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On the solvability of implicit nonlinear systems in the vectorial case

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Abstract. We continue the study of a functional analytic method based on Baire category theorem for handling existence of almost everywhere solutions of a large class of partial differential equations and systems, which are nonlinear in the highest derivatives. We consider in this paper the solvability in the vectorial case of some implicit nonlinear systems of arbitrary order. The results have applications to the calculus of variations, nonlinear elasticity, problems of phase transitions or optimal design.

1. Introduction

The aim of this paper is to prove some existence results for systems of the form

$$\begin{cases} u \in W^{N,\infty}(\Omega;\mathbb{R}^m) \\ F_i\left(x, D^{[N-1]}u(x), D^N u(x)\right) = 0, \quad \text{a.e. } x \in \Omega, \quad i = 1,\dots,I \\ D^\alpha u(x) = D^\alpha \varphi(x), \quad x \in \partial\Omega, \quad \alpha = 0,\dots,N-1, \end{cases} \tag{1.1}$$

where Ω is an open set of $\mathbb{R}^n$ $(n \geq 1)$, F_i for $i = 1,\dots,I$, $(I \geq 1)$ are given real functions and $u : \mathbb{R}^n \to \mathbb{R}^m$ $(m \geq 1)$ is the unknown in the Sobolev class $W^{N,\infty}(\Omega;\mathbb{R}^m)$. For the partial derivatives of u up to the order N $(\alpha = 1,2,\dots,N \geq 1)$ we use the symbols (see the next section for some details more)

$$D^\alpha u = \left(\frac{\partial^\alpha u^i}{\partial x_{j_1}...\partial x_{j_N}}\right)_{1\leq j_1,\dots,j_N\leq n}^{1\leq i\leq m}, \quad \alpha = 1,2,\dots,N,$$

$$D^{[N-1]}u = \left(u,\dots,D^\alpha u,\dots,D^{N-1}u\right).$$

Finally $\varphi \in W^{N,\infty}(\Omega;\mathbb{R}^m)$ is a fixed boundary datum.

Recently in a series of papers the authors exploited a method to obtain existence of solutions to differential problems of the type (1.1) in the vector valued case $m \geq 1$. In particular in [**18**], [**19**], [**20**], [**21**], the authors studied the first order case $(N = 1)$, while in [**22**] they considered the second order one $(N = 2)$. The method, based on *Baire category theorem*, was originated by A. Cellina [**11**] to study existence of some ordinary differential inclusion, i.e. $n = 1$. Other researches for the scalar case,

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following the work of A. Cellina, were those of F.S. De Blasi and G. Pianigiani [**25**], [**26**], and A. Bressan and F. Flores [**7**]. Again in the scalar case we mention some recent result by M.A. Sychev [**49**] and S. Zagatti [**50**]. Coming back to the vectorial setting $N \geq 1$, there is an explicit construction of a solution for a particular system (1.1) by A. Cellina and S. Perrotta [**12**]. Moreover a generalization of the second order case $N = 2$, with similar assumptions than in the author's paper [**22**], have been proposed by L. Poggiolini [**46**] for the general vector-valued case with higher derivatives, i.e. $m \geq 1$ and $N \geq 1$.

In this paper we consider a general framework, with general assumption, which well fit into applications to the calculus of variations, nonlinear elasticity, problems of phase transitions or optimal design. Some model results proved in this paper are stated in Section 3; some other model results for singular values are stated below in this introduction. We have in progress a book [**23**] on this subject that will explain in details these applications, as well as some other related existence results for general systems of the type (1.1).

We propose below two theorems for *singular values*, consequence of the general results proved in this paper, respectively for first and for second order problems. For a given matrix $\xi \in \mathbb{R}^{n\times n}$ we denote by $0 \leq \lambda_1(\xi) \leq ... \leq \lambda_n(\xi)$ its singular values, i. e. the eigenvalues of the symmetric matrix $(\xi^t\xi)^{1/2}$.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set, $a_i : \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R}$, $i = 1,\dots,n$, be continuous functions satisfying $c \leq a_1(x,s) \leq \dots \leq a_n(x,s)$ for a positive constant c and for every $(x,s) \in \overline{\Omega} \times \mathbb{R}^n$. Let $\varphi \in C^1(\overline{\Omega};\mathbb{R}^n)$ (or piecewise C^1) satisfy*

$$\prod_{i=\nu}^{n}\lambda_i(D\varphi(x)) < \prod_{i=\nu}^{n}a_i(x,\varphi(x)), \quad x \in \Omega, \quad \nu = 1,\dots,n.$$

Then there exists a function $u \in W^{1,\infty}(\Omega;\mathbb{R}^n)$ such that

$$\begin{cases} \lambda_i(Du(x)) = a_i(x,u(x)), & a.e.\ x \in \Omega, \quad i = 1,\dots,n \\ u(x) = \varphi(x), & x \in \partial\Omega. \end{cases} \tag{1.2}$$

It is interesting to see an implication of Theorem 1.1 when $n = 2$; in this case we have

$$|\xi|^2 = \sum_{i,j=1}^{2}\xi_{ij}^2 = (\lambda_1(\xi))^2 + (\lambda_2(\xi))^2, \quad |\det\xi| = \lambda_1(\xi)\lambda_2(\xi).$$

The differential problem (1.2) can be equivalently formulated in the form

$$\begin{cases} |Du(x)|^2 = a_1^2 + a_2^2, & \text{a.e. } x \in \Omega \\ |\det Du(x)| = a_1a_2, & \text{a.e. } x \in \Omega \\ u(x) = \varphi(x), & x \in \partial\Omega. \end{cases} \tag{1.3}$$

Therefore system (1.3) can be seen as a combination of the *vectorial eikonal equation* and of the *equation of prescribed absolute value of the Jacobian determinant.*

We consider below the singular values of a *symmetric* matrix $\xi \in \mathbb{R}_s^{n\times n}$, $0 \leq \lambda_1(\xi) \leq \dots \leq \lambda_n(\xi)$, which are now, because of the symmetry of the matrix, the absolute value of the eigenvalues.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^n$ be an open set, $a_i : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, $i = 1,...,n$ be continuous functions satisfying*

$$0 < c \leq a_1(x,s,p) \leq \dots \leq a_n(x,s,p)$$

for a constant c and for every $(x,s,p) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$. Let $\varphi \in C^2_{piec}\left(\overline{\Omega}\right)$ be such that

$$\lambda_i\left(D^2\varphi(x)\right) < a_i\left(x,\varphi(x),D\varphi(x)\right), \quad a.e.\ x \in \Omega, \quad i=1,\ldots,n \tag{1.4}$$

(in particular $\varphi \equiv 0$). Then there exists a function $u \in W^{2,\infty}(\Omega)$ such that

$$\begin{cases} \lambda_i\left(D^2u(x)\right) = a_i\left(x,u(x),Du(x)\right), & a.e.\ x \in \Omega, \quad i=1,\ldots,n \\ u(x)=\varphi(x),\ Du(x)=D\varphi(x), & x \in \partial\Omega. \end{cases}$$

As a consequence we find that the following Dirichlet-Neumann problem (1.5) admits a solution.

Corollary 1.3. *Let $\Omega \subset \mathbb{R}^n$ be open. Let $f : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be continuous and such that $f(x,s,p) \geq f_0 > 0$ for some constant f_0 and for every $(x,s,p) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$. Let $\varphi \in C^2(\overline{\Omega})$ (or $C^2_{piec}(\overline{\Omega})$) satisfy*

$$\left|\det D^2\varphi(x)\right| < f\left(x,\varphi(x),D\varphi(x)\right), \quad x \in \overline{\Omega}.$$

Then there exists a function $u \in W^{2,\infty}(\Omega)$ such that

$$\begin{cases} \left|\det D^2u(x)\right| = f\left(x,u(x),Du(x)\right), & a.e.\ x \in \Omega, \\ u=\varphi,\ Du=D\varphi, & on\ \partial\Omega. \end{cases} \tag{1.5}$$

The above Theorems 1.1 and 1.2 are consequences of the general results proved in this paper (see Theorem 6.1 and, in the case independent of lower order terms, Theorem 3.1). When the differential problems above are independent of lower order terms, the first order case has been established in [**18**], [**19**], [**20**], [**21**] when $n=2$ and, with the same proof, in [**24**] for the general case. When $n=3$, $a_i \equiv 1$ and $\varphi \equiv 0$, the result can be found in Cellina-Perrotta [**12**]; see also Celada-Perrotta [**10**].

Motivated by an application to nonlinear elasticity, more precisely in the study of a problem of *potential wells*, S. Muller and V. Sverak [**44**], [**45**] recently obtained very interesting attainment results, that in particular cases can be compared with ours, at least in a model case of first order vectorial problem $m \geq 1$, $N=1$, without x dependence and without lower order terms, by using Gromov's method of *convex integration* (see M. Gromov [**32**] and D. Spring [**47**]). Some related results obtained by mean of Gromov method are also due to P. Celada and S. Perrotta [**10**].

The use of *viscosity solutions* to solve this kind of problems is classical and much older, although it seems related essentially with the scalar case. The bibliography in this subject is very wide; there are several excellent books and articles in this field and we can mention here only a few of them: M. Bardi-I. Capuzzo Dolcetta [**3**], G. Barles [**4**], S.H. Benton [**6**], I. Capuzzo Dolcetta-L.C. Evans [**8**], I. Capuzzo Dolcetta-P.L. Lions [**9**], M.G. Crandall-L.C. Evans-P.L. Lions [**13**], M.G. Crandall-H. Ishii-P.L. Lions [**14**], M.G. Crandall-P.L. Lions [**15**], A. Douglis [**27**], W.H. Fleming-H.M. Soner [**28**], H. Frankowska [**30**], E. Hopf [**34**], H. Ishii [**35**], S.N. Kruzkov [**38**], P.D. Lax [**39**], P.L. Lions [**40**] and A.I. Subbotin [**48**].

2. Notations

We introduce some notations and definitions to handle the higher order case. Our presentation is slightly different from those of [**2**], [**33**], [**46**]. We start first with some notations, with the aim to write in a simple way the matrix D^Nu of all partial derivatives of order N of a map $u : \mathbb{R}^n \to \mathbb{R}^m$.

Let $N,n,m\geq 1$ be integers. We denote by $\mathbb{R}_s^{m\times n^N}$ the set of matrices

$$A=\left(A^i_{j_1\dots j_N}\right)^{1\leq i\leq m}_{1\leq j_1,\dots,j_N\leq n}\in\mathbb{R}^{m\times n^N}$$

such that for every permutation σ of $\{j_1,\dots,j_N\}$ we have

$$A^i_{\sigma(j_1\dots j_N)}=A^i_{j_1\dots j_N}.$$

When $N=1$ we have $\mathbb{R}_s^{m\times n}=\mathbb{R}^{m\times n}$, while if $m=1$ and $N=2$ we get $\mathbb{R}_s^{n^2}=\mathbb{R}_s^{n\times n}$, i.e. the usual set of symmetric matrices.

As already stated in the introduction, for $u:\mathbb{R}^n\to\mathbb{R}^m$ we write

$$D^N u=\left(\frac{\partial^N u^i}{\partial x_{j_1}\dots\partial x_{j_N}}\right)^{1\leq i\leq m}_{1\leq j_1,\dots,j_N\leq n}\in\mathbb{R}_s^{m\times n^N}.$$

We also use the notation

$$D^{[N]}u=\left(u,Du,\dots,D^N u\right)\in\mathbb{R}^m\times\mathbb{R}^{m\times n}\times\dots\times\mathbb{R}_s^{m\times n^N},$$

which stands for the matrix of all partial derivatives of u up to the order N. We shall write

$$\mathbb{R}_s^{m\times M}=\mathbb{R}^m\times\mathbb{R}^{m\times n}\times\mathbb{R}_s^{m\times n^2}\times\dots\times\mathbb{R}_s^{m\times n^{(N-1)}},$$

where

$$M=1+n+\dots+n^{(N-1)}=\frac{n^N-1}{n-1};$$

hence

$$D^{[N]}u=\left(D^{[N-1]}u,D^N u\right)\in\mathbb{R}_s^{m\times M}\times\mathbb{R}_s^{m\times n^N}.$$

Given $\alpha\in\mathbb{R}^n$, we denote by $\alpha^{\otimes_N}=\alpha\otimes\alpha\dots\otimes\alpha$ (N times); it is a matrix in $\mathbb{R}_s^{n^N}$. Therefore a generic matrix of rank one in $\mathbb{R}_s^{m\times n^N}$ has the form

$$\beta\otimes\alpha^{\otimes_N}=\left(\beta_i\alpha_{j_1}\dots\alpha_{j_N}\right)^{1\leq i\leq m}_{1\leq j_1,\dots,j_N\leq n},$$

where $\beta\in\mathbb{R}^m$ and $\alpha\in\mathbb{R}^n$.

We now give the definitions of quasiconvexity and of rank one convexity in the higher order case. Let $f:\mathbb{R}_s^{m\times n^N}\to\mathbb{R}$ be a continuous function. We say that f is *quasiconvex* if

$$\int_\Omega f\left(\xi+D^N u(x)\right)dx\geq f(\xi)\operatorname{meas}\Omega,$$

for every $\xi\in\mathbb{R}_s^{m\times n^N}$ and every $u\in W_0^{N,\infty}(\Omega;\mathbb{R}^m)$. We say that f is *rank one convex* if the function of one real variable

$$F(t)=f\left(\xi+tw\otimes v^{\otimes_N}\right)$$

is convex in $t\in\mathbb{R}$ for every $\xi\in\mathbb{R}_s^{m\times n^N}$, $w\in\mathbb{R}^m$ and $v\in\mathbb{R}^n$. It has been established by N.G. Meyers [**42**] (c.f. also [**2**]) that quasiconvexity implies rank one convexity. In the case $N=2$ it is proved in [**33**] that if the function is quasiconvex or rank one convex then it is automatically locally Lipschitz (as well as for $N=1$).

Semicontinuity results for quasiconvex integrands have been studied by C.B. Morrey [**43**], who introduced the concept of quasiconvexity, and then by N.G. Meyers [**42**], E. Acerbi and N. Fusco [**1**], P. Marcellini [**41**] and many others since then. In particular, in the higher order case we refer to N.G. Meyers [**42**], N. Fusco [**31**] and M. Guidorzi and L. Poggiolini [**33**].

In a similar way as we defined the different notions of convexity for functions we may define these notions for sets. In particular we say that a set $K\subset\mathbb{R}_s^{m\times n^N}$

is *rank one convex* if for every $t \in [0,1]$ and every $A, B \in K$ with $\operatorname{rank}(A-B) \leq 1$ then $tA + (1-t)B \in K$. Moreover, for a set $E \subset \mathbb{R}_s^{m \times n^N}$ we define

$$\operatorname{Rco} E = \left\{ \begin{array}{c} \xi \in \mathbb{R}^{m \times n} : f(\xi) \leq 0, \quad \forall f : \mathbb{R}_s^{m \times n^N} \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}\,, \\ f\,|_E = 0\,, \;\; f \text{ rank one convex} \end{array} \right\},$$

called the *rank one convex hull* of E, and

$$\overline{\operatorname{Qco} E} = \left\{ \begin{array}{l} \xi \in \mathbb{R}_s^{m \times n^N} : \;\; f(\xi) \leq 0, \;\; \forall f : \mathbb{R}_s^{m \times n^N} \to \mathbb{R} \\ f\,|_E = 0\,, \quad f \;\; \text{quasiconvex and continuous} \end{array} \right\},$$

called the (closure of the) *quasiconvex hull* of E.

3. Statement of some model results

Among some other existence results presented in Sections 5, 6, four model theorems, consequence of the theory developed in the next sections, are stated below. Note that the boundary condition is to be interpreted as

$$u - \varphi \in W_0^{N,\infty}(\Omega; \mathbb{R}^m).$$

The first result that we state in this section and that we will prove in this paper is the following Theorem 3.1. It has applications in particular in *singular values problems* (see [**20**], [**21**], [**22**], [**10**] or [**23**]).

THEOREM 3.1. *Let $\Omega \subset \mathbb{R}^n$ be open. Let $F_i : \mathbb{R}_s^{m \times n^N} \to \mathbb{R}$, $i = 1, 2, ..., I$, be quasiconvex, locally Lipschitz and positively homogeneous of degree $\alpha_i > 0$. Let $a_i > 0$ and*

$$E = \left\{ \xi \in \mathbb{R}_s^{m \times n^N} : \; F_i(\xi) = a_i, \;\; i = 1, 2, ..., I \right\}.$$

Assume that $\operatorname{Rco} E$ *is compact and satisfies*

$$\operatorname{Rco} E = \left\{ \xi \in \mathbb{R}_s^{m \times n^N} : \; F_i(\xi) \leq a_i, \;\; i = 1, 2, ..., I \right\}.$$

Let $\varphi \in C_{piec}^N(\overline{\Omega}; \mathbb{R}^m)$ be such that

$$F_i(D^N \varphi(x)) < a_i, \qquad a.e. \; x \in \Omega, \;\; i = 1, ..., I.$$

Then there exists (a dense set of) $u \in W^{N,\infty}(\Omega; \mathbb{R}^m)$ such that

$$\left\{ \begin{array}{l} F_i(D^N u(x)) = a_i, \quad a.e. \; x \in \Omega, \;\; i = 1, ..., I \\ D^\alpha u(x) = D^\alpha \varphi(x), \quad x \in \partial\Omega, \quad \alpha = 0, ..., N-1. \end{array} \right.$$

THEOREM 3.2. *Let $\Omega \subset \mathbb{R}^n$ be open. Let $F_i : \mathbb{R}_s^{m \times n^N} \to \mathbb{R}$, $i = 1, 2, ..., I$, be convex and let*

$$E = \left\{ \xi \in \mathbb{R}_s^{m \times n^N} : \; F_i(\xi) = 0, \quad i = 1, 2, ..., I \right\}.$$

Assume that $\operatorname{Rco} E$ *is compact and satisfy*

$$\operatorname{Rco} E = \left\{ \xi \in \mathbb{R}_s^{m \times n^N} : \; F_i(\xi) \leq 0, \quad i = 1, 2, ..., I \right\}.$$

Let $\varphi \in C_{piec}^N(\overline{\Omega}; \mathbb{R}^m)$ satisfy

$$F_i(D^N \varphi(x)) < 0, \qquad a.e. \; x \in \Omega, \;\; i = 1, ..., I$$

or $\varphi \in W^{N,\infty}(\Omega; \mathbb{R}^m)$ satisfy

$$F_i(D^N \varphi(x)) \leq -\theta, \qquad a.e. \; x \in \Omega, \;\; i = 1, ..., I$$

for a certain $\theta > 0$. Then there exists (a dense set of) $u \in W^{N,\infty}(\Omega;\mathbb{R}^m)$ such that

$$\begin{cases} F_i(D^N u(x)) = 0, & a.e.\ x \in \Omega,\quad i = 1, ..., I \\ D^\alpha u(x) = D^\alpha \varphi(x), & x \in \partial\Omega,\quad \alpha = 0, ..., N-1. \end{cases}$$

The generalization of the previous results to the case with explicit dependence on lower order terms is not as simple, from the technical point of view, as it could seem at a first glance. However, in particular (see Section 6 for more general results) we can obtain the following two theorems.

THEOREM 3.3. *Let $\Omega \subset \mathbb{R}^n$ be open. Let $F_i : \overline{\Omega} \times \mathbb{R}_s^{m\times M} \times \mathbb{R}_s^{m\times n^N} \to \mathbb{R}$, $F_i = F_i(x,s,\xi)$, $i = 1, ..., I$, be continuous with respect to $(x,s) \in \overline{\Omega} \times \mathbb{R}_s^{m\times M}$ and quasiconvex, locally Lipschitz and positively homogeneous of degree $\alpha_i > 0$ with respect to the last variable $\xi \in \mathbb{R}_s^{m\times n^N}$.*

Let $a_i : \overline{\Omega} \times \mathbb{R}_s^{m\times M} \to \mathbb{R}$, $i = 1, ..., I$, be continuous and satisfy for a certain $a_0 > 0$

$$a_i(x,s) \geq a_0 > 0,\ i = 1, ..., I,\ \forall (x,s) \in \overline{\Omega} \times \mathbb{R}_s^{m\times M}.$$

Assume that, for every $(x,s) \in \overline{\Omega} \times \mathbb{R}_s^{m\times M}$

$$\begin{aligned} &\operatorname{Rco}\left\{\xi \in \mathbb{R}_s^{m\times n^N} : F_i(x,s,\xi) = a_i(x,s),\quad i = 1, ..., I\right\} \\ &= \left\{\xi \in \mathbb{R}_s^{m\times n^N} : F_i(x,s,\xi) \leq a_i(x,s),\quad i = 1, ..., I\right\} \end{aligned}$$

and is bounded in $\mathbb{R}_s^{m\times n^N}$ uniformly with respect to (x,s) in a bounded set of $\overline{\Omega} \times \mathbb{R}_s^{m\times M}$. If $\varphi \in C_{piec}^N(\overline{\Omega};\mathbb{R}^m)$ satisfies

$$F_i(x, D^{[N-1]}\varphi(x), D^N\varphi(x)) < a_i\left(x, D^{[N-1]}\varphi(x)\right),\ a.e.\ x \in \Omega,\ i = 1, ..., I,$$

then there exists (a dense set of) $u \in W^{N,\infty}(\Omega;\mathbb{R}^m)$ such that

$$\begin{cases} F_i(x, D^{[N-1]}u(x), D^N u(x)) = a_i\left(x, D^{[N-1]}u(x)\right),\ a.e.\ x \in \Omega,\ i = 1, ..., I \\ D^\alpha u(x) = D^\alpha \varphi(x),\ x \in \partial\Omega,\ \alpha = 0, ..., N-1. \end{cases}$$

THEOREM 3.4. *Let $\Omega \subset \mathbb{R}^n$ be open. Let $F_i : \overline{\Omega} \times \mathbb{R}_s^{m\times M} \times \mathbb{R}_s^{m\times n^N} \to \mathbb{R}$, $F_i = F_i(x,s,\xi)$, $i = 1, ..., I$, be continuous with respect to $(x,s) \in \overline{\Omega} \times \mathbb{R}_s^{m\times M}$ and convex with respect to the last variable $\xi \in \mathbb{R}_s^{m\times n^N}$. Assume that, for every $(x,s) \in \overline{\Omega} \times \mathbb{R}_s^{m\times M}$*

$$\begin{aligned} &\operatorname{Rco}\left\{\xi \in \mathbb{R}_s^{m\times n^N} : F_i(x,s,\xi) = 0,\quad i = 1, ..., I\right\} \\ &= \left\{\xi \in \mathbb{R}_s^{m\times n^N} : F_i(x,s,\xi) \leq 0,\quad i = 1, ..., I\right\} \end{aligned}$$

and is bounded in $\mathbb{R}_s^{m\times n^N}$ uniformly with respect to (x,s) in a bounded set of $\overline{\Omega} \times \mathbb{R}_s^{m\times M}$. Assume also that for every $(x,s) \in \overline{\Omega} \times \mathbb{R}_s^{m\times M}$ there exists $\xi_0 = \xi_0(x,s) \in \mathbb{R}_s^{m\times n^N}$ such that

$$F_i(x,s,\xi_0) < 0,\quad i = 1, ..., I.$$

Let $\varphi \in C_{piec}^N(\overline{\Omega};\mathbb{R}^m)$ satisfy

$$F_i(x, D^{[N-1]}\varphi(x), D^N\varphi(x)) < 0,\quad a.e.\ x \in \Omega,\ i = 1, ..., I,$$

or $\varphi \in W^{N,\infty}(\Omega;\mathbb{R}^m)$ be such that

$$F_i(x, D^{[N-1]}\varphi(x), D^N\varphi(x)) \leq -\theta,\quad a.e.\ x \in \Omega,\ i = 1, ..., I,$$

for a certain $\theta > 0$. *Then there exists (a dense set of)* $u \in W^{N,\infty}(\Omega; \mathbb{R}^m)$ *such that*

$$\begin{cases} F_i(x, D^{[N-1]}u(x), D^N u(x)) = 0, & a.e.\ x \in \Omega,\ i = 1, ..., I \\ D^\alpha u(x) = D^\alpha \varphi(x), & x \in \partial\Omega,\ \alpha = 0, ..., N-1. \end{cases}$$

Proofs of Theorems 3.1, 3.2, 3.3, 3.4 will be given in Section 5.

4. Problems without lower order terms

4.1. Weakly extreme sets and the relaxation property. We start with the definition of *weakly extreme set* E_{ext} of a given set $E \subset \mathbb{R}_s^{m \times n^N}$.

Definition 4.1 (Weakly extreme set). *A set* E_{ext} *is said to be weakly extreme for a subset* E *of* $\mathbb{R}_s^{m \times n^N}$ *if* $E_{\text{ext}} \subset E$ *and if for every* $\varepsilon > 0$ *there exists* $\delta = \delta(\varepsilon) > 0$ *such that, for every* $u, u_\nu \in W^{N,\infty}(\Omega; \mathbb{R}^m)$ *satisfying*

$$\begin{cases} u_\nu \overset{*}{\rightharpoonup} u & in\ W^{N,\infty}(\Omega; \mathbb{R}^m) \\ D^N u_\nu(x) \in \overline{\text{Qco}\, E}, & a.e\ in\ \Omega, \end{cases} \tag{4.1}$$

then the following implication holds for ν *sufficiently large*

$$\int_\Omega \text{dist}\left(D^N u(x); E_{\text{ext}}\right) dx \leq \delta \;\; \Rightarrow \int_\Omega \text{dist}\left(D^N u_\nu(x); E_{\text{ext}}\right) dx \leq \varepsilon. \tag{4.2}$$

In some cases we can choose in the previous Definition 4.1 the set E_{ext} equal to E; by the following result we give sufficient conditions in order to make this choice.

Theorem 4.1. *Let* $\Omega \subset \mathbb{R}^n$ *be open and bounded. Assume* $E \subset \mathbb{R}_s^{m \times n^N}$ *is compact and that there exist* $F_i,\ i = 1, 2, ..., I$, *quasiconvex and locally Lipschitz functions such that*

$$E = \left\{ \xi \in \mathbb{R}_s^{m \times n^N} : \;\; F_i(\xi) = 0,\;\; i = 1, 2, ..., I \right\}. \tag{4.3}$$

Then the set E_{ext} *in Definition 4.1 can be chosen equal to* E.

Proof. Under assumption (4.3) we will prove that for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that, for every $u, u_\nu \in W^{N,\infty}(\Omega; \mathbb{R}^m)$ satisfying (4.1), for ν sufficiently large the following implication holds

$$\int_\Omega \text{dist}\left(D^N u(x); E\right) dx \leq \delta \;\; \Rightarrow \int_\Omega \text{dist}\left(D^N u_\nu(x); E\right) dx \leq \varepsilon. \tag{4.4}$$

Step 1 : We first observe that, by definition of $\overline{\text{Qco}\, E}$ and because of the weak* convergence, from (4.1) we also have

$$D^N u(x) \in \overline{\text{Qco}\, E}, \quad \text{a.e in } \Omega.$$

We next fix the constants. Since $\overline{\text{Qco}\, E}$ is compact we can find $\beta > 0$ such that

$$\eta \in \overline{\text{Qco}\, E} \;\; \Rightarrow \;\text{dist}(\eta; E) \leq \beta. \tag{4.5}$$

We also see that, by definition of $\overline{\text{Qco}\, E}$ (since $F_i = 0$ on E), we have $F_i(\eta) \leq 0$ for every $\eta \in \overline{\text{Qco}\, E}$ and since F_i is Lipschitz on $\overline{\text{Qco}\, E}$ we can find $\alpha > 0$ such that

$$0 \leq -F_i(\eta) \leq \alpha \cdot \text{dist}(\eta; E), \quad \forall \eta \in \overline{\text{Qco}\, E}\ ,\ \forall\, i = 1, 2, ..., I. \tag{4.6}$$

We finally observe that by continuity of the distance function and of the F_i we can find for every $\sigma > 0$, $\rho = \rho(\sigma) > 0$ such that for every $\eta \in \overline{\text{Qco}\, E}$

$$\left. \begin{array}{c} 0 \leq -F_i(\eta) < \rho \\ \forall\, i = 1, 2, ..., I \end{array} \right\} \;\; \Rightarrow \;\; \text{dist}(\eta; E) < \sigma\,; \tag{4.7}$$

in fact the choice

$$\rho\left(\sigma\right)=\min_{\substack{\operatorname{dist}\left(\eta;E\right)\geq\sigma\\ \eta\in\overline{\operatorname{Qco}E}}}\ \max_{i=1,2,\ldots,I}\left\{-F_i(\eta)\right\}$$

leads immediately to (4.7).

Step 2 : We now fix $\varepsilon>0$ and we choose

$$\sigma=\frac{\varepsilon}{2\operatorname{meas}\Omega}\ ,\quad \delta\left(\varepsilon\right)=\frac{\varepsilon\rho\left(\sigma\right)}{4I\alpha\beta}\ .$$

With this choice we will prove that, for ν sufficiently large, (4.4) holds.

In fact, by weak lower semi continuity, by (4.6) and by the fact that $D^N u_\nu\left(x\right)\in\overline{\operatorname{Qco}E}$ we get

$$\begin{aligned}0&\geq\liminf_{\nu\to\infty}\int_\Omega F_i\left(D^N u_\nu\left(x\right)\right)dx\geq\int_\Omega F_i\left(D^N u\left(x\right)\right)dx\\&\geq-\alpha\int_\Omega\operatorname{dist}\left(D^N u\left(x\right);E\right)dx\geq-\alpha\delta\,,\qquad\forall\, i=1,2,...,I.\end{aligned}$$

Therefore, for ν large enough, we have

$$\int_\Omega F_i\left(D^N u_\nu\left(x\right)\right)dx\geq-2\alpha\delta=-\frac{\varepsilon\rho}{2I\beta}\ ,\quad\forall\, i=1,2,...,I. \tag{4.8}$$

Setting

$$\Omega_{i,\nu}=\left\{x\in\Omega:\ -F_i\left(D^N u_\nu\left(x\right)\right)\geq\rho\right\},$$

we get

$$\rho\cdot\operatorname{meas}\Omega_{i,\nu}\leq-\int_\Omega F_i\left(D^N u_\nu\left(x\right)\right)dx\leq\frac{\varepsilon\rho}{2I\beta}$$

and hence

$$\operatorname{meas}\Omega_{i,\nu}\leq\frac{\varepsilon}{2I\beta}\,. \tag{4.9}$$

Letting $\Omega_\nu=\cup_{i=1}^I\Omega_{i,\nu}$, we find

$$\operatorname{meas}\Omega_\nu\leq\frac{\varepsilon}{2\beta}\,. \tag{4.10}$$

We now use (4.7) on $\Omega\diagdown\Omega_\nu$ to obtain

$$\operatorname{dist}\left(D^N u_\nu\left(x\right);E\right)\leq\sigma=\frac{\varepsilon}{2\operatorname{meas}\Omega}\ ,\quad\text{a.e. } x\in\Omega\diagdown\Omega_\nu$$

and hence, by using (4.5),

$$\begin{aligned}\int_\Omega\operatorname{dist}\left(D^N u_\nu\left(x\right);E\right)dx&=\int_{\Omega\diagdown\Omega_\nu}\operatorname{dist}\left(D^N u_\nu\left(x\right);E\right)dx\\&\quad+\int_{\Omega_\nu}\operatorname{dist}\left(D^N u_\nu\left(x\right);E\right)dx\\&\leq\frac{\varepsilon\operatorname{meas}\left(\Omega\diagdown\Omega_\nu\right)}{2\operatorname{meas}\Omega}+\beta\operatorname{meas}\Omega_\nu\leq\varepsilon\,,\end{aligned}$$

which is the claimed result. □

We now state the main hypothesis that should satisfy $\overline{\mathrm{Qco}\, E}$ or rather a subset K of it; it says that for every boundary datum which is a polynomial of degree at most N we can find an approximate solution of our problem that remains almost everywhere in $\mathrm{int}\, \overline{\mathrm{Qco}\, E}$.

DEFINITION 4.2 (Relaxation property). *Let E be a set of $\mathbb{R}_s^{m\times n^N}$. A subset $K \subset \overline{\mathrm{Qco}\, E}$ is said to have the relaxation property with respect to E if the following condition is satisfied: for every $\xi \in \mathrm{int}\, K$, for every bounded open set Ω of $\mathbb{R}^n$, there exists a sequence $u_\nu \in C^N_{piec}\left(\overline{\Omega}; \mathbb{R}^m\right)$ such that (defining $u_\xi(x)$ by $D^N u_\xi(x) = \xi$)*

$$\begin{cases} D^\alpha u_\nu(x) = D^\alpha u_\xi(x) \quad \text{on } \partial\Omega, \quad \alpha = 0, ..., N-1 \\ u_\nu \overset{*}{\rightharpoonup} u_\xi \quad \text{in } W^{N,\infty}(\Omega; \mathbb{R}^m) \\ D^N u_\nu(x) \in E \cup \mathrm{int}\, K, \quad \text{a.e in } \Omega \\ \int_\Omega \mathrm{dist}\left(D^N u_\nu(x); E\right) dx \to 0 \quad \text{as } \nu \to \infty. \end{cases}$$

REMARK 4.1. *(i) Below we will give some cases where the relaxation property can be established.*

(ii) What is interesting about defining the relaxation property for K and not for $\overline{\mathrm{Qco}\, E}$ (in some sense the relaxation property could be taken as the definition of the quasiconvex hull) is that we do not need to compute this last quantity and we can choose for example $K = \mathrm{Rco}\, E$ (or even a subset), our approximation property (c.f. Definition 4.3) then ensures that K has the relaxation property (c.f. also [**21**] *or Lemma 2.2 and Lemma 3.4 of* [**22**]*).*

4.2. An abstract result. The main result of this section is

THEOREM 4.2. *Let $\Omega \subset \mathbb{R}^n$ be open and let $E \subset \mathbb{R}_s^{m\times n^N}$ be compact. Let us assume that $K \subset \mathbb{R}_s^{m\times n^N}$ has the relaxation property with respect to E_{ext}. Let $\varphi \in C^N_{piec}\left(\overline{\Omega}; \mathbb{R}^m\right)$ be such that*

$$D^N \varphi(x) \in E_{\mathrm{ext}} \cup \mathrm{int}\, K, \quad \text{a.e in } \Omega.$$

Then there exists (a dense set of) $u \in W^{N,\infty}(\Omega; \mathbb{R}^m)$ such that

$$\begin{cases} D^N u(x) \in E_{\mathrm{ext}}, \quad \text{a.e. } x \in \Omega \\ D^\alpha u(x) = D^\alpha \varphi(x) \quad \text{on } \partial\Omega, \quad \alpha = 0, ..., N-1. \end{cases} \tag{4.11}$$

REMARK 4.2. *(i) Since $E_{\mathrm{ext}} \subset E$, in particular we obtain that $u \in W^{N,\infty}(\Omega; \mathbb{R}^m)$ satisfies*

$$\begin{cases} D^N u(x) \in E, \quad \text{a.e. } x \in \Omega \\ D^\alpha u(x) = D^\alpha \varphi(x) \quad \text{on } \partial\Omega, \quad \alpha = 0, ..., N-1. \end{cases} \tag{4.12}$$

Of course conclusion (4.11) is sharper than (4.12).

(ii) The boundary condition is to be interpreted as

$$u - \varphi \in W_0^{N,\infty}(\Omega; \mathbb{R}^m).$$

(iii) The conclusion of the theorem is in fact more precise. The solution found is in the C^{N-1} closure of the set

$$\left\{ \begin{array}{c} u \in C^N_{piec}\left(\overline{\Omega}; \mathbb{R}^m\right) : D^\alpha u(x) = D^\alpha \varphi(x) \quad \text{on } \partial\Omega, \quad \alpha = 0, ..., N-1 \\ \text{and} \quad D^N u(x) \in E_{\mathrm{ext}} \cup \mathrm{int}\, K, \quad \text{a.e in } \Omega \end{array} \right\}.$$

PROOF. Step 1 : We observe first that $\Omega \subset \mathbb{R}^n$ can be assumed bounded, without loss of generality. We then let V be the closure in the $C^{N-1}\left(\overline{\Omega};\mathbb{R}^m\right)$ norm of

$$\left\{ \begin{array}{c} u \in C^N_{piec}\left(\overline{\Omega};\mathbb{R}^m\right): \ D^\alpha u(x) = D^\alpha \varphi(x) \quad \text{on } \partial\Omega, \quad \alpha = 0,...,N-1 \\ \text{and} \ \ D^N u(x) \in E_{\text{ext}} \cup \operatorname{int} K, \quad \text{a.e in } \Omega \end{array} \right\}.$$

Note that $\varphi \in V$, that V is a complete metric space when endowed with the C^{N-1} norm and that (by weak lower semicontinuity and since $E_{\text{ext}} \cup K \subset \overline{\operatorname{Qco} E_{\text{ext}}}$)

$$V \subset \left\{ u \in \varphi + W_0^{N,\infty}(\Omega;\mathbb{R}^m): \ D^N u(x) \in \overline{\operatorname{Qco} E_{\text{ext}}}\,, \quad \text{a.e in } \Omega \right\}$$

Step 2 : Let for $u \in V$

$$I(u) = \inf_{\{u_\nu\}} \ \liminf_{u_\nu \overset{*}{\rightharpoonup} u, \ u_\nu \in V} \left[-\int_\Omega \operatorname{dist}\left(D^N u_\nu(x); E_{\text{ext}}\right) dx \right].$$

Observe that I is lower semicontinuous on V, i.e. more precisely

$$\liminf_{u_\nu \overset{*}{\rightharpoonup} u, \ u_\nu \in V} I(u_\nu) \ge I(u)\,, \quad \forall u \in V. \tag{4.13}$$

Since $I \le 0$ on V, by taking $u_\nu \equiv u$ in the definition of I we see that

$$u \in V, \ I(u) = 0 \ \ \Rightarrow \ \ D^N u(x) \in E_{\text{ext}}\,, \text{ a.e. in } \Omega. \tag{4.14}$$

The above implication admits a converse (see (4.17) below). Indeed, by definition of I, for every $u \in V$ and for every $\varepsilon > 0$ we can find $u_\nu \in V$ (and hence in particular $D^N u_\nu(x) \in \overline{\operatorname{Qco} E_{\text{ext}}}$), $u_\nu \overset{*}{\rightharpoonup} u$ such that

$$I(u) \ge -\varepsilon - \int_\Omega \operatorname{dist}\left(D^N u_\nu(x); E_{\text{ext}}\right) dx\,. \tag{4.15}$$

By the Definition 4.1 of E_{ext}, there exists $\delta = \delta(\varepsilon) > 0$ such that the following implication holds for ν sufficiently large

$$\int_\Omega \operatorname{dist}\left(D^N u(x); E_{\text{ext}}\right) dx \le \delta \quad \Rightarrow \quad \int_\Omega \operatorname{dist}\left(D^N u_\nu(x); E_{\text{ext}}\right) dx \le \varepsilon. \tag{4.16}$$

Therefore, by combining (4.15), (4.16) we get that, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\int_\Omega \operatorname{dist}\left(D^N u(x); E_{\text{ext}}\right) dx \le \delta \ \ \Rightarrow \ \ I(u) \ge -2\varepsilon. \tag{4.17}$$

Step 3 : Let

$$V^k = \{u \in V : I(u) > -1/k\}\,.$$

The set V^k is open in V (c.f. (4.13)). Furthermore it is dense in V. This follows from the relaxation property and we will prove this fact below. If this property has been established we deduce from Baire category theorem that $\cap V^k \subset \{u \in V: \ I(u) = 0\}$ is dense in V. Thus the result by (4.14).

It remains to prove that for any $u \in V$ and any $\varepsilon > 0$ sufficiently small we can find $u_\varepsilon \in V^k$ so that

$$\|u_\varepsilon - u\|_{N-1,\infty} \le \varepsilon.$$

We will prove this property under the further assumption that $u \in C^N_{piec}$ and

$$D^N u(x) \in E_{\text{ext}} \cup \operatorname{int} K, \text{ a.e in } \Omega.$$

The general case will follow by definition of V. By working on each piece where $u \in C^N$ we can assume, without loss of generality, that $u \in C^N\left(\overline{\Omega};\mathbb{R}^m\right)$ and $D^N u(x) \in E_{\text{ext}} \cup \operatorname{int} K$. We introduce the notations

$$\Omega_0 = \left\{x \in \Omega : D^N u(x) \in E_{\text{ext}}\right\}, \quad \Omega_1 = \Omega - \Omega_0.$$

It is clear, by continuity, that Ω_0 is closed and hence Ω_1 is open.

Before proceeding further we fix the constants. We let k be an integer and choose $0 < \varepsilon < 1/(2k)$, $\delta = \delta(\varepsilon)$ be as in Definition 4.1 and $\beta > 0$ be such that ($K \subset \overline{\operatorname{Qco} E}$ being compact)

$$\xi \in K \;\Rightarrow\; \operatorname{dist}(\xi; E_{\text{ext}}) \leq \beta.$$

By approximation (c.f. the Appendix of [**23**]) we can find $u_s \in C^N\left(\overline{\Omega}_1;\mathbb{R}^m\right)$, an integer $I = I(\varepsilon)$ and $\Omega_{s,i} \subset \Omega_1$, $1 \leq i \leq I$, disjoint open sets such that

$$\begin{cases} u_s \equiv u, \quad \text{near } \partial\Omega_1 \\ \left\|u_s - u\right\|_{N,\infty} \leq \frac{\varepsilon}{2} \\ D^N u_s(x) \in \operatorname{int} K, \quad \text{a.e. } x \in \Omega_1 \\ \operatorname{meas}\left(\Omega_1 - \cup_{i=1}^I \Omega_{s,i}\right) \leq \frac{\delta}{2\beta} \\ D^N u_s(x) = \xi_{s,i} = \text{ constant}, \quad x \in \Omega_{s,i}. \end{cases}$$

We next use the relaxation property (since $\xi_{s,i} \in \operatorname{int} K$) to find $u_{i,\nu} \in C^N_{piec}\left(\overline{\Omega}_{s,i};\mathbb{R}^m\right)$ satisfying

$$\begin{cases} D^\alpha u_{i,\nu} = D^\alpha u_s, \text{ on } \partial\Omega_{s,i}, \quad \alpha = 0, ..., N-1 \\ \left\|u_s - u_{i,\nu}\right\|_{N-1,\infty} \leq \frac{\varepsilon}{2}, \quad \text{in } \Omega_{s,i} \\ D^N u_{i,\nu}(x) \in E_{\text{ext}} \cup \operatorname{int} K \\ \int_{\Omega_{s,i}} \operatorname{dist}\left(D^N u_{i,\nu}(x); E_{\text{ext}}\right) dx \leq \frac{\delta}{2} \cdot \frac{\operatorname{meas}(\Omega_{s,i})}{\operatorname{meas}(\Omega_1)}. \end{cases}$$

Now we can define

$$u_\varepsilon(x) = \begin{cases} u(x) & \text{if } x \in \Omega_0 \\ u_s(x) & \text{if } x \in \Omega_1 - \cup_{i=1}^I \Omega_{s,i} \\ u_{i,\nu}(x) & \text{if } x \in \Omega_{s,i}. \end{cases}$$

Observe that $u_\varepsilon \in C^N_{piec}\left(\overline{\Omega};\mathbb{R}^m\right)$,

$$\begin{cases} D^\alpha u_\varepsilon = D^\alpha u, \quad \text{on } \partial\Omega, \quad \alpha = 0, ..., N-1 \\ \left\|u_\varepsilon - u\right\|_{N-1,\infty} \leq \varepsilon, \quad \text{in } \Omega \\ D^N u_\varepsilon(x) \in E_{\text{ext}} \cup \operatorname{int} K, \quad \text{a.e. } x \in \Omega \end{cases}$$

and that

$$\begin{aligned}\int_{\Omega} \operatorname{dist}\left(D^N u_\varepsilon(x);E_{\text{ext}}\right)dx &= \int_{\Omega_0} \operatorname{dist}\left(D^N u_\varepsilon(x);E_{\text{ext}}\right)dx \\ &+ \int_{\Omega_1} \operatorname{dist}\left(D^N u_\varepsilon(x);E_{\text{ext}}\right)dx \\ &= \int_{\Omega_1} \operatorname{dist}\left(D^N u_\varepsilon(x);E_{\text{ext}}\right)dx \\ &= \int_{\Omega_1-\cup\Omega_{s,i}} \operatorname{dist}\left(D^N u_\varepsilon(x);E_{\text{ext}}\right)dx \\ &+ \sum_{i=1}^{I}\int_{\Omega_{s,i}} \operatorname{dist}\left(D^N u_\varepsilon(x);E_{\text{ext}}\right)dx \\ &\le \frac{\delta}{2}+\frac{\delta}{2}\le\delta.\end{aligned}$$

Hence combining (4.17) and the above inequality we get

$$I(u_\varepsilon)\ge -2\varepsilon > -\frac{1}{k}\ ,$$

which implies that $u_\varepsilon \in V^k$. The claimed density has therefore been established and the proof is thus complete. □

Direct consequence of Theorems 4.1 and 4.2 is the following

Corollary 4.3. *Let $\Omega\subset\mathbb{R}^n$ be open. Let $E\subset\mathbb{R}_s^{m\times n^N}$ be compact and satisfy (4.3), i.e. there exist F_i, $i=1,2,...,I$, quasiconvex and locally Lipschitz functions such that*

$$E=\left\{\xi\in\mathbb{R}_s^{m\times n^N}\ :\ F_i(\xi)=0,\quad i=1,2,...,I\right\}.$$

Assume $K\subset\overline{\operatorname{Qco}E}\subset\mathbb{R}_s^{m\times n^N}$ has the relaxation property with respect to E. Let $\varphi\in C_{piec}^N\left(\overline{\Omega};\mathbb{R}^m\right)$ be such that

$$D^N\varphi(x)\in E\cup\operatorname{int}K,\quad a.e\ in\ \Omega;$$

then there exists (a dense set of) $u\in W^{N,\infty}(\Omega;\mathbb{R}^m)$ such that

$$\begin{cases} D^N u(x)\in E, & a.e.\ x\in\Omega\\ D^\alpha u(x)=D^\alpha\varphi(x) & on\ \partial\Omega,\quad \alpha=0,...,N-1,\end{cases}$$

or equivalently $u\in W^{N,\infty}(\Omega;\mathbb{R}^m)$ satisfies the differential problem

$$\begin{cases} F_i(D^N u(x))=0, & a.e.\ x\in\Omega,\quad i=1,2,...,I\\ D^\alpha u(x)=D^\alpha\varphi(x) & on\ \partial\Omega,\quad \alpha=0,...,N-1.\end{cases}$$

4.3. The key approximation lemma. The following lemma will be useful to establish the relaxation property.

Lemma 4.4. *Let $\Omega\subset\mathbb{R}^n$ be an open set with finite measure. Let $t\in[0,1]$ and $A,B\in\mathbb{R}_s^{m\times n^N}$ with rank $\{A-B\}=1$. Let φ be such that*

$$D^N\varphi(x)=tA+(1-t)B,\quad \forall x\in\overline{\Omega}.$$

Then, for every $\varepsilon > 0$, *there exist* $u \in C_{piec}^{N}\left(\overline{\Omega};\mathbb{R}^{m}\right)$ *and disjoint open sets* $\Omega_A, \Omega_B \subset \Omega$, *so that*

$$\begin{cases} \left|\operatorname{meas}\Omega_A - t\operatorname{meas}\Omega\right|, \ \left|\operatorname{meas}\Omega_B - (1-t)\operatorname{meas}\Omega\right| \le \varepsilon \\ u \equiv \varphi \text{ near } \partial\Omega \\ \|u-\varphi\|_{N-1,\infty} \le \varepsilon \\ D^N u(x) = \begin{cases} A & \text{in } \Omega_A \\ B & \text{in } \Omega_B \end{cases} \\ \operatorname{dist}\left(D^N u(x), \operatorname{co}\{A,B\}\right) \le \varepsilon \quad \text{a.e. in } \Omega \end{cases}$$

REMARK 4.3. *(i) By* $\operatorname{co}\{A,B\} = [A,B]$ *we mean the closed segment joining* A *to* B.

(ii) It is interesting to note that when $n = 1$ *the construction is exact, i.e.* $\overline{\Omega} = \overline{\Omega}_A \cup \overline{\Omega}_B$.

PROOF. We divide the proof into two steps (unfortunately, since the notations are not easy, every time we will point out the case $n, m \ge 1$ and $N = 1$ as well as the case $m = 1$ and $N = 2$).

Step 1: Let us first assume that the matrix has the form

$$A - B = \alpha \otimes e_1^{\otimes N}$$

where $e_1 = (1, 0, ..., 0) \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}^m$. For example when $N = 1$ we have

$$A - B = \begin{pmatrix} \alpha_1 & 0 & ... & 0 \\ \alpha_2 & 0 & ... & 0 \\ ... & ... & ... & ... \\ \alpha_m & 0 & ... & 0 \end{pmatrix} \in \mathbb{R}^{m\times n}.$$

or when $m = 1$ and $N = 2$ we can write

$$A - B = \begin{pmatrix} \alpha & 0 & ... & 0 \\ 0 & 0 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 0 \end{pmatrix} \in \mathbb{R}_s^{n\times n}.$$

We can express Ω as union of cubes with faces parallel to the coordinate axes and a set of small measure. Then, by posing $u \equiv \varphi$ on the set of small measure, and by dilatations and translations, we can reduce ourselves to work with Ω equal to the unit cube.

Let Ω_ε be a set compactly contained in Ω and let $\eta \in C_0^\infty(\Omega)$ and $L > 0$ be such that

$$\begin{cases} \operatorname{meas}(\Omega - \Omega_\varepsilon) \le \varepsilon \\ 0 \le \eta(x) \le 1, \ \ \forall x \in \Omega \\ \eta(x) = 1, \ \ \forall x \in \Omega_\varepsilon \\ \left|D^k \eta(x)\right| \le \frac{L}{\varepsilon^k}, \ \ \forall x \in \Omega - \Omega_\varepsilon \text{ and } \forall k = 1, ..., N. \end{cases} \tag{4.18}$$

Let us define a function $v : [0,1] \to \mathbb{R}^m$ in the following way: given $\delta > 0$, divide the interval $(0,1)$ into two finite unions I, J of disjoint open subintervals such that

$$\begin{cases} \overline{I} \cup \overline{J} = [0,1], \ \ I \cap J = \emptyset \\ \operatorname{meas} I = t, \ \operatorname{meas} J = 1 - t \\ v^{(N)}(x_1) = \begin{cases} (1-t)\,\alpha & \text{if } x_1 \in I \\ -t\,\alpha & \text{if } x_1 \in J \end{cases} \\ \left|v^{(k)}(x_1)\right| \le \delta, \ \ \forall x_1 \in (0,1), \ \ \forall k = 0, 1, ..., N-1. \end{cases}$$

In particular $v^{(N)}(x_1)$ can assume the two values $(1-t)\,\alpha$ and $-t\,\alpha$, and at the same time $v^{(k)}(x_1)$ can be small for every k up to order $N-1$, i.e. in absolute value less than or equal to δ, since 0 is a convex combination of the two values, with coefficients t and $1-t$.

We define u as a convex combination of $v+\varphi$ (by abuse of notations now the function $v:\mathbb{R}^n\to\mathbb{R}^m$ depends explicitly only on the first variable) and φ in the following way

$$u=\eta(v+\varphi)+(1-\eta)\varphi=\eta v+\varphi.$$

Choosing $\delta>0$ sufficiently small (of the order ε^N), we find that u satisfies the conclusions of the lemma, with

$$\Omega_A=\{x\in\Omega_\varepsilon:\ x_1\in I\}\,,\qquad \Omega_B=\{x\in\Omega_\varepsilon:\ x_1\in J\}\,.$$

In fact $u\equiv\varphi$ near $\partial\Omega$ and for $k=0,1,...,N-1$ we have, for every $x\in\Omega$,

$$\left|D^k u-D^k\varphi\right|\le\sum_{l=0}^{k}\left|D^l\eta\right|\left|D^{k-l}v\right|\le\sum_{l=0}^{k}\frac{L}{\varepsilon^l}\delta$$

and hence $\|u-\varphi\|_{N-1,\infty}\le\varepsilon$. Since in Ω_ε we have $\eta\equiv1$ we deduce that

$$D^N u=D^N v+D^N\varphi=D^N v+tA+(1-t)B=\begin{cases}A & \text{in }\Omega_A\\ B & \text{in }\Omega_B\end{cases}.$$

Finally it remains to show that

$$\operatorname{dist}\left(D^N u(x),\operatorname{co}\{A,B\}\right)\le\varepsilon\qquad\text{a.e. in }\Omega.$$

We have that

$$D^N u=\eta D^N v+D^N\varphi+R\left(D^1\eta,...,D^N\eta,D^0v,...,D^{N-1}v\right)$$

where (choosing δ smaller if necessary)

$$\left|R\left(D^1\eta,...,D^N\eta,D^0v,...,D^{N-1}v\right)\right|\le\gamma\sum_{l=1}^{N}\left|D^l\eta\right|\left|D^{N-l}v\right|\le\varepsilon.$$

In the case $n,m\ge1$ and $N=1$ we have $R\,(D\eta,v)=v\otimes D\eta$, while when $m=1$ and $N=2$

$$R\left(D^1\eta,D^2\eta,D^0v,D^1v\right)=D^1v\otimes D^1\eta+D^1\eta\otimes D^1v+v\,D^2\eta.$$

Since both $D^Nv+D^N\varphi$ $(=A$ or $B)$ and $D^N\varphi=tA+(1-t)B$ belong to $\operatorname{co}\{A,B\}$ we obtain that

$$\eta D^Nv+D^N\varphi=\eta\left(D^Nv+D^N\varphi\right)+(1-\eta)\,D^N\varphi\in\operatorname{co}\{A,B\}\,;$$

since the remaining term is arbitrarily small we deduce the result i.e.

$$\operatorname{dist}\left(D^Nu;\operatorname{co}\{A,B\}\right)\le\varepsilon.$$

Step 2: Let us assume now that $A-B$ is any matrix of rank one of $\mathbb{R}_s^{m\times n^N}$ and therefore it can be written as $A-B=\alpha\otimes v^{\otimes_N}$, i.e.

$$(A-B)^i_{j_1...j_N}=\alpha_i v_{j_1}...v_{j_N}$$

for a certain $\alpha\in\mathbb{R}^m$ and $v\in\mathbb{R}^n$ (v not necessarily e_1 as in Step 1). Replacing α by $|v|^N\alpha$ we can assume that $|v|=1$. We can then find $R=(r_{ij})\in SO\,(n)\subset\mathbb{R}^{n\times n}$

(i.e. a rotation) so that $v = e_1 R$, and hence $e_1 = vR^t$. We then set $\widetilde{\Omega} = R^t \Omega$ and for $1 \leq i \leq m$, $1 \leq j_1, ..., j_N \leq n$ we let

$$\widetilde{A}^i_{j_1 ... j_N} = \sum_{k_1,...,k_N=1}^{n} A^i_{k_1...k_N} r_{j_1k_1} ... r_{j_Nk_N}$$
$$\widetilde{B}^i_{j_1 ... j_N} = \sum_{k_1,...,k_N=1}^{n} B^i_{k_1...k_N} r_{j_1k_1} ... r_{j_Nk_N} .$$

For example if $n, m \geq 1$ and $N = 1$ we have

$$\widetilde{A}^i_j = \sum_{k=1}^{n} A^i_k r_{jk} \ , \quad \widetilde{B}^i_j = \sum_{k=1}^{n} B^i_k r_{jk} \ , \ \text{ i.e. } \ \widetilde{A} = AR^t \ \text{ and } \ \widetilde{B} = BR^t.$$

While if $m = 1$ and $N = 2$ we have

$$\widetilde{A}_{j_1j_2} = \sum_{k_1,k_2=1}^{n} A_{k_1k_2} r_{j_1k_1} r_{j_2k_2} \ , \ \widetilde{B}_{j_1j_2} = \sum_{k_1,k_2=1}^{n} B_{k_1k_2} r_{j_1k_1} r_{j_2k_2} \ ,$$

i.e. we get $\widetilde{A} = RAR^t$ and $\widetilde{B} = RBR^t$. We observe that by construction that

$$\widetilde{A} - \widetilde{B} = \alpha \otimes e_1^{\otimes N}.$$

Indeed, since $e_1 = vR^t$, we have

$$\begin{aligned}\left(\widetilde{A} - \widetilde{B}\right)^i_{j_1...j_N} &= \sum_{k_1,...,k_N=1}^{n} \alpha_i v_{k_1} ... v_{k_N} r_{j_1k_1} ... r_{j_Nk_N} \\ &= \alpha_i \sum_{k_1,...,k_N=1}^{n} (v_{k_1} r_{j_1k_1}) ... (v_{k_N} r_{j_Nk_N}) \\ &= \alpha_i (e_1)_{j_1} ... (e_1)_{j_N} .\end{aligned}$$

We can therefore apply Step 1 to $\widetilde{\Omega}$ and to $\widetilde{\varphi}(y) = \varphi(Ry)$ and find $\widetilde{\Omega}_{\widetilde{A}}$, $\widetilde{\Omega}_{\widetilde{B}}$ and $\widetilde{u} \in C^N_{piec}(\overline{\widetilde{\Omega}}; \mathbb{R}^m)$ with the claimed properties. By setting

$$\begin{cases} u(x) = \widetilde{u}(R^t x), & x \in \Omega \\ \Omega_A = R\widetilde{\Omega}_{\widetilde{A}}, & \Omega_B = R\widetilde{\Omega}_{\widetilde{B}} \end{cases}$$

we get the result. □

4.4. Sufficient conditions for the relaxation property. We will now give some conditions that can ensure the relaxation property (c.f. Definition 4.2), which is the main condition to prove existence of solutions. We give three types of results arranged by increasing order of difficulty. The first one will apply to the case of one quasiconvex function (c.f. Theorem 6.2).

THEOREM 4.5. *Let $E \subset \mathbb{R}^{m \times n^N}_s$ be closed and such that $\partial \operatorname{Rco} E \subset E$ and* $\operatorname{Rco} E$ *is bounded in at least one direction of rank one, then* $\operatorname{Rco} E$ *has the relaxation property with respect to E.*

REMARK 4.4. *By* $\operatorname{Rco} E$ *is bounded in at least one direction of rank one, we mean that there exists $\eta \in \mathbb{R}^{m \times n^N}_s$ with rank $\{\eta\} = 1$ such that, for every $\xi \in$* $\operatorname{int} \operatorname{Rco} E$*, there exist $t_1 < 0 < t_2$ with $\xi + t_1 \eta$, $\xi + t_2 \eta \in \partial \operatorname{Rco} E \subset E$.*

Proof. The proof is elementary. Let $\xi \in \operatorname{int} \operatorname{Rco} E$ then by boundedness we can find $t_1 < 0 < t_2$ such that

$$\begin{cases} \xi_t = \xi + t\eta \in \operatorname{int} \operatorname{Rco} E, & \forall t \in (t_1, t_2) \\ \xi_{t_1}, \xi_{t_2} \in \partial \operatorname{Rco} E \subset E. \end{cases}$$

The approximation lemma (c.f. Lemma 4.4) (with $A = \xi_{t_1+\varepsilon}$ and $B = \xi_{t_2-\varepsilon}$ for ε small enough and $\xi = \frac{t_2-\varepsilon}{t_2-t_1-2\varepsilon} A + \frac{-(t_1+\varepsilon)}{t_2-t_1-2\varepsilon} B$) leads immediately to the result. □

We now consider the more difficult case (which corresponds to the case of systems or the case of one non quasiconvex equation) where $\partial \operatorname{Rco} E \nsubseteq E$. To handle this case we need to assume more structure on $\operatorname{Rco} E$ or more generally on a subset $K(E)$ of E (note that, to emphasize the dependence of K on E we will use the notation $K = K(E)$.

Definition 4.3 (Approximation property). *Let $E \subset K(E) \subset \mathbb{R}_s^{m \times n^N}$. The sets E and $K(E)$ are said to have the approximation property if for every $\delta > 0$, there exist closed sets E_δ and $K(E_\delta)$ such that*

(i) $E_\delta \subset K(E_\delta) \subset \operatorname{int} K(E)$ for every $\delta > 0$

(ii) $\operatorname{dist}(\eta; E) < \delta$ for every $\eta \in E_\delta$

(iii) for every $\xi \in \operatorname{int} K(E)$ there exists $\delta = \delta(\xi) > 0$ such that $\xi \in K(E_\delta)$.

Remark 4.5. *The above definition is similar to the so called in-approximation of convex integration (c.f. S. Muller and V. Sverak* [**44**], [**45**]*).*

Theorem 4.6. *Let $E \subset \mathbb{R}_s^{m \times n^N}$ be compact and $\operatorname{Rco} E$ has the approximation property with $K(E_\delta) = \operatorname{Rco} E_\delta$; then $\operatorname{Rco} E$ has the relaxation property with respect to E.*

Proof. The proof is a direct consequence of the next theorem. Since if $\xi \in \operatorname{Rco} E_\delta$ for some δ, we can find an integer $I = I(\xi)$ such that $\xi \in R_I \operatorname{co} E_\delta$. We have therefore with the notations of the following theorem

$$\begin{aligned} E_\delta^i &= R_i \operatorname{co} E_\delta,\ i = 1, ..., I \\ K(E_\delta) &= \operatorname{Rco} E_\delta \end{aligned}$$

and all the hypotheses of that theorem are satisfied by definition of $R_i \operatorname{co} E_\delta$. □

In some applications, such as the one on singular values, one may want to work not with the whole of $\operatorname{Rco} E$ but rather on a subset $K(E)$ which is not necessary rank one convex. The next theorem provides some answer to this type of problems.

Theorem 4.7. *Let $E \subset K(E) \subset \mathbb{R}_s^{m \times n^N}$ be compact and have the approximation property. Let $\delta > 0$ and assume that for every $\xi \in K(E_\delta)$ (as in Definition 4.3) there exist an integer $I = I(\xi)$ and E_δ^i, $i = 0, 1, ..., I$, such that*

(i) $E_\delta^0 = E_\delta \subset E_\delta^1 \subset ... \subset E_\delta^I \subset K(E_\delta)$

(ii) $\xi \in E_\delta^I$

(iii) there exist $\xi_1, \xi_2 \in E_\delta^{I-1}$ with $\operatorname{rank}[\xi_1 - \xi_2] \leq 1$ such that $\xi \in [\xi_1, \xi_2] \subset K(E_\delta)$

(iv) for every $\eta \in E_\delta^i$, $i = 1, ..., I-1$, there exist $\eta_1, \eta_2 \in E_\delta^{i-1}$ with $\operatorname{rank}[\eta_1 - \eta_2] \leq 1$ such that $\eta \in [\eta_1, \eta_2] \subset K(E_\delta)$.

Then $K(E)$ has the relaxation property with respect to E.

Remark 4.6. *The four conditions presented in the above theorem capture the essential features of* $\operatorname{Rco} E$ *but are more general and thus more flexible. Examples of application to singular values will be given in the book* [**23**] *(see also* [**21**], [**24**]*).*

Proof. Let $\varepsilon > 0$, we wish to show that for every $\xi \in \operatorname{int} K(E)$ and $\Omega \subset \mathbb{R}^n$ a bounded open set, we can find $u_\nu \in C^N_{piec}(\overline{\Omega}; \mathbb{R}^m)$ such that (defining φ by $D^N\varphi(x) = \xi$)

$$\begin{cases} D^\alpha u_\nu(x) = D^\alpha \varphi(x) \quad \text{on } \partial\Omega, \quad \alpha = 0, ..., N-1 \\ u_\nu \overset{*}{\rightharpoonup} \varphi \ \text{ in } W^{N,\infty} \\ D^N u_\nu(x) \in E \cup \operatorname{int} K(E), \quad \text{a.e in } \Omega \\ \int_\Omega \operatorname{dist}\left(D^N u_\nu(x); E\right) dx \le \varepsilon \text{ as } \nu \to \infty\,. \end{cases}$$

Since $K(E)$ has the approximation property and is compact, we can find, for every $\delta > 0$ sufficiently small, a closed set E_δ, with $\xi \in K(E_\delta)$ and $K(E_\delta)$ is compactly contained in $\operatorname{int} K(E)$. It will be sufficient to find a C^N_{piec}vector valued function u and an open set $\widetilde{\Omega} \subset \Omega$ so that

$$\begin{cases} \operatorname{meas}(\Omega - \widetilde{\Omega}) \le \varepsilon \\ u \equiv \varphi \ \text{ near } \partial\Omega \\ \|u - \varphi\|_{N-1,\infty} \le \varepsilon \\ \operatorname{dist}\left(D^N u(x); E_\delta\right) \le \varepsilon, \quad \text{a.e. } x \in \widetilde{\Omega} \\ \operatorname{dist}\left(D^N u(x); K(E_\delta)\right) \le \varepsilon, \quad \text{a.e. } x \in \Omega. \end{cases}$$

The fact that $K(E_\delta) \subset\subset \operatorname{int} K(E)$ and the last inequality imply that $D^N u(x) \in \operatorname{int} K(E)$ for ε sufficiently small. Finally since E_δ is close to E for δ sufficiently small we will have indeed obtained the theorem.

By hypothesis $\xi \in E^I_\delta$ for a certain I. We proceed by induction on I.

Step 1: We start with $I = 1$. We can therefore write

$$\xi = \lambda A + (1-\lambda)B, \ \operatorname{rank}\{A - B\} = 1, \quad A, B \in E_\delta.$$

We then use the approximation lemma (c.f. Lemma 4.4) to get the claimed result by setting $\widetilde{\Omega} = \Omega_A \cup \Omega_B$ and since $\operatorname{co}\{A, B\} = [A; B] \subset K(E_\delta)$ and hence for ε sufficiently small

$$\operatorname{dist}\left(D^N u(x); K(E_\delta)\right) \le \varepsilon.$$

Step 2: For $I > 1$ we consider $\xi \in E^I_\delta$. Therefore there exist $A, B \in \mathbb{R}^{m \times n^N}_s$ such that

$$\begin{cases} \xi = \lambda A + (1-\lambda)B, \quad \operatorname{rank}\{A - B\} = 1 \\ A, B \in E^{I-1}_\delta\,. \end{cases}$$

We then apply the approximation lemma (c.f. Lemma 4.4) and find that there exist a C^N_{piec} vector valued function v and Ω_A, Ω_B disjoint open sets such that

$$\begin{cases} \operatorname{meas}(\Omega - (\Omega_A \cup \Omega_B)) \le \varepsilon/2 \\ v \equiv \varphi \quad \text{near } \partial\Omega \\ \|v - \varphi\|_{N-1,\infty} \le \varepsilon/2 \\ D^N v(x) = \begin{cases} A \text{ in } \Omega_A \\ B \text{ in } \Omega_B \end{cases} \\ \operatorname{dist}\left(D^N v(x); K(E_\delta)\right) \le \varepsilon, \quad \text{in } \Omega. \end{cases}$$

We now use the hypothesis of induction on Ω_A, Ω_B and A, B. We then can find $\widetilde{\Omega}_A, \widetilde{\Omega}_B$ $v_A \in C^N_{piec}$ in Ω_A, $v_B \in C^N_{piec}$ in Ω_B satisfying

$$\begin{cases} \operatorname{meas}(\Omega_A - \widetilde{\Omega}_A),\ \ \operatorname{meas}(\Omega_B - \widetilde{\Omega}_B) \leq \varepsilon/2 \\ v_A \equiv v \text{ near } \partial\Omega_A,\ v_B \equiv v \text{ near } \partial\Omega_B \\ \|v_A - v\|_{N-1,\infty} \leq \varepsilon/2 \text{ in } \overline{\Omega}_A,\ \ \|v_B - v\|_{N-1,\infty} \leq \varepsilon/2 \text{ in } \overline{\Omega}_B \\ \operatorname{dist}(D^N v_A; E_\delta) \leq \varepsilon,\ \text{a.e. in } \widetilde{\Omega}_A,\ \ \operatorname{dist}(D^N v_B; E_\delta) \leq \varepsilon,\ \text{a.e. in } \widetilde{\Omega}_B \\ \operatorname{dist}(D^N v_A; K(E_\delta)) \leq \varepsilon,\ \text{a.e. in } \Omega_A,\ \ \operatorname{dist}(D^N v_B; K(E_\delta)) \leq \varepsilon,\ \text{a.e. in } \Omega_B. \end{cases}$$

Letting $\widetilde{\Omega} = \widetilde{\Omega}_A \cup \widetilde{\Omega}_B$ and

$$u(x) = \begin{cases} v(x) & \text{in } \Omega - (\Omega_A \cup \Omega_B) \\ v_A(x) & \text{in } \Omega_A \\ v_B(x) & \text{in } \Omega_B \end{cases}$$

we have indeed obtained the result. □

5. Proofs of the model results

We start with the following

THEOREM 5.1. *Let $\Omega \subset \mathbb{R}^n$ be open. Let $F_i : \mathbb{R}^{m\times n^N}_s \to \mathbb{R}$, $i = 1, 2, ..., I$, be quasiconvex and locally Lipschitz functions and let*

$$E = \left\{ \xi \in \mathbb{R}^{m\times n^N}_s : F_i(\xi) = 0,\ \ i = 1, 2, ..., I \right\}.$$

Assume that $\operatorname{Rco} E$ *is compact and strongly star shaped with respect to a fixed* $\xi_0 \in \operatorname{int}\operatorname{Rco} E$ *(i.e. for every* $\xi \in \operatorname{Rco} E$ *and every* $t \in (0, 1]$ *then* $t\xi_0 + (1 - t)\xi \in \operatorname{int}\operatorname{Rco} E$*). Let* $\varphi \in C^N_{piec}(\overline{\Omega}; \mathbb{R}^m)$ *satisfy*

$$D^N\varphi(x) \in E \cup \operatorname{int}\operatorname{Rco} E,\ \ a.e.\ x \in \Omega.$$

Then there exists (a dense set of) $u \in W^{N,\infty}(\Omega; \mathbb{R}^m)$ *such that*

$$\begin{cases} F_i(D^N u(x)) = 0, & a.e.\ x \in \Omega,\ \ i = 1, ..., I \\ D^\alpha u(x) = D^\alpha \varphi(x), & x \in \partial\Omega,\ \ \alpha = 0, ..., N - 1. \end{cases}$$

PROOF. The result is a consequence of Corollary 4.3 when we set $K = K(E) = \operatorname{Rco} E$. The only hypothesis that remains to be proved is that $\operatorname{Rco} E$ has the relaxation property with respect to E. This will follow from Theorem 4.6 and from the fact that $\operatorname{Rco} E$ is strongly star shaped.

We thus let for $\delta \in (0, 1]$

$$E_\delta = \delta\xi_0 + (1 - \delta)E$$

and observe that (by induction)

$$K(E_\delta) = \operatorname{Rco} E_\delta = \delta\xi_0 + (1 - \delta)\operatorname{Rco} E \subset \operatorname{int}\operatorname{Rco} E, \quad \forall\delta \in (0, 1].$$

Therefore E and $\operatorname{Rco} E$ have the approximation property (c.f. Definition 4.3) and hence Theorem 4.6 applies. □

Theorem 5.1 has as direct consequences Theorem 3.1 of Section 3.

Proof. (Of Theorem 3.1) In order to apply Theorem 5.1 the only thing to be checked is that $\operatorname{Rco} E$ is strongly star shaped with respect to $0 \in \mathbb{R}_s^{m \times n^N}$. This is elementary. Indeed $0 \in \operatorname{int} \operatorname{Rco} E$, since $F_i(0) = 0$ for every $i = 1, ..., I$. Moreover we have for every $\xi \in \operatorname{Rco} E$ and $t \in (0, 1]$

$$F_i((1-t)\,\xi) = (1-t)^{\alpha_i}\, F_i(\xi) \leq (1-t)^{\alpha_i}\, a_i < a_i, \quad i = 1, 2, ..., I.$$

□

Corollary 5.2. *Let $\Omega \subset \mathbb{R}^n$ be open. Let $F_i : \mathbb{R}_s^{m \times n^N} \to \mathbb{R},\ i = 1, 2, ..., I,$ be quasiconvex and locally Lipschitz functions and let*

$$E = \left\{ \xi \in \mathbb{R}_s^{m \times n^N} : \ F_i(\xi) = 0, \quad i = 1, 2, ..., I \right\}.$$

Assume that $\operatorname{Rco} E$ *is compact and* $\operatorname{Rco} E = \operatorname{co} E$. *Let $\varphi \in C_{piec}^N(\overline{\Omega}; \mathbb{R}^m)$ verify*

$$D^N \varphi\,(x) \in E \cup \operatorname{int} \operatorname{Rco} E, \quad a.e.\ x \in \Omega$$

or $\varphi \in W^{N,\infty}(\Omega; \mathbb{R}^m)$ satisfy

$$D^N \varphi\,(x) \ \textit{compactly contained in } \operatorname{int} \operatorname{Rco} E, \quad a.e.\ x \in \Omega.$$

Then there exists (a dense set of) $u \in W^{N,\infty}(\Omega; \mathbb{R}^m)$ such that

$$\begin{cases} F_i(D^N u(x)) = 0, & a.e.\ x \in \Omega,\ i = 1, ..., I \\ D^\alpha u(x) = D^\alpha \varphi(x), & x \in \partial\Omega, \quad \alpha = 0, ..., N-1. \end{cases}$$

Proof. We start by considering the case where $\varphi \in C_{piec}^N(\overline{\Omega}; \mathbb{R}^m)$. Assume that $\operatorname{int} \operatorname{Rco} E \neq \emptyset$, otherwise the corollary is trivial. So to apply Theorem 5.1 it is only necessary to show that $\operatorname{Rco} E$ is strongly star shaped with respect to any $\xi_0 \in \operatorname{int} \operatorname{Rco} E$. This is however a trivial property of convex sets.

The case $\varphi \in W^{N,\infty}(\Omega; \mathbb{R}^m)$ is deduced from the preceding one by applying an approximation Theorem that can be found in the appendix of [**23**]; this result allows to replace the boundary condition φ by $\psi \in C_{piec}^N(\overline{\Omega}; \mathbb{R}^m)$ with

$$D^N \psi\,(x) \in \operatorname{int} \operatorname{Rco} E, \quad \text{a.e. } x \in \Omega.$$

□

Theorem 3.2 is a direct consequence of the preceding Corollary 5.2. We now state an extension of Theorem 5.1, that can be used when dealing with the problem of *potential wells* (see [**21**], [**44**], [**45**]).

Theorem 5.3. *Let $\Omega \subset \mathbb{R}^n$ be open. Let $F_i^\delta : \mathbb{R}_s^{m \times n^N} \to \mathbb{R},\ i = 1, ..., I,$ be quasiconvex, locally Lipschitz and continuous with respect to $\delta \in [0, \delta_0]$, for some $\delta_0 > 0$. Assume that*

$$(i) \quad \operatorname{Rco} \left\{ \xi \in \mathbb{R}_s^{m \times n^N} : \ F_i^\delta(\xi) = 0,\ i = 1, ..., I \right\} = \left\{ \xi \in \mathbb{R}_s^{m \times n^N} : \ F_i^\delta(\xi) \leq 0,\ i = 1, ..., I \right\}, \quad \forall \delta \in [0, \delta_0]$$

and it is compact;

$$(ii) \quad \left\{ \xi \in \mathbb{R}_s^{m \times n^N} : \ F_i^\delta(\xi) = 0,\ i = 1, ..., I \right\} \subset \left\{ \xi \in \mathbb{R}_s^{m \times n^N} : \ F_i^0(\xi) < 0,\ i = 1, ..., I \right\}, \quad \forall\, 0 < \delta \leq \delta_0.$$

If $\varphi \in C_{piec}^{N}(\overline{\Omega};\mathbb{R}^m)$ *satisfies*

$$F_i^0(D^N\varphi(x)) < 0, \qquad \forall x \in \Omega, \quad i = 1, ..., I,$$

then there exists (a dense set of) $u \in W^{N,\infty}(\Omega;\mathbb{R}^m)$ *such that*

$$\begin{cases} F_i^0(D^N u(x)) = 0, & a.e.\ x \in \Omega, \quad i = 1, ..., I \\ D^\alpha u(x) = D^\alpha \varphi(x), & x \in \partial\Omega, \quad \alpha = 0, ..., N-1. \end{cases}$$

Moreover u *is in the* C^{N-1} *closure of*

$$\left\{ \begin{array}{c} u \in C_{piec}^{N}\left(\overline{\Omega};\mathbb{R}^m\right) : D^\alpha u = D^\alpha \varphi, \ \text{on } \partial\Omega, \ \alpha = 0, ..., N-1, \\ \text{and } F_i^0(D^N u(x)) < 0, \quad a.e \text{ in } \Omega \end{array} \right\}.$$

Proof. The proof is a direct consequence of the abstract results of the preceding section. Indeed let

$$E = \left\{ \xi \in \mathbb{R}_s^{m\times n^N} : \ F_i^0(\xi) = 0, \ \ i = 1, ..., I \right\}$$

$$E_\delta = \left\{ \xi \in \mathbb{R}_s^{m\times n^N} : \ F_i^\delta(\xi) = 0, \ \ i = 1, ..., I \right\}$$

$$K\left(E\right) = \operatorname{Rco} E = \left\{ \xi \in \mathbb{R}_s^{m\times n^N} : \ F_i^0(\xi) \le 0, \ \ i = 1, ..., I \right\}$$

$$K\left(E_\delta\right) = \operatorname{Rco} E_\delta = \left\{ \xi \in \mathbb{R}_s^{m\times n^N} : \ F_i^\delta(\xi) \le 0, \ \ i = 1, ..., I \right\}.$$

Note that, since

$$\left\{ \xi \in \mathbb{R}_s^{m\times n^N} : \ F_i^0(\xi) < 0, \ \ i = 1, ..., I \right\}$$

is rank one convex and open, we have

$$\begin{aligned} &\operatorname{Rco} \left\{ \xi \in \mathbb{R}_s^{m\times n^N} : F_i^0(\xi) < 0, \ i = 1, ..., I \right\} \\ &= \left\{ \xi \in \mathbb{R}_s^{m\times n^N} : F_i^0(\xi) < 0, \ i = 1, ..., I \right\} \\ &\subset \operatorname{int} \left\{ \xi \in \mathbb{R}_s^{m\times n^N} : F_i^0(\xi) \le 0, \ i = 1, ..., I \right\} = \operatorname{int} \operatorname{Rco} E. \end{aligned}$$

Applying therefore hypotheses (i) and (ii) we deduce that

$$\begin{aligned} K\left(E_\delta\right) = \operatorname{Rco} E_\delta &\subset \left\{ \xi \in \mathbb{R}_s^{m\times n^N} : F_i^0(\xi) < 0, \ i = 1, ..., I \right\} \\ &\subset \operatorname{int} \operatorname{Rco} E = \operatorname{int} K\left(E\right). \end{aligned}$$

Then $\operatorname{Rco} E$ has the relaxation property (since it has by construction the approximation property and therefore, by applying Theorem 4.6, it has necessarily the claimed property). Thus our result follows from Theorem 4.2. □

Now we turn our attention to the case with dependence on lower order terms. The first result that we will prove is Theorem 3.3 of Section 3.

Proof. (Of Theorem 3.3) The claim follows from Theorem 6.1 that is stated below in Section 6; we only need to construct functions F_i^δ that satisfy all the hypotheses of this theorem. We therefore let for $\delta \in [0, 1)$

$$F_i^\delta(x, s, \xi) = F_i(x, s, \xi) - (1-\delta)^{\alpha_i} \, a_i\left(x, s\right).$$

The first obvious claim is that

$$F_i^{\delta'}(x, s, \xi) < F_i^\delta(x, s, \xi), \ \text{ whenever } 0 \le \delta' < \delta < 1$$

which implies (ii) of Theorem 6.1. Therefore the only hypothesis that remains to be checked is that

$$\begin{aligned} &\operatorname{Rco}\left\{\xi \in \mathbb{R}_s^{m\times n^N} : F_i^\delta(x,s,\xi)=0, \quad i=1,...,I\right\} \\ &= \left\{\xi \in \mathbb{R}_s^{m\times n^N} : F_i^\delta(x,s,\xi)\le 0, \quad i=1,...,I\right\}, \quad \forall \delta\in[0,1)\,. \end{aligned}$$

For $\delta = 0$ this is the assumption of the present theorem. Since (x, s) act only as parameters, we will drop below the dependence on these variables. We let

$$\begin{aligned} E_\delta &= \left\{\xi \in \mathbb{R}_s^{m\times n^N} : F_i^\delta(\xi)=0, \quad i=1,...,I\right\} \\ &= \left\{\xi \in \mathbb{R}_s^{m\times n^N} : F_i(\xi)=(1-\delta)^{\alpha_i}\, a_i, \quad i=1,...,I\right\} \end{aligned}$$

and

$$E = \left\{\xi \in \mathbb{R}_s^{m\times n^N} : F_i(\xi)=a_i,\ i=1,...,I\right\}.$$

Observe that by homogeneity we have $E_\delta = (1-\delta)\, E$ and hence (by induction)

$$\begin{aligned} \operatorname{Rco} E_\delta &= (1-\delta)\operatorname{Rco} E \\ &= (1-\delta)\left\{\xi \in \mathbb{R}_s^{m\times n^N} : F_i(\xi)\le a_i, \quad i=1,...,I\right\} \\ &= \left\{\xi \in \mathbb{R}_s^{m\times n^N} : F_i^\delta(\xi)\le 0, \quad i=1,...,I\right\} \end{aligned}$$

which is (i) of Theorem 6.1. □

It remains to prove Theorem 3.4 of Section 3.

PROOF. (Of Theorem 3.4) We start by noticing that the case $\varphi \in W^{N,\infty}(\Omega;\mathbb{R}^m)$ follows from the case $\varphi \in C^N_{piec}(\overline{\Omega};\mathbb{R}^m)$ via the use of an approximation result of function in $W^{N,\infty}(\Omega;\mathbb{R}^m)$ by mean of piecewise smooth functions. This approximation result is proved in the appendix of the book [**23**]. We then proceed as in the proof of the above Theorem 3.3. We wish to find F_i^δ satisfying the hypotheses of Theorem 6.1 below. We let

$$F_i^\delta(x,s,\xi) = F_i(x,s,\frac{\xi}{1-\delta}-\frac{\delta}{1-\delta}\xi_0).$$

Since (x, s) are only parameters, we will drop below the dependence on these variables. We let

$$E = \left\{\xi \in \mathbb{R}_s^{m\times n^N} : F_i(\xi)=0,\ i=1,...,I\right\}$$

and for $\delta \in (0, 1)$ we have $E_\delta = \delta\xi_0 + (1-\delta)\, E$. Note that as above

$$\begin{aligned} \operatorname{Rco} E_\delta &= \delta\xi_0 + (1-\delta)\operatorname{Rco} E \\ &= \left\{\eta \in \mathbb{R}_s^{m\times n^N} : \eta = \delta\xi_0 + (1-\delta)\,\xi \text{ with } F_i(\xi)\le 0,\ i=1,...,I\right\} \\ &= \left\{\eta \in \mathbb{R}_s^{m\times n^N} : F_i^\delta(\eta)\le 0,\ i=1,...,I\right\} \end{aligned}$$

which is (i) of Theorem 6.1. Therefore the only thing that remains to be checked is that

$$\begin{aligned} E_\delta &= \left\{\xi \in \mathbb{R}_s^{m\times n^N} : F_i^\delta(\xi)=0,\ i=1,...,I\right\} \\ &\subset \left\{\xi \in \mathbb{R}_s^{m\times n^N} : F_i^{\delta'}(\xi)<0,\ i=1,...,I\right\}, \forall\, 0\le\delta' <\delta<1. \end{aligned}$$

So let

$$t=\frac{1-\delta}{1-\delta'}\in(0,1)$$

and $\xi\in E_\delta$. Bearing in mind that $F_i(\xi_0)<0$ and the convexity of F_i we have for every $i=1,...,I$

$$\begin{aligned}0=F_i^\delta(\xi)=tF_i^\delta(\xi)&>tF_i^\delta(\xi)+(1-t)\,F_i(\xi_0)\\ &\geq tF_i(\frac{\xi}{1-\delta}-\frac{\delta}{1-\delta}\xi_0)+(1-t)\,F_i(\xi_0)\\ &\geq F_i(\frac{t\xi}{1-\delta}+\left(1-t-\frac{t\delta}{1-\delta}\right)\xi_0)=F_i^{\delta'}(\xi)\end{aligned}$$

thus the result. □

6. Other differential problems with lower order terms

The previous results generalize to the case with dependence on lower order terms. More precisely, a generalization of Theorem 5.3 to the case of explicit dependence on lower order terms is the following.

Theorem 6.1 (Attainment for general systems). *Let $\Omega\subset\mathbb{R}^n$ be open. Let $F_i^\delta:\overline{\Omega}\times\mathbb{R}_s^{m\times M}\times\mathbb{R}_s^{m\times n^N}\to\mathbb{R}$, $F_i^\delta=F_i^\delta(x,s,\xi)$, $i=1,...,I$, be quasiconvex and locally Lipschitz with respect to $\xi\in\mathbb{R}_s^{m\times n^N}$ and continuous with respect to $(x,s)\in\overline{\Omega}\times\mathbb{R}_s^{m\times M}$ and with respect to $\delta\in[0,\delta_0]$, for some $\delta_0>0$. Assume that, for every $(x,s)\in\overline{\Omega}\times\mathbb{R}_s^{m\times M}$,*

$$\begin{aligned}(i)\quad &\operatorname{Rco}\left\{\xi\in\mathbb{R}_s^{m\times n^N}:F_i^\delta(x,s,\xi)=0,\ i=1,...,I\right\}\\ &=\left\{\xi\in\mathbb{R}_s^{m\times n^N}:F_i^\delta(x,s,\xi)\leq 0,\ i=1,...,I\right\},\ \forall\delta\in[0,\delta_0]\end{aligned}$$

and it is bounded in $\mathbb{R}_s^{m\times n^N}$ uniformly with respect to (x,s) in a bounded set of $\overline{\Omega}\times\mathbb{R}_s^{m\times M}$;

$$\begin{aligned}(ii)\quad &\left\{\xi\in\mathbb{R}_s^{m\times n^N}:F_i^\delta(x,s,\xi)=0,\ i=1,...,I\right\}\\ &\subset\left\{\xi\in\mathbb{R}_s^{m\times n^N}:F_i^{\delta'}(x,s,\xi)<0,\ i=1,...,I\right\},\ \forall\,0\leq\delta'<\delta\leq\delta_0.\end{aligned}$$

If $\varphi\in C_{piec}^N(\overline{\Omega};\mathbb{R}^m)$ satisfies

$$F_i^0(x,D^{[N-1]}\varphi(x),D^N\varphi(x))<0,\quad a.e.\ x\in\Omega,\ i=1,...,I,$$

then there exists (a dense set of) $u\in W^{N,\infty}(\Omega;\mathbb{R}^m)$ such that

$$\begin{cases}F_i^0(x,D^{[N-1]}u(x),D^Nu(x))=0,\quad a.e.\ x\in\Omega,\ i=1,...,I\\ D^\alpha u(x)=D^\alpha\varphi(x),\quad x\in\partial\Omega,\ \alpha=0,...,N-1.\end{cases}$$

The proof of Theorem 6.1 is not simple from the technical point of view. However it follows the lines of a similar proof given by the authors in [**22**] for the second order case $N=2$ (see also L. Poggiolini [**46**]). For this reason, and since we have here a limited room, we will not give in this paper the details of the proof of Theorem 6.1, but we refer to the book [**23**]. If $I=1$ we obtain a simpler result as follows.

Corollary 6.2. *Let $\Omega \subset \mathbb{R}^n$ be open. Let $F : \overline{\Omega} \times \mathbb{R}_s^{m\times M} \times \mathbb{R}_s^{m\times n^N} \to \mathbb{R}$ be continuous and quasiconvex and locally Lipschitz in the last variable. Assume that $\left\{\xi \in \mathbb{R}_s^{m\times n^N} : F(x,s,\xi) \le 0\right\}$ is bounded in $\mathbb{R}_s^{m\times n^N}$ uniformly with respect to (x,s) in a bounded set of $\overline{\Omega} \times \mathbb{R}_s^{m\times M}$. If $\varphi \in C_{piec}^N(\overline{\Omega};\mathbb{R}^m)$ is such that*

$$F\left(x, D^{[N-1]}\varphi(x), D^N\varphi(x)\right) \le 0, \quad \forall x \in \Omega,$$

then there exists (a dense set of) $u \in W^{N,\infty}(\Omega;\mathbb{R}^m)$ such that

$$\begin{cases} F\left(x, D^{[N-1]}u(x), D^N u(x)\right) = 0, & a.e.\ x \in \Omega \\ D^\alpha u(x) = D^\alpha \varphi(x),\ x \in \partial\Omega, & \alpha = 0,...,N-1. \end{cases}$$

Proof. The proof follows from Theorem 6.1. Indeed let for $\delta \ge 0$

$$F^\delta(x,s,\xi) = F(x,s,\xi) + \delta\,,$$
$$E_\delta = \left\{\xi \in \mathbb{R}_s^{m\times n^N} : F^\delta(x,s,\xi) = 0\right\}.$$

We then have $\operatorname{Rco} E_\delta = \left\{\xi \in \mathbb{R}_s^{m\times n^N} : F^\delta(x,s,\xi) \le 0\right\}$ and therefore all the hypotheses of the previous theorem are satisfied. The fact that the compatibility on the boundary datum is not given by a strict inequality is in this case acceptable, since we can choose $u = \varphi$ on the set where $F\left(x, D^{[N-1]}\varphi(x), D^N\varphi(x)\right) = 0$. □

Remark 6.1. *(i) The required compactness can be weakened and it is sufficient to assume that there exists $\eta \in \mathbb{R}_s^{m\times n^N}$ with $\operatorname{rank}\{\eta\} = 1$ such that $F(x,s,\xi+t\eta) \to +\infty$ as $|t| \to \infty$, for every $(x,s,\xi) \in \Omega \times \mathbb{R}_s^{m\times M} \times \mathbb{R}_s^{m\times n^N}$.*

(ii) Observe that the vectorial problem can here be obtained from the scalar one since $\varphi \in C_{piec}^N(\overline{\Omega};\mathbb{R}^m)$. Indeed choosing the $(m-1)$ first components of u equal to the $(m-1)$ first components of φ, we would reduce the problem to a scalar one. If in addition $N = 1$ i.e. the first order case, the problem is then reduced to a convex scalar problem (since quasiconvexity of F implies convexity with respect to the last vector of Du).

We give here an example of applications of the above theorems. The example is a scalar problem which is an Nth order version of the *eikonal equation.*

Corollary 6.3. *Let $\Omega \subset \mathbb{R}^n$ be open. Let $a : \overline{\Omega} \times \mathbb{R}_s^M \to \mathbb{R}_+$ be continuous and $\varphi \in C_{piec}^N(\overline{\Omega})$ satisfy*

$$\left|D^N\varphi(x)\right| \le a\left(x, D^{[N-1]}\varphi(x)\right), \quad a.e.\ x \in \Omega,$$

or $\varphi \in W^{N,\infty}(\Omega)$ such that

$$\left|D^N\varphi(x)\right| \le a\left(x, D^{[N-1]}\varphi(x)\right) - \theta, \quad a.e.\ x \in \Omega,$$

for a certain $\theta > 0$. Then there exists (a dense set of) $u \in W^{N,\infty}(\Omega)$ verifying

$$\begin{cases} \left|D^N u(x)\right| = a\left(x, D^{[N-1]}u(x)\right), & a.e.\ x \in \Omega \\ D^\alpha u(x) = D^\alpha \varphi(x),\ x \in \partial\Omega, & \alpha = 0,...,N-1. \end{cases}$$

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