having the property (O.3) is said to be <u>regular</u>. Morrey ([Mo1],[Mo2]) has isolated a necessary and sufficient condition on f, called <u>quasiconvexity</u>, in order that (O.3) holds. A function f is said to be quasiconvex if

$$\int_D f(x_0,u_0,F_0 + \nabla\phi(x))dx \geqslant f(x_0,u_0,F_0)\,\text{meas}\,D, \tag{O.4}$$

for every bounded domain $D \subset \mathbb{R}^n$, for every $(x_0,u_0,F_0) \in \Omega \times \mathbb{R}^m \times \mathbb{R}^{nm}$ and for every $\phi \in W_0^{1,\infty}(D;\mathbb{R}^m)$.

For more details about the exact conditions imposed on f in order that quasiconvexity and weak lower semicontinuity be equivalent, as well as for precise references concerning the next remarks see [Mo1],[Mo2],[Me1],[AF1] or [Da4].

<u>Remarks:</u>

(i) If $n = 1$ or $m = 1$, then the convexity of f (with respect to the last argument) is equivalent to the quasiconvexity of f, while if $n,m > 1$, then one only has that: convexity $\Rightarrow$ quasiconvexity.

(ii) If $m = n$ and there exists $g: \mathbb{R} \to \mathbb{R}$ continuous so that $f(\nabla u) = g(\det\nabla u)$, then the quasiconvexity of f is equivalent to the convexity of g.

(iii) If f is quasiconvex then f is rank one convex, i.e.,

$$f(\lambda F + (1-\lambda)G) \leqslant \lambda f(F) + (1-\lambda)f(G), \tag{O.5}$$

for every $\lambda \in [0,1]$, $F,G \in \mathbb{R}^{nm}$ with rank $(F - G) \leqslant 1$. If $f \in C^2(\mathbb{R}^{nm})$ then (O.5) is equivalent to the <u>Legendre-Hadamard</u> (or ellipticity) condition

$$\sum_{i,j,\alpha,\beta} \frac{\partial^2 f(F)}{\partial F_{i\alpha}\partial F_{j\beta}} \lambda_i\lambda_j\mu_\alpha\mu_\beta \geqslant 0 \quad \text{for all } \lambda \in \mathbb{R}^n,\ \mu \in \mathbb{R}^m.$$

Without the quasiconvexity hypothesis on f, then, in general (P) does not have a solution in $W^{1,\alpha}$ and one is thus led to look for "generalized solutions" of (P). One possible approach for definitions of such solutions is the so called relaxation theorem that we describe now. Let Qf be the lower quasiconvex envelope of f, i.e.,

$$Qf(x,u,\cdot) = \sup\{\Phi(x,u,\cdot) \leqslant f(x,u,\cdot) \text{ and } \Phi(x,u,\cdot) \text{ quasi-convex}\}. \tag{0.6}$$

Let

$$\bar{I}(u,\Omega) = \int_\Omega Qf(x,u(x),\nabla u(x))\,dx \tag{0.7}$$

and define

(QP) $\inf\{\bar{I}(u,\Omega)\,;\, u \in u_0 + W_0^{1,\alpha}(\Omega;\mathbb{R}^m)\}$

We will then prove that

Theorem:

(i) (QP) is a regular problem

(ii) $\inf(P) = \inf(QP)$

(iii) For every $\bar{u} \in W^{1,\alpha}(\Omega;\mathbb{R}^m)$, there exists a sequence $\{u^\nu\}_{\nu=1}^\infty$, $u^\nu \in W^{1,\alpha}(\Omega;\mathbb{R}^m)$ such that

$$\begin{cases} u^\nu = \bar{u} \text{ on } \partial\Omega, \quad \nu \geqslant 1, & (0.8)\\ u^\nu \rightharpoonup \bar{u} \text{ in } W^{1,\alpha}, \text{ as } \nu \to \infty, & (0.9)\\ I(u^\nu,\Omega) \to \bar{I}(\bar{u},\Omega), \text{ as } \nu \to \infty . & (0.10)\end{cases}$$

The theorem therefore shows that (P) and (QP) are "equivalent" with respect to weak convergence and then one may define the solutions of (QP) as "generalized solutions" of (P).

The above theorem was proved first by L.C. Young ([Yo1],[Yo2]) in the case $m = n = 1$, using his notions of generalized curves. Following the same approach it was then established by Ekeland ([Ek1]) and by Ekeland and Témam ([ET1]) in the case $m = 1$ or $n = 1$ (see also [BL1],[MS1]). The general case $(m,n > 1)$ was established in [Da3]. It has since then also been proved, using different methods and somehow weaker hypotheses on f, by Acerbi and Fusco [AF1].

In the last section of this article, we will give an example of applications of the above result to phase transitions for gases satisfying the Van der Waal's equation of state.

To end this introduction we give an elementary example illustrating the theorem.

Example:

Let $m = n = 1$, $\Omega = (0,1)$, $u_0(0) = u_0(1) = 0$ and

$$f(u,u') = (1 + u^2)(1 + (u'^2 - 1)^2). \tag{0.11}$$

So let

$$I(u) = \int_0^1 (1 + (u(x))^2)(1 + ((u'(x))^2 - 1)^2)dx, \tag{0.12}$$

and

$$\text{(P)} \quad \inf\{I(u) : u \in W_0^{1,\infty}(0,1)\}.$$

As mentioned earlier, in the case $m = 1$ or $n = 1$ the notions of convexity and quasiconvexity are equivalent. It is easy to see that $f(u,\cdot)$ is not convex, therefore

$$Qf(u,u') = \begin{cases} f(u,u') & \text{if } |u'| \geqslant 1 \\ (1 + u^2) & \text{if } |u'| \leqslant 1. \end{cases} \tag{0.13}$$

Let

$$\bar{I}(u) = \int_0^1 Qf(u(x),u'(x))dx \tag{0.14}$$

and

$$\text{(QP)} \quad \text{Inf}\{\bar{I}(u) : u \in W_0^{1,\infty}(0,1)\}.$$

So define for $k = 0,1,2,\ldots,(\nu-1)$

$$u^\nu(x) = \begin{cases} x - \frac{k}{\nu} & \text{if } x \in [\frac{2k}{2\nu}, \frac{2k+1}{2\nu}] \\ -x + \frac{k+1}{\nu} & \text{if } x \in [\frac{2k+1}{2\nu}, \frac{2k+2}{2\nu}]. \end{cases} \tag{0.15}$$

We thus have $u^\nu \in W^{1,\infty}(0,1)$ and

$$\begin{cases} 0 \leqslant u^\nu(x) \leqslant \frac{1}{2\nu}, & x \in [0,1] \\ u^\nu(0) = u^\nu(1) = 0 \\ |(u^\nu(x))'| = 1 & \text{a.e. in } (0,1). \end{cases} \tag{0.16}$$

It is then easy to see that u^ν is a minimising sequence of (P), since

$$1 = \bar{I}(0) = \inf(QP) \leqslant \inf(P) \leqslant I(u^\nu) \leqslant 1 + \frac{1}{4\nu^2} \tag{0.17}$$

We then get the theorem, i.e.,

$$\begin{cases} \inf(P) = \inf(QP) = 1 \\ u^{\nu}(0) = u^{\nu}(1) = 0 \\ u^{\nu} \overset{*}{\rightharpoonup} 0 \quad \text{in} \quad W^{1,\infty}(0,1), \quad \text{as} \quad \nu \to \infty, \\ I(u^{\nu}) \to \bar{I}(0), \quad \text{as} \quad \nu \to \infty. \end{cases} \tag{0.18}$$

(By $\overset{*}{\rightharpoonup}$, we denote weak $*$ convergence.)

I. PRELIMINARY RESULTS

We have seen in the introduction the definition of quasiconvexity; we now turn our attention to a special type of functions Φ, called <u>quasiaffine</u> or <u>null-Lagrangians</u>, which are such that Φ and $-\Phi$ are quasiconvex. The next theorem characterizes such functions (for a proof see [Ba1],[Da4] ...).

<u>Theorem 1:</u>

The following properties are equivalent:

(i) Φ is quasiaffine (or null Lagrangian).

(ii) For every bounded domain $\Omega \subset \mathbb{R}^n$, for every $F \in \mathbb{R}^{nm}$ and for every $\phi \in W_0^{1,\infty}(\Omega;\mathbb{R}^m)$

$$\int_{\Omega} \Phi(F + \nabla\phi(x))\,dx = \Phi(F)\,\text{meas}\,\Omega. \tag{1.1}$$

(iii) Let $\text{adj}_s F$ denote the matrix of all $s \times s$ $(1 \leqslant s \leqslant \inf\{n,m\})$ subdeterminants of the matrix $F \in \mathbb{R}^{nm}$, then

$$\Phi(F) = A + \sum_{s=1}^{\inf\{n,m\}} \langle B_s\,;\,\text{adj}_s F\rangle_{\sigma(s)}, \tag{1.2}$$

where

$$\sigma(s) = \binom{m}{s}\binom{n}{s} = \frac{m!\,n!}{s!\,(m-s)!\,s!\,(n-s)!}$$

$\langle .,. \rangle_{\sigma(s)}$ denotes the scalar product in $\mathbb{R}^{\sigma(s)}$, $A \in \mathbb{R}$ and $B_s \in \mathbb{R}^{\sigma(s)}$ are constants.

<u>Remarks:</u>

(i) If $m = n = 2$ then (1.2) can be rewritten as

$$\Phi(F) = A + \sum_{i,j=1}^{2} B_1^{ij} F_{ij} + B_2 \det F.$$

(ii) The quasiaffine functions are the only weakly continuous functions.

(iii) The Euler Lagrange equations for the functional $\int_\Omega \Phi(\nabla u(x))dx$ reduce to an identity if Φ is a null Lagrangian.

We now turn our attention to lower quasiconvex envelopes (for a proof of the next theorem, see [Da3],[Da4]).

Theorem 2:

Let $f\colon \Omega \times \mathbb{R}^m \times \mathbb{R}^{nm} \to \mathbb{R}$ be continuous and such that

$$|f(x,u,F)| \leqslant a(x) + b|u|^\alpha + c|F|^\alpha,$$

for some $\alpha \geqslant 1$, $a \in L^1(\Omega)$, $b \geqslant 0$ and $c \geqslant 0$. Then

$$Qf(x_0,u_0,F_0) = \sup\{\Phi(x_0,u_0,F_0) \leqslant f(x_0,u_0,F_0) : \Phi(x_0,u_0,\cdot) \text{ quasiconvex}\}$$

$$= \inf_\phi\{\frac{1}{\operatorname{meas}D}\int_D f(x_0,u_0,F_0 + \nabla\phi(y))dy : \phi \in W_0^{1,\infty}(D;\mathbb{R}^m)\}$$

where D is any bounded domain of $\mathbb{R}^n$. Furthermore Qf is continuous.

II. RELAXATION THEOREMS

In this section we will assume that $\Omega \subset \mathbb{R}^n$ is a bounded open set with Lipschitz boundary. We then have the following result established in [Da3].

Theorem 3:

Let $f\colon \mathbb{R}^{nm} \to \mathbb{R}$ be continuous and such that

$$\text{(H1)}\quad a + \sum_{\nu=1}^N b_\nu|\Phi_\nu(F)|^{\beta_\nu} \leqslant f(F) \leqslant c + \sum_{\nu=1}^N d_\nu|\Phi_\nu(F)|^{\beta_\nu}$$

for every $F \in \mathbb{R}^{nm}$, for some $a,c \in \mathbb{R}$, $N \geqslant 1$ (an integer), $\beta_\nu > 1$, $d_\nu \geqslant b_\nu > 0$ and where $\Phi_\nu\colon \mathbb{R}^{nm} \to \mathbb{R}$, $\nu = 1,\ldots,N$ are quasiaffine functions.

Then for every $\bar{u} \in W^{1,\infty}(\Omega;\mathbb{R}^m)$, there exists $\{u^s\}_{s=1}^\infty$, $u^s \in W^{1,\infty}(\Omega;\mathbb{R}^m)$ such that

$$\begin{cases} u^s = \bar{u} \text{ on } \partial\Omega & (2.1)\\ \Phi_\nu(\nabla u^s) \rightharpoonup \Phi_\nu(\nabla \bar{u}) \text{ in } L^{\beta_\nu}(\Omega),\ \nu = 1,..,N,\ s \to \infty & (2.2)\\ \int_\Omega f(\nabla u^s(x))dx \to \int_\Omega Qf(\nabla \bar{u}(x))dx. & (2.3)\end{cases}$$

Remarks:

(i) The case where f satisfies a condition

$$a + b|F|^{\beta} \leqslant f(F) \leqslant c + d|F|^{\beta}$$

is a particular case of (H1); one needs to choose $N = nm$, $\beta_\nu = \beta > 1$, $d_\nu = d \geqslant b_\nu = b > 0$ for $\nu = 1,\ldots,N$ and for $F = (F_{ij})_{1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m}$

$$\begin{cases} \Phi_1(F) = F_{11}, \ldots. \ \Phi_n(F) = F_{n1}, \\ \Phi_{n+1}(F) = F_{12}, .. \ \Phi_{2n}(F) = F_{n2}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots \ \Phi_{nm}(F) = F_{nm} \ . \end{cases}$$

Furthermore in this case condition (2.2) is equivalent to

$u^s \rightharpoonup \bar{u}$ in $W^{1,\beta}(\Omega;\mathbb{R}^m)$.

(ii) Similarly if $n = m$ and f satisfies

$$a + b|\det F|^{\beta} \leqslant f(F) \leqslant c + d|\det F|^{\beta}$$

then by choosing $N = 1$ and $\Phi_\nu(F) = \det F$, we get (H1).

(iii) One may note however that the problem of minimal hypersurface in parametric form is not contained in Hypothesis (H1), since in this case $\beta_\nu = 1$. However one can still prove a relaxation result for this problem (see [Da2]).

We now show that as corollaries of Theorem 3, we may allow f to depend not only on ∇u but also on x and on u.

Corollary 4:

Let $f: \Omega \times \mathbb{R}^{nm} \to \mathbb{R}$ be continuous and such that

$$\text{(H2)} \quad a(x) + \sum_{\nu=1}^{N} b_\nu |\Phi_\nu(F)|^{\beta_\nu} \leqslant f(x,F) \leqslant c(x) + \sum_{\nu=1}^{N} d_\nu |\Phi_\nu(F)|^{\beta_\nu}$$

for every $F \in \mathbb{R}^{nm}$, $a, c \in L^1(\Omega)$, b_ν, d_ν and Φ_ν as in (H1).

Then for every $\bar{u} \in W^{1,\infty}(\Omega;\mathbb{R}^m)$, there exists $\{u^s\}_{s=1}^{\infty}$, $u^s \in W^{1,\infty}(\Omega;\mathbb{R}^m)$ such that

$$\begin{cases} u^s = \bar{u} \text{ on } \partial\Omega, & (2.4) \\ \Phi_\nu(\nabla u^s) \rightharpoonup \Phi_\nu(\nabla \bar{u}) \text{ in } L^{\beta_\nu}(\Omega),\ \nu = 1,\dots,N, & (2.5) \\ \int_\Omega f(x,\nabla u^s(x))\,dx \to \int_\Omega Qf(x,\nabla \bar{u}(x))\,dx. & (2.6) \end{cases}$$

Corollary 5:

Let $f\colon \Omega \times \mathbb{R}^m \times \mathbb{R}^{nm} \to \mathbb{R}$ be continuous and such that

(H3) $a_1(x) + c_1|F|^\beta \leqslant f(x,u,F) \leqslant a_2(x) + b|u|^\beta + c_2|F|^\beta$

for every $F \in \mathbb{R}^{nm}$, $a_1, a_2 \in L^1(\Omega)$, $c_2 \geqslant c_1 > 0$, $b \geqslant 0$ and $\beta > 1$. Then for every $\bar{u} \in W^{1,\beta}(\Omega;\mathbb{R}^m)$, there exists $\{u^s\}_{s=1}^\infty$, $u^s \in W^{1,\beta}(\Omega;\mathbb{R}^m)$ such that

$$\begin{cases} u^s = \bar{u} \text{ on } \partial\Omega, & (2.7) \\ u^s \rightharpoonup \bar{u} \text{ in } W^{1,\beta}(\Omega;\mathbb{R}^m), & (2.8) \\ \int_\Omega f(x,u^s(x),\nabla u^s(x))\,dx \to \int_\Omega Qf(x,\bar{u}(x),\nabla \bar{u}(x))\,dx. & (2.9) \end{cases}$$

We now sketch the proofs of Corollaries 4 and 5.

Proof of Corollary 4:

Fix $\varepsilon > 0$ and let

$$K(\bar{u}) = 1 + \int_\Omega Qf(x,\nabla \bar{u}(x))\,dx + \int_\Omega |a_1(x)|\,dx, \tag{2.10}$$

$$A = \{\phi \in W^{1,\infty}\colon \sum_{\nu=1}^N b_\nu \|\Phi_\nu(\nabla\phi)\|^{\beta_\nu}_{L^{\beta_\nu}} \leqslant K(\bar{u})\}. \tag{2.11}$$

Let Ω be approximated by a union of hypercubes Ω_p, $1 \leqslant p \leqslant P$, of edge length $\frac{1}{p}$ and define

$$x_p = \frac{1}{\text{meas}\,\Omega_p} \int_{\Omega_p} x\,dx \tag{2.12}$$

Choose p sufficiently large so that

$$\Big|\int_{\Omega - \bigcup_{p=1}^P \Omega_p} f(x,\nabla \bar{u}(x))\,dx\Big| \leqslant \frac{\varepsilon}{5}, \tag{2.13}$$

$$\Big| \int_{\Omega - \bigcup_{p=1}^{P} \Omega_p} Qf(x,\nabla\bar{u}(x))\,dx \Big| \leq \frac{\varepsilon}{5}, \tag{2.14}$$

$$\Big| \int_{\bigcup_{p=1}^{P}\Omega_p} Qf(x,\nabla\bar{u}(x))\,dx - \sum_{p=1}^{P} \int_{\Omega_p} Qf(x_p,\nabla\bar{u}(x))\,dx \Big| \leq \frac{\varepsilon}{5}, \tag{2.15}$$

$$\Big| \int_{\bigcup_{p=1}^{P}\Omega_p} f(x,\nabla\phi(x))\,dx - \sum_{p=1}^{P} \int_{\Omega_p} f(x_p,\nabla\phi(x))\,dx \Big| \leq \frac{\varepsilon}{5}, \text{ for all } \phi \in A. \tag{2.16}$$

Having fixed P in this way, we may use Theorem 3 on each Ω_p, $1 \leq p \leq P$, and deduce that there exist $u_p^s \in W^{1,\infty}(\Omega_p;\mathbb{R}^m)$ such that for s sufficiently large

$$u_p^s = \bar{u} \text{ on } \partial\Omega_p \tag{2.17}$$

$$\Phi_\nu(\nabla u_p^s) \rightharpoonup \Phi_\nu(\nabla\bar{u}) \text{ in } L^{\beta_\nu}(\Omega_p),\ 1 \leq \nu \leq N \tag{2.18}$$

$$\Big| \int_{\Omega_p} f(x_p,\nabla u_p^s(x))\,dx - \int_{\Omega_p} Qf(x_p,\nabla\bar{u}(x))\,dx \Big| \leq \frac{\varepsilon}{5P}. \tag{2.19}$$

Defining $u^s = u_p^s$ in Ω_p and $u^s = \bar{u}$ in $\Omega - \bigcup\Omega_p$ we immediately get (2.4) and (2.5). Therefore it remains to show (2.6). Observe first that, by (H2), $u^s \in A$. Combining (2.13) - (2.16) with (2.19) we immediately get the result.

<u>Proof of Corollary 5:</u>

Fix $\varepsilon > 0$. Apply Corollary 4 to $f(x,\bar{u}(x),\cdot)$ and $Qf(x,\bar{u}(x),\cdot)$ to get $u^s \in W^{1,\infty}(\Omega;\mathbb{R}^m)$ such that

$$\begin{cases} u^s = \bar{u} \text{ on } \partial\Omega, & (2.20)\\ u^s \rightharpoonup \bar{u} \text{ in } W^{1,\beta}(\Omega;\mathbb{R}^m), & (2.21)\\ \int_\Omega f(x,\bar{u}(x),\nabla u^s(x))\,dx \to \int_\Omega Qf(x,\bar{u}(x),\nabla\bar{u}(x))\,dx. & (2.22)\end{cases}$$

Using Rellich's theorem we have (up to a subsequence)

$$u^s \to \bar{u} \quad \text{a.e. and in} \quad L^{\beta}(\Omega). \tag{2.23}$$

Hypothesis (H3), combined with (2.22) and (2.23), gives the result.

III. APPLICATIONS

Before giving an example of applications of Theorem 3, we will give a representation of the lower quasiconvex envelope Qf for a certain particular function.

Proposition 6:

Let $n = m$, $f\colon \mathbb{R}^{n^2} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be continuous and such that

$$f(F) = g(\det F), \tag{3.1}$$

for every $F \in \mathbb{R}^{n^2}$. Then

$$Qf(F) = g^{**}(\det F), \tag{3.2}$$

where g^{**} denotes the lower convex envelope of g, i.e.

$$g^{**} = \sup\{\Phi \leqslant f;\ \Phi \text{ convex}\}.$$

Remark:

We have mentioned in the introduction that if f has the form (3.1), then it has been shown by Morrey ([Mo1], see also [Da4]) that the quasiconvexity of f and the convexity of g are equivalent. The above result has also been proved in [AF1] using the Relaxation Theorem.

In order to prove the above proposition, we need two elementary lemmas (the proof of the first one being obvious).

Lemma 7:

Let $F \in \mathbb{R}^{n^2}$ with rank $F \leqslant n-1$. Let $\theta > 0$ be fixed. Then there exists $G \in \mathbb{R}^{n^2}$, with rank $G = n-1$ and $\|F-G\| = \sum_{i,j} |F_{ij} - G_{ij}| \leqslant \theta$.

Lemma 8:

Let $\lambda \in (0,1)$, $d,\delta \in \mathbb{R}$ be fixed. Let $F \in \mathbb{R}^{n^2}$, with rank $F \geqslant n-1$, be such that

$$\det F = \lambda d + (1-\lambda)\delta. \tag{3.3}$$

Then there exist $a,b \in \mathbb{R}^n$ such that

$$\begin{cases} \det(F + \lambda a \otimes b) = \delta, & (3.4) \\ \det(F - (1-\lambda) a \otimes b) = d, & (3.5) \end{cases}$$

where $(a \otimes b)_{ij} = a_i b_j$, $1 \leqslant i, j \leqslant n$.

Proof of Lemma 8:

Since rank $F \geqslant n-1$, we may suppose (without loss of generality) that

$$\det \hat{F}_{11} = \begin{vmatrix} F_{22} & \cdots & F_{2n} \\ \vdots & & \vdots \\ F_{n2} & \cdots & F_{nn} \end{vmatrix} \neq 0. \tag{3.6}$$

We may then choose a and b in such a way that

$$\begin{cases} a_j = b_j = 0, & 2 \leqslant j \leqslant n \\ b_1 = 1 \\ a_1 = \dfrac{\delta - \det F}{\lambda \det \hat{F}_{11}} \end{cases}$$

It is then easy to show that (3.4) and (3.5) hold.

Proof of Proposition 6:

Step 1: Observe that since $g^{**} \leqslant f$ and g^{**} is convex (and thus quasiconvex) we trivially have

$$g^{**} \leqslant Qf \leqslant f = g. \tag{3.7}$$

Step 2: We need to show the converse of (3.7). First define

$$h(\delta) = \sup\{Qf(F) : \det F = \delta\}. \tag{3.8}$$

We immediately deduce that

$$\begin{aligned} Qf(F) &\leq h(\det F) = \sup\{Qf(G) : \det G = \det F\} \\ &\leq \sup\{f(G) : \det G = \det F\} \\ &\leq \sup\{g(\det G) : \det G = \det F\} \\ &\leq g(\det F) = f(F). \end{aligned} \tag{3.9}$$

Therefore if we show that h is convex, we will obtain that

$$Qf(F) \leq h(\det F) \leq g^{**}(\det F) \leq f(F) \tag{3.10}$$

and thus combining (3.7) and (3.10), we will obtain the result.

Step 3: It therefore remains to show that for $\lambda \in (0,1)$, $d, \delta \in \mathbb{R}$, we have

$$h(\lambda d + (1-\lambda)\delta) \leq \lambda h(d) + (1-\lambda)h(\delta). \tag{3.11}$$

Fix $\varepsilon > 0$. Then, by definition (3.8), there exists $F_1 \in \mathbb{R}^{n^2}$ such that

$$\det F_1 = \lambda d + (1-\lambda)\delta, \tag{3.12}$$

$$h(\lambda d + (1-\lambda)\delta) \leq \frac{\varepsilon}{2} + Qf(F_1). \tag{3.13}$$

Observe that there is no loss of generality if we suppose rank $F_1 \geq n - 1$, otherwise use Lemma 7 and the continuity of Qf (c.f. Theorem 2) to get $F \in \mathbb{R}^{n^2}$ with

$$\det F = \lambda d + (1-\lambda)\delta, \quad \text{rank } F \geq n - 1, \tag{3.14}$$

$$h(\lambda d + (1-\lambda)\delta) \leq \varepsilon + Qf(F). \tag{3.15}$$

Note that we trivially have

$$\begin{aligned} h(\lambda d + (1-\lambda)\delta) \leq \varepsilon + Qf(\lambda[F - (1-\lambda)a \otimes b] \\ + (1-\lambda)[F + \lambda a \otimes b]), \end{aligned} \tag{3.16}$$

for every $a, b \in \mathbb{R}^n$. Using the rank one convexity of Qf (c.f. (0.5) in the introduction: Qf quasiconvex $\Rightarrow$ Qf rank one convex), we have

$$\begin{aligned} h(\lambda d + (1-\lambda)\delta) \leq \varepsilon + \lambda Qf(F - (1-\lambda)a \otimes b) \\ + (1-\lambda)Qf(F + \lambda a \otimes b), \end{aligned} \tag{3.17}$$

for every $a, b \in \mathbb{R}^n$. We now make a particular choice of a and b as in Lemma 8, and we get

$$\begin{aligned} h(\lambda d + (1-\lambda)\delta) \leq \varepsilon + \lambda \sup\{Qf(G) : \det G = d\} \\ + (1-\lambda)\sup\{Qf(H) : \det H = \delta\}, \end{aligned} \tag{3.18}$$

i.e.,

$$h(\lambda d + (1 - \lambda)\delta) \leqslant \varepsilon + \lambda h(d) + (1 - \lambda)h(\delta) \; ;$$

ε being arbitrary, we have indeed established the proposition.

Remark:

One can establish in the same way a similar result for functions $f(F) = g(\Phi(F))$ where Φ is any quasiaffine (null Lagrangian) function.

We conclude this article with an example of application of the relaxation theorem. We consider a gas an an elastic material which occupies in a reference configuration a volume $\Omega \subset \mathbb{R}^3$. We use Lagrangian coordinates. In a deformed configuration the particle $x \in \Omega$ occupies the position $u(x) \in \mathbb{R}^3$. We assume furthermore that the displacement is prescribed on the boundary of Ω, i.e., $u = u_0$ on $\partial\Omega$. The energy of the deformation is then, in absence of external forces,

$$I(u) = \int_\Omega \Big(\int_1^{\det \nabla u(x)} P(v)\,dv \Big) dx \tag{3.19}$$

where P is the pressure and v the specific volume. We will assume that P satisfies the Van der Waal's equation of state, i.e.,

$$P(v) = \frac{nRT}{v - nb} - \frac{an^2}{v^2} \, , \tag{3.20}$$

where T denotes the temperatures, a,b,R are constants and n is the number of moles (see [Cal] for reference). We let

$$f(\nabla u) = g(\det \nabla u) = \int_1^{\det \nabla u(x)} P(v)\,dv. \tag{3.21}$$

It is well known that for some values of the temperature T, the pressure P, in (3.20), is not a monotone function of v and thus g is not convex (and hence f is not quasiconvex). Therefore there is a range of values of v where the Euler-Lagrange equations associated to (3.19) are not elliptic. We may then consider the quasiconvex envelope of f, $Qf = g^{**}$ (Proposition 6). So let

$$\bar{I}(u) = \int_\Omega Qf(\nabla u(x))\,dx = \int_\Omega g^{**}(\det \nabla u(x))\,dx. \tag{3.22}$$

If one considers the corresponding pressure associated to Qf, one gets the usual Maxwell line (see for more details [Da4] p.98). We then have the following

Theorem 9:

Let Ω be diffeomorphic to a sphere. Let $u_0 \in C^2(\bar{\Omega};\mathbb{R}^3)$ be a homeomorphism and let

(P) $\inf\{I(u) : u = u_0 \text{ on } \partial\Omega,\ u \in C^1(\bar{\Omega};\mathbb{R}^3)\}$

(QP) $\inf\{\bar{I}(u) : u = u_0 \text{ on } \partial\Omega,\ u \in C^1(\bar{\Omega};\mathbb{R}^3)\}$.

Then

(i) (QP) has a solution $\bar{u} \in C^1$ with $\bar{u} = u_0$ on $\partial\Omega$ and $\det \nabla\bar{u}(x) > 0$ in $\bar{\Omega}$.

(ii) $\inf(P) = \inf(QP)$.

(iii) Let $\bar{u}$ be a solution of (QP) (i.e., $\bar{u} = u_0$ on $\partial\Omega$, $\bar{u} \in C^1$ and $\det\nabla\bar{u} > 0$ in $\bar{\Omega}$), then there exists a minimizing sequence of (P) with $\det \nabla u_s > 0$ in $\bar{\Omega}$, $u_s = u_0$ on $\partial\Omega$ such that

$$\begin{cases} \det \nabla u_s \rightharpoonup \det \nabla \bar{u} \quad \text{in } L^1(\Omega), & (3.23) \\ I(u_s) \to \bar{I}(\bar{u}) & (3.24) \end{cases}$$

Proof:

(i) The existence of a solution is established in [Da1].

(ii),(iii) In order to apply Theorem 3, one needs to modify the integrand f, which does not satisfy the continuity and coercivity conditions assumed in Theorem 3. One changes the integrand in the following manner

$$\int_\Omega g(\det \nabla u(x))\,dx = \int_{u_0(\Omega)} g\left(\frac{1}{\det \nabla x(u)}\right) \det \nabla x(u)\,du \tag{3.25}$$

and observe that if

$$h(t) = g(t^{-1})t \quad \text{for} \quad t > 0, \tag{3.26}$$

then

$$h^{**}(t) = g^{**}(t^{-1})t. \tag{3.27}$$

We then apply Theorem 3 to h and h^{**}, and by inverting the minimizing sequences, we obtain (3.23) and (3.24). The fact that $\det \nabla u_s > 0$ in $\bar{\Omega}$ (in (3.23)) requires a more detailed analysis which is carried out in [Da1].

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