On a Conjugate Gradient/Newton/Penalty Method for the Solution of Obstacle Problems. Application to the Solution of an Eikonal System with Dirichlet Boundary Conditions

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Dedicated to M. Hestenes, C. Lanczos, E. Stieffel,
and to all those individuals
who made conjugate gradient the universally used method it is today.

Dedicated to G. H. Golub for his 70th birthday.

1 Introduction and historical perspective

As the dedication suggests, many scientists in several countries have contributed significantly during these last fifty years to make *conjugate gradient* the efficient and versatile method it is today. However, this article is not limited to conjugate gradient and from that point of view, it owes a lot to *M. Hestenes* for his major contributions, not only to conjugate gradient, but also to *calculus of variations* and *augmented Lagrangian methods.* Indeed, in this article after investigating the solution of *obstacle problems* for *elliptic* and *parabolic* linear *operators* by a methodology combining *penalty, Newton's method* and *conjugate gradient algorithms,* we shall apply the above methodology to the search for *non-negative solutions* to an *Eikonal system* with Dirichlet boundary conditions, after transforming it into a (non-convex) problem from *Calculus of Variations;* in order to solve this non-convex minimization problem we shall apply an *operator-splitting method,* and as shown in, e.g., [1], [2] operator splitting methods have close links with augmented Lagrangian methods and therefore with M. Hestenes contributions to Applied and Computational Mathematics. Since, in this article, we make a systematic use of *conjugate gradient algorithms,* and of *cyclic reduction methods* for the *fast solution of discrete linear elliptic problems,* a special credit has to be given to *G. H. Golub* for his outstanding contributions to these topics (among many other contributions, which include, in fact, complementarity methods for the solution of discrete obstacle problems). We deeply regret he could not attend the Jyväskylä Conference on "50 Years of Conjugate Gradients".

To conclude this section on a more personal note, we shall mention that the initial intention of the second author was to write a survey on some conjugate gradient success stories of the past thirty years, particularly in Fluid Mechanics and in Control; however, after the first author of this article brought to the attention of his colleagues a (challenging) *system of Eikonal equations* with Dirichlet boundary conditions, modeling *anisotropic microstructural phenomena from Material Science,* it was decided to report on this new problem, where *conjugate gradient algorithms for the fast solution of obstacle problems* play a fundamental role.

2 An obstacle problem for the Laplace operator: Motivation and generalizations

In Section 9 of this article, we will show that *non-negative* solutions to the following *system* of *Eikonal equations*, with *Dirichlet boundary conditions,*

$$u \in H_0^1(\Omega), \quad \left|\frac{\partial u}{\partial x_i}\right| = 1 \quad a.e., \ \forall i = 1, \ldots, d, \tag{2.1}$$

can be obtained via an algorithm (of the operator-splitting type) requiring at each iteration (in fact time step, and among other things) the solution of an *obstacle problem* of the following type:

$$\int_\Omega \nabla u \cdot \nabla (v - u)\, \mathrm{d}x \geq \int_\Omega \mathbf{p} \cdot \nabla (v - u)\, \mathrm{d}x \quad \forall v \in K^+; \quad u \in K^+. \tag{2.2}$$

In (2.1), (2.2):
- Ω is a bounded domain of $\mathbb{R}^d$ ($d \geq 1$) with boundary Γ and $\mathrm{d}x = \mathrm{d}x_1 \ldots \mathrm{d}x_d$;
- $\mathbf{p}$ is a given vector-valued function of $(L^2(\Omega))^d$;
- $K^+ = \{v \mid v \in H_0^1(\Omega), \ v \geq 0, \ \text{a.e. on } \Omega\}$.

Let us mention immediately that in the context of problem (2.1) the over-relaxation methods with projection, and Uzawa type algorithms, discussed in, e.g., [3] and [4] perform pretty poorly proving too slow. Looking for alternatives it was decided to investigate an approach combining (exterior) *penalty* and *Newton's method.* Actually, with this combination one has been able to simulate (see [5], [6] for details) the vibrations of strings and beams, when their motion is constrained by *obstacles*; however, the above were obstacle problems in *one space dimension,* where the linear systems resulting from the combination penalty/Newton (after appropriate finite element discretizations) could be solved easily by *direct methods* taking advantage of the sparsity and band structure of the corresponding matrices. Since some of the finite element meshes (or finite difference grids) associated to the solution of problem (2.1) involve more than 10^6 points, it is clear that, as today, direct methods are not a feasible option as components of the penalty/Newton solution of problem (2.2) (for most practitioners at least). It will be shown in this article that replacing the above direct methods by a well-chosen *conjugate gradient algorithm*, leads to a *fast converging* iterative method, the only requirement being the access to a *fast solver* for *linear second order elliptic boundary value problems*. Since vector $\mathbf{p}$ in (2.2) is provided by the solution of an initial value problem coupled – in some sense – to (2.2), we shall take as model problem the *parabolic variational inequality* below (where $0 < T \leq +\infty$):

$$u(0) = u_0 \ (\in K), \tag{2.3}$$

$$\left.\begin{aligned} &\text{for } t \in (0,T), \ \mathbf{u}(t) \in K \text{ and}, \forall v \in K, \\ &\left\langle \frac{\partial}{\partial t} u(t), v - u(t) \right\rangle + \int_\Omega \nabla u(t) \cdot \nabla (v - u(t))\,dx \\ &\qquad \geq \langle f(t), v - u(t) \rangle; \end{aligned}\right\} \tag{2.4}$$

in (2.3), (2.4), $f \in L^2(0,T; H^{-1}(\Omega))$, $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$, and the convex K is defined by

$$K = \{ v \mid v \in H_0^1(\Omega), \quad v \geq \psi \text{ a.e. on } \Omega \}, \tag{2.5}$$

with $\psi \in C^0(\overline{\Omega}) \cap H^1(\Omega)$ and $\psi \leq 0$ on the boundary Γ of Ω. It follows from, e.g., [7] that the obstacle problem (2.3), (2.4) has a unique solution in $L^2(0,T; H_0^1(\Omega)) \cap C^0([0,T]; L^2(\Omega))$.

3 Backward Euler time discretization of (2.3) and (2.4)

Let Δt (> 0) be a time-discretization step. The *backward Euler time discretization* of problem (2.3), (2.4) leads to

$$u^0 = u_0; \tag{3.1}$$

then, for $n \geq 0$, u^n being known, we obtain u^{n+1} from the solution of

$$\left.\begin{aligned} &u^{n+1} \in K; \ \forall v \in K \\ &\int_\Omega \frac{u^{n+1} - u^n}{\Delta t}(v - u^{n+1})\,dx + \int_\Omega \nabla u^{n+1} \cdot \nabla (v - u^{n+1})\,dx \\ &\quad \geq \langle f^{n+1}, v - u^{n+1} \rangle. \end{aligned}\right\} \tag{3.2}$$

The *elliptic variational inequality problem* (3.2) has a unique solution. Problem (3.2) is, $\forall n \geq 0$, of the following type

$$\int_\Omega u(v-u)\,\mathrm{d}x + \mu \int_\Omega \nabla u \cdot \nabla(v-u)\,\mathrm{d}x \geq \langle f, v-u\rangle \quad \forall v \in K; \quad u \in K \tag{3.3}$$

with μ a *positive constant* and $f \in H^{-1}(\Omega)$.

4 Penalty approximation of (3.3)

Let ε be a *positive parameter* and denote $\sup(0,-v)$ by v^-. We approximate problem (3.3) by

$$\left.\begin{aligned}&\int_\Omega u^\varepsilon v\,\mathrm{d}x + \mu\int_\Omega \nabla u^\varepsilon \cdot \nabla v\,\mathrm{d}x - \varepsilon^{-1}\int_\Omega ((u^\varepsilon-\psi)^-)^2 v\,\mathrm{d}x = \langle f, v\rangle,\\ &\forall v \in H_0^1(\Omega);\quad u^\varepsilon \in H_0^1(\Omega).\end{aligned}\right\} \tag{4.1}$$

Problem (4.1) is obtained by *penalization* of the constraint $v-\psi \geq 0$ a.e. in Ω, via the introduction of the functional $(\varepsilon^{-1}/3)\int_\Omega((v-\psi)^-)^3\,\mathrm{d}x$ and its differentiation with respect to v. Indeed, (4.1) is the *Euler-Lagrange equation* of the following problem of the *Calculus of Variations*:

$$J_\varepsilon(u^\varepsilon) \leq J_\varepsilon(v) \quad \forall v \in H_0^1(\Omega); \quad u^\varepsilon \in H_0^1(\Omega), \tag{4.2}$$

with

$$J_\varepsilon(v) = \frac{1}{2}\int_\Omega v^2\,\mathrm{d}x + \frac{\mu}{2}\int_\Omega |\nabla v|^2\,\mathrm{d}x - \langle f, v\rangle + \frac{\varepsilon^{-1}}{3}\int_\Omega((v-\psi)^-)^3\,\mathrm{d}x.$$

Using *Convex Analysis* methods (see, e.g., [8], [9]) we can easily show that problem (4.1), (4.2) has a unique solution; moreover $\lim_{\varepsilon\to 0^+}\|u^\varepsilon - u\|_{H^1(\Omega)} = 0$, with u the solution of problem (3.3).

5 Newton's method for the solution of (4.1) and (4.2)

Let us drop the superscript ε; problem (4.1) can also be written as:

$$F_\varepsilon(u) = 0, \tag{5.1}$$

where $F_\varepsilon \colon H_0^1(\Omega) \to H^{-1}(\Omega)$ is defined by

$$F_\varepsilon(v) = v - \mu\Delta v - \varepsilon^{-1}((v-\psi)^-)^2 - f, \ \forall v \in H_0^1(\Omega). \tag{5.2}$$

Applying Newton's method to the solution of problem (5.1) leads (with obvious notation) to:

$$u^0 \text{is given in } H_0^1(\Omega) \quad (u^0 \in K \textit{ is recommended}); \tag{5.3}$$

for $m \geq 0$, u^m *being known, compute* u^{m+1} *via*

$$u^{m+1} = u^m - F_\varepsilon'(u^m)^{-1} F_\varepsilon(u^m), \tag{5.4}$$

where

$$\left.\begin{aligned}&\forall v, w \in H_0^1(\Omega), \ \langle F_\varepsilon'(u^m)v, w\rangle = \\ &\int_\Omega vw \, dx + \mu \int_\Omega \nabla v \cdot \nabla \, dx + 2\varepsilon^{-1} \int_\Omega (u^m - \psi)^- vw \, dx.\end{aligned}\right\} \tag{5.5}$$

It follows from (5.2), (5.5) that (5.4) is equivalent to the following (well-posed) *linear variational problem:*

$$\left.\begin{aligned}&\overline{u}^m \in H_0^1(\Omega); \ \forall v \in H_0^1(\Omega), \\ &\int_\Omega \overline{u}^m v \, dx + \mu \int_\Omega \nabla \overline{u}^m \cdot \nabla v \, dx + 2\varepsilon^{-1} \int_\Omega (u^m - \psi)^- \overline{u}^m v \, dx \\ &= \int_\Omega u^m v \, dx + \mu \int_\Omega \nabla u^m \cdot \nabla v \, dx - \varepsilon^{-1} \\ &\times \int_\Omega ((u^m - \psi)^-)^2 v \, dx - \langle f, v\rangle,\end{aligned}\right\} \tag{5.6}$$

with $\overline{u}^m = u^m - u^{m+1}$. The *finite element implementation* of Newton's algorithm (5.3), (5.6), and the *conjugate gradient solution* of the discrete analogues of problem (5.6) will be discussed in the following Sections 6 and 7.

6 Finite element implementation of the Newton's algorithm (5.3), (5.6)

From now on, we suppose that Ω is a bounded polygonal domain of $\mathbb{R}^2$; we introduce next a finite element triangulation $\mathcal{T}_h$ of $\overline{\Omega}$ (with h the length of the largest edge(s) of the triangles of $\mathcal{T}_h$) and approximate $H_0^1(\Omega)$, K, (3.3), (4.1), and the Newton's algorithm (5.3), (5.6) by, respectively,

$$V_{0h} = \{v_h \mid v_h \in C^0(\overline{\Omega}), \quad v_h|_T \in P_1, \ \forall T \in \mathcal{T}_h, \ v_h = 0 \text{ on } \Gamma\} \tag{6.1}$$

(with P_1 the space of the polynomials in two variables of degree ≤ 1),

$$K_h = \{v_h \mid v_h \in V_{0h}, \quad v_h(P) \geq \psi(P), \quad \forall P \in \Sigma_{0h}\} \tag{6.2}$$

(with $\Sigma_{0h} = \{P \mid P \text{ is a vertex of } \mathcal{T}_h,\ P \notin \Gamma\}$),

$$\left.\begin{array}{l} u_h \in K_h; \quad \forall v_h \in K_h \\ \displaystyle\int_\Omega u_h(v_h - u_h)\,\mathrm{d}x + \mu \int_\Omega \nabla u_h \cdot \nabla(v_h - u_h)\,\mathrm{d}x \geq \langle f, v_h - u_h\rangle, \end{array}\right\} \tag{6.3}$$

$$\left.\begin{array}{l} \displaystyle\int_\Omega u_h^\varepsilon v_h\,\mathrm{d}x + \mu \int_\Omega \nabla u_h^\varepsilon \cdot \nabla v_h\,\mathrm{d}x - (\varepsilon^{-1}/3) \\ \quad \times \sum_{P\in\Sigma_{0h}} A_P((u_h^\varepsilon(P) - \psi(P))^-)^2 v_h(P) \\ = \langle f, v\rangle \quad \forall v_h \in V_{0h}; \ u_h^\varepsilon \in V_{0h}, \end{array}\right\} \tag{6.4}$$

and (dropping some of the subscripts h):

$$u^0 \textit{ is given in } V_{0h} \ (\textit{in } K_h \textit{ if possible}); \tag{6.5}$$

for $m \geq 0$, u^m being known, denote $u^m - u^{m+1}$ by $\overline{u}^m$ and solve

$$\left.\begin{array}{l} \overline{u}^m \in V_{0h}; \quad \forall v \in V_{0h} \\ \displaystyle\int_\Omega \overline{u}^m v\,\mathrm{d}x + \mu \int_\Omega \nabla\overline{u}^m \cdot \nabla v\,\mathrm{d}x + \frac{2\varepsilon^{-1}}{3} \\ \quad \times \sum_{P\in\Sigma_{0h}} A_P(u^m(P) - \psi(P))^- \overline{u}^m(P) v(P) \\ \displaystyle = \int_\Omega u^m v\,\mathrm{d}x + \mu \int_\Omega \nabla u^m \cdot \nabla v\,\mathrm{d}x - \frac{\varepsilon^{-1}}{3} \\ \quad \displaystyle\times \sum_{P\in\Sigma_{0h}} A_P((u^m(P) - \psi(P))^-)^2 v(P) - \langle f, v\rangle; \end{array}\right\} \tag{6.6}$$

in (6.4), (6.6), A_P is the measure of the polygonal union of the triangles of $\mathcal{T}_h$ which have P as a common vertex (the penalty related terms in (6.4) and (6.6) have been obtained by approximating $\int_\Omega ((u^\varepsilon - \psi)^-)^2 v\,\mathrm{d}x$, and other similar integrals, by the *trapezoidal rule*, in order to "diagonalize" the matrices associated to the penalty treatment of the condition $u \geq \psi$ on Ω; actually, the trapezoidal rule can also be employed to approximate the various $L^2(\Omega)$-scalar products encountered in (6.3), (6.4) and (6.6)).

7 Conjugate gradient solution of problem (6.6)

Problem (6.6) is equivalent to a linear system of the following form

$$AX + \varepsilon^{-1}DX = b, \tag{7.1}$$

where, in (7.1), A is an $N \times N$ matrix, symmetric and positive definite, D is a diagonal $N \times N$ matrix, positive semi-definite, and $b \in \mathbb{R}^N$. Define Y by

$$Y = \varepsilon^{-1}D^{1/2}X; \tag{7.2}$$

system (7.1) can be reformulated as

$$\varepsilon Y + D^{1/2}A^{-1}D^{1/2}Y = D^{1/2}A^{-1}b. \tag{7.3}$$

Matrix $\varepsilon I + D^{1/2}A^{-1}D^{1/2}$ being *symmetric* and *positive definite* it makes sense to attempt solving problem (7.3) by a *conjugate gradient algorithm*. *Without preconditioning*, such an algorithm reads as follows (where $V \cdot W = \sum_{i=1}^{N} V_i W_i$, $\forall V = \{V_i\}_{i=1}^{N}$, $W = \{W_i\}_{i=1}^{N} \in \mathbb{R}^N$):

$$Y^0 \textit{ is given in } \mathbb{R}^N \quad (Y^0 = 0, \textit{ for example}); \tag{7.4}$$

solve

$$Ar^0 = D^{1/2}Y^0 - b, \tag{7.5}$$

and set

$$g^0 = \varepsilon Y^0 + D^{1/2}r^0, \tag{7.6}$$

$$w^0 = g^0. \tag{7.7}$$

For $k \geq 0$, assuming that Y^k, g^k, w^k are known, solve

$$A\bar{r}^k = D^{1/2}w^k, \tag{7.8}$$

and set

$$\bar{g}^k = \varepsilon w^k + D^{1/2}\bar{r}^k. \tag{7.9}$$

Compute

$$\rho_k = g^k \cdot g^k / \bar{g}^k \cdot w^k, \tag{7.10}$$

$$Y^{k+1} = Y^k - \rho_k w^k, \tag{7.11}$$

$$g^{k+1} = g^k - \rho_k \bar{g}^k. \tag{7.12}$$

If $g^{k+1} \cdot g^{k+1}/g^0 \cdot g^0 \leq \eta$, take $X = A^{-1}(D^{1/2}Y^{k+1} - b)$; else, compute

$$\gamma_k = g^{k+1} \cdot g^{k+1}/g^k \cdot g^k, \tag{7.13}$$

$$w^{k+1} = g^{k+1} + \gamma_k w^k. \tag{7.14}$$

Do $k = k+1$ and return to (7.8).

It follows from (7.5) and (7.8) that each iteration of algorithm (7.4)–(7.14) requires the solution of a linear system associated to matrix A, i.e., in the context of the discrete obstacle problem (6.3), of a linear system associated to a discrete analogue of operator $I - \mu\Delta$ with Dirichlet boundary conditions, a classical problem indeed. Concerning the *speed of convergence* of algorithm (7.4)–(7.14), it can be shown that, in the neighborhood of the solution of problem (6.4), the *condition number* ν of the corresponding matrix $\varepsilon I + D^{1/2}A^{-1}D^{1/2}$ is $O(\varepsilon^{-1/2})$, implying (from the classical relation $\|Y^k - Y\| \leq C\left(\frac{\sqrt{\nu}-1}{\sqrt{\nu}+1}\right)^k \|Y^0 - Y\|$, $\forall k \geq 1$) that, in fact, the speed of convergence of (7.4)–(7.14) is "controlled" by $\varepsilon^{-1/4}$, a not too small number, even if ε is small (of the order of 10^{-4}, for example). From this observation, we can expect a reasonably fast convergence of algorithm (7.4)–(7.14); the numerical experiments to be reported in Section 8 confirm this prediction.

Remark 7.1. The methodology discussed in Sections 4 to 7 can be easily modified to handle problem (2.2). Actually, it will be quite successful (as we shall see in Section 13) at finding non-negative solutions to problem (2.1), a much more complicated obstacle problem indeed.

8 Application to the solution of a time dependent obstacle problem: Numerical experiments and results

In order to validate the methodology discussed in Sections 2 to 7, we consider the variant of problem (2.3), (2.4) with $\Omega = (0,1) \times (0,1)$, $f = C$, and $K = \{v \mid v \in H_0^1(\Omega),\ v(x) \leq \delta(x,\Gamma),\ \text{a.e., on } \Omega\}$; here $\delta(x,\Gamma)$ = distance from x to the boundary Γ of Ω. The above problem is well documented and related to the elasto-plastic torsion of a infinitely long cylinder of cross section Ω, C being a torsion angle per unit length (see, e.g., [3], [4] for details). To approximate the solution of this problem we have used a finite element approximation like the one discussed in Section 6, with V_{0h} and K_h defined from a uniform triangulation $\mathcal{T}_h$ of Ω, allowing the use of fast elliptic solvers in algorithm (7.4)–(7.14).

ε	N_{Newton}	N_{CG}	L^2-error
0.0032	4	3	8.9604339×10^{-2}
0.0016	5	3	6.9899789×10^{-2}
0.0008	5	4	5.3565620×10^{-2}
0.0004	5	4	4.0485722×10^{-2}
0.0002	6	4	3.0261847×10^{-2}
0.0001	6	5	2.2398988×10^{-2}
0.00005	7	5	1.6440081×10^{-2}
0.000025	9	6	1.1985428×10^{-2}
0.0000125	11	6	8.6906453×10^{-3}

Table 1

All calculations have been done with $\Delta t = 2.5 \times 10^{-4}$, $h = 1/256$, and $C = 10$. The Newton's iterations have been stopped when $\sum_{i,j} |\bar{u}_{m,ij}| \leq 10^{-4}$. In the conjugate gradient algorithm we have taken 10^{-6} for η in the stopping criterion. Finally concerning the time discretization scheme itself, we consider that a steady state has been reached when $\sum_{i,j} |u_{ij}^{n+1} - u_{ij}^{n}| \leq 10^{-4}$. In Tab. 1, we have shown the maximal numbers of iterations in the Newton's iteration and the conjugate gradient algorithm and the L^2-error (compared with the solution obtained with same parameters by the relaxation method discussed in [4]) with different choices of the penalty parameter ε. It is clear from Tab. 1 that both the Newton's and conjugate gradient methods have fast convergence properties. There is no doubt that the fact that we initialize Newton's method with the solution obtained at the previous time step is an important factor of this good convergence property. However, when applying a similar initialization strategy for SOR projection when solving problem (2.1) the convergence is quite slow while the penalty/Newton/conjugate gradient approach performs very efficiently: the good initial guess does not explain everything. Our final comment is that the L^2-error in Tab. 1 behaves essentially like $\sqrt{\varepsilon}$ which is a result we were expecting.

9 A system of Eikonal equations with Dirichlet conditions: Related variational problems

Motivated by the analysis of nonlinear models from *μ-Mechanics* and *Material Sciences*, the first author of this article has been lead to investigate (see [10] for details) the properties of the solution of the following nonlinear boundary value problem:

$$\text{Find } u \in H_0^1(\Omega) \text{ such that } \left|\frac{\partial u}{\partial x_i}\right| = 1 \text{ a.e. on } \Omega, \quad \forall i = 1, \ldots, d, \tag{9.1}$$

where, in (9.1), Ω is a bounded domain of $\mathbb{R}^d$ $(d \geq 1)$ with the boundary Γ; problem (9.1) can be viewed as a boundary value problem for a *system of Eikonal equations.* Since problem (9.1) has an infinity of solutions, we are going to focus on those solutions which *maximize* (or "nearly" maximize) functional $v \to \int_\Omega v\,\mathrm{d}x$. This leads us to consider the following problem of the *Calculus of Variations*:

$$u \in E; \quad \int_\Omega u\,\mathrm{d}x \geq \int_\Omega v\,\mathrm{d}x \quad \forall v \in E, \tag{9.2}$$

with $E = \{v \mid v \in H_0^1(\Omega), \left|\frac{\partial v}{\partial x_i}\right| = 1 \text{ a.e.}, \forall i = 1, \ldots, d\}$. If $d = 2$ and $\Omega = \{x \mid x = (x_1, x_2), \ |x_1 \pm x_2| < 1\}$, the unique solution of problem (9.2) is given by $u(x) = 1 - |x_1| - |x_2|$ for all $x \in \Omega$; on the other hand, problem (2.1) has no solution in general. We can easily show that if u is a solution of (9.2), then

$u(x) \geq 0$ on $\overline{\Omega}$, moreover, since $|\nabla v|^2 = d$ a.e. in Ω if $v \in E$, problem (9.2) is equivalent to:

$$u \in E^+; \quad J(u) \leq J(v), \quad \forall v \in E^+, \tag{9.3}$$

where

$$E^+ = \{v \mid v \in E, \quad v \geq 0 \text{ on } \overline{\Omega}\}, \tag{9.4}$$

and

$$J(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 \, dx - C\int_\Omega v \, dx, \tag{9.5}$$

where, in (9.5), C is an (arbitrary) *positive constant.* There are, in principle, several (if not many) ways to solve *numerically* problem (9.2), (9.3); motivated by some earlier work on the numerical solution of a *Ginzburg-Landau* equation (see refs. [11], [12]), we are going to treat the constraints $\left|\frac{\partial v}{\partial x_i}\right| = 1$, $\forall i = 1, \ldots, d$, by a (exterior) *penalty* method. Moreover, in order to "control" mesh-related oscillations, we are going to bound $\|\Delta v\|_{L^2(\Omega)}$. This leads us to approximate the functional $J(\cdot)$ in (9.3), (9.5) by $J^\varepsilon(\cdot)$ defined as follows:

$$J^\varepsilon(v) = (\varepsilon_1/2)\int_\Omega |\Delta v|^2 \, dx + J(v) + (\varepsilon_2^{-1}/4)\sum_{i=1}^{d}\int_\Omega \left(\left|\frac{\partial v}{\partial x_i}\right|^2 - 1\right)^2 dx; \tag{9.6}$$

in (9.6), $\varepsilon = \{\varepsilon_1, \varepsilon_2\}$, ε_1 and ε_2 being both *positive* and *"small".* We then approximate (9.2), (9.3) by

$$u^\varepsilon \in K^+; \quad J^\varepsilon(u^\varepsilon) \leq J^\varepsilon(v), \quad \forall v \in H^2(\Omega) \cap K^+, \tag{9.7}$$

with

$$K^+ = \{v \mid v \in H_0^1(\Omega), \quad v \geq 0 \text{ a.e. on } \Omega\}. \tag{9.8}$$

Proving that problem (9.7) (a "beautiful" problem from Calculus of Variations, indeed) has at least a solution is an (almost) elementary exercise. The *numerical solution* of the (*obstacle*) problem (9.7) is the main objective of the remainder of this article.

Remark 9.1. *Augmented Lagrangian methods* (closely related to those discussed in, e.g., [1]–[4], [13]) can be applied to the solution of problem (9.7). Preliminary results look promising and will be reported elsewhere.

Remark 9.2. Suppose that one looks for a solution of problem (9.1) as close as possible (in $H_0^1(\Omega)$) of $\varphi_d \in H_0^1(\Omega)$; in that case, we should take in (9.5), (9.6), the functional $J(\cdot)$ defined by $J(v) = (1/2)\int_\Omega |\nabla v|^2\,dx - \int_\Omega \nabla\varphi_d \cdot \nabla v\,dx$.

10 An equivalent variational formulation of problem (9.7) and an associated boundary value problem

Let us denote $(L^2(\Omega))^d$ by Λ and ∇u^ε by $\mathbf{p}^\varepsilon$; problem (9.7) is clearly *equivalent* to

$$\mathbf{p}^\varepsilon \in \Lambda; \quad j^\varepsilon(\mathbf{p}^\varepsilon) \le j^\varepsilon(\mathbf{q}), \ \forall \mathbf{q} \in \Lambda, \tag{10.1}$$

with $\mathbf{q} = \{q_i\}_{i=1}^d$, and, for all $\mathbf{q} \in \Lambda$

$$\begin{aligned} j^\varepsilon(\mathbf{q}) &= (1/2)\int_\Omega |\mathbf{q}|^2\,dx - C\int_\Omega \nabla\varphi_1 \cdot \mathbf{q}\,dx \\ &\quad + (\varepsilon_2^{-1}/4)\sum_{i=1}^d \int_\Omega (|q_i|^2 - 1)^2\,dx + I_+(\mathbf{q}); \end{aligned} \tag{10.2}$$

in (10.2), φ_1 is the unique solution in $H_0^1(\Omega)$ of the Dirichlet problem $-\Delta\varphi_1 = 1$ in Ω, $\varphi_1 = 0$ on Γ, and $I_+(\cdot)$ is the functional defined as follows:

$$I_+(\mathbf{q}) = \begin{cases} (\varepsilon_1/2)\displaystyle\int_\Omega |\nabla \cdot \mathbf{q}|^2\,dx & \text{if } \mathbf{q} \in \nabla(H^2(\Omega) \cap K^+), \\ +\infty & \text{elsewhere.} \end{cases} \tag{10.3}$$

It is easy to show that the functional $I_+(\cdot)$ is *convex, proper*, and *lower semi-continuous* over the space Λ. The *variational formulation* of the (formal) *Euler-Lagrange equation* associated to problem (10.1) reads as follows:

$$\left.\begin{aligned} &\textstyle\int_\Omega \mathbf{p}^\varepsilon \cdot \mathbf{q}\,dx + \varepsilon_2^{-1}\sum_{i=1}^d \int_\Omega (|p_i^\varepsilon|^2 - 1)p_i^\varepsilon q_i\,dx + \langle \partial I_+(\mathbf{p}^\varepsilon), \mathbf{q}\rangle \\ &= C\textstyle\int_\Omega \nabla\varphi_1 \cdot \mathbf{q}\,dx, \quad \forall \mathbf{q} \in \Lambda; \quad \mathbf{p}^\varepsilon \in \Lambda, \end{aligned}\right\} \tag{10.4}$$

with $\partial I_+(\cdot)$ the *sub-differential* of $I_+(\cdot)$. We associate to equation (10.4) the following *initial value problem*

$$\left.\begin{aligned} &\mathbf{p}^\varepsilon(0) = \mathbf{p}_0, \\ &\textstyle\int_\Omega (\partial \mathbf{p}^\varepsilon/\partial t) \cdot \mathbf{q}\,dx + \int_\Omega \mathbf{p}^\varepsilon \cdot \mathbf{q}\,dx \\ &\quad + \varepsilon_2^{-1}\textstyle\sum_{i=1}^d \int_\Omega (|p_i^\varepsilon|^2 - 1)p_i^\varepsilon q_i\,dx + \langle \partial I_+(\mathbf{p}^\varepsilon), \mathbf{q}\rangle \\ &= C\textstyle\int_\Omega \nabla\varphi_1 \cdot \mathbf{q}\,dx, \quad \forall \mathbf{q} \in \Lambda, \ t > 0. \end{aligned}\right\} \tag{10.5}$$

From now on, our goal is to "capture" the *steady state solutions* of (10.5); to achieve this objective, we shall time-discretize (10.5) by an *operator-splitting* *scheme à la Marchuk-Yanenko* (simply because it is the simplest splitting-scheme we can think of; more sophisticated splitting-schemes will be discussed elsewhere).

11 Time discretization of problem (10.5) by operator splitting

Let Δt (> 0) be a time-discretization step; dropping the superscript ε, we time-discretize (10.5) by the following *operator-splitting scheme* (of the *Marchuk-Yanenko* type):

$$\mathbf{p}^0 = \mathbf{p}_0 \quad (= \mathbf{0}, \text{ or } \nabla\varphi_1, \quad \text{for example}), \tag{11.0}$$

then, for $n \geq 0$, assuming $\mathbf{p}^n$ known, solve successfully:

$$\left.\begin{aligned} &(\Delta t)^{-1}\int_\Omega (\mathbf{p}^{n+1/2} - \mathbf{p}^n)\cdot\mathbf{q}\,\mathrm{d}x + \int_\Omega \mathbf{p}^{n+1/2}\cdot\mathbf{q}\,\mathrm{d}x \\ &+\varepsilon_2^{-1}\textstyle\sum_{i=1}^d \displaystyle\int_\Omega (|p_i^{n+1/2}|^2 - 1)p_i^{n+1/2}q_i\,\mathrm{d}x = C\int_\Omega \nabla\varphi_1\cdot\mathbf{q}\,\mathrm{d}x, \\ &\forall\mathbf{q}\in\Lambda;\ \mathbf{p}^{n+1/2}\in\Lambda, \end{aligned}\right\} \tag{11.1}$$

$$\left.\begin{aligned} &(\Delta t)^{-1}\int_\Omega (\mathbf{p}^{n+1} - \mathbf{p}^{n+1/2})\cdot\mathbf{q}\,\mathrm{d}x + \langle\partial I_+(\mathbf{p}^{n+1}),\mathbf{q}\rangle = 0, \\ &\forall\mathbf{q}\in\Lambda;\ \mathbf{p}^{n+1}\in\Lambda. \end{aligned}\right\} \tag{11.2}$$

Problem (11.1) has a *unique* solution "as soon" as $\Delta t \leq \varepsilon_2$; moreover, (11.1) can be solved "pointwise", since for all $i = 1,\ldots,d$, and a.e. on Ω, $p_i^{n+1/2}(x)$ is the solution of a cubic equation of the following type

$$(1 - \Delta t\varepsilon_2^{-1} + \Delta t)z + \Delta t\varepsilon_2^{-1}z^3 = \text{RHS}. \tag{11.3}$$

If $\varepsilon_2 \geq \Delta t$, problem (11.3) has a unique solution which can be easily solved by *Newton's method.* On the other hand, the solution of problem (11.2) is given by

$$\mathbf{p}^{n+1} = \nabla u^{n+1}, \tag{11.4}$$

function u^{n+1} being the solution of the following *well-posed elliptic variational inequality*

$$\left.\begin{aligned} &\int_\Omega \nabla u^{n+1}\cdot\nabla(v - u^{n+1})\,\mathrm{d}x + \Delta t\varepsilon_1\int_\Omega \Delta u^{n+1}\Delta(v - u^{n+1})\,\mathrm{d}x \\ &\geq \int_\Omega \mathbf{p}^{n+1/2}\cdot\nabla(v - u^{n+1})\,\mathrm{d}x,\ \ \forall v\in H^2(\Omega)\cap K^+; \\ &\quad u^{n+1}\in H^2(\Omega)\cap K^+. \end{aligned}\right\} \tag{11.5}$$

In order to facilitate the numerical solution of problem (11.5), we perform a (variational) crime by *"approximately" factoring* the above problem as follows (other approximate factorizations are possible):

$$\left.\begin{aligned} &\omega^{n+1} \in K^+; \ \forall v \in K^+, \\ &\int_\Omega \nabla\omega^{n+1} \cdot \nabla(v - \omega^{n+1})\, dx \geq \int_\Omega \mathbf{p}^{n+1/2} \cdot \nabla(v - \omega^{n+1})\, dx, \end{aligned}\right\} \quad (11.6)$$

$$u^{n+1} - \Delta t \varepsilon_1 \Delta u^{n+1} = \omega^{n+1} \text{ in } \Omega, \ u^{n+1} = 0 \text{ on } \Gamma. \quad (11.7)$$

From the *non-negativity* of ω^{n+1}, the *maximum principle for second order elliptic operators implies* (via (11.7)) the *non-negativity* of u^{n+1}.

Remark 11.1. There is no basic difficulty in solving problem (11.5) numerically, but the approximate factorization (11.6), (11.7) of the above problem produces smooth non-negative solutions, as (11.5) does, while leading to faster algorithms. Indeed, $u^{n+1} \in H^3(\Omega) \cap K^+$ if Γ is smooth enough.

12 Finite element implementation of scheme (11.0), (11.1), (11.6), (11.7), (11.4)

Since any solution of problem (9.1) is *continuous* and *piecewise affine*, it makes sense to compute these solutions via *globally continuous, piecewise affine finite element approximations.* From now on, we shall assume that Ω is a bounded *polygonal* domain of $\mathbb{R}^2$ (the non-polygonal case is almost as easy to treat). Let $\mathcal{T}_h$ be a *finite element triangulation* of $\overline{\Omega}$, like those considered in Section 6. We approximate $L^2(\Omega)$ (and $H^1(\Omega)$), $H_0^1(\Omega)$, K^+, and Λ by:

$$V_h = \{v_h \mid v_h \in C^0(\overline{\Omega}), \quad v_h|_T \in P_1, \ \forall T \in \mathcal{T}_h\}, \quad (12.1)$$

$$V_{0h} = V_h \cap H_0^1(\Omega) = \{v_h \mid v_h \in V_h, \quad v_h = 0 \text{ on } \Gamma\}, \quad (12.2)$$

$$K_h^+ = \{v_h \mid v_h \in V_{0h}, \ v_h(P) \geq 0, \ \forall P \text{ vertex of } \mathcal{T}_h\}, \quad (12.3)$$

and

$$\Lambda_h = \{\mathbf{q}_h \mid \mathbf{q}_h \in \Lambda, \quad \mathbf{q}_h|_T \in \mathbb{R}^2, \ \forall T \in \mathcal{T}_h\}, \quad (12.4)$$

respectively, with P_1 the space of linear polynomials in x_1, x_2. We clearly have $\nabla V_h \subset \Lambda_h$ and $\dim(\Lambda_h) = 2\,\mathrm{Card}(\mathcal{T}_h)$. A *finite element implementation* of scheme (11.0), (11.1), (11.6), (11.7), (11.4), corresponding to (12.1)–(12.4), reads as follows (with $m(T) = \int_T dx$, and other obvious notation):

$$\mathbf{p}_h^0 = \mathbf{p}_{0h}, \quad \textit{with } \mathbf{p}_{0h} \textit{ given in } \Lambda_h; \quad (12.5)$$

for $n \geq 0$, $\mathbf{p}_h^n$ being known, compute $\mathbf{p}_h^{n+1/2}$, ω_h^{n+1}, u_h^{n+1}, and $\mathbf{p}_h^{n+1}$ as follows:

$$\left.\begin{aligned}(p_{iT}^{n+1/2} - p_{iT}^n)/\Delta t + p_{iT}^{n+1/2} + \varepsilon_2^{-1}(|p_{iT}^{n+1/2}|^2 - 1)p_{iT}^{n+1/2}\\ = [C/m(T)] \int_T \frac{\partial \varphi_1}{\partial x_i}\, dx, \ \forall i = 1, 2, \ \forall T \in \mathcal{T}_h,\end{aligned}\right\} \tag{12.6}$$

$$\left.\begin{aligned}&\omega_h^{n+1} \in K_h^+; \ \forall v_h \in K_h^+,\\ &\int_\Omega \nabla \omega_h^{n+1} \cdot \nabla(v_h - \omega_h^{n+1})\, dx \geq \int_\Omega \mathbf{p}_h^{n+1/2} \cdot \nabla(v_h - \omega_h^{n+1})\, dx,\end{aligned}\right\} \tag{12.7}$$

$$\left.\begin{aligned}&\mathbf{p}_h^{n+1} = \nabla u_h^{n+1}, \quad \textit{with } u_h^{n+1} \in V_{0h}, \textit{ and } \forall v_h \in V_{0h},\\ &\int_\Omega (u_h^{n+1} v_h + \Delta t \varepsilon_1 \nabla u_h^{n+1} \cdot \nabla v_h)\, dx = \int_\Omega \omega_h^{n+1} v_h\, dx.\end{aligned}\right\} \tag{12.8}$$

The cubic equations in (12.6) are particular cases of problem (11.3); they have a unique solution if $\Delta t \leq \varepsilon_2$, and can be easily solved by Newton's method. The *discrete obstacle problem* (12.7) is a pretty classical one, in principle; however, in the context of (the difficult) problem (9.7), it turns out that the solution methods based on *over-relaxation with projection* and *Uzawa algorithms with projections* (see, e.g., refs [3], [4] for details) have troubles to achieve fast convergence, forcing us to look for more efficient alternatives. Such an alternative is provided by the *penalty/Newton/conjugate gradient method* discussed in Sections 4 to 8. For the test cases to be considered in Section 13, the above method never failed and converged, always, in few iterations, even for small values of its penalty parameter. Finally, problem (12.8) is a fairly standard one (see, e.g., [4, Appendix 1] and the references therein), and does not deserve further comments, therefore, other than: a sufficient condition to have a *discrete maximum principle* is to have the angles of $\mathcal{T}_h$ not exceeding $\frac{\pi}{2}$ (a classical result indeed; see, e.g., [14]). To summarize:

We have reduced the approximate solution of problem (9.2) to that of a sequence of simple cubic equations and of, essentially, discrete linear elliptic equations.

Remark 12.1. Let us denote by $\{u_h^\varepsilon\}_h^\varepsilon$ the family of approximate solutions provided by algorithm (12.5)–(12.8). The role of the *biharmonic regularization (à la Tychonoff)* in problems (12.7), (12.8) is to enhance the *compactness* properties of $\{u_h^\varepsilon\}_h^\varepsilon$ (and both things being related) control spatial oscillations of short wave length (of order h, typically).

Remark 12.2. As can be expected, *biharmonic regularizations* are not new; they have been used in, e.g., [15] for the solution of boundary control problems for the wave equation, and in [16] for the solutions of problems such as (9.1), precisely (by methods quite different from those discussed in this article). Indeed, on the basis of *boundary layer thickness considerations* (similar to those in [15]) we can expect that the optimal value of $\Delta t\varepsilon_1$ in (12.8) has to be of order h^2; the numerical experiments to be reported in Section 13 will validate this prediction, showing that a near optimal value of $\Delta t\varepsilon_1$ is $h^2/36$, for all h's sufficiently small.

Remark 12.3. With minor modifications, the methodology discussed above can be applied to those variants of (9.1), where the boundary condition $u = 0$ on Γ is replaced by $u = g$, with g a *Lipschitz continuous function* such that $|g(x) - g(y)| \leq |x - y|$, $\forall x, y \in \Gamma$, and $\max_{\{x,y\}\in\Gamma}[g(x) - g(y)]$ not too large. Actually, for validation purposes, some of the test problems to be considered in Section 13 will involve non-zero Dirichlet data (i.e., $g \neq 0$) and will correspond to situations, where a unique maximal (or minimal) solution to problem (9.1) exists.

13 Numerical Experiments

For the test problems discussed below we took $\Omega = (0,1) \times (0,1)$ and used a uniform triangulation of Ω. From a computational point of view, this implies that the finite element approximations reduce to finite difference ones, allowing, for example, the use of *cyclic reduction based* on fast Poisson and Helmholtz solvers for the solution of the discrete elliptic problems encountered at each iteration of the numerical procedure discussed in the above sections. The main goal of the first and second test problems is to validate the numerical methodology, since both of them have closed form solutions. The third test problem is the one of interest, since the boundary conditions being incompatible with the constraints $|\partial u/\partial x_1| = |\partial u/\partial x_2| = 1$ a.e., we can expect a non-smooth behavior of the solutions (*fractal*, in fact) near the boundary.

First two test problems: They are defined by

$$\left|\frac{\partial u}{\partial x_1}\right| = \left|\frac{\partial u}{\partial x_2}\right| = 1 \quad \text{a.e. in } \Omega, \quad u = g \text{ on } \Gamma, \tag{13.1}$$

with

$$\left.\begin{aligned} g(x_1, 0) = g(x_1, 1) = \min(x_1, 1 - x_1), 0 \leq x_1 \leq 1, \\ g(0, x_2) = g(1, x_2) = \min(x_2, 1 - x_2), 0 \leq x_2 \leq 1. \end{aligned}\right\}$$

Here, the maximal (which turns out to be also a viscosity) solution of problem (13.1) is given by

$$u_{\max}(x_1, x_2) = 1 - \left|\frac{1}{2} - x_1\right| - \left|\frac{1}{2} - x_2\right| = \min(x_1, 1 - x_1) + \min(x_2, 1 - x_2).$$

The minimal solution is given by

$$u_{\min}(x_1, x_2) = \begin{cases} |x_1 - x_2|, \ \ 0 \le x_1, \ \ x_2 \le \frac{1}{2}, \ \ \text{and} \ \ \frac{1}{2} \le x_1, \ \ x_2 \le 1, \\ |1 - x_1 - x_2|, \ \ \frac{1}{2} \le x_1 \le 1, \ \ 0 \le x_2 \le \frac{1}{2}, \\ \qquad \qquad \text{and} \ \ 0 \le x_1 \le \frac{1}{2}, \ \ \frac{1}{2} \le x_2 \le 1. \end{cases}$$

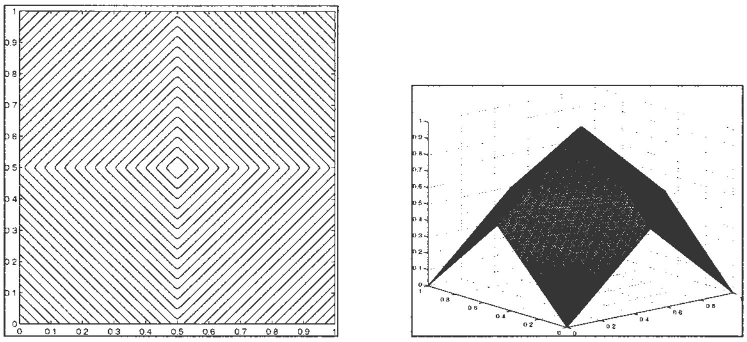

Figure 1. Contours (left) and graph (right) of the computed maximal solution of problem (13.1) ($h = 1/128$).

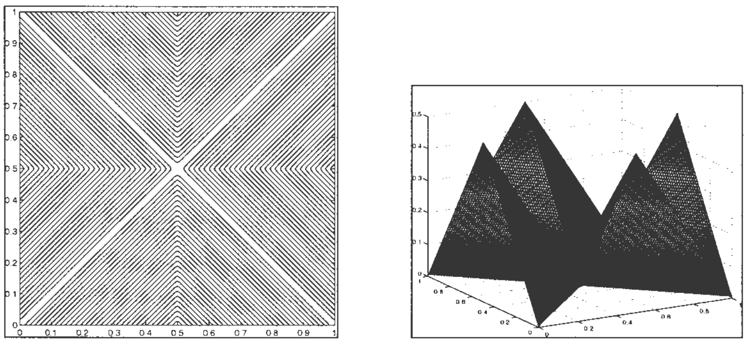

Figure 2. Contours (left) and graph (right) of the computed minimal solution of problem (13.1) ($h = 1/128$).

The computed solution u_h obtained with $h = 1/128$, $\Delta t \varepsilon_1 = h^2/36$, $\varepsilon_2 = 0.001$, $\Delta t = 0.0001$, and $C = 10$ for the maximal case (resp., $C = -10$ for the minimal case) has been visualized in Fig. 1 (resp., Fig. 2); it coincides quite well with the exact solution, since approximation errors are $\|u_h - u\|_{0,\Omega} = 3.268 \times 10^{-4}$ and $\|u_h - u\|_{\infty,\Omega} = 5.086 \times 10^{-3}$ (resp., $\|u_h - u\|_{0,\Omega} = 7.133 \times 10^{-4}$ and $\|u_h - u\|_{\infty,\Omega} = 5.715 \times 10^{-3}$).

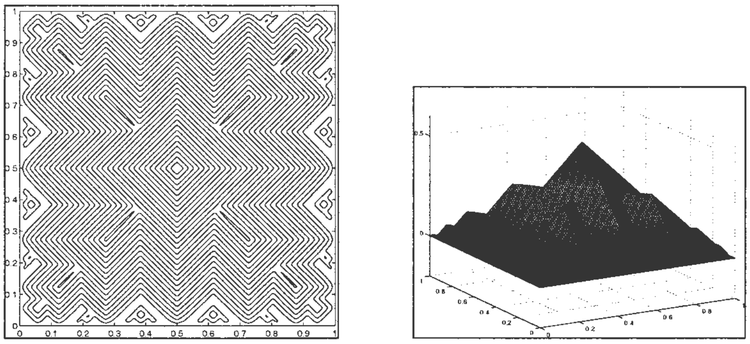

Figure 3. Contours (left) and graph (right) of the computed maximal solution of problem (2.1), (9.1) ($h = 1/512$).

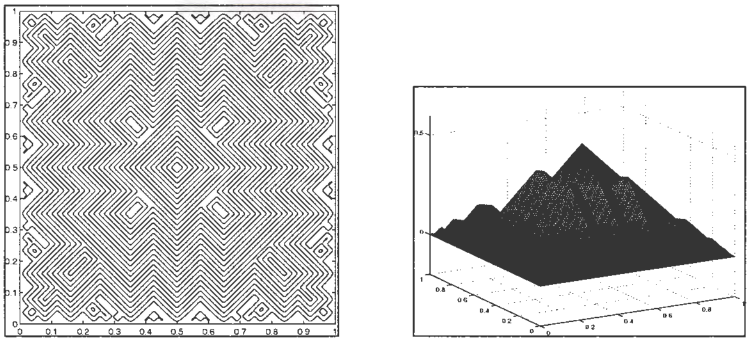

Figure 4. Contours (left) and graph (right) of the computed maximal solution of problem (2.1), (9.1) ($h = 1/1024$).

Third test problem: In this case we look for a nearly maximal solution of problem (2.1), (9.1), recalling that in this case maximal solutions do not exist. We have visualized in Figs. 3 and 4 the graphs and contours of the discrete solutions computed with $h = 1/512$ and $1/1024$. The other parameters are $\Delta t \varepsilon_1 = h^2/9$, $\varepsilon_2 = 0.001$, $C = 10$ and $\Delta t = 0.0001$. These figures show that as one refines the mesh new structures appear clearly showing (as expected) a fractal behavior near the boundary (and particularly in the corners). Incidentally, the L^1-norms corresponding to $h = 1/512$ and $h = 1/1024$ are 0.142263 and 0.143316, respectively, while the maximal values (reached at $x_1 = x_2 = 1/2$) are 0.52457 and 0.51494 (they should be equal to 0.5, but realize that we are solving a highly non-smooth problem).

To further validate the methodology discussed in Sections 9 to 12, we have applied it to the solution of a *genuine Eikonal equation*, namely

$$|\nabla u| = 1 \text{ a.e. in } \Omega, \quad u = g \text{ on } \Gamma. \tag{13.2}$$

If one looks for the maximal solution (the *distance function* to the boundary if $g = 0$ in (13.2)), the same methodology used for problem (2.1), (9.1) can be used except that we replace $\sum_{i=1}^{d}\int_\Omega\left(\left|\frac{\partial v}{\partial x_i}\right|^2-1\right)^2 dx$ in (9.6) by $\int_\Omega(|\nabla v|^2-1)^2\,dx$, which is in fact simpler. If one looks for a minimal solution, we will take $C<0$ in (9.5). The resulting Eikonal solver has been validated via the following test problems (both with $\Omega=(0,1)\times(0,1)$):

First test problem: We take $g=0$ in (13.2), implying that the maximal solution is given by $u(x)=\delta(x,\Gamma)$, where $\delta(x,\Gamma)$ is the distance from x to the boundary Γ. This solution has been captured very quickly with $h=1/256$, $\Delta t\varepsilon_1=h^2/49$, $\varepsilon_2=0.001$, $C=10$, and Δt =0.0001. The approximation errors are $\|u_h-u\|_{0,\Omega}=1.638\times10^{-3}$ and $\|u_h-u\|_{\infty,\Omega}=9.811\times10^{-3}$. Also for this solution, we have $u(1/2,1/2)=1/2$ and $\int_\Omega u\,dx=1/6$ (= 0.16666...), the computed values being 0.50789761 and 0.16737264. The graph and contours of the computed solution are shown in Fig. 5.

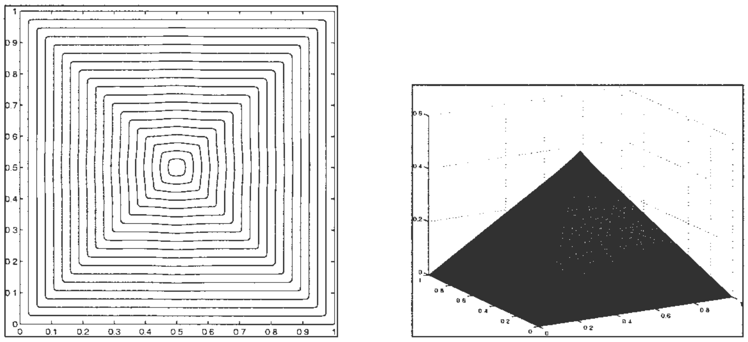

Figure 5. Contours (left) and graph (right) of the computed maximal solution of problem (13.2) with $g=0$ ($h=1/256$).

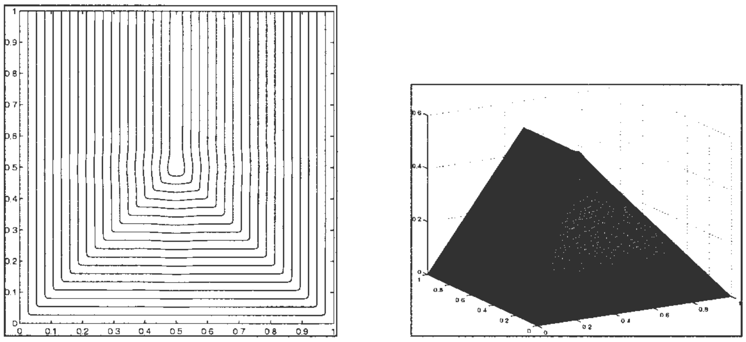

Figure 6. Contours (left) and graph (right) of the computed maximal solution of problem (13.2) with g given in (13.3) ($h=1/256$).

Second test problem: Here g in (13.2) is given by

$$\left.\begin{aligned} g(x_1,0) &= 0,0 \le x_1 \le 1, \\ g(x_1,1) &= \min(x_1, 1-x_1), 0 \le x_1 \le 1, \\ g(0,x_2) = g(1,x_2) &= 0,0 \le x_2 \le 1, \end{aligned}\right\} \tag{13.3}$$

implying that the maximal solution of (13.2) is given by

$$u(x_1,x_2) = \begin{cases} x_2, & x_1 \ge x_2 \text{ and } 1-x_1 \ge x_2, \\ x_1, & x_1 \le x_2 \text{ and } x_1 \le \dfrac{1}{2}, \\ 1-x_1, & 1-x_1 \le x_2 \text{ and } x_1 \ge \dfrac{1}{2}. \end{cases}$$

The parameters for this test case are $\Delta t\varepsilon_1 = h^2/49$, $\varepsilon_2 = 0.001$, $C = 10$, and Δt =0.0001. Despite the (small) overshots shown in Fig. 6, the exact solution has been captured quite accurately, since $\|u_h - u\|_{0,\Omega} = 1.1158 \times 10^{-3}$ for $h = 1/256$. The maximal value of $u(x_1,x_2)$ is 1/2, whereas the computed value is 0.50642583; also $\int_\Omega u\,dx = 5/24$ (= 0.2083333...), whereas $\int_\Omega u_h\,dx = 0.2088403$. The contours of the approximated maximal solution obtained with $h = 1/256$ are shown in Fig. 6.

14 Final comments

For those wondering if the solutions computed via "our" *biharmonic regularization technique* are *viscosity solutions*, the answer is definitely yes. To be convinced by this strong statement, let's consider the following *Eikonal equation*

$$|\nabla\psi|^2 = f \ (\ge 0) \text{ a.e. in } \Omega, \quad \psi = g \text{ on } \Gamma. \tag{14.1}$$

We associate to (14.1) the following penalized/regularized problem:

$$\psi_\varepsilon \in V_g; \ J^\varepsilon(\psi_\varepsilon) \le J^\varepsilon(\varphi), \quad \forall \varphi \in V_g, \tag{14.2}$$

where $V_g = \{\varphi \mid \varphi \in H^2(\Omega), \ \varphi = g \text{ on } \Omega\}$, and

$$\begin{aligned} J^\varepsilon(\varphi) = {} & \frac{\varepsilon_1}{2}\int_\Omega |D^2\varphi|^2\,dx + \frac{1}{2}\int_\Omega |\nabla\varphi|^2\,dx - C\int_\Omega \varphi\,dx \\ & + \frac{\varepsilon_2^{-1}}{4}\int_\Omega (|\nabla\varphi|^2 - f)^2\,dx, \end{aligned} \tag{14.3}$$

with

$$D^2\varphi = (\partial^2\varphi/\partial x_i\partial x_j)_{1\le i,j\le d}.$$

The Euler-Lagrange equation associated to (14.3) reads as follows:

$$\left.\begin{aligned} &\psi_\varepsilon \in V_g;\ \forall \varphi \in V_0\ \ (= H^2(\Omega) \cap H_0^1(\Omega)),\\ &\varepsilon_1 \int_\Omega D^2\psi_\varepsilon : D^2\varphi \,\mathrm{d}x + \int_\Omega \nabla\psi_\varepsilon \cdot \nabla\varphi \,\mathrm{d}x \\ &\quad + \varepsilon_2^{-1} \int_\Omega (|\nabla\psi_\varepsilon|^2 - f)\nabla\psi_\varepsilon \cdot \nabla\varphi \,\mathrm{d}x = C \int_\Omega \varphi \,\mathrm{d}x. \end{aligned}\right\} \tag{14.4}$$

Suppose now that $d = 2$ and that Ω is simply connected. Introduce then $\mathbf{u}_\varepsilon = \left\{\frac{\partial\psi_\varepsilon}{\partial x_2}, -\frac{\partial\psi_\varepsilon}{\partial x_1}\right\}$ and $\mathbf{v} = \left\{\frac{\partial\varphi}{\partial x_2}, -\frac{\partial\varphi}{\partial x_1}\right\}$; we clearly have $\mathbf{u}_\varepsilon \in (H^1(\Omega))^2$, $\mathbf{v} \in (H^1(\Omega))^2$, $\nabla \cdot \mathbf{u}_\varepsilon = \nabla \cdot \mathbf{v} = 0$, and $\mathbf{u}_\varepsilon \cdot \mathbf{n} = \mathrm{d}g/\mathrm{d}s$, $\mathbf{v} \cdot \mathbf{n} = 0$ on Γ, where s is a curvilinear coordinate on Γ, Γ being oriented counterclockwise. Problem (14.4) takes then the equivalent (kind of "velocity-pressure") following formulation:

$$\left.\begin{aligned} &\mathbf{u}_\varepsilon \in W_g,\\ &\varepsilon_1 \int_\Omega \nabla\mathbf{u}_\varepsilon : \nabla\mathbf{v} \,\mathrm{d}x + \int_\Omega \mathbf{u}_\varepsilon \cdot \mathbf{v} \,\mathrm{d}x + \varepsilon_2^{-1} \int_\Omega (|\mathbf{u}_\varepsilon|^2 - f)\mathbf{u}_\varepsilon \cdot \mathbf{v} \,\mathrm{d}x \\ &\quad + \int_\Omega p\nabla \cdot \mathbf{v} \,\mathrm{d}x = C \int_\Omega \mathbf{u}_1 \cdot \mathbf{v} \,\mathrm{d}x, \quad \forall \mathbf{v} \in W_0,\\ &\nabla \cdot \mathbf{u}_\varepsilon = 0 \ \text{in}\ \Omega, \end{aligned}\right\} \tag{14.5}$$

where $W_0 = \{\mathbf{v} \mid \mathbf{v} \in (H^1(\Omega))^2,\ \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma\}$, $W_g = \{\mathbf{v} \mid \mathbf{v} \in (H^1(\Omega))^2,\ \mathbf{v} \cdot \mathbf{n} = \mathrm{d}g/\mathrm{d}s \text{ on } \Gamma\}$, and $\mathbf{u}_1 = \left\{\frac{\partial\varphi_1}{\partial x_2}, -\frac{\partial\varphi_1}{\partial x_1}\right\}$ with $-\Delta\varphi_1 = 1$ in Ω, $\varphi = 0$ on Γ.

Problem (14.5) is clearly a *Stokes* variant of a *Ginzburg-Landau* equation, ε_1 playing here the role of a *viscosity coefficient*. We have shown thus that our *biharmonic regularization procedure* is indeed a *viscosity solution method* in the *sense of Stokes* (of course, once $\mathbf{u}_\varepsilon$ is known, we obtain ψ_ε via the solution of $-\Delta\psi_\varepsilon = \frac{\partial u_{2\varepsilon}}{\partial x_1} - \frac{\partial u_{1\varepsilon}}{\partial x_2}$ in Ω, $\psi_\varepsilon = g$ on Γ); the case where Ω has holes is a little more complicated to handle, but can be treated essentially in a similar way.

Our very last comments will be about fractals and geometry: in the September 2002 issue of SIAM NEWS, Ph. J. Davis wrote "Since 1977, when Mandelbrot published his *Fractals: Form, Chance and Dimension*, geometry, to a very large constituency, has meant fractals and their production by iteration." There is no doubt (in our mind, at least; see Figs. 3 and 4) that this article is another contribution to fractal geometry.

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