

# Attainment of minima and implicit partial differential equations

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*This paper is dedicated to the memory of Ennio De Giorgi, master in Mathematics and in life.*

## 1. INTRODUCTION

Convexity played a crucial role in the scientific interests and researches of Ennio De Giorgi in the fields of functional analysis and in the direct methods of calculus of variations. Of course, the lack of convexity too, as a main tool for counterexamples and extensions.

More modestly we (as well as several other students of him) took advantage of his ideas, opinions and joint discussions. Integrals of the calculus of variations, with not convex integrand, were one of these themes of research. Maybe, if there is a contribution by us in this field in our joint discussions with Ennio, is to have attracted his point of view mainly in the direction of problems for *vector-valued applications* (although this has been always a main interest of De Giorgi, since his celebrated counterexamples to regularity for systems of PDE), for which *lack of convexity* (in some cases) and lack of *quasiconvexity* (in some other cases), are more appropriate points of view to analyze the problem of attainment.

Nonconvex problems of the calculus of variations have been for the authors of this note a main motivation to introduce the study of what we called *implicit partial differential equations*. These are differential problems for equations and systems of any order, which are strongly nonlinear with respect to the highest derivatives, with assumptions that exclude the linear case. The authors' monograph [10], that should appear soon, will treat in details this subject.

In this paper we present some model results of existence for implicit PDE and examples of applications of these results to attainment for some problems in the calculus of variations with lack of convexity. Precisely, in Section 2 we study an approximation problem by piecewise affine functions that is useful in the following sections. We deal in Section 3 with a model problem of implicit PDE in the scalar case, while in Section 4 we deal with a vector-valued case. Finally in Sections 5 and 6 we study two cases of attainment of minimizers, respectively in the scalar and in the vector-valued cases.

We refer to the monograph [10] for more details and for a large list of references. Of course, some references are also presented in the next sections, related to the single results.

## 2. APPROXIMATION BY PIECEWISE AFFINE FUNCTIONS

Let  $\Omega$  be an open set of  $\mathbb{R}^n$ . We say that a function  $v \in W^{1,\infty}(\Omega)$  is *piecewise affine* in  $\Omega$  if there exists a (at most) countable partition of  $\Omega$  into open sets  $\Omega_k$ ,  $k \in \mathbb{N}$  and a set of measure zero, i.e.

$$\begin{aligned} \Omega_h \cap \Omega_k &= \emptyset, \quad \forall h, k \in \mathbb{N}, \quad h \neq k, \\ \text{meas} \left( \Omega - \bigcup_{k \in \mathbb{N}} \Omega_k \right) &= 0, \end{aligned}$$

such that  $v$  is affine on each  $\Omega_k$ , i.e. there exist  $\xi_k \in \mathbb{R}^n$  and  $q_k \in \mathbb{R}$  such that

$$v(x) = \langle \xi_k; x \rangle + q_k, \quad \forall x \in \Omega_k, \quad k \in \mathbb{N}.$$

**Theorem 1.** *Let  $\Omega$  be an open set of  $\mathbb{R}^n$ . Let  $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$  be a continuous function and let  $u \in W^{1,\infty}(\Omega)$  be such that*

$$F(x, u(x), Du(x)) \leq 0, \quad \text{a.e. } x \in \Omega. \quad (1)$$

*Then, for every  $\varepsilon > 0$ , there exists a function  $v \in W^{1,\infty}(\Omega)$  and an open set  $\Omega' \subset \Omega$  ( $\Omega' = \Omega$  if the strict inequality in (1) holds) such that*

$$F(x, u(x), Du(x)) = 0 \quad \text{a.e. } x \in \Omega - \Omega' \quad (2)$$

and

$$\begin{cases} v \text{ is piecewise affine on } \Omega'; \\ v = u \text{ on } \partial\Omega; \\ \|v - u\|_{L^\infty(\Omega)} < \varepsilon; \\ F(x, v(x), Dv(x)) < 0, \text{ a.e. } x \in \Omega'; \\ v(x) = u(x) \text{ a.e. } x \in \Omega - \Omega'; \end{cases} \quad (3)$$

and thus  $Dv(x) = Du(x)$  and  $F(x, v(x), Dv(x)) = 0$  for almost every  $x \in \Omega - \Omega'$ . Moreover, if the measure of the set

$$\{x \in \Omega : F(x, u(x), Du(x)) < 0\} \quad (4)$$

is finite, then we can also choose the open set  $\Omega'$  above such that

$$\text{meas}(\Omega') < \varepsilon + \text{meas}\{x \in \Omega : F(x, u(x), Du(x)) < 0\}.$$

The proof of Theorem 1 is adapted from an idea of De Blasi-Pianigiani [11], in the context of differential inclusions, and by Sychev [20] and Zagatti [22], related to some scalar non convex minimization problems in the calculus of variations.

**Proof.** The function  $u$  is locally Lipschitz continuous in  $\Omega$ , thus by Rademacher theorem (see for example Theorem 2.2.1 of [23])  $u$  is (classically) differentiable for almost every  $x \in \Omega$ . Let  $x_0$  be a point of  $\Omega$  where  $u$  is differentiable; then

$$u(x) = u(x_0) + \langle Du(x_0); x - x_0 \rangle + o(\|x - x_0\|_\infty), \quad x \in \Omega; \quad (5)$$

here  $\|\cdot\|_\infty$  denotes the  $\infty$ -norm on  $\mathbb{R}^n$ , i.e.

$$\|x\|_\infty = \max\{|x_i| : i = 1, 2, \dots, n\}, \quad \forall x \equiv (x_i)_{i=1,2,\dots,n} \in \mathbb{R}^n.$$

In our context the  $\infty$ -norm has the advantage, with respect to the euclidean 2-norm  $\|\cdot\|_2 = |\cdot|$ , to be piecewise affine on  $\mathbb{R}^n$ . Any other piecewise affine norm on  $\mathbb{R}^n$  (for example the 1-norm) would give the same effect. Note that we have

$$\|D\|x\|_\infty\|_\infty = 1, \quad \text{a.e. } x \in \mathbb{R}^n, \quad (6)$$

as well as  $\|D\|x\|_\infty\|_p = 1$ , a.e.  $x \in \mathbb{R}^n$  for every  $p \in [1, +\infty]$ .

Let us denote by  $\overline{B}_r(\xi)$  the closed ball in the  $\infty$ -norm  $\|\cdot\|_\infty$  of center at  $\xi \in \mathbb{R}^n$  and radius  $r > 0$ . Assume that  $u$  is differentiable at  $x_0 \in \Omega$  and that

$$F(x_0, u(x_0), Du(x_0)) < 0.$$

Since  $F$  is a continuous function, then there exists  $\gamma \in (0, 1)$  (depending on  $u$  and  $x_0$ ) such that

$$F(x, s, \xi) < 0 \quad (7)$$

for all  $(x, s, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$  such that

$$\|x - x_0\|_\infty \leq \gamma, \quad |s - u(x_0)| \leq \gamma, \quad \|\xi - Du(x_0)\|_\infty \leq 2\gamma. \quad (8)$$

Moreover there exists  $\delta > 0$  (depending on  $x_0$ ) such that

$$\begin{cases} \overline{B}_\delta(x_0) = \{x \in \mathbb{R}^n : \|x - x_0\|_\infty \leq \delta\} \subset \Omega, \\ \frac{|o(\|x - x_0\|_\infty)|}{\|x - x_0\|_\infty} \leq \gamma, \quad \forall x \in \overline{B}_\delta(x_0), \quad x \neq x_0; \end{cases} \quad (9)$$

we can also assume that

$$\begin{cases} \delta \leq \gamma; \\ \delta \leq \frac{\gamma}{2\|Du(x_0)\|_1 + 4\gamma}. \end{cases} \quad (10)$$

For every  $r > 0$ ,  $r \leq \min\{\delta; 1/2\}$  let us define in  $\Omega$  the piecewise affine function  $v_{x_0}^r$  by

$$v_{x_0}^r(x) = u(x_0) + \langle Du(x_0); x - x_0 \rangle + \gamma \cdot (r - 2\|x - x_0\|_\infty), \quad x \in \Omega.$$

Since  $0 < r \leq 1/2$ , we have

$$|v_{x_0}^r(x) - u(x_0)| \leq \|Du(x_0)\|_1 \cdot \|x - x_0\|_\infty + \frac{\gamma}{2} + 2\gamma\|x - x_0\|_\infty,$$

and, if  $x \in \overline{B}_\delta(x_0)$ , by the second inequality of (10),

$$|v_{x_0}^r(x) - u(x_0)| \leq \delta(\|Du(x_0)\|_1 + 2\gamma) + \frac{\gamma}{2} \leq \gamma, \quad \forall x \in \overline{B}_\delta(x_0).$$

By (6) we have

$$\|Dv_{x_0}^r(x) - Du(x_0)\|_\infty = 2\gamma\| -D \|x - x_0\|_\infty \|_\infty = 2\gamma, \quad \text{a.e. } x \in \Omega$$

and thus by (7), (8), for the same values of  $r$  we obtain

$$F(x, v_{x_0}^r(x), Dv_{x_0}^r(x)) < 0, \quad \text{a.e. } x \in \overline{B}_\delta(x_0). \quad (11)$$

Let us define the set

$$G(x_0, r) = \{x \in \overline{B}_\delta(x_0) \subset \Omega : v_{x_0}^r(x) \geq u(x)\}. \quad (12)$$

Then  $G(x_0, r)$  is a closed set contained in the ball  $\overline{B}_r(x_0)$  and containing  $\overline{B}_{r/3}(x_0)$ , i.e.

$$\overline{B}_{r/3}(x_0) \subset G(x_0, r) \subset \overline{B}_r(x_0). \quad (13)$$

In fact if  $x \in \overline{B}_\delta(x_0)$  but  $x \notin \overline{B}_r(x_0)$  (that is  $r < \|x - x_0\|_\infty \leq \delta$ ) then by the second inequality of (9) we have

$$\begin{aligned} v_{x_0}^r(x) - u(x) &= \gamma \cdot (r - 2\|x - x_0\|_\infty) - o(\|x - x_0\|_\infty) < \\ &= -\gamma\|x - x_0\|_\infty - o(\|x - x_0\|_\infty) = \end{aligned}$$

$$-\|x - x_0\|_\infty \cdot \left( \gamma + \frac{o(\|x - x_0\|_\infty)}{\|x - x_0\|_\infty} \right) \leq 0;$$

thus  $v_{x_0}^r(x) - u(x) < 0$  and  $x \notin G(x_0, r)$ . While, if  $x \in \overline{B}_{r/3}(x_0)$ , since  $r/3 \geq \|x - x_0\|_\infty$ , again by the second of (9) we obtain

$$\begin{aligned} v_{x_0}^r(x) - u(x) &= \gamma \cdot (r - 2\|x - x_0\|_\infty) - o(\|x - x_0\|_\infty) \geq \\ &\gamma \|x - x_0\|_\infty - o(\|x - x_0\|_\infty) = \\ &\|x - x_0\|_\infty \left( \gamma - \frac{o(\|x - x_0\|_\infty)}{\|x - x_0\|_\infty} \right) \geq 0 \end{aligned}$$

and  $x \in G(x_0, r)$ . Thus (13) is proved.

For every  $x \in G(x_0, r) \subset \overline{B}_r(x_0)$ , since  $\gamma < 1$ , we also have

$$\begin{aligned} |v_{x_0}^r(x) - u(x)| &= |\gamma \cdot (r - 2\|x - x_0\|_\infty) - o(\|x - x_0\|_\infty)| \leq \\ &\gamma \cdot (r + 2\|x - x_0\|_\infty) + |o(\|x - x_0\|_\infty)| \leq 4\gamma r < 4r. \end{aligned}$$

Thus, if  $\varepsilon > 0$  is as in the statement of the theorem, then

$$r \leq \varepsilon/4 \Rightarrow \|v_{x_0}^r - u\|_{L^\infty(G(x_0, r))} < \varepsilon. \quad (14)$$

By (13) for every  $r \in (0, \delta)$  we have

$$G(x_0, r) \subset \overline{B}_r(x_0) \subset \overline{B}_\delta(x_0), \quad \overline{B}_r(x_0) \neq \overline{B}_\delta(x_0),$$

and thus, by the definition (12) and the continuity of  $u$  and  $v_{x_0}^r$

$$\partial G(x_0, r) = \{x \in \overline{B}_\delta(x_0) : v_{x_0}^r(x) = u(x)\}. \quad (15)$$

With the definition

$$w(x) = u(x) - u(x_0) - \langle Du(x_0); x - x_0 \rangle + 2\gamma \|x - x_0\|_\infty, \quad x \in \Omega,$$

then the boundary  $\partial G(x_0, r)$  of  $G(x_0, r)$  can be represented by

$$\partial G(x_0, r) = \{x \in \overline{B}_\delta(x_0) : w(x) = \gamma r\},$$

i.e., for every  $r \in (0, \delta)$ ,  $\partial G(x_0, r)$  is a level set of a  $W^{1,\infty}(\Omega)$  function. Then it is possible to see (see for example Chapter 10 of author's book [10]) that there exists a sequence of real numbers  $r_h$  such that

$$\begin{cases} r_h \rightarrow 0 \text{ as } h \rightarrow +\infty, \\ 0 < r_h \leq \min\{\delta; 1/2; \varepsilon/4\}, \forall h \in \mathbb{N}, \\ \text{meas}(\partial G(x_0, r_h)) = 0, \forall h \in \mathbb{N}. \end{cases} \quad (16)$$

Let us consider the measurable subset of  $\Omega$

$$M = \{x_0 \in \Omega : u \text{ differentiable at } x_0, F(x_0, u(x_0), Du(x_0)) < 0\}$$

and the family of open sets

$$\mathcal{G} = \{\text{int } G(x_0, r_h) : x_0 \in M, r_h \text{ as in (16)}\}.$$

By Vitali's covering theorem (for example in the form proved in Chapter 10 of author's book [10]), there exists in  $\mathcal{G}$  a (at most) countable subcollection  $\mathcal{G}'$  of sets with disjoint closures

$$\mathcal{G}' = \{\text{int } G(x_k, r_k) \in \mathcal{G}\}, \quad G(x_h, r_h) \cap G(x_k, r_k) = \emptyset, \text{ if } h \neq k,$$

such that the open set  $\Omega' \subset \Omega$

$$\Omega' = \bigcup_{\text{int } G(x_k, r_k) \in \mathcal{G}'} \text{int } G(x_k, r_k), \quad (17)$$

cover  $M$  up to a set of zero measure, i.e.  $\text{meas}(M - \Omega') = 0$ .

Let us define the function  $v$  in  $\Omega$  by

$$\begin{cases} v(x) = u(x) & \text{if } x \in \Omega - \Omega', \\ v(x) = v_{x_k}^{r_k}(x) & \text{if } x \in G(x_k, r_k). \end{cases} \quad (18)$$

Then  $v \in W^{1,\infty}(\Omega)$  (in particular by (15)  $v$  is continuous in  $\Omega$ ) and satisfies the stated properties (3). In fact, since every  $v_{x_k}^{r_k}(x)$  is piecewise affine on the set  $\text{int } G(x_k, r_k)$  and satisfies the constraint in (11), and since the sets  $G(x_k, r_k)$  are disjoint, then  $v$  is piecewise affine on the open subset  $\Omega'$  in (17) and

$$F(x, v(x), Dv(x)) < 0, \quad \text{a.e. } x \in \Omega'.$$

While, if  $x \in \Omega - \Omega'$  and if  $u$  is differentiable at  $x$ , since  $v(x) = u(x)$  then  $Dv(x) = Du(x)$  and we must have

$$F(x, u(x), Du(x)) = 0,$$

because otherwise the condition  $F(x, u(x), Du(x)) < 0$  would imply  $x \in M \subset \Omega'$ .

Moreover  $v = u$  on  $\partial\Omega$ , since in (18) every set  $G(x_k, r_k)$  is (compactly) contained in  $\Omega$ . Finally  $\|v - u\|_{L^\infty(\Omega)} < \varepsilon$  by (14).

Finally, since the set in (4) is measurable, it can be approximated by a sequence of open sets containing it, and it is enough to change  $\Omega$  into one of these approximating open sets and then apply the above proof. ■

**Corollary 2.** *Let  $\Omega$  be an open set of  $\mathbb{R}^n$ . Let  $A, B$  be disjoint sets of  $\mathbb{R}^n$ , with  $A$  open and  $B$  possibly empty. Let  $u \in W^{1,\infty}(\Omega)$  such that*

$$Du(x) \in A \cup B, \quad \text{a.e. } x \in \Omega.$$

*Then, for every  $\varepsilon > 0$ , there exists a function  $v \in W^{1,\infty}(\Omega)$  and an open set  $\Omega' \subset \Omega$  ( $\Omega' = \Omega$  if  $B = \emptyset$ ) such that*

$$\begin{cases} v \text{ is piecewise affine on } \Omega'; \\ v = u \text{ on } \partial\Omega; \\ \|v - u\|_{L^\infty(\Omega)} < \varepsilon; \\ Dv(x) \in A, \text{ a.e. } x \in \Omega'; \\ Dv(x) = Du(x) \in B, \text{ a.e. } x \in \Omega - \Omega'. \end{cases} \quad (19)$$

*Moreover, if the measure of the set  $\{x \in \Omega : Du(x) \in A\}$  is finite, then we can choose  $\Omega'$  such that*

$$\text{meas}(\Omega') < \varepsilon + \text{meas}\{x \in \Omega : Du(x) \in A\}.$$

**Proof.** We apply Theorem 1 to the continuous function  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  defined by

$$F(\xi) = -\text{dist}(\xi; \mathbb{R}^n - A).$$

We have  $F(\xi) \leq 0$  for all  $\xi \in \mathbb{R}^n$  and thus

$$F(Du(x)) \leq 0, \quad \text{a.e. } x \in \Omega. \quad (20)$$

By Theorem 1 for every  $\varepsilon > 0$  there exists a function  $v \in W^{1,\infty}(\Omega)$  and an open set  $\Omega' \subset \Omega$  ( $\Omega' = \Omega$  if the strict inequality in (20) holds) such that

$$F(Dv(x)) = 0 \quad \text{a.e. } x \in \Omega - \Omega'$$

and

$$\begin{cases} v \text{ is piecewise affine on } \Omega'; \\ v = u \text{ on } \partial\Omega; \\ \|v - u\|_{L^\infty(\Omega)} < \varepsilon; \\ F(Dv(x)) < 0, \text{ a.e. } x \in \Omega'; \\ v(x) = u(x) \text{ a.e. } x \in \Omega - \Omega'; \end{cases}$$

thus  $Dv(x) = Du(x)$  and  $F(Dv(x)) = 0$  for almost every  $x \in \Omega - \Omega'$ . Since  $F(\xi) < 0$  if and only if  $\xi \in A$ , we also have

$$Dv(x) \in A, \text{ a.e. } x \in \Omega', \quad Dv(x) = Du(x) \in B, \text{ a.e. } x \in \Omega - \Omega',$$

and (19) holds. ■

### 3. IMPLICIT PARTIAL DIFFERENTIAL EQUATIONS IN THE SCALAR CASE

*Implicit partial differential equations and systems* are treated in details in the author's monograph [10]. Here we give two examples of model results, the first one in the scalar case, the second one (presented in the next section) in the vector-valued case. For more general framework we refer to [10].

The main result of this section is the following one.

**Theorem 3.** *Let  $\Omega \subset \mathbb{R}^n$  be open and  $E \subset \mathbb{R}^n$ . Let  $\varphi \in W^{1,\infty}(\Omega)$  satisfies*

$$D\varphi(x) \in E \cup \text{int co } E, \quad \text{a.e. } x \in \Omega \quad (21)$$

(where  $\text{int co } E$  stands for the interior of the convex hull of  $E$ ); then there exists (a dense set of)  $u \in W^{1,\infty}(\Omega)$  such that

$$\begin{cases} Du(x) \in E, & \text{a.e. } x \in \Omega \\ u(x) = \varphi(x), & x \in \partial\Omega. \end{cases} \quad (22)$$

Theorem 3 has been established in [8], [9] (c.f. also Bressan-Flores [2], De Blasi-Pianigiani [11]), using Baire category theorem; however we will present here a different approach. In fact, in the case that the boundary datum is affine, the theorem can be obtained by elementary means, i.e. the method of *pyramids* introduced by Cellina [5] (c.f. also Friesecke [12]). For the sake of completeness we will present this method in the following Lemma 4. Then the proof of Theorem 3, following an idea of De Blasi-Pianigiani [11], Sychev [20] and Zagatti [22], will be given at the end of this section and will be consequence of Lemma 4.

**Lemma 4.** *Let  $\Omega \subset \mathbb{R}^n$  be open and  $E \subset \mathbb{R}^n$ . Let  $\varphi$  be affine, i.e.  $\varphi(x) = \langle \xi; x \rangle + q$  for some  $\xi \in \mathbb{R}^n$  and  $q \in \mathbb{R}$ , with*

$$D\varphi = \xi \in E \cup \text{int co } E.$$

Then there exists  $u \in W^{1,\infty}(\Omega)$  such that

$$\begin{cases} Du(x) \in E, & \text{a.e. } x \in \Omega \\ u(x) = \varphi(x), & x \in \partial\Omega. \end{cases}$$

**Proof.** If  $\xi \in E$  the result is trivial, therefore we assume that  $\xi \in \text{int co } E$ . By a translation we can also assume that  $\xi = 0$ . We divide the proof into four steps.

Step 1: We first show that we can find a number  $m \geq n + 1$  of vectors  $\xi_1, \xi_2, \dots, \xi_m \in E \subset \mathbb{R}^n$  such that

$$0 \in \text{int co } \{\xi_1, \xi_2, \dots, \xi_m\}; \quad (23)$$

more precisely we can find vectors  $\xi_1, \xi_2, \dots, \xi_m \in E$  that span the whole of  $\mathbb{R}^n$  and such that

$$0 = \sum_{i=1}^m s_i \xi_i \quad (24)$$

for some strictly positive coefficients  $s_i$ ,  $i = 1, 2, \dots, m$ , with  $\sum_{i=1}^m s_i = 1$ .

We propose below a proof of this elementary application of Carathéodory theorem. Since  $0 \in \text{int co } E$ , we can find a cube  $C_\varepsilon$  of side  $2\varepsilon$ , with  $\varepsilon > 0$  sufficiently small, so that

$$C_\varepsilon = \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n : |x_i| < \varepsilon, i = 1, 2, \dots, n\} \subset \text{int co } E.$$

We denote by  $\eta_1, \dots, \eta_{2^n}$  the vertices of the cube  $C_\varepsilon$ . Then

$$0 \in \text{int co } \{\eta_1, \dots, \eta_{2^n}\} = C_\varepsilon \subset \text{int co } E. \quad (25)$$

Note that, since  $\sum_{i=1}^{2^n} \eta_i = 0$ , then the 0-vector can be expressed by the following elementary convex combination

$$0 = \sum_{i=1}^{2^n} \frac{1}{2^n} \eta_i. \quad (26)$$

Next we use Carathéodory theorem to write

$$\eta_i = \sum_{k=1}^{n+1} t_{i,k} \xi_{i,k}, \quad \forall i = 1, 2, \dots, n,$$

where  $\xi_{i,k} \in E$ ,  $t_{i,k} \geq 0$  and  $\sum_{k=1}^{n+1} t_{i,k} = 1$ . Some of the coefficients  $t_{i,k}$  may be equal to zero; then, by a change of notations with a renaming of vectors, for every  $i = 1, 2, \dots, n$  we limit ourselves to consider strictly positive  $t_{i,k}$ . Since the corresponding set of vectors of  $\mathbb{R}^n$

$$\{\xi_{i,k}, k = 1, \dots, n+1, i = 1, \dots, 2^n\} \quad (27)$$

generate by convex combinations at least the whole cube  $C_\varepsilon$ , then the span of the set in (27) is  $\mathbb{R}^n$ , i.e. linear combinations of elements of (27) give all possible vectors of  $\mathbb{R}^n$ . By (26) we obtain

$$0 = \sum_{i=1}^{2^n} \frac{1}{2^n} \eta_i = \sum_{i=1}^{2^n} \sum_{k=1}^{n+1} \frac{1}{2^n} t_{i,k} \xi_{i,k} = \sum_{i,k} s_{i,k} \xi_{i,k},$$

where

$$s_{i,k} = \frac{t_{i,k}}{2^n} > 0 \quad \forall i, k, \quad \sum_{i,k} s_{i,k} = 1,$$

which proves (24). By combining (25) with

$$\text{co}\{\eta_1, \eta_2, \dots, \eta_{2^n}\} \subset \text{co}\{\xi_{i,k}, \quad k = 1, \dots, n+1, \quad i = 1, \dots, 2^n\}$$

we also have (23).

Step 2: Now we show that we can find  $u \in W_0^{1,\infty}(\Omega)$  such that

$$Du(x) \in \{\xi_1, \xi_2, \dots, \xi_m\}, \quad \text{a.e. } x \in \Omega,$$

where  $\Omega$  is a special domain of  $\mathbb{R}^n$ . We will construct such a  $u$  explicitly (as a function whose graph is a *pyramid*) when  $\Omega$  is a particular domain (see the set  $\Omega = G(x_0, r)$  below in (29)) and then by Vitali Covering Theorem we will deduce the result for general  $\Omega$ .

For every  $x_0 \in \mathbb{R}^n$  and  $r > 0$  let us consider the function  $v_{x_0}^r$ , that we call briefly a *pyramid*, since its graph in the case  $n = 2$  is a pyramid, defined by

$$v_{x_0}^r(x) = r - \max_i \{\langle \xi_i; x - x_0 \rangle : i = 1, 2, \dots, m\}, \quad x \in \mathbb{R}^n. \quad (28)$$

Of course we have

$$Dv_{x_0}^r(x) \in \{\xi_1, \xi_2, \dots, \xi_m\}, \quad \text{a.e. } x \in \mathbb{R}^n.$$

Thus  $v_{x_0}^r$  is a solution to the problem

$$\begin{cases} Du(x) \in \{\xi_1, \xi_2, \dots, \xi_m\} \subset E, & \text{a.e. } x \in G(x_0, r) \\ u(x) = 0, & x \in \partial G(x_0, r), \end{cases}$$

where  $G(x_0, r)$  is the domain

$$G(x_0, r) = \{x \in \mathbb{R}^n : v_{x_0}^r(x) \geq 0\}. \quad (29)$$

Step 3: Let us prove that, for every  $x_0 \in \mathbb{R}^n$  and  $r > 0$ , the set  $G(x_0, r)$  defined in (29) is bounded.

Let us assume, by contradiction, that for some  $x_0 \in \mathbb{R}^n$  and  $r > 0$  the set  $G(x_0, r)$  is not bounded. Then there exists in  $\mathbb{R}^n$  a sequence  $x_k, k \in \mathbb{N}$ , such that

$$\lim_{k \rightarrow \infty} |x_k| = +\infty; \quad \langle \xi_i; x_k - x_0 \rangle \leq r, \quad \forall i = 1, 2, \dots, m, \quad \forall k \in \mathbb{N}.$$

Let us denote by  $y_k$  the corresponding normalized vectors

$$y_k = \frac{x_k}{|x_k|}.$$

Then, up to a subsequence that we still denote by  $y_k$ , we have that  $y_k \rightarrow y_0$  for some  $y_0 \in \mathbb{R}^n$ , with  $|y_0| = 1$ . We go to the limit as  $k \rightarrow +\infty$  in both sides of the inequality

$$\left\langle \xi_i; y_k - \frac{x_0}{|x_k|} \right\rangle \leq \frac{r}{|x_k|}, \quad \forall i = 1, 2, \dots, m, \quad \forall k \in \mathbb{N},$$

and we get  $\langle \xi_i; y_0 \rangle \leq 0$ , for every  $i = 1, 2, \dots, m$ . By using (24) we obtain

$$0 = \langle 0; y_0 \rangle = \sum_{i=1}^m s_i \langle \xi_i; y_0 \rangle$$



and since

$$s_i > 0, \quad \langle \xi_i; y_0 \rangle \leq 0, \quad \forall i = 1, 2, \dots, m,$$

we deduce that

$$\langle \xi_i; y_0 \rangle = 0, \quad \forall i = 1, 2, \dots, m.$$

Recall that the span of the set  $\{\xi_1, \xi_2, \dots, \xi_m\}$  is the whole of  $\mathbb{R}^n$ . Therefore there exist real coefficients  $c_i, i = 1, 2, \dots, m$ , such that  $y_0 = \sum_{i=1}^m c_i \xi_i$ . We obtain the contradiction

$$1 = |y_0|^2 = \langle y_0; y_0 \rangle = \sum_{i=1}^m c_i \langle \xi_i; y_0 \rangle = 0.$$

Step 4: Now let us prove that we can find  $u \in W_0^{1,\infty}(\Omega)$  such that

$$Du(x) \in \{\xi_1, \xi_2, \dots, \xi_m\}, \quad \text{a.e. } x \in \Omega,$$

where  $\Omega$  is a general domain of  $\mathbb{R}^n$ , by covering  $\Omega$  with the domains  $G(x_0, r)$  defined above in (29).

Let us consider again the function  $v_{x_0}^r$  defined in (28). Since  $v_{x_0}^r(x_0) = r > 0$ , we have

$$x_0 \in \text{int } G(x_0, r), \quad \forall x_0 \in \mathbb{R}^n.$$

Moreover, by Step 3, for every  $x_0 \in \Omega$  and  $r > 0$  the set  $G(x_0, r)$  is bounded. By dilations we can easily see that  $G(x_0, r)$  is contained in a ball of center  $x_0$ , with a radius that converges to zero as  $r \rightarrow 0$ . Thus

$$G(x_0, r) \subset \Omega$$

if  $r$  is sufficiently close to zero. Then the family

$$\mathcal{G} = \{G(x_0, r) : x_0 \in \Omega, r > 0\},$$

with  $r$  sufficiently small in dependence of  $x_0$ , cover  $\Omega$ . This  $\mathcal{G}$  is a family of translated and dilated sets of

$$G = G(0, 1) = \left\{ x \in \mathbb{R}^n : \max_i \{\langle \xi_i; x \rangle : i = 1, 2, \dots, m\} \leq 1 \right\}.$$

We are under the assumptions of the Standard Vitali Covering Theorem, since  $G$  is a closed set of positive measure. Therefore we can cover  $\Omega$ , up to a set of null measure, by a countable subfamily  $\mathcal{G}'$  of disjoint sets of  $\mathcal{G}$ ; i.e.

$$\mathcal{G}' = \{G(x_k, r_k) \in \mathcal{G} : k = 1, 2, \dots\},$$

such that

$$G(x_h, r_h) \cap G(x_k, r_k) = \emptyset, \quad \text{if } h \neq k, \quad \Omega' = \bigcup_{G(x_k, r_k) \in \mathcal{G}'} G(x_k, r_k)$$

and  $\text{meas}(\Omega - \Omega') = 0$ . We define in  $\Omega$  a function  $u$  by

$$u(x) = \begin{cases} 0 & \text{if } x \in \Omega - \Omega', \\ v_{x_k}^{r_k}(x) & \text{if } x \in G(x_k, r_k). \end{cases}$$

Then, since  $v_{x_k}^{r_k}(x) = 0$  if  $x \in \partial G(x_k, r_k)$ , then  $u$  is a continuous function in  $\Omega$ ; moreover the gradient of  $u$  belongs to the finite set  $\{\xi_1, \xi_2, \dots, \xi_m\}$  and thus  $u \in W^{1,\infty}(\Omega)$ . Our lemma is proved. ■

**Proof.** (Theorem 3). By assumption the boundary datum  $\varphi$  satisfies the condition

$$D\varphi(x) \in E \cup \text{int co } E, \quad \text{a.e. } x \in \Omega.$$

We apply Corollary 2, with  $A = \text{int co } E$  and  $B = E$ ; then there exists a function  $\varphi' \in W^{1,\infty}(\Omega)$  and an open set  $\Omega' \subset \Omega$  such that

$$\begin{cases} \varphi' \text{ is piecewise affine on } \Omega'; \\ \varphi' = \varphi \text{ on } \partial\Omega; \\ D\varphi'(x) \in \text{int co } E, \quad \text{a.e. } x \in \Omega'; \\ D\varphi'(x) = D\varphi(x) \in E, \quad \text{a.e. } x \in \Omega - \Omega'. \end{cases} \quad (30)$$

Since  $\varphi'$  is piecewise affine on  $\Omega'$ , there exists a (at most) countable partition of  $\Omega'$  into open sets  $\Omega_k$ ,  $k \in \mathbb{N}$  and a set of measure zero, i.e.

$$\begin{aligned} \Omega_h \cap \Omega_k &= \emptyset, \quad \forall h, k \in \mathbb{N}, \quad h \neq k, \\ \text{meas} \left( \Omega' - \bigcup_{k \in \mathbb{N}} \Omega_k \right) &= 0, \end{aligned}$$

such that  $\varphi'$  is affine on each  $\Omega_k$ , i.e. there exist  $\xi_k \in \mathbb{R}^n$  and  $q_k \in \mathbb{R}$  such that

$$\varphi'(x) = \langle \xi_k; x \rangle + q_k, \quad \forall x \in \Omega_k, \quad k \in \mathbb{N}.$$

By Lemma 4, for every  $k \in \mathbb{N}$  there exists a function  $u_k \in W^{1,\infty}(\Omega_k)$  such that

$$\begin{cases} Du_k(x) \in E, \quad \text{a.e. } x \in \Omega_k \\ u_k(x) = \varphi'(x), \quad x \in \partial\Omega_k. \end{cases}$$

Then the function  $u(x)$  defined in  $\Omega$  by

$$u(x) = \begin{cases} u_k(x) & \text{if } x \in \Omega_k \\ \varphi'(x) & \text{if } x \in \Omega - \bigcup_{k \in \mathbb{N}} \Omega_k \end{cases}$$

satisfies the conditions

$$\begin{cases} u \in W^{1,\infty}(\Omega) \\ Du(x) \in E, \quad \text{a.e. } x \in \Omega \\ u(x) = \varphi'(x) = \varphi(x), \quad x \in \partial\Omega \end{cases}$$

and thus it is a solution to the differential problem (22). ■

#### 4. IMPLICIT PARTIAL DIFFERENTIAL EQUATIONS IN THE VECTOR-VALUED CASE

We study in this section a differential problem related to a second order *implicit* partial differential equation of the type

$$F(D^2w(x)) = 0,$$

where  $F: \mathbb{R}_s^{2 \times 2} \rightarrow \mathbb{R}$  is the convex function defined by

$$F(\xi) = (\text{trace } \xi - a)(\text{trace } \xi - b)$$

and  $\text{trace } \xi$  is the *trace* of the square matrix  $\xi \in \mathbb{R}_s^{2 \times 2}$ , the set of *symmetric* matrices  $2 \times 2$ . We will prove the following result, related to a problem with constraint on the determinant of  $\xi$ .

**Theorem 5.** Let  $\Omega \subset \mathbb{R}^2$  be open,  $a, b \in \mathbb{R}$  with  $a < b$  and  $\varphi \in C_{piec}^2(\overline{\Omega})$  satisfy

$$\begin{cases} a \leq \Delta\varphi(x, y) \leq b, & \text{a.e. } (x, y) \in \Omega, \\ \det [D^2\varphi(x, y) - \frac{a}{2}I] > 0, & \text{a.e. } (x, y) \in \Omega \end{cases}$$

where  $I$  is the identity matrix of  $\mathbb{R}^{2 \times 2}$ . Then there exists  $w \in \varphi + W_0^{2, \infty}(\Omega)$  such that

$$\begin{cases} \Delta w(x, y) \in \{a, b\}, & \text{a.e. } (x, y) \in \Omega, \\ \det [D^2w(x, y) - \frac{a}{2}I] \geq 0, & \text{a.e. } (x, y) \in \Omega. \end{cases}$$

The proof is constructive and is based on the method of *confocal ellipses* of Murat-Tartar [21]. The theorem below has been essentially established in [7]. We will prove the theorem first in a particular case, by an explicit construction, and then show how we can reduce the theorem to this special case.

**Lemma 6.** Let

$$\varphi(x, y) = \frac{\lambda}{2}x^2 + \frac{\mu}{2}y^2$$

where  $\lambda, \mu > 0$  and  $0 < \lambda + \mu < 1$ . Let

$$\alpha = \frac{1-\lambda}{\lambda} \cdot \frac{1-(\lambda+\mu)}{2-(\lambda+\mu)}, \quad \beta = \frac{1-\mu}{\mu} \cdot \frac{1-(\lambda+\mu)}{2-(\lambda+\mu)}$$

and

$$\Omega = \left\{ (x, y) \in \mathbb{R}^2 : \frac{x^2}{1+\alpha} + \frac{y^2}{1+\beta} < 1 \right\}.$$

Then there exists  $w \in \varphi + W_0^{2, \infty}(\Omega)$  such that

$$\begin{cases} \Delta w(x, y) \in \{0, 1\}, & \text{a.e. } (x, y) \in \Omega, \\ \det D^2w(x, y) \geq 0, & \text{a.e. } (x, y) \in \Omega. \end{cases}$$

**Proof.** (Lemma 6) We divide the proof into two parts.

Part 1: We first discuss the simpler case when  $\lambda = \mu$ ; this corresponds to the construction of Hashin-Shtrikman [14]. In the notations of our lemma we have  $0 < \lambda < 1/2$ , letting  $r^2 = x^2 + y^2$ ,

$$\begin{aligned} \Omega &= \left\{ (x, y) \in \mathbb{R}^2 : r^2 < \frac{1}{2\lambda} \right\} \\ \varphi(x, y) &= \frac{\lambda}{2}r^2, \quad D\varphi(x, y) = \lambda(x, y). \end{aligned}$$

We then let

$$\begin{aligned} \Omega_1 &= \left\{ (x, y) \in \mathbb{R}^2 : r^2 < \frac{1-2\lambda}{2\lambda} \right\} \\ \Omega_2 &= \left\{ (x, y) \in \mathbb{R}^2 : \frac{1-2\lambda}{2\lambda} < r^2 < \frac{1}{2\lambda} \right\} \end{aligned}$$

and define

$$w(x, y) = \begin{cases} f\left(\frac{1-2\lambda}{2\lambda}\right) & \text{if } r^2 < \frac{1-2\lambda}{2\lambda} \\ f(r^2) & \text{if } \frac{1-2\lambda}{2\lambda} < r^2 < \frac{1}{2\lambda} \end{cases}$$

where  $f$  is given by  $f(1/2\lambda) = 1/4$  and

$$f'(r^2) = \frac{1}{8\lambda} \left( 2\lambda - \frac{1-2\lambda}{r^2} \right) \quad \text{if } \frac{1-2\lambda}{2\lambda} < r^2 < \frac{1}{2\lambda}.$$

Note that we then have

$$Dw(x, y) = \begin{cases} (0, 0) & \text{in } \Omega_1 \\ 2f'(r^2)(x, y) & \text{in } \Omega_2 \end{cases}$$

while in  $\Omega_2$

$$D^2w(x, y) = \begin{pmatrix} 2f'(r^2) + 4x^2f''(r^2) & 4xyf''(r^2) \\ 4xyf''(r^2) & 2f'(r^2) + 4y^2f''(r^2) \end{pmatrix}.$$

Observe that  $w \in W^{2,\infty}(\Omega)$  since  $f \in C^1\left(\left[0, \sqrt{1/2\lambda}\right]\right)$ . Furthermore, since  $f(1/2\lambda) = 1/4$  and  $f'(1/2\lambda) = \lambda/2$ , we deduce that

$$w(x, y) = \varphi(x, y) \quad \text{and} \quad Dw(x, y) = D\varphi(x, y) \quad \text{for every } (x, y) \in \partial\Omega.$$

Finally we also obtain

$$\begin{aligned} \Delta w &= \begin{cases} 0 & \text{in } \Omega_1 \\ 1 & \text{in } \Omega_2 \end{cases} \\ \det D^2w &= 4(f'(r^2))^2 + 8r^2f'(r^2)f''(r^2) \geq 0 \quad \text{in } \Omega. \end{aligned}$$

(As a curiosity note that  $\det D^2w \in C^0(\bar{\Omega})$  although  $w \notin C^2(\bar{\Omega})$ ). This concludes Part 1.

Part 2: From now on we will assume that  $\lambda \neq \mu$ . The result of this part is obtained from the preceding one by applying a conformal transformation of the ellipse onto the disk. We will do this explicitly, proceeding in five steps.

We will first define, in Step 1, two functions  $u, v \in W^{1,\infty}(\Omega)$  and show, in Step 2 to Step 4, that they satisfy

$$\begin{cases} u_x + v_y \in \{0, 1\}, \quad u_y - v_x = 0 \quad \text{a.e. in } \Omega \\ \det(Du, Dv) = u_xv_y - u_yv_x \geq 0 \quad \text{a.e. in } \Omega \\ u(x, y) = \lambda x, \quad v(x, y) = \mu y \quad \text{on } \partial\Omega. \end{cases} \quad (31)$$

We will then construct in Step 5 the solution,  $w$ , to our problem by setting

$$Dw = (u, v).$$

Step 1: By hypothesis  $\Omega$  is given by the ellipse

$$\Omega = \left\{ (x, y) \in \mathbb{R}^2 : \frac{x^2}{1+\alpha} + \frac{y^2}{1+\beta} < 1 \right\}. \quad (32)$$

We then define

$$\Omega_1 = \left\{ (x, y) \in \mathbb{R}^2 : \frac{x^2}{\alpha} + \frac{y^2}{\beta} < 1 \right\}, \quad \Omega_2 = \Omega - \overline{\Omega_1} \quad (33)$$

$$u(x, y) = \begin{cases} 0 & \text{in } \Omega_1 \\ f(\rho)x & \text{in } \Omega_2 \end{cases}, \quad v(x, y) = \begin{cases} 0 & \text{in } \Omega_1 \\ g(\rho)y & \text{in } \Omega_2 \end{cases} \quad (34)$$

where for  $\rho \in [0, 1]$  we let

$$\begin{cases} f(\rho) = c \left[ \left( \frac{\rho + \beta}{\rho + \alpha} \right)^{1/2} - \left( \frac{\beta}{\alpha} \right)^{1/2} \right], \\ g(\rho) = -c \left[ \left( \frac{\rho + \alpha}{\rho + \beta} \right)^{1/2} - \left( \frac{\alpha}{\beta} \right)^{1/2} \right], \\ c = \frac{\lambda}{\left( \frac{1 + \beta}{1 + \alpha} \right)^{1/2} - \left( \frac{\beta}{\alpha} \right)^{1/2}}, \end{cases} \quad (35)$$

and  $\rho$  is defined implicitly by

$$\frac{x^2}{\rho + \alpha} + \frac{y^2}{\rho + \beta} = 1 \quad (36)$$

(i.e.,  $\Omega_2$  corresponds to  $\rho \in (0, 1)$ ). It remains to be checked that  $u$  and  $v$  have all the asserted properties (31).

Step 2: Recall that  $\lambda, \mu > 0$ ,  $\lambda \neq \mu$  and  $\lambda + \mu < 1$ , and that

$$\alpha = \frac{1 - \lambda}{\lambda} \cdot \frac{1 - (\lambda + \mu)}{2 - (\lambda + \mu)}, \quad \beta = \frac{1 - \mu}{\mu} \cdot \frac{1 - (\lambda + \mu)}{2 - (\lambda + \mu)}. \quad (37)$$

Direct computations lead to

$$\left( \frac{\alpha}{1 + \alpha} \right)^{1/2} \left( \frac{\beta}{1 + \beta} \right)^{1/2} = 1 - (\lambda + \mu), \quad (38)$$

$$\frac{\lambda}{\left( \frac{1 + \beta}{1 + \alpha} \right)^{1/2} - \left( \frac{\beta}{\alpha} \right)^{1/2}} + \frac{\mu}{\left( \frac{1 + \alpha}{1 + \beta} \right)^{1/2} - \left( \frac{\alpha}{\beta} \right)^{1/2}} = 0, \quad (39)$$

$$\frac{\lambda}{\left( \frac{1 + \beta}{1 + \alpha} \right)^{1/2} - \left( \frac{\beta}{\alpha} \right)^{1/2}} = \frac{(\alpha\beta)^{1/2}}{\alpha - \beta}. \quad (40)$$

Step 3: We start with some preliminary computations. We have

$$f'(\rho) = \frac{c}{2} \frac{\alpha - \beta}{(\rho + \alpha)^{3/2}(\rho + \beta)^{1/2}}, \quad g'(\rho) = \frac{c}{2} \frac{\alpha - \beta}{(\rho + \beta)^{3/2}(\rho + \alpha)^{1/2}}. \quad (41)$$

We next compute  $\rho_x = \frac{\partial \rho}{\partial x}$  and  $\rho_y = \frac{\partial \rho}{\partial y}$ . Using (36) we have

$$x^2(\rho + \beta) + y^2(\rho + \alpha) = (\rho + \alpha)(\rho + \beta). \quad (42)$$

Differentiating with respect to  $x$  and then  $y$  we get

$$\rho_x = \frac{2x(\rho + \beta)}{2\rho + \alpha + \beta - x^2 - y^2}, \quad \rho_y = \frac{2y(\rho + \alpha)}{2\rho + \alpha + \beta - x^2 - y^2}, \quad (43)$$

which implies that

$$\frac{\rho_x}{\rho_y} = \frac{x}{y} \cdot \frac{\rho + \beta}{\rho + \alpha}. \quad (44)$$

By combining (42) and (43) we also obtain that

$$\frac{x\rho_x}{(\rho + \alpha)^{3/2}(\rho + \beta)^{1/2}} + \frac{y\rho_y}{(\rho + \alpha)^{1/2}(\rho + \beta)^{3/2}} = \frac{2}{(\rho + \alpha)^{1/2}(\rho + \beta)^{1/2}}. \quad (45)$$

Step 4: We are now in a position to conclude that:

(i) If  $(x, y) \in \partial\Omega_1$ , then  $\rho = 0$  and from (35) we get  $f(0) = g(0) = 0$ , i.e.  $u = v = 0$  on  $\partial\Omega_1$ .

(ii) If  $(x, y) \in \partial\Omega$ , then  $\rho = 1$  and by definition of  $c$  we get  $f(1) = \lambda$  and thus  $u = \lambda x$  on  $\partial\Omega$ . Similarly we find using (39) in (35) that

$$g(1) = -c \left[ \left( \frac{1 + \alpha}{1 + \beta} \right)^{1/2} - \left( \frac{\alpha}{\beta} \right)^{1/2} \right] = \mu$$

and hence  $v = \mu y$  on  $\partial\Omega$ .

(iii) We now prove that  $u_y - v_x \equiv 0$  in  $\Omega_2$ ; indeed,

$$u_y - v_x = xf'(\rho)\rho_y - yg'(\rho)\rho_x = 0 \quad (46)$$

since (41) and (44) hold.

(iv) We then show that  $u_x + v_y \equiv 1$  in  $\Omega_2$ . We have

$$\begin{aligned} u_x + v_y &= f(\rho) + xf'(\rho)\rho_x + g(\rho) + yg'(\rho)\rho_y \\ &= c \left\{ \left( \frac{\rho + \beta}{\rho + \alpha} \right)^{1/2} - \left( \frac{\rho + \alpha}{\rho + \beta} \right)^{1/2} + \left( \frac{\alpha}{\beta} \right)^{1/2} - \left( \frac{\beta}{\alpha} \right)^{1/2} \right\} \\ &\quad + \frac{c}{2}(\alpha - \beta) \left\{ \frac{x\rho_x}{(\rho + \alpha)^{3/2}(\rho + \beta)^{1/2}} + \frac{y\rho_y}{(\rho + \alpha)^{1/2}(\rho + \beta)^{3/2}} \right\}. \end{aligned}$$

Using (45) we get

$$u_x + v_y = c \frac{\alpha - \beta}{(\alpha\beta)^{1/2}} = 1$$

where we have used (40) and the definition of  $c$  (c.f. (35)) in the last identity.

(v) We finally establish that

$$\det(Du, Dv) = u_x v_y - u_y v_x \geq 0 \quad \text{a.e. in } \Omega_2. \quad (47)$$

We have

$$u_x v_y - u_y v_x = f(\rho)g(\rho) + y\rho_y f(\rho)g'(\rho) + x\rho_x f'(\rho)g(\rho). \quad (48)$$

Observe first that  $(\alpha - \beta)c \geq 0$ , by (35) and (40). Therefore, using (41), we deduce that  $f'(\rho), g'(\rho) \geq 0$  and hence by (35), that  $f(\rho), g(\rho) \geq 0$  for every  $\rho \in [0, 1]$ . Therefore if we can show that

$$x\rho_x, y\rho_y \geq 0 \quad \text{for every } \rho \in [0, 1], \quad (49)$$

we obtain from (48) the claimed result.

In fact, (49) is an immediate consequence of (42) and (43). Indeed from (42) we have that

$$\frac{\rho + \alpha}{\rho + \beta} y^2 = \rho + \alpha - x^2, \quad \frac{\rho + \beta}{\rho + \alpha} x^2 = \rho + \beta - y^2,$$

which implies that

$$2\rho + \alpha + \beta - x^2 - y^2 \geq 0. \quad (50)$$

Combining (43) and (50) we obtain (49) and thus the proof of (31) is complete.

Step 5: We may now define the solution to our problem by setting, for  $c$  a constant defined below,

$$w(x, y) = \begin{cases} c & \text{in } \Omega_1 \\ c + \int_0^y sg(\rho(0, s)) ds + \int_0^x tf(\rho(t, y)) dt & \text{in } \Omega_2. \end{cases}$$

Before defining the constant  $c$ , let us check that  $Dw = (u, v)$ . The result is obvious in  $\Omega_1$ ; so let us compute  $w_x$  and  $w_y$  in  $\Omega_2$ . The first one is immediately obtained

$$w_x = xf(\rho(x, y)) = u.$$

The second one leads to

$$w_y = yg(\rho(0, y)) + \int_0^x tf'(\rho(t, y))\rho_y(t, y) dt.$$

Using (46) we find

$$w_y = yg(\rho(0, y)) + \int_0^x yg'(\rho(t, y))\rho_x(t, y) dt = yg(\rho(x, y)) = v.$$

Therefore, since  $\Omega$  is connected,  $Dw = (u, v)$  and  $(u, v) = (\lambda x, \mu y)$  on  $\partial\Omega$ , we can adjust the constant  $c$  so that

$$w(x, y) = \varphi(x, y) = \frac{\lambda}{2}x^2 + \frac{\mu}{2}y^2 \quad \text{on } \partial\Omega.$$

Note also that the continuity of  $w$  on  $\partial\Omega_1$  is also clear since on  $\partial\Omega_1$  we have  $\rho = 0$  and  $f(0) = g(0) = 0$ . This concludes the proof of the lemma. ■

**Proof.** (Theorem 5) We divide the proof into two steps.

Step 1: We first show that, without losing generality, we can assume that  $a = 0$  and  $b = 1$ . Let

$$\varphi'(x, y) = \frac{1}{b-a} \left[ \varphi(x, y) - \frac{a}{4}(x^2 + y^2) \right].$$

We thus have

$$D^2\varphi'(x, y) = \frac{1}{b-a} \left[ D^2\varphi(x, y) - \frac{a}{2}I \right]$$

and hence, by hypothesis, we conclude that

$$\begin{cases} \Delta\varphi' = \frac{1}{b-a} (\Delta\varphi - a) \in \frac{1}{b-a} [0, b-a] = [0, 1] \\ \det D^2\varphi' = \left( \frac{1}{b-a} \right)^2 \det \left[ D^2\varphi - \frac{a}{2}I \right] > 0. \end{cases}$$

So assume that the theorem has been proved whenever  $a = 0$  and  $b = 1$ ; we can therefore find  $w' \in \varphi' + W_0^{2,\infty}(\Omega)$  such that

$$\begin{cases} \Delta w'(x, y) \in \{0, 1\}, \text{ a.e. } (x, y) \in \Omega, \\ \det [D^2w'(x, y)] \geq 0, \text{ a.e. } (x, y) \in \Omega. \end{cases} \quad (51)$$

Thus setting

$$w(x, y) = (b-a)w'(x, y) + \frac{a}{4}(x^2 + y^2)$$

we obtain

$$\begin{cases} \Delta w(x, y) = (b-a)\Delta w'(x, y) + a \in \{a, b\} \\ \det [D^2w - \frac{a}{2}I] = (b-a)^2 \det D^2w' \geq 0. \end{cases}$$

Step 2: So from now on we will assume that  $a = 0$  and  $b = 1$ . We now show by several restrictions that  $\Omega$  and  $\varphi$  can be chosen as in the lemma.

(i) It is clear that by covering  $\Omega$  by a countable union of bounded open sets we can assume that  $\Omega$  is bounded. By restricting our attention to each piece where  $\varphi$  is  $C^2$  we can assume that  $\varphi \in C^2(\overline{\Omega})$ .

(ii) We may also make the hypothesis that

$$0 < \Delta\varphi < 1 \quad \text{in } \overline{\Omega};$$

otherwise we choose  $w = \varphi$  on the closed set where  $\Delta\varphi = 0$  or  $\Delta\varphi = 1$  and we then work on the complement of this set which is open.

(iii) We may then assume (see Chapter 10 of author's book [10]) that  $\varphi$  is a piecewise polynomial of degree two. Restricting our attention to each of these pieces we can therefore reduce our considerations to polynomial of degree two, i.e.

$$\varphi(x, y) = \frac{\alpha}{2}x^2 + \frac{\beta}{2}y^2 + \gamma xy + \delta_1x + \delta_2y + \delta_3.$$

By changing  $\varphi$  to  $\varphi' = \frac{\alpha}{2}x^2 + \frac{\beta}{2}y^2 + \gamma xy$ , solving the problem with  $\varphi'$  and then adding to the solution the term  $(\delta_1x + \delta_2y + \delta_3)$  we can assume that

$$\begin{aligned} \varphi(x, y) &= \frac{\alpha}{2}x^2 + \frac{\beta}{2}y^2 + \gamma xy \\ &= \frac{1}{2} \left\langle \begin{pmatrix} \alpha & \gamma \\ \gamma & \beta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}; \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle. \end{aligned}$$



(iv) We then diagonalize the above matrix and write

$$\begin{pmatrix} \alpha & \gamma \\ \gamma & \beta \end{pmatrix} = R^t \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} R$$

where  $R \in SO(2)$  is a rotation and  $R^t$  is its transpose. It is clear that

$$\begin{cases} \alpha\beta - \gamma^2 = \lambda\mu > 0 \\ 0 < \alpha + \beta = \lambda + \mu < 1. \end{cases}$$

Hence we can write

$$\varphi(x, y) = \frac{1}{2} \left\langle \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} R \begin{pmatrix} x \\ y \end{pmatrix}; R \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle.$$

Changing coordinates, writing  $\begin{pmatrix} x' \\ y' \end{pmatrix} = R \begin{pmatrix} x \\ y \end{pmatrix}$ , we have reduced the problem to

$$\varphi(x, y) = \frac{\lambda}{2}x^2 + \frac{\mu}{2}y^2$$

with  $\lambda, \mu > 0$  and  $0 < \lambda + \mu < 1$ .

(v) So now  $\varphi$  is as in the lemma and it therefore remains to show that  $\Omega$  can be taken to be the ellipse of the lemma. Indeed use the standard Vitali covering theorem to cover  $\Omega$ , up to a set of measure zero, by translating and dilating such ellipses. In fact, translations are allowed because the differential equation does not involve explicitly the variable  $(x, y)$ . For dilations we make the transformations  $(x', y') = (Rx, Ry)$ , with  $R > 0$ , on the independent variable and  $u'(x, y) = \frac{1}{R^2}u(x', y') = \frac{1}{R^2}u(Rx, Ry)$  on the dependent variable. Since  $\varphi$  is quadratic, then  $\varphi'(x, y) = \frac{1}{R^2}\varphi'(Rx, Ry) = \varphi'(x, y)$ ; finally  $D^2w'(x, y) = D^2w(Rx, Ry)$ .

(vi) We may therefore solve the problem on each ellipse and define  $w = \varphi$  on the set of measure zero.

The theorem has thus been proved. ■

## 5. ATTAINMENT OF MINIMA IN THE SCALAR CASE

We consider the following minimization problem

$$\inf \left\{ \int_{\Omega} f(Du(x)) \, dx : u \in u_0 + W_0^{1,\infty}(\Omega) \right\},$$

where  $\Omega$  is a bounded open set of  $\mathbb{R}^n$ ,  $n \geq 2$ ,  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is a lower semicontinuous function and  $u_0$  is a given boundary datum.

We say that a bounded open set  $\Omega$  of  $\mathbb{R}^n$ ,  $n \geq 2$ , is uniformly convex, if there exists a positive constant  $c$  such that for every  $x_0 \in \partial\Omega$  and every hyperplane  $\pi_{x_0}$  containing  $x_0$  the following inequality holds

$$\text{dist}(x; \pi_{x_0}) \geq c \cdot |x - x_0|^2, \quad \forall x \in \partial\Omega.$$

Note that, for every  $x_0 \in \partial\Omega$ ,  $\pi_{x_0}$  is a supporting hyperplane, i.e. it is a hyperplane passing through  $x_0$  and leaving the set  $\Omega$  on one of the two half spaces delimited by  $\pi_{x_0}$ .

Denoting by  $f^{**}$  the convex envelope of  $f$  (i.e. the largest function which is less than or equal to  $f$  on  $\mathbb{R}^n$ ), we assume that  $f^{**}$  is *affine on each connected component* of the (open) set

$$A = \{\xi \in \mathbb{R}^n : f(\xi) > f^{**}(\xi)\}; \quad (52)$$

i.e. there exist a partition of  $A$  into disjoint open sets  $A_j \subset \mathbb{R}^n$ , vectors  $\eta_j \in \mathbb{R}^n$  and constants  $q_j$  such that

$$f^{**}(\xi) = \langle \eta_j; \xi \rangle + q_j, \quad \forall \xi \in A_j. \quad (53)$$

We also assume that each connected component  $A_j$  is a bounded set.

Then the following existence theorem holds.

**Theorem 7.** *Let  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 2$ , be bounded, open and uniformly convex. Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a lower semicontinuous function, bounded from below and satisfying the assumptions stated above, in particular (53). Let  $u_0 \in C^2(\bar{\Omega})$ . Then the problem*

$$\inf \left\{ \int_{\Omega} f(Du(x)) \, dx : u \in u_0 + W_0^{1,\infty}(\Omega) \right\} \quad (54)$$

*attains its minimum.*

Note that, if the boundary datum  $u_0$  is affine in  $\Omega$ , then the attainment result of Theorem 7 holds for every bounded open set  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 2$ , not necessarily uniformly convex.

Theorem 7 is a generalization of some analogous results obtained for  $n \geq 2$ , under more restrictive assumptions, by Marcellini [17], Mascolo-Schianchi [18], Cellina [5] and Friesecke [12]. Theorem 7 has been recently proved by Sychev [20] in the form presented here and also, under some further coercivity condition on the integrand  $f$ , for a wider class of boundary data (see also Zagatti [22] and Celada-Perrotta [3]). In particular Mascolo and Schianchi pointed out the condition of affinity of the function  $f^{**}$  on the set where  $f \neq f^{**}$ , while Cellina (see also [4]) and Friesecke proved the necessity and sufficiency of this condition of affinity for linear boundary data  $u_0$ .

**Proof.** Step 1: The proof of Theorem 7 makes use of the associated relaxed variational problem

$$\inf \left\{ \int_{\Omega} f^{**}(Du(x)) \, dx : u \in u_0 + W_0^{1,\infty}(\Omega) \right\}. \quad (55)$$

A classical result in the context of the so called “*bounded slope condition*”, due to Hartman-Stampacchia [13] and Miranda [19], ensures that the functional integral in (55) has a Lipschitz-continuous minimizer  $u^{**}$  in the class  $u_0 + W_0^{1,\infty}(\Omega)$  (at this step, and in fact only here, we use the assumptions of uniform convexity of  $\Omega$  and the smoothness of  $u_0$ ). Of course, if  $u_0$  is affine in  $\Omega$ , then  $u^{**} = u_0$  is a minimizer, independently of the uniform convexity of  $\Omega$ ; this proves the note after the statement of the Theorem.

Step 2: We consider the set  $A = \{\xi \in \mathbb{R}^n : f(\xi) > f^{**}(\xi)\}$  in (52). Since  $f$  is a lower semicontinuous function and  $f^{**}$  a continuous one, the set  $A$  is open. Let us denote by  $A_j$ ,  $j = 1, 2, \dots$ , the connected components of the open set  $A \subset \mathbb{R}^n$  (we consider here the case that  $\{A_j\}$  is a countable family of sets, the other possibility that  $\{A_j\}$  is a finite set being similar, and in fact simpler to treat). Therefore  $\{A_j : j \in \mathbb{N}\}$  is a collection of disjoint open sets whose union is equal to  $A$ .

By assumption, for every  $j \in \mathbb{N}$  there exist vectors  $\eta_j \in \mathbb{R}^n$  and constants  $q_j$  such that

$$f^{**}(\xi) = \langle \eta_j; \xi \rangle + q_j, \quad \forall \xi \in A_j.$$

By defining again, if necessary, the indices and by taking as new set  $A_j$  a suitable union of  $A_k$  for some  $k$ , we can assume that these support hyperplanes are different for different indices. Then  $\{A_j : j \in \mathbb{N}\}$  still is a collection of disjoint open sets (not necessarily connected) whose union is equal to  $A$ . With this definition, if we denote by

$$E_j = \{\xi \in \mathbb{R}^n : f^{**}(\xi) = \langle \eta_j; \xi \rangle + q_j\}, \quad (56)$$

then we have  $A_j \subset E_j$  for every  $j \in \mathbb{N}$ , and

$$A_k \cap E_j = \emptyset, \quad \text{if } k \neq j. \quad (57)$$

Finally we denote by  $A_0$  the empty set.

Step 3: Now we build a sequence  $u_j^{**} \in u_0 + W_0^{1,\infty}(\Omega)$ ,  $j = 1, 2, \dots$ , of solutions of the relaxed problem (55) with the properties

$$\begin{cases} Du_j^{**}(x) \notin A_k, \text{ a.e. } x \in \Omega, & \forall k = 0, 1, \dots, j-1 \\ \text{a.e. } x \in \Omega, \text{ if } Du_j^{**}(x) \in A_k \text{ then } u_j^{**}(x) = u^{**}(x), & \forall k = j, j+1, \dots \end{cases} \quad (58)$$

We proceed by induction (finite induction, if the set  $\{A_j\}$  is a finite collection of sets). For  $j = 1$  we define  $u_1^{**} = u^{**}$ . By induction we assume that  $u_j^{**} \in u_0 + W_0^{1,\infty}(\Omega)$ , for some  $j = 0, 1, 2, \dots$ , is a minimizer of the variational problem (55), satisfying (58), and we proceed to define  $u_{j+1}^{**}$ .

The set  $A_j$  being open, the compatibility assumption (with  $B_j = \mathbb{R}^n - A_j$ )

$$Du_j^{**}(x) \in A_j \cup B_j = \mathbb{R}^n, \text{ a.e. } x \in \Omega,$$

of Corollary 2 in Section 2 is satisfied. Therefore, by the thesis of that corollary, there exist a function  $v_j \in u_j^{**} + W_0^{1,\infty}(\Omega) = u_0 + W_0^{1,\infty}(\Omega)$  and an open set  $\Omega_j \subset \Omega$  such that  $v_j$  is piecewise affine in  $\Omega_j$  and

$$\begin{cases} Dv_j(x) \in A_j, & \text{a.e. } x \in \Omega_j \\ v_j(x) = u_j^{**}(x), \quad Dv_j(x) = Du_j^{**}(x) \in B_j, & \text{a.e. } x \in \Omega - \Omega_j. \end{cases} \quad (59)$$

Moreover, again by Corollary 2, we can choose the open set  $\Omega_j$  such that

$$\text{meas}(\Omega_j) < 2^{-j} + \text{meas}\{x \in \Omega : Du_j^{**}(x) \in A_j\}. \quad (60)$$

By assumption (53) the function  $f^{**}$  is affine on  $A_j$  and  $f^{**}(\xi) = \langle \eta_j; \xi \rangle + q_j$  for every  $\xi \in A_j$ . We can consider the set  $E_j$  in (56). By the convexity of  $f^{**}$  we have

$$E_j = \{\xi \in \mathbb{R}^n : f^{**}(\xi) - \langle \eta_j; \xi \rangle - q_j \leq 0\};$$

thus  $E_j$ , being a level set of a convex function, is a convex set. Since by assumption every connected component of  $A$  is a bounded set, it is easy to see that every point  $\xi \in A_j$  is convex combination of two points of the boundary  $\partial A_j \subset E_j - A_j$ . Moreover  $A_j \subset E_j$  is an open set; thus we have

$$A_j \subset \text{int } E_j = \text{int co}(E_j - A_j).$$

By the compatibility condition in (59) we deduce that

$$Dv_j(x) \in A_j \subset \text{int co}(E_j - A_j), \quad \text{a.e. } x \in \Omega_j.$$

Therefore we can apply Theorem 3 with the set  $E_j - A_j$  and obtain, for every  $j \in \mathbb{N}$ , the existence of a function  $u_j \in W^{1,\infty}(\Omega_j)$  such that

$$\begin{cases} Du_j(x) \in E_j - A_j, & \text{a.e. } x \in \Omega_j \\ u_j(x) = v_j(x), & x \in \partial\Omega_j. \end{cases} \quad (61)$$

Now we are ready to prove that a new minimizer in the class of Lipschitz-continuous functions  $u_0 + W_0^{1,\infty}(\Omega)$  of the integral in (55) is given by

$$u_{j+1}^{**}(x) = \begin{cases} u_j(x), & \text{if } x \in \Omega_j \\ u_j^{**}(x), & \text{if } x \in \Omega - \Omega_j \end{cases} \quad (62)$$

(thus in particular  $u_{j+1}^{**} = v_j = u_j^{**} = u_0$  for every  $x \in \partial\Omega$ ). In fact, by the affinity (53) of  $f^{**}$  on  $E_j$  and the boundary conditions in (59), (61) which state that  $u_{j+1}^{**} = v_j = u_j^{**}$  on  $\partial\Omega_j$ , we get

$$\begin{aligned} & \int_{\Omega} f^{**}(Du_{j+1}^{**}(x)) \, dx \\ &= \int_{\Omega - \Omega_j} f^{**}(Du_{j+1}^{**}(x)) \, dx + \int_{\Omega_j} f^{**}(Du_{j+1}^{**}(x)) \, dx \\ &= \int_{\Omega - \Omega_j} f^{**}(Du_j^{**}(x)) \, dx + \int_{\Omega_j} \{ \langle \eta_j; Du_{j+1}^{**}(x) \rangle + q_j \} \, dx \\ &= \int_{\Omega - \Omega_j} f^{**}(Du_j^{**}(x)) \, dx + \int_{\Omega_j} \{ \langle \eta_j; Du_j^{**}(x) \rangle + q_j \} \, dx; \end{aligned}$$

by the convexity inequality  $f^{**}(\xi) \geq \langle \eta_j; \xi \rangle + q_j$ , valid for every  $\xi \in \mathbb{R}^n$ , finally we obtain

$$\begin{aligned} & \int_{\Omega} f^{**}(Du_{j+1}^{**}(x)) \, dx \\ & \leq \int_{\Omega} f^{**}(Du_j^{**}(x)) \, dx \\ & = \min \left\{ \int_{\Omega} f^{**}(Du(x)) \, dx : u \in u_0 + W_0^{1,\infty}(\Omega) \right\}. \end{aligned}$$

Recalling (61) and (59), we also have

$$\begin{cases} Du_{j+1}^{**}(x) = Du_j(x) \in E_j - A_j & \text{a.e. } x \in \Omega_j \\ Du_{j+1}^{**}(x) = Du_j^{**}(x) \in B_j = \mathbb{R}^n - A_j, & \text{a.e. } x \in \Omega - \Omega_j. \end{cases}$$

In particular  $Du_{j+1}^{**} \notin A_j$  a.e.  $x \in \Omega$ . Moreover, by (57), for almost every  $x \in \Omega_j$  we have

$$Du_{j+1}^{**}(x) \in E_j \quad \Rightarrow \quad Du_{j+1}^{**}(x) \notin A_k, \quad \text{if } k \neq j,$$

while in  $\Omega - \Omega_j$  we can use the induction hypotheses (58) on  $Du_j^{**}$ . We obtain

$$\begin{cases} Du_{j+1}^{**}(x) \notin A_k, & \text{a.e. } x \in \Omega, \quad \forall k = 0, 1, \dots, j \\ \text{a.e. } x \in \Omega, & \text{if } Du_{j+1}^{**}(x) \in A_k \text{ then } u_{j+1}^{**}(x) = u^{**}(x), \quad \forall k = j+1, \dots \end{cases}$$

and (58) is proved too.

Step 4: If  $\{A_j\}$  is a finite collection of sets, with  $j = 1, 2, \dots, J$ , then by (58)  $u_{J+1}^{**}$  has the property

$$Du_{J+1}^{**}(x) \notin A = \bigcup_{k=1}^J A_k, \quad \text{a.e. } x \in \Omega,$$

and thus, since  $f = f^{**}$  on  $\mathbb{R}^n - A$ ,

$$f(Du_{J+1}^{**}(x)) = f^{**}(Du_{J+1}^{**}(x)), \quad \text{a.e. } x \in \Omega. \quad (63)$$

We obtain that  $u_{J+1}^{**}$  is a minimizer for the original variational problem (54); in fact, since  $f^{**} \leq f$  on  $\mathbb{R}^n$ , by (63) we have

$$\begin{aligned} \int_{\Omega} f(Du_{J+1}^{**}(x)) \, dx &= \int_{\Omega} f^{**}(Du_{J+1}^{**}(x)) \, dx \\ &= \min \left\{ \int_{\Omega} f^{**}(Du(x)) \, dx : u \in u_0 + W_0^{1,\infty}(\Omega) \right\} \\ &\leq \inf \left\{ \int_{\Omega} f(Du(x)) \, dx : u \in u_0 + W_0^{1,\infty}(\Omega) \right\}. \end{aligned}$$

In this case the next Step 5 becomes unnecessary.

Step 5: We are now considering the case that  $\{A_j : j \in \mathbb{N}\}$  is a countable collection of sets of  $\mathbb{R}^n$ . By Step 3,  $\{u_j^{**}\}_{j \in \mathbb{N}}$  is a sequence of minimizers of the relaxed problem (55), satisfying (58). We will pass to the limit as  $j \rightarrow +\infty$ .

First, since  $u^{**} \in W^{1,\infty}(\Omega)$ , there exists  $M > 0$  such that  $\|Du^{**}\|_{L^\infty(\Omega)} \leq M$ . The sequence  $u_j^{**}$  has been defined by induction by mean of values of  $u^{**}$  (see in particular (61), (62)); thus we also have

$$\|Du_j^{**}\|_{L^\infty(\Omega)} \leq M, \quad \forall j \in \mathbb{N}. \quad (64)$$

Note that it is not sufficient to use below only the weak\* convergence in  $W^{1,\infty}(\Omega)$  of a subsequence of  $\{u_j^{**}\}_{j \in \mathbb{N}}$ . Instead we consider the following inclusions, valid almost everywhere in  $\Omega$ . Before let us recall that, by the definition (62) of  $u_{j+1}^{**}$ ,

$$\{x \in \Omega : Du_{j+1}^{**}(x) \neq Du_j^{**}(x)\} \subset \Omega_j, \quad \forall j \in \mathbb{N}.$$

We obtain, for every  $j, k \in \mathbb{N}$ ,

$$\begin{aligned} &\{x \in \Omega : |Du_{j+k}^{**}(x) - Du_j^{**}(x)| > 0\} \\ &\subset \left\{ x \in \Omega : \sum_{i=1}^k |Du_{j+i}^{**}(x) - Du_{j+i-1}^{**}(x)| > 0 \right\} \\ &\subset \Omega_j \cup \Omega_{j+1} \cup \dots \cup \Omega_{j+k-1}. \end{aligned}$$

By (64) and by (60) we deduce that

$$\begin{aligned} \|Du_{j+k}^{**} - Du_j^{**}\|_{L^1(\Omega)} &\leq M \cdot \text{meas} \left( \bigcup_{i=j}^{j+k-1} \Omega_i \right) \\ &\leq M \cdot \sum_{i=j}^{\infty} (2^{-i} + \text{meas} \{x \in \Omega : Du_i^{**}(x) \in A_i\}). \end{aligned}$$

By the second assertion of (58), if  $Du_i^{**}(x) \in A_i$  then we also have  $Du^{**}(x) = Du_i^{**}(x) \in A_i$ . Thus

$$\|Du_{j+k}^{**} - Du_j^{**}\|_{L^1(\Omega)} \leq M \left( 2^{-j+1} + \sum_{i=j}^{\infty} \text{meas} \{x \in \Omega : Du^{**}(x) \in A_i\} \right)$$

and the quantity in the right hand side goes to zero as  $j \rightarrow +\infty$ , since the corresponding series converges (note that the  $A_i$  are disjoint sets):

$$\sum_{i=1}^{\infty} \text{meas} \{x \in \Omega : Du^{**}(x) \in A_i\} \leq \text{meas}(\Omega) < +\infty.$$

Taking into account the boundary condition  $u_j^{**} = u_0$  on  $\partial\Omega$ , we see that  $\{u_j^{**}\}_{j \in \mathbb{N}}$  is a Cauchy sequence in  $W^{1,1}(\Omega)$  and converges to a function  $u \in u_0 + W_0^{1,\infty}(\Omega)$  (by (64)). Moreover, up to a subsequence, its gradient  $u_j^{**}$  pointwise converges almost everywhere in  $\Omega$  to  $Du$  and, by the first assertion in (58), we have

$$Du(x) \notin A = \bigcup_{k=1}^{\infty} A_k, \quad \text{a.e. } x \in \Omega.$$

Thus  $f(Du(x)) = f^{**}(Du(x))$  almost everywhere in  $\Omega$  and we can conclude, similarly to the previous Step 4, that  $u$  is a minimizer of the original variational problem (54). ■

The above theorem should not be misleading; the general rule for minimization of non convex variational problems is that there is non existence of minimizers. This is classical in the one-dimensional case  $n = 1$  for integrals of the type

$$\int_{\Omega} f(u(x), u'(x)) \, dx$$

(when  $n = 1$  note the necessity of the dependence of the integrand  $f$  on  $u$ , other than  $u'$ , to exhibit an example of lack of existence of minimizers). A well known example due to Bolza is the following.

**Example 1.** *The problem*

$$\inf \left\{ \int_0^1 f(u(x), u'(x)) \, dx : u \in W_0^{1,\infty}(0,1) \right\},$$

where  $f(s, \xi) = (\xi^2 - 1)^2 + s^2$ , has no solution.

For  $n \geq 2$  the lack of minimizers for integrals of the type (54) has been pointed out by Marcellini in [16]. Other examples are known, such as Example 1 of Section 5.2.2.2 of [6]. As already said, necessary conditions for existence of solutions with affine boundary data  $u_0$  are due to Cellina [4] and Friesecke [12]. Here we present an example of non existence of minimizers for a variational problem of the type

$$\inf \left\{ \int_{\Omega} f(Du(x)) \, dx : u \in u_0 + W_0^{1,\infty}(\Omega) \right\},$$

where  $\Omega$  is a subset of  $\mathbb{R}^2$  and  $f$  is a coercive nonconvex function with the property that  $f^{**}$  is *piecewise affine* on the set

$$A = \{\xi \in \mathbb{R}^n : f(\xi) > f^{**}(\xi)\}$$

(but not affine on each connected component of  $A$ ). This shows the necessity of (53); in particular the necessity of the assumption that  $f^{**}$  is affine on each connected component of the set where  $f \neq f^{**}$ .

**Proposition 8.** *Let  $\Omega$  be a disk of  $\mathbb{R}^2$ . Let*

$$f(\xi) = f(\xi_1, \xi_2) = \max \left\{ |\xi_1| + (\xi_2^2 - 1)^2; |\xi|^2 - 2 \right\}.$$

*Then the problem*

$$\inf \left\{ \int_{\Omega} f(Du(x)) dx : u \in W_0^{1,\infty}(\Omega) \right\},$$

*has no solution.*

**Proof.** Note first that  $f$  is a nonnegative coercive nonconvex function on  $\mathbb{R}^2$ ; in particular, inside the ball of center at  $\xi = 0$  and radius 1 the function  $f$  is given by

$$f(\xi) = |\xi_1| + (\xi_2^2 - 1)^2.$$

Moreover, on the set of  $\mathbb{R}^2$  where  $f \neq f^{**}$  then  $f^{**}$  is piecewise affine (is affine separately for some  $(\xi_1, \xi_2)$  either with  $\xi_1 > 0$  or with  $\xi_1 < 0$ ).

We now prove that the variational problem

$$\inf \left\{ \int_{\Omega} f(Du(x)) dx : u \in W_0^{1,\infty}(\Omega) \right\},$$

(with the zero boundary condition) has no solution.

Note that the boundary datum has constant zero gradient, which is convex combination (with coefficients equal to 1/2) of the two points  $\bar{\xi} = (0, 1)$ ,  $\bar{\eta} = (0, -1)$  where the function  $f$  is equal to zero. With a technique that in the book by the authors [10] has a great relevance in many steps, we can show that, for every  $\varepsilon > 0$ , there exists  $u \in W^{1,\infty}(\Omega)$  and there exist disjoint open sets  $\Omega_{\bar{\xi}}, \Omega_{\bar{\eta}} \subset \Omega$  such that

$$\left\{ \begin{array}{l} \left| \text{meas } \Omega_{\bar{\xi}} - t \text{ meas } \Omega \right|, \left| \text{meas } \Omega_{\bar{\eta}} - (1-t) \text{ meas } \Omega \right| \leq \varepsilon \\ u(x) = 0, \quad \forall x \in \partial\Omega \\ |u(x)| \leq \varepsilon, \quad \forall x \in \Omega \\ Du(x) = \begin{cases} \bar{\xi} & \text{a.e. in } \Omega_{\bar{\xi}} \\ \bar{\eta} & \text{a.e. in } \Omega_{\bar{\eta}} \end{cases} \\ \text{dist}(Du(x), \text{co}\{\bar{\xi}, \bar{\eta}\}) \leq \varepsilon, \quad \text{a.e. in } \Omega, \end{array} \right.$$

where  $\text{co}\{\bar{\xi}, \bar{\eta}\} = [\bar{\xi}, \bar{\eta}]$  is the convex hull of  $\{\bar{\xi}, \bar{\eta}\}$ , that is the closed segment joining  $\bar{\xi}$  and  $\bar{\eta}$ . When  $\varepsilon = 1/k$ , with  $k \in \mathbb{N}$ , we obtain a sequence  $u = u_k$  that converges uniformly to zero and, by the fact that  $f(\bar{\xi}) = f(\bar{\eta}) = 0$ ,

$$\lim_{k \rightarrow \infty} \int_{\Omega} f(Du_k(x)) dx = 0.$$

Since  $f \geq 0$ , we deduce that

$$\inf \left\{ \int_{\Omega} f(Du(x)) \, dx : u \in W_0^{1,\infty}(\Omega) \right\} = 0.$$

A minimizer does not exist. In fact, by contradiction, if there is  $\bar{u}$  such that

$$\int_{\Omega} f(D\bar{u}(x)) \, dx = 0,$$

then

$$0 = \int_{\Omega} f(D\bar{u}(x)) \, dx \geq \int_{\Omega} \left\{ |\bar{u}_{x_1}| + (\bar{u}_{x_2}^2 - 1)^2 \right\} dx \geq 0, \quad (65)$$

and  $\bar{u}_{x_1} = 0$ . Thus  $\bar{u} = \bar{u}(x_2)$  depends only on  $x_2$ . Since  $\bar{u} = 0$  on  $\partial\Omega$ , we deduce that  $\bar{u}$  is identically equal to zero in  $\Omega$ , that contradicts the condition  $|\bar{u}_{x_2}| = 1$ , which also comes from (65). ■

## 6. ATTAINMENT OF MINIMA IN THE VECTOR-VALUED CASE

We now apply Theorem 5 to a problem of *optimal design* introduced by Kohn-Strang [15]. We consider the two dimensional case  $n = 2$  and  $m = 2$  (that finds its origins in a variational problem in optimal design) related to the lower semicontinuous (nonconvex) function  $f : \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}$

$$f(\xi) = \begin{cases} 1 + |\xi|^2 & \text{if } \xi \neq 0 \\ 0 & \text{if } \xi = 0. \end{cases} \quad (66)$$

Here for simplicity we limit ourselves to  $m = 2$ ; see [7] and Kohn-Strang [15] for a discussion of the case  $m > 2$ ; see also Allaire-Francfort [1] for  $m, n \geq 2$ . The problem is then to find a minimizer, which is the desired optimal design, of

$$\inf \left\{ \int_{\Omega} f(Du(x)) \, dx : u \in u_0 + W_0^{1,\infty}(\Omega; \mathbb{R}^2) \right\}.$$

We then have the following theorem.

**Theorem 9.** *Let  $\Omega$  be a bounded open set of  $\mathbb{R}^2$ . Let  $u_0 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a linear boundary datum, such that  $Du_0 \in \mathbb{R}_s^{2 \times 2}$  is a constant symmetric matrix satisfying the conditions*

$$0 < \text{trace } Du_0 < 1, \quad \det Du_0 > 0.$$

*Let  $f$  be as in (66). Then the nonconvex variational problem*

$$\inf \left\{ \int_{\Omega} f(Du(x)) \, dx : u \in u_0 + W_0^{1,\infty}(\Omega; \mathbb{R}^2) \right\}$$

*has a solution  $\bar{u} \in u_0 + W_0^{1,\infty}(\Omega; \mathbb{R}^2)$ . Moreover there exists  $w \in W^{2,\infty}(\Omega)$ , such that  $\bar{u} = Dw$ , satisfying*

$$\begin{cases} \Delta w(x) \in \{0, 1\}, & \text{a.e. } x \in \Omega, \\ \det D^2 w(x) \geq 0, & \text{a.e. } x \in \Omega. \end{cases}$$

We have as an immediate Corollary the following result (c.f. Theorem 6.1 in [7]).



**Corollary 10.** *Let  $\Omega$  be a bounded open set of  $\mathbb{R}^2$ . Let  $u_0 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a linear boundary datum, such that  $Du_0 \in \mathbb{R}^{2 \times 2}$  is a constant matrix satisfying one of the following conditions*

- (i)  $Du_0 = 0$
- (ii)  $|Du_0|^2 + 2|\det Du_0| \geq 1$
- (iii)  $\det Du_0 \neq 0$ .

Let  $f$  be as in (66). Then the nonconvex variational problem

$$\inf \left\{ \int_{\Omega} f(Du(x)) dx : u \in u_0 + W_0^{1,\infty}(\Omega; \mathbb{R}^2) \right\}$$

has a solution  $\bar{u} \in u_0 + W_0^{1,\infty}(\Omega; \mathbb{R}^2)$ .

**Remark 1.** *In fact (c.f. Theorem 6.1 in [7]) the three conditions (i), (ii), (iii) are also necessary for the existence of minimizers of the problem; in the sense that if  $Du_0$  does not satisfy any of the three conditions then no minimizer in the class  $u_0 + W_0^{1,\infty}(\Omega; \mathbb{R}^2)$  exists.*

**Proof.** (Theorem 9) We first characterize the quasiconvex envelope  $Qf$  of  $f$  ( $Qf : \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}$  is the largest quasiconvex function less than or equal to  $f$  on  $\mathbb{R}^{2 \times 2}$ ; see for more details Chapter 5), computed in [15], it is given by

$$Qf(\xi) = \begin{cases} 1 + |\xi|^2 & \text{if } |\xi|^2 + 2|\det \xi| \geq 1 \\ 2\sqrt{|\xi|^2 + 2|\det \xi|} - 2|\det \xi| & \text{if } |\xi|^2 + 2|\det \xi| < 1. \end{cases} \quad (67)$$

If  $\xi \in \mathbb{R}_s^{2 \times 2}$  is such that  $\det \xi \geq 0$  then the formula takes the simpler form (since then  $|\xi|^2 + 2|\det \xi| = (\text{trace } \xi)^2$ )

$$Qf(\xi) = \begin{cases} 1 + |\xi|^2 & \text{if } |\text{trace } \xi| \geq 1 \\ 2(|\text{trace } \xi| - \det \xi) & \text{if } |\text{trace } \xi| < 1. \end{cases} \quad (68)$$

To avoid the trivial situation  $Qf(Du_0) = f(Du_0)$ , we assume that

$$|Du_0|^2 + 2|\det Du_0| < 1; \quad (69)$$

which, with our hypotheses, is equivalent to

$$0 < \text{trace } Du_0 < 1. \quad (70)$$

Since by hypothesis  $Du_0$  is a symmetric  $2 \times 2$  matrix, this implies that there exists a polynomial of degree 2 such that

$$u_0 = \begin{pmatrix} \varphi_x \\ \varphi_y \end{pmatrix}, \quad \text{with } \det D^2\varphi(x) = \det Du_0 > 0.$$

We therefore consider vector-valued functions  $u$  which are gradients of functions  $v \in W^{2,\infty}(\Omega)$ ; i.e.:

$$u = \begin{pmatrix} u^1 \\ u^2 \end{pmatrix} = \begin{pmatrix} v_x \\ v_y \end{pmatrix};$$

and we obtain, if  $\det Du \geq 0$ , that

$$Du = \begin{pmatrix} v_{xx} & v_{xy} \\ v_{xy} & v_{yy} \end{pmatrix}, \quad |Du|^2 + 2|\det Du| = (v_{xx} + v_{yy})^2 = (\Delta v)^2. \quad (71)$$

Since  $\det D^2v = v_{xx} \cdot v_{yy} - (v_{xy})^2 \geq 0$ , then  $v_{xx} \cdot v_{yy} \geq 0$ ; therefore

$$\Delta v = v_{xx} + v_{yy} = 0 \iff \begin{cases} v_{xx} = 0 \\ v_{yy} = 0 \end{cases} \quad \text{and thus also } v_{xy} = 0.$$

Thus  $\Delta v = 0$  if and only if  $Dv = 0$ . The compatibility condition on the boundary datum  $\varphi$ , by (69), (70) and (71) becomes

$$\varphi \in C^2(\bar{\Omega}) \quad \text{and} \quad 0 < \Delta\varphi(x) < 1, \quad \det D^2\varphi(x) > 0. \quad (72)$$

We can apply Theorem 5 with  $a = 0$  and  $b = 1$ . Therefore there exists  $w \in \varphi + W_0^{2,\infty}(\Omega)$  such that

$$\begin{cases} \Delta w(x) \in \{0, 1\}, & \text{a.e. } x \in \Omega, \\ \det D^2w(x) \geq 0, & \text{a.e. } x \in \Omega. \end{cases} \quad (73)$$

Since either  $\Delta w = 0$  or  $\Delta w = 1$ , a.e. in  $\Omega$ , by (67), (71) we obtain

$$Qf(D^2w(x)) = f(D^2w(x)), \quad \text{a.e. } x \in \Omega. \quad (74)$$

Now we can prove that the function

$$\bar{u} = \begin{pmatrix} w_x \\ w_y \end{pmatrix}$$

is a minimizer of the integral

$$\int_{\Omega} f(Du(x)) \, dx$$

in the class of functions  $u \in W^{1,\infty}(\Omega; \mathbb{R}^2)$  such that  $u = u_0 = D\varphi$  on  $\partial\Omega$ . Recall that by (68) we have, if  $\det \xi$ ,  $\text{trace } \xi \geq 0$

$$Qf(\xi) = 2\sqrt{|\xi|^2 + 2|\det \xi|} - 2|\det \xi| = 2\text{trace } \xi - 2\det \xi.$$

Thus  $Qf$  is a so called *quasi-affine function*; it has the property that the integral of  $Qf(D\bar{u}(x))$  depends exclusively on the boundary datum of  $\bar{u}(x)$ ; more precisely, since  $\bar{u} = u_0$  on  $\partial\Omega$  and  $\det D\bar{u}$ ,  $\det Du_0$ ,  $\text{trace } D\bar{u}$ ,  $\text{trace } Du_0 \geq 0$

$$\begin{aligned} \int_{\Omega} Qf(D\bar{u}(x)) \, dx &= \int_{\Omega} \{2\text{trace } D\bar{u}(x) - 2\det D\bar{u}(x)\} \, dx \\ &= \int_{\Omega} \{2\text{trace } Du_0 - 2\det Du_0\} \, dx \\ &= \int_{\Omega} Qf(Du_0) \, dx. \end{aligned}$$

By (74) we also have

$$Qf(D\bar{u}(x)) = f(D\bar{u}(x)), \quad \text{a.e. } x \in \Omega.$$

Hence we can use the inequality of quasiconvexity and we obtain

$$\begin{aligned}
\int_{\Omega} f(D\bar{u}(x)) \, dx &= \int_{\Omega} Qf(D\bar{u}(x)) \, dx \\
&= \int_{\Omega} Qf(Du_0) \, dx \\
&= \min \left\{ \int_{\Omega} Qf(Du(x)) \, dx : u \in u_0 + W_0^{1,\infty}(\Omega; \mathbb{R}^2) \right\} \\
&\leq \inf \left\{ \int_{\Omega} f(Du(x)) \, dx : u \in u_0 + W_0^{1,\infty}(\Omega; \mathbb{R}^2) \right\}.
\end{aligned}$$

We have therefore proved the claimed result. ■

We may now conclude with the proof of Corollary 10.

**Proof.** Since the function  $f$  is invariant by orthogonal transformations, we can assume that  $Du_0$  is a diagonal matrix with  $\det Du_0$ ,  $\text{trace } Du_0 \geq 0$ .

If  $Qf(Du_0) = f(Du_0)$  the result is trivial by choosing  $\bar{u} = u_0$ . So we assume that  $Qf(Du_0) \neq f(Du_0)$  (this implies that the conditions (i) and (ii) do not hold) which combined with the facts that  $Du_0$  is a diagonal matrix and  $\det Du_0$ ,  $\text{trace } Du_0 \geq 0$  lead to  $0 < \text{trace } Du_0 < 1$  and by hypothesis (iii) we then have  $\det Du_0 > 0$ . We can then apply Theorem 9 to get the conclusion of the Corollary. ■

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