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Manifold constrained variational problems

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Abstract. The integral representation for the relaxation of a class of energy functionals where the admissible fields are constrained to remain on a C^1 m-dimensional manifold $\mathcal{M} \subset \mathbb{R}^d$ is obtained. If $f : \mathbb{R}^{d\times N} \to [0, \infty)$ is a continuous function satisfying $0 \leq f(\xi) \leq C(1 + |\xi|^p)$, for $C > 0$, $p \geq 1$, and for all $\xi \in \mathbb{R}^{d\times N}$, then

$$\begin{aligned}\mathcal{F}(u, \Omega) := \inf_{\{u_n\}} \Big\{ &\liminf_{n\to\infty} \int_\Omega f(\nabla u_n)\, dx : u_n \rightharpoonup u \text{ in } W^{1,p}, \\ &u_n(x) \in \mathcal{M} \text{ a.e. } x \in \Omega, n \in \mathbb{N} \Big\} \\ = &\int_\Omega Q_T f(u, \nabla u)\, dx,\end{aligned}$$

where $\Omega \subset \mathbb{R}^N$ is open, bounded, and $Q_T f(y_0, \xi)$ is the tangential quasiconvexification of f at $y_0 \in \mathcal{M}$, $\xi = (\xi^1, ..., \xi^N), \xi^i$ belong to the tangent space to $\mathcal{M}$ at y_0, $i = 1, ..., N$.

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1 Introduction

Several interesting questions in materials science require a good understanding of constrained variational problems. Here our motivation to address nonconvex manifold constrained energy relaxation is originated mainly by questions of equilibria for liquid crystals and magnetostrictive materials, a common characteristic being the study of problems of the type

$$\min_{u\in\mathcal{M}} I(u), \quad I(u) := \int_\Omega f(\nabla u)\,dx,$$

where the class of admissible fields is constrained to remain with values on the C^∞ $(d-1)$-dimensional manifold $\mathcal{S}^{d-1} := \{u \in \mathbb{R}^d : |u| = 1\}$. Here $\Omega \subset \mathbb{R}^N$ is an open, bounded domain, which represents the reference configuration of a certain material body, and the bulk energy density $f : \mathbb{M}^{d\times N} \to [0, +\infty)$ is a continuous function satisfying

$$0 \le f(\xi) \le C(1 + |\xi|^p),$$

for some $C > 0, p \ge 1,$ and all $\xi \in \mathbb{M}^{d\times N}$, where $\mathbb{M}^{d\times N}$ denotes the set of all $d \times N$ matrices.

Often lack of convexity prevents f to meet the requirements ensuring weak lower semicontinuity of $I(\cdot)$ (see [1], [2], [14], [15]), and so we introduce the relaxed energy

$$\mathcal{F}(u) := \inf_{\{u_n\}} \left\{ \liminf_{n\to\infty} \int_\Omega f(\nabla u_n)\,dx : u_n \rightharpoonup u \text{ in } W^{1,p}(\Omega;\mathbb{R}^d), \right.$$
$$\left. u_n(x) \in \mathcal{M} \text{ a.e. } x \in \Omega \right\} \tag{1.1}$$

where $\mathcal{M}$ is a C^1 m-dimensional manifold in $\mathbb{R}^d$.

One of the main objectives of relaxation theory is to find the new relaxed (effective) bulk energy density or, equivalently, to give an integral representation for $\mathcal{F}(\cdot)$. In the non-constrained case, i.e., when $\mathcal{M} = \mathbb{R}^d$, it has been shown by Dacorogna [4] that

$$\mathcal{F}(u) = \int_\Omega Qf(\nabla u)\,dx,$$

where the quasiconvex envelope Qf of f is defined by

$$Qf(\xi) := \inf \left\{ \int_{\mathrm{Q}} f(\xi + \nabla\phi(x))\,dx : \phi \in W_0^{1,\infty}(\mathrm{Q};\mathbb{R}^d) \right\}, \tag{1.2}$$

and $\mathrm{Q} := (0,1)^N$. We recall that $\mathcal{F}(\cdot)$ is now a $W^{1,p}$-sequentially weakly lower semicontinuous functional, and (see [4])

$$\inf_{u\in\mathcal{C}} \int_\Omega f(\nabla u)\,dx = \min_{u\in\mathcal{C}} \int_\Omega Qf(\nabla u)\,dx,$$

whenever $\mathcal{C}$ is a $W^{1,p}(\Omega;\mathbb{R}^d)$ weakly compact class of admissible fields.

In this paper we identify the relaxed energy $\mathcal{F}(\cdot)$ in (1.1) when $u \in \mathcal{C}$ are constrained to remain with values on a C^1 m-dimensional manifold $\mathcal{M} \subset \mathbb{R}^d$. Precisely, we show that the relaxed energy in (1.1) can be represented as

$$\mathcal{F}(u) = \int_\Omega Q_T f(u, \nabla u)\, dx, \tag{1.3}$$

where the tangential quasiconvexification $Q_T f$ is defined in Sect. 2 (see Definition 2.1). We derive the formula (see Proposition 2.2)

$$Q_T f(y, \xi) = Q\bar{f}(y, \xi)$$

with

$$\bar{f}(y, \xi) := f(P_y \xi), \tag{1.4}$$

where P_y is the orthogonal projection of $\mathbb{R}^d$ onto the tangent space to $\mathcal{M}$ at y, $T_y(\mathcal{M})$, $P_y\xi := (P_y\xi^1, \dots, P_y\xi^N)$, and ξ^i stands for the i^{th} column of the matrix $\xi \in \mathbb{R}^{d\times N}$. In particular, this concludes (1.3) for globally parametrized constraint manifolds. When $\mathcal{M}$ is the unit sphere, $\mathcal{M} = \mathcal{S}^{d-1}$, then (1.4) reduces to

$$\bar{f}(y, \xi) := f((\mathbb{I}_{d\times d} - y \otimes y)\xi)$$

for $y \in \mathcal{S}^{d-1}$, $\xi = (\xi^1, \dots, \xi^N)$, with the column vectors ξ^i being tangent to the unit sphere at the point y, $i = 1, \dots, N$ (see Example 2.4). This example is relevant in the study of equilibria for liquid crystals, in ferromagnetism, and for magnetostrictive materials (see [6], [16]). For a wide literature concerning existence and regularity of (local) minima of energy functionals where the admissible fields are constrained to have values on the sphere, see [3], [11], [12], [5], [13], and the references therein.

In Proposition 2.5 we prove $W^{1,p}(\Omega; \mathcal{M})$-sequentially weak lower semicontinuity of

$$u \mapsto \int_\Omega Q_T f(u, \nabla u)\, dx.$$

The statement and proof of the relaxation result are presented in Sect. 3. The main challenge of the analysis is to ensure the locality of the relaxed energy with respect to the domain of integration, and this, in turn, allows us to use the blow-up technique to identify the effective energy density. We will not prove directly the locality property (although it will be an obvious consequence of the representation (1.3)), instead we will introduce an auxiliary functional $\mathcal{F}_\infty$ where the approximating sequences to a macroscopic field u are required to converge uniformly to u on a compact subset of the manifold, outside which they must coincide with u. Locality for $\mathcal{F}_\infty$ is proven in Proposition 3.4, and then we go on to showing that, in fact, $\mathcal{F}$ and $\mathcal{F}_\infty$ agree.

2 The tangential quasiconvexification

To motivate the introduction of the notion of tangential quasiconvexification, first we derive (1.3) in the simple case where $\mathcal{M}$ may be globally parametrized. Suppose that there exists a single C^1 chart $\Psi : \mathcal{M}^* \to \mathbb{R}^d$, where $\mathcal{M}^* \subset \mathbb{R}^m$ is an open set, such that

$$\mathcal{M} = \Psi(\mathcal{M}^*).$$

Further, for simplicity, assume that $\mathcal{M}^*$ is the entire space $\mathbb{R}^m$. Let $u \in W^{1,p}(\Omega; \mathcal{M})$. With $u^* := \Psi^{-1} \circ u$ we have

$$u = \Psi \circ u^*, \quad \nabla u = \nabla\Psi(u^*)\nabla u^*,$$

and so

$$\begin{aligned} \int_\Omega f(\nabla u)\, dx &= \int_\Omega f(\nabla\Psi(u^*)\nabla u^*)\, dx \\ &= \int_\Omega \hat{f}(u^*, \nabla u^*)\, dx, \end{aligned}$$

where $\hat{f} : \mathbb{R}^m \times \mathbb{M}^{m\times N} \to [0, +\infty)$ is given by

$$\hat{f}(y^*, \xi^*) := f(\nabla\Psi(y^*)\xi^*).$$

Well known relaxation theorems (see [1], [4]) yield that

$$\begin{aligned} \mathcal{F}(u) :&= \inf_{\{u_n^*\}} \left\{ \liminf_{n\to\infty} \int_\Omega \hat{f}(u_n^*, \nabla u_n^*)\, dx : u_n^* \rightharpoonup u^* \text{ in } W^{1,p}(\Omega; \mathbb{R}^m) \right\} \\ &= \int_\Omega Q\hat{f}(u^*, \nabla u^*)\, dx, \end{aligned} \tag{2.1}$$

where (see (1.2)) $Q\hat{f}(y^*, \xi^*) := Q\hat{f}(y^*, \cdot)(\xi^*)$. If $\mathcal{M}^*$ is not the entire space then some additional technical difficulties occur, and if $\mathcal{M}$ is not parametrizable by a single chart then previously known results fail to apply. Nevertheless, formula (2.1) indicates how the tangential quasiconvexification Q_T could be constructed from local charts.

In what follows, let $f : \mathbb{M}^{d\times N} \to [0, +\infty)$ be a Borel measurable function, and let $\mathcal{M} \subset \mathbb{R}^d$ be a C^1 m-dimensional manifold, $1 \le m \le d$. The tangent space to $\mathcal{M}$ at y, where $y \in \mathcal{M}$, is denoted $T_y(\mathcal{M})$.

Definition 2.1. Let $y \in \mathcal{M}$ and $\xi \in [T_y(\mathcal{M})]^N$. The tangential quasiconvexification of f at ξ relative to y is defined by

$$Q_T f(y, \xi) := \inf \left\{ \int_{\mathrm{Q}} f(\xi + \nabla\varphi(x))\, dx : \varphi \in W_0^{1,\infty}(\mathrm{Q}; T_y(\mathcal{M})) \right\}.$$

In order to provide alternative characterizations of the tangential quasi-convexification of f, we remark that to each point $y \in \mathcal{M}$ there corresponds a C^1 projection Π_y of a neighborhood $U(y)$ of y in $\mathbb{R}^d$ onto $\mathcal{M}$. This projection has the property that for each $z \in U(y)$, $\Pi_y(z)$ is the unique point z' in $\mathcal{M} \cap U(z)$ with

$$P_y(z' - z) = 0 \qquad \text{(in particular, } z' = z \text{ if } z \in \mathcal{M}\text{)},$$

where, as in (1.4), P_y is the orthogonal projection of $\mathbb{R}^d$ onto the tangent space $T_y(\mathcal{M})$. The projection Π_y is constructed via the Implicit Function Theorem. Consider a local chart $\Psi : \mathcal{M}^* \to \mathbb{R}^d$ such that $\mathcal{M}^* \subset \mathbb{R}^m$ is an open set and $\Psi(\mathcal{M}^*) = \mathcal{M} \cap \tilde{U}(y)$, where $\tilde{U}(y)$ is an open neighborhood of y in $\mathbb{R}^d$. Set $y^* := \Psi^{-1}(y)$, and denote by $\nabla\Psi[y^*]$ the derivative of Ψ at y^* considered as a linear map $\nabla\Psi[y^*] : \mathbb{R}^m \to T_y(\mathcal{M})$. Since $\nabla\Psi[y^*]$ is an isomorphism of $\mathbb{R}^m$ onto the tangent space $T_y(\mathcal{M})$, there is an inverse mapping $\Lambda_y : T_y(\mathcal{M}) \to \mathbb{R}^m$ such that $\Lambda_y \circ \nabla\Psi[y^*]$ is the identity on $\mathbb{R}^m$. Applying the Implicit Function Theorem to the function

$$G : \mathcal{M}^* \times \tilde{U}(y) \to \mathbb{R}^m, \quad (z^*, z) \mapsto \Lambda_y(P_y(\Psi(z^*) - z)),$$

we may find a neighborhood $U(y)$ of y, with $U(y) \subset \tilde{U}(y)$, and a unique ($C^1$) function $\Upsilon : U(y) \to \mathcal{M}^*$ such that

$$G(\Upsilon(z), z) = 0 \quad \text{for all } z \in U(y).$$

We set

$$\Pi_y(z) := \Psi(\Upsilon(z)). \tag{2.2}$$

We observe that

$$P_y(\Pi_y(z) - z) = 0 \quad \text{and } \nabla\Pi_y(y) = P_y.$$

Using the above notation, we provide alternative characterizations for $Q_T f$.

Proposition 2.2. *For any $y \in \mathcal{M}$ and $\xi \in [T_y(\mathcal{M})]^N$ we have*

(i)

$$Q_T f(y, \xi) = Q\hat{f}(y^*, \xi^*), \tag{2.3}$$

where

$$\xi^* := \Lambda_y \xi$$

and

$$\hat{f}(y^*, \eta^*) := f(\nabla\Psi(y^*)\eta^*)$$

for all $\eta^* \in \mathbb{M}^{m\times N}$;

(ii)

$$Q_T f(y,\xi) = Q\bar{f}(y,\xi), \tag{2.4}$$

where

$$\bar{f}(y,\xi) := f(P_y\xi),$$

with $P_y\xi := (P_y\xi^1, \dots, P_y\xi^N)$ *and* ξ^i *stands for the* i^{th} *column of the matrix* $\xi \in \mathbb{R}^{d\times N}$.

Proof. (i) We have

$$\begin{aligned} Q\hat{f}(y^*,\xi^*) &= \inf\left\{\int_Q \hat{f}(y^*,\xi^* + \nabla\phi(x))\,dx : \phi \in W_0^{1,\infty}(Q;\mathbb{R}^m)\right\} \\ &= \inf\left\{\int_Q f(\nabla\Psi[y^*]\xi^* + \nabla\Psi[y^*]\nabla\phi(x))dx : \right. \\ &\qquad\qquad \left. \phi \in W_0^{1,\infty}(Q;\mathbb{R}^m)\right\} \qquad (2.5) \\ &= \inf\left\{\int_Q f(\xi + \nabla\Psi[y^*]\nabla\phi(x))\,dx : \phi \in W_0^{1,\infty}(Q;\mathbb{R}^m)\right\}. \end{aligned}$$

Therefore, if $\phi \in W_0^{1,\infty}(Q;\mathbb{R}^m)$ then

$$\varphi := \nabla\Psi[y^*]\circ\phi \ \in W_0^{1,\infty}(Q;T_y(\mathcal{M}))$$

and we obtain

$$Q_T f(y,\xi) \le Q\hat{f}(y^*,\xi^*).$$

Conversely, if $\varphi \in W_0^{1,\infty}(Q;T_y(\mathcal{M}))$ then

$$\phi := \Lambda_y \circ \varphi \in W_0^{1,\infty}(Q;\mathbb{R}^m)$$

and we conclude that

$$Q\hat{f}(y^*,\xi^*) \le Q_T f(y,\xi).$$

(ii) If $\varphi \in W_0^{1,\infty}(Q;T_y(\mathcal{M}))$ then $\varphi \in W_0^{1,\infty}(Q;\mathbb{R}^d)$ and $P_y\varphi = \varphi$, so that

$$\bar{f}(y,\xi+\varphi) = f(y,\xi+\varphi),$$

and thus

$$Q\bar{f}(y,\xi) \le Q_T f(y,\xi).$$

Conversely, if $\bar{\varphi} \in W_0^{1,\infty}(Q;\mathbb{R}^d)$, then $\varphi := P_y \circ \bar{\varphi} \in W_0^{1,\infty}(Q;T_y(\mathcal{M}))$ and

$$f(y,\xi+\nabla\varphi) = \bar{f}(y,\xi+\nabla\varphi);$$

hence

$$Q_T f(y,\xi) \le Q\bar{f}(y,\xi). \qquad \square$$

Remark 2.3 (A formula exploiting the normal field). Let $\mathcal{M}$ be a $(N-1)$-dimensional C^1 manifold and $y \in \mathcal{M}$. If $\nu(y)$ is a normal vector to $\mathcal{M}$ at y, then the expression $y \mapsto \nu(y) \otimes \nu(y)$ does not depend on the orientation of $\mathcal{M}$ and clearly

$$P_y z = (\mathbb{I}_{d\times d} - \nu(y) \otimes \nu(y))\, z \quad \text{for all } z \in \mathbb{R}^d \text{ and } y \in \mathcal{M}.$$

Hence in the notation of Proposition 2.2 (ii),

$$\bar{f}(y,\xi) = f((\mathbb{I}_{d\times d} - \nu(y) \otimes \nu(y))\, \xi).$$

Example 2.4 (The Unit Sphere). Let $\mathcal{M} = \mathcal{S}^{d-1} := \{x \in \mathbb{R}^d : |x| = 1\}$, so here $d = N$, $m = d-1$. We claim that if $y \in \mathcal{S}^{d-1}$ and $\xi \in [T_y(\mathcal{S}^{d-1})]^N$ then

$$Q_T f(y,\xi) = Q\bar{f}(y,\xi),$$

where

$$\bar{f}(y,\xi) := f((\mathbb{I}_{d\times d} - y \otimes y)\xi).$$

This may be seen directly from Remark 2.3. Alternatively, we use Proposition 2.2 to derive this formula in terms of parametrization charts for $\mathcal{M}$. Let R be a rotation such that $Re_d = y$, where $\{e_1, e_2, \ldots, e_d\}$ stands for the canonical basis of $\mathbb{R}^d$, and consider the chart

$$\Psi(x_1,\ldots,x_{d-1}) = R\left(x_1,\ldots,x_{d-1}, \sqrt{1-(x_1^2+\ldots+x_{d-1}^2)}\right), \quad (x_1,\ldots,x_{d-1}) \in B_{d-1}(0,1).$$

By (2.5) we have that

$$Q_T f(y,\xi) = \inf\left\{ \int_{\mathrm{Q}} f(\xi + \nabla\Psi(0)\nabla\phi(x))dx : \phi \in W_0^{1,\infty}(\mathrm{Q};\mathbb{R}^{d-1}) \right\}, \tag{2.6}$$

where

$$\nabla\Psi(0) = RH, \qquad \text{with} \quad H := \begin{pmatrix} 1 \cdots\cdots 0 \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \\ 0 \cdots\cdots 1 \\ 0 \cdots\cdots 0 \end{pmatrix}_{d\times(d-1)}.$$

Given $\phi \in W_0^{1,\infty}(\mathrm{Q};\mathbb{R}^{d-1})$, define $\bar{\phi} \in W_0^{1,\infty}(\mathrm{Q};\mathbb{R}^d)$ by

$$\bar{\phi}(x) = RH\,\phi(x).$$

Then

$$\xi + \nabla\Psi(0)\nabla\phi(x) = (\mathbb{I} - y \otimes y)(\xi + \nabla\bar{\phi}(x))$$

because $\xi \in [T_y(\mathcal{S}^{d-1})]^N$, so $(y \otimes y)\xi = 0$, and

$$\begin{aligned}(\mathbb{I} - y \otimes y)\nabla\bar{\phi} &= R(\mathbb{I} - e_d \otimes e_d)R^T\nabla\bar{\phi} \\ &= RHH^TR^T\nabla\bar{\phi} \\ &= RH\nabla\phi \\ &= \nabla\Psi(0)\nabla\phi.\end{aligned} \tag{2.7}$$

Conversely, given $\bar{\phi} \in W_0^{1,\infty}(\mathrm{Q};\mathbb{R}^d)$ we set

$$\phi = H^TR^T\bar{\phi}$$

so that $\phi \in W_0^{1,\infty}(\mathrm{Q};\mathbb{R}^{d-1})$, and it can be seen easily that (2.7) still holds. This, together with (2.6), implies that

$$\begin{aligned}Q_Tf(y,\xi) &= \inf\left\{\int_{\mathrm{Q}} f((\mathbb{I} - y \otimes y)(\xi + \nabla\bar{\phi}(x))\,dx : \bar{\phi} \in W_0^{1,\infty}(\mathrm{Q};\mathbb{R}^d)\right\} \\ &= Q\bar{f}(y,\xi).\end{aligned}$$

Proposition 2.5 (Lower Semicontinuity). *Let $f : \mathbb{M}^{d\times N} \to [0,+\infty)$ be a continuous function such that*

$$0 \le f(\xi) \le C(1 + |\xi|^p), \quad p \ge 1,$$

for some $C > 0$ and all $\xi \in \mathbb{M}^{d\times N}$. Let

$$J(u) := \int_{\Omega} Q_Tf(u,\nabla u)\,dx, \quad u \in W^{1,p}(\Omega;\mathcal{M}).$$

Then $J(\cdot)$ is $W^{1,p}(\Omega;\mathcal{M})$- sequentially weakly lower semicontinuous.

Proof. For all $y \in \mathcal{M}$ let $\Pi_y : U(y) \to U(y) \cap \mathcal{M}$ be the C^1 projection onto $\mathcal{M}$ introduced in (2.2). Let $\{\eta\}_{\eta\in\mathcal{G}}$ be a partition of unity subordinate to $\{U(y) : y \in \mathcal{M}\}$ (see [17]), so that

(i) $\eta \ge 0, \quad \eta \in C_0^\infty(\mathbb{R}^d;[0,1])$;
(ii) for every $\eta \in \mathcal{G}$ there exists $y \in \mathcal{M}$ such that $\operatorname{supp}\eta \subset U(y)$;
(iii) if $E \subset \mathcal{M}$ is compact then $\operatorname{supp}\eta \cap E \neq \emptyset$ for only finitely many $\eta \in \mathcal{G}$;
(iv) $\sum_{\eta\in\mathcal{G}} \eta(y) = 1$ if $y \in \mathcal{M}$.

Consider $u, u_n \in W^{1,p}(\Omega;\mathcal{M})$ such that $u_n \rightharpoonup u \in W^{1,p}$, and fix $\varepsilon > 0$. By Lebesgue's Monotone Convergence Theorem, we may find a compact set $E \subset \mathcal{M}$ such that

$$\int_\Omega Q_T f(u,\nabla u)\,dx \le \int_{\Omega\cap E} Q_T f(u,\nabla u)\,dx + \varepsilon. \tag{2.8}$$

By (iii) and (iv) above, we may choose $k \in \mathbb{N}$ such that

$$\chi_E(y) = \sum_{i=1}^k \eta_i(y)\chi_E(y), \quad \text{where } \eta_i \in \mathcal{G},$$

and so, using (ii), with

$$\operatorname{supp}\eta_i \subset U(y_i), \qquad \Pi_i := \Pi_{y_i},$$

we obtain

$$\begin{aligned}\int_{\Omega\cap E} Q_T f(u,\nabla u)\,dx &= \sum_{i=1}^k \int_{\Omega\cap E} \eta_i(u) Q_T f(u,\nabla u)\,dx\\ &= \sum_{i=1}^k \int_{\Omega\cap E} QH_i(u,\nabla u)\,dx,\end{aligned}$$

where, according to Definition 2.1 and Remark 2.2 (iii),

$$H_i(y,\xi) := \eta_i(y) f(\nabla\Pi_i(y)\xi).$$

Well known lower semicontinuity results for Carathéodory integrands yield (see [1], [4])

$$\begin{aligned}\int_{\Omega\cap E} Q_T f(u,\nabla u)\,dx &\le \sum_{i=1}^k \liminf_{n\to+\infty} \int_{\Omega\cap E} QH_i(u_n,\nabla u_n)\,dx\\ &\le \liminf_{n\to+\infty} \sum_{i=1}^k \int_{\Omega\cap E} QH_i(u_n,\nabla u_n)\,dx\\ &= \liminf_{n\to+\infty} \sum_{i=1}^k \int_{\Omega\cap E} \eta_i(u_n) Q_T f(u_n,\nabla u_n)\,dx\\ &\le \liminf_{n\to+\infty} \int_\Omega Q_T f(u_n,\nabla u_n)\,dx,\end{aligned}$$

which, together with (2.8), concludes the proof. □

3 Relaxation

The main result of this section is the integral representation of the relaxed energy, precisely

Theorem 3.1. *If* $f : \mathbb{M}^{d\times N} \to [0,+\infty)$ *is a continuous function satisfying*

$$0 \le f(\xi) \le C(1+|\xi|^p)$$

for some $p \ge 1, C > 0,$ *and all* $\xi \in \mathbb{M}^{d\times N}$, *and if* $\mathcal{M}$ *is a* C^1 m*-dimensional manifold then*

$$\mathcal{F}(u) = \int_\Omega Q_T f(u,\nabla u)\,dx,$$

where

$$\mathcal{F}(u) := \inf_{\{u_n\}}\left\{\liminf_{n\to+\infty}\int_\Omega f(\nabla u_n)\,dx : u_n \rightharpoonup u \;\; in \;\; W^{1,p}(\Omega;\mathcal{M})\right\}.$$

Proposition 2.5 entails

$$\mathcal{F}(u) \ge J(u), \tag{3.1}$$

and a typical procedure to obtain the converse inequality would involve a “localization” of $\mathcal{F}(u)$, i.e., the introduction of the functional

$$\mathcal{F}(u;A) := \inf_{\{u_n\}}\left\{\liminf_{n\to\infty}\int_A f(\nabla u_n)\,dx : u_n \rightharpoonup u \right.$$
$$\left. \text{in } W^{1,p}(\Omega;\mathbb{R}^d), u_n(x)\in\mathcal{M} \text{ a.e. } x\in\Omega\right\},$$

where $A \in \mathcal{A}(\Omega)$ and $\mathcal{A}(\Omega)$ denotes the class of all open subsets of Ω. We would then go on to showing that $\mathcal{F}(u;\cdot)$ is the trace on $\mathcal{A}(\Omega)$ of a finite Radon measure, absolutely continuous with respect to the N-dimensional Lebesgue measure $\mathcal{L}^N$ in $\mathbb{R}^N$. Finally, establishing the converse of (3.1) would be equivalent to proving that the Radon Nikodym derivative of $\mathcal{F}(u,\cdot)$ with respect to $\mathcal{L}^N$ is bounded above by the tangential quasiconvexification, i.e.,

$$\frac{d\mathcal{F}(u;\cdot)}{d\mathcal{L}^N}(x_0) \le Q_T f(u(x_0),\nabla u(x_0)) \text{ for } \mathcal{L}^N \; a.e.\, x_0\in\Omega.$$

Now, showing that $\mathcal{F}(u;\cdot)$ is the trace on $\mathcal{A}(\Omega)$ of a finite Radon measure is equivalent to establishing

(i) (the subadditivity property)

$$\mathcal{F}(u;V) \le \mathcal{F}(u;V') + \mathcal{F}(u;V\setminus\overline{V''}) \tag{3.2}$$

where $V, V', V'' \in \mathcal{A}(\Omega)$, $V'' \subset\subset V' \subset\subset V$;

(ii) there exists a finite Radon measure ν in Ω such that

$$\mathcal{F}(u, A) \le \nu(A) \text{ for all } A \in \mathcal{A}(\Omega).$$

Using the growth condition imposed on f, (ii) follows immediately by setting

$$\nu := C(1 + |\nabla u|^p)\mathcal{L}^N \lfloor \Omega.$$

Here we use the notation $\mu\lfloor A$ to denote the restriction of a Borel measure μ to a Borel set A, i.e., $\mu\lfloor A(E) := \mu(E \cap A)$.

The main difficulty lies on part (i), which requires the ability to connect admissible sequences for $V \setminus \overline{V''}$ and for V' across a suitable transition layer on $V' \setminus \overline{V''}$ and without increasing the total limiting energy. Rather than proving this property directly on $\mathcal{F}(u, \cdot)$, we introduce the auxiliary functional

$$\mathcal{F}_\infty : W^{1,p}(\Omega; \mathcal{M}) \times \mathcal{A}(\Omega) \to [0, +\infty)$$

defined by

$$\begin{aligned} \mathcal{F}_\infty(u; A) := \inf_{\{u_n\}} \Bigg\{ \liminf_{n \to +\infty} \int_A f(\nabla u_n)\, dx : u_n \rightharpoonup u \text{ in } W^{1,p}(\Omega; \mathcal{M}), \\ u_n \to u \text{ uniformly, there exists a compact subset} \\ E \subset \mathcal{M} \text{ such that } u_n(x) = u(x) \text{ if } x \notin E \Bigg\}. \end{aligned}$$

Our aim is to show that

$$\mathcal{F}(u) = \mathcal{F}_\infty(u; \Omega) = \int_\Omega Q_T f(u, \nabla u)\, dx,$$

where the last equality for $\mathcal{F}_\infty(u; \Omega)$ will be obtained following the scheme outlined above for $\mathcal{F}(u; \Omega)$. The lemma below will be instrumental in the proof of (3.2) for $\mathcal{F}_\infty(u; \cdot)$.

Lemma 3.2. *Let $E \subset \mathcal{M}$ be a compact subset. There exist $\delta > 0, C > 0$, and a uniformly continuously differentiable mapping*

$$\Phi : D_\delta \times [0, 1] \to \mathcal{M},$$

where

$$\begin{aligned} D_\delta = \{(y_0, y_1) \in \mathcal{M} \times \mathcal{M} : \operatorname{dist}(y_0, E) < \delta, \\ \operatorname{dist}(y_1, E) < \delta, \quad |y_0 - y_1| < \delta\}, \end{aligned}$$

such that

$$\Phi(y_0, y_1, 0) = y_0, \qquad \Phi(y_0, y_1, 1) = y_1$$

and

$$\frac{\partial \Phi}{\partial t}(y_0, y_1, t) \le C\, |y_1 - y_0|\, .$$

Proof. Cover E with a finite collection of convex open sets $U(y_i) \subset \mathbb{R}^d$, where $y_i \in E \cap U(y_i), \quad i = 1, \ldots, k$, and consider the C^1 projections onto $\mathcal{M}$ introduced in (2.2) (if necessary, the original neighborhoods $U(y_i)$ are reduced so as to be rendered convex)

$$\Pi_i : U(y_i) \to U(y_i) \cap \mathcal{M}.$$

Define

$$\widetilde{\Phi}_i : (U_i \cap \mathcal{M}) \times (U_i \cap \mathcal{M}) \times [0,1] \to U_i \cap \mathcal{M},$$
$$(y_0, y_1, t) \mapsto \Pi_i((1-t)y_0 + ty_1).$$

It is clear that $\widetilde{\Phi}_i(y_0, y_1, 0) = y_0$, and $\widetilde{\Phi}_i(y_0, y_1, 1) = y_1$. Since E is a compact set, we may shrink slightly the open sets U_i so as to keep a covering of E, i.e., we choose $\delta_1 > 0$ such that

$$E \subset \cup_{i=1}^k V_i, \qquad V_i := \{y \in U_i : \operatorname{dist}(y, \partial U_i) > \delta_1\},$$

and, similarly, fix $0 < \delta_2 < 2\delta_1$ verifying

$$E \subset \cup_{i=1}^k W_i, \qquad W_i := \{y \in V_i : \operatorname{dist}(y, \partial V_i) > \delta_2\}.$$

Let

$$C_1 := \max\left\{1, \quad \|\Pi_i\|_{W^{1,\infty}(\tilde{U})} : i = 1, \ldots, k\right\}$$

where

$$\widetilde{U}_i := \left\{y \in U_i : \operatorname{dist}(y, \partial U_i) \geq \frac{\delta_2}{2}\right\}.$$

Set

$$\delta := \frac{\delta_2}{4C_1^k}.$$

We claim that if $y_0, y_1 \in \mathcal{M}$ are such that $\operatorname{dist}(y_0, E) < \delta$, $\operatorname{dist}(y_1, E) < \delta$, and $|y_0 - y_1| < \delta$, then there exists an index $i \in \{1, \ldots, k\}$ such that

$$(y_0, y_1) \in \widetilde{U}_i \times \widetilde{U}_i \tag{3.3}$$

and

$$\left|\widetilde{\Phi}_i(y_0, y_1, t) - y_0\right| < C_1 |y_0 - y_1|. \tag{3.4}$$

Indeed, if $\operatorname{dist}(y_0, E) < \delta$ then we may find

$$e \in E \subset \cup_{i=1}^k W_i$$

such that $|y_0 - e| < \delta < \delta_2$, $|y_1 - e| < 2\delta < \delta_2$. Let $i \in \{1, \ldots, k\}$ be such that $e \in W_i$. Since $\operatorname{dist}(e, \partial V_i) > \delta_2$, we have that $y_0, y_1 \in V_i$, and since $\operatorname{dist}(y_0, \partial U_i), \operatorname{dist}(y_1, \partial U_i) > \delta_1 > \frac{\delta_2}{2}$, we conclude that $y_0, y_1 \in \widetilde{U}_i$. This proves claim (3.3). Also,

$$\begin{aligned}\left|\widetilde{\Phi}_i(y_0, y_1, t) - y_0\right| &= |\Pi_i((1-t)y_0 + ty_1) - \Pi_i(y_0)| \\ &\le C_1 \, |y_1 - y_0| \,,\end{aligned}$$

and we obtain (3.4).

For $i = 1, \ldots, k$, let $\eta_i \in C_0^\infty(U_i; [0,1])$ be a cut-off function such that $\eta_i = 1$ in $\widetilde{U}_i$ and $\eta_i(y) = 0$ if $\operatorname{dist}(y, \partial U_i) < \delta_2/4$. Clearly

$$\eta_i(y) = 0 \Rightarrow y \notin \widetilde{U}_i, \quad \eta_i(y) > 0 \Rightarrow B\,(y, \delta_2/4) \subset U_i.$$

Set

$$\bar{\eta}_i(y_0, y_1) := \eta_i(y_0)\, \eta_i(y_1),$$

and define

$$\Phi_1 : \left(D_\delta \cap (\widetilde{U}_1 \times \widetilde{U}_1)\right) \times [0,1] \to U_1 \cap \mathcal{M}, \quad \Phi_1 := \widetilde{\Phi}_1.$$

Suppose that $(y_0, y_1) \in (\widetilde{U}_1 \times \widetilde{U}_1) \cap (\mathcal{M} \times \mathcal{M})$, $|y_0 - y_1| < \delta$, and $\bar{\eta}_2(y_0, y_1) > 0$. If $t \in [0,1]$ then by (3.4)

$$|\Phi_1(y_0, y_1, t) - y_0| \le C_1 \delta < \frac{\delta_2}{4},$$

and, since $\eta_2(y_0) > 0$, $B\,(y_0, \delta_2/4) \subset U_2$. We deduce that

$$\Phi_1(y_0, y_1, t) \in U_2.$$

We may then define

$$\Phi_2 : \left(D_\delta \cap \left[(\widetilde{U}_1 \times \widetilde{U}_1) \cup (\widetilde{U}_2 \times \widetilde{U}_2)\right]\right) \times [0,1] \to \mathcal{M},$$

$$\Phi_2(y_0, y_1, t) := \begin{cases} \Phi_1(y_0, y_1, t) \\ \quad \text{if} \quad \bar{\eta}_2(y_0, y_1) = 0 \\ \\ \Pi_2\left(\dfrac{\bar{\eta}_1(y_0, y_1)\Phi_1(y_0, y_1, t) + \bar{\eta}_2(y_0, y_1)\widetilde{\Phi}_2(y_0 y_1, t)}{\bar{\eta}_1(y_0, y_1) + \bar{\eta}_2(y_0, y_1)}\right) \\ \quad \text{if} \quad \bar{\eta}_2(y_0, y_1) > 0. \end{cases}$$

Note that $\Phi_2(y_0, y_1, 0) = y_0,\ \Phi_2(y_0, y_1, 1) = y_1$, and if

$$\bar{\eta}_2(y_0^n, y_1^n) \to 0^+ \ \text{ as } \ y_0^n \to y_0,\ y_1^n \to y_1, \ \text{ with } \ \bar{\eta}_1(y_0, y_1) > 0,$$

then

$$\Phi_2(y_0^n, y_1^n, t) \to \Pi_2(\Phi_1(y_0, y_1, t)) = \Phi_1(y_0, y_1, t),$$

and it follows easily that Φ_2 is smooth. In addition

$$|\Phi_2(y_0, y_1, t) - y_0| < C_1^2 \delta$$

and so, if $\eta_3(y_0) > 0$ then

$$\Phi_2(y_0, y_1, t) \in B\left(y_0, C_1^2\delta\right) \subset B\left(y_0, \delta_2/4\right) \subset U_3;$$

hence $\Pi_3(\theta\Phi_2 + (1-\theta)\widetilde{\Phi}_3)$ is well defined for all $\theta \in [0,1]$. In this manner, recursively we define

$$\Phi_i : \Big(D_\delta \cap \bigcup_{j \le i} (\widetilde{U}_j \times \widetilde{U}_j)\Big) \times [0,1] \to \mathcal{M}$$

by

$$\Phi_i(y_0, y_1, t) := \begin{cases} \Phi_{i-1}(y_0, y_1, t) & \text{if} \quad \bar{\eta}_i(y_0, y_1) = 0, \\ \Pi_i\left(\frac{(\bar{\eta}_1 + \ldots + \bar{\eta}_{i-1})\Phi_{i-1} + \bar{\eta}_i \widetilde{\Phi}_i}{\bar{\eta}_1 + \ldots + \bar{\eta}_i}\right) & \text{if} \quad \bar{\eta}_i(y_0, y_1) > 0. \end{cases}$$

In light of (3.3), $\Phi := \Phi_k$ fulfills the desired requirements. □

Remark 3.3. It follows from the construction of Φ in the above lemma that

$$|\Phi(y_0, y_1, t) - y_0| < C\,|y_0 - y_1|$$

for some $C > 0$, and for all $(y_0, y_1, t) \in D_\delta \times [0,1]$.

Proposition 3.4. *If $u \in W^{1,p}(\Omega; \mathcal{M})$ then $\mathcal{F}_\infty(u; \cdot)$ is the trace in $\mathcal{A}(\Omega)$ of a finite Radon measure, absolutely continuous with respect to the N-dimensional Lebesgue measure $\mathcal{L}^N$.*

Proof. The last part of the statement is trivial, since due to the growth condition on f, we have that

$$\mathcal{F}_\infty(u; A) \le C \int_A (1 + |\nabla u|^p)\, dx = C\,(1 + |\nabla u|^p)\mathcal{L}^N \lfloor \Omega(A). \tag{3.5}$$

We claim that the subadditivity property holds (see (3.2)), i.e.,

$$\mathcal{F}_\infty(u; V) \le \mathcal{F}_\infty(u; V') + \mathcal{F}_\infty(u; V \setminus \overline{V''}) \tag{3.6}$$

whenever $V, V', V'' \in \mathcal{A}(\Omega)$, $V'' \subset\subset V' \subset\subset V$. We may assume that V' has a smooth boundary. Fix $\varepsilon > 0$ and let $u_n \in W^{1,p}(\Omega; \mathcal{M})$ be such that $u_n \to u$ uniformly, $u_n \rightharpoonup u$ in $W^{1,p}$, $u_n = u$ if $u(x) \notin E_1$, $E_1 \subset \mathcal{M}$ compact, and

$$\mathcal{F}_\infty(u; V \setminus \overline{V''}) + \varepsilon \ge \lim_{n \to +\infty} \int_{V \setminus \overline{V''}} f(\nabla u_n)\, dx.$$

Similarly, let $v_n \in W^{1,p}(\Omega;\mathcal{M})$ be such that $v_n \to u$ uniformly and weakly in $W^{1,p}$, $v_n = u$ if $u(x) \notin E_2$, $E_2 \subset \mathcal{M}$ compact, with

$$\mathcal{F}_\infty(u;V') + \varepsilon \geq \lim_{n\to+\infty} \int_{V'} f(\nabla v_n)\,dx.$$

Without loss of generality, we may assume that (up to a subsequence)

$$(|\nabla u_n|^p + |\nabla v_n|^p)\mathcal{L}^N \lfloor (V' \setminus \overline{V''}) \overset{*}{\rightharpoonup} \nu,$$

where ν is a finite, Radon measure, and we choose $\delta_0, 0 < \delta_0 < \operatorname{dist}(V'', \partial V')$, such that

$$\nu(S) = 0, \quad S := \left\{x \in V' \setminus \overline{V''} : \operatorname{dist}(x, \partial V') = \delta_0\right\}.$$

Given $j \in \mathbb{N}$ we consider a smooth cut-off function $\eta_j \in C_0^\infty(V';[0,1])$ with $\|\nabla \eta_j\|_\infty \leq Cj$ and

$$\eta_j = \begin{cases} 1 & \text{if } \operatorname{dist}(x,\partial V') > \delta_0 + 1/j, \\ 0 & \text{if } \operatorname{dist}(x,\partial V') < \delta_0 - 1/j. \end{cases}$$

Setting $E := E_1 \cup E_2 \subset \mathcal{M}$, E is compact and we consider $\delta > 0$ and a function $\Phi : D_\delta \times [0,1] \to \mathcal{M}$ satisfying the properties of Lemma 3.2. Choose n large enough so that

$$\|u_n - u\|_{L^\infty(V\setminus\overline{V''})}, \quad \|v_n - u\|_{L^\infty(V')}, \quad \|u_n - v_n\|_{L^\infty(V'\setminus\overline{V''})} < \delta.$$

If $u(x) \in E$ then $\operatorname{dist}(u_n(x), E)$, $\operatorname{dist}(v_n(x), E) < \delta$, and we set

$$w_{n,j}(x) := \Phi(u_n(x), v_n(x), \eta_j(x)).$$

If $u(x) \notin E$, then define

$$w_{n,j}(x) := u(x).$$

By Remark 3.3 it follows that

$$\lim_j \lim_n \|w_{n,j} - u\|_{L^\infty(V)} = 0,$$

and

$$\begin{aligned}\limsup_j\ \limsup_n \int_\Omega |\nabla w_{n,j}|^p\,dx \\ &\leq C + \limsup_j\ \limsup_n \int_{L_j} C(1 + j^p|u_n - v_n|^p \\ &+ |\nabla u_n|^p + |\nabla v_n|^p)\,dx < +\infty,\end{aligned}$$

where

$$L_j := \left\{ x \in V' \setminus \overline{V''} : \delta_0 - \frac{1}{j} < \operatorname{dist}(x, \partial V') < \delta_0 + \frac{1}{j} \right\}.$$

Therefore,

$$\begin{aligned}
\mathcal{F}_\infty(u; V) &\le \liminf_{j\to+\infty} \liminf_{n\to\infty} \int_V f(\nabla w_{n,j})\, dx \\
&\le \mathcal{F}_\infty(u; V \setminus \overline{V''}) + \mathcal{F}_\infty(u; V') + 2\varepsilon \\
&\quad + \limsup_{j\to+\infty} \limsup_{n\to+\infty} \int_{L_j} C(1 + j^p |u_n - v_n|^p + |\nabla u_n|^p \\
&\quad + |\nabla v_n|^p)\, dx \\
&\le \mathcal{F}_\infty(u; V \setminus \overline{V''}) + \mathcal{F}_\infty(u; V') + 2\varepsilon + \limsup_{j\to+\infty} \nu(\overline{L_j}) \\
&= \mathcal{F}_\infty(u; V \setminus \overline{V''}) + \mathcal{F}_\infty(u; V') + 2\varepsilon
\end{aligned}$$

because

$$\lim_{j\to+\infty} \nu(\overline{L_j}) = \nu(S) = 0.$$

This proves (3.6).

Now we conclude the proof of Proposition 3.4. Up to the extraction of a subsequence, we assume that

$$\mathcal{F}_\infty(u; \Omega) = \lim_{n\to+\infty} \int_\Omega f(\nabla u_n)\, dx,$$

where

$$u_n \in W^{1,p}(\Omega, \mathcal{M}),\ \|u_n - u\|_{L^\infty(\Omega)} \to 0,\ u_n(x) = u(x) \ \text{ if } \ u(x) \notin E,$$

and $E \subset \mathcal{M}$ is a compact set. Moreover,

$$f(\nabla u_n)\mathcal{L}^N \lfloor \Omega \overset{*}{\rightharpoonup} \mu,$$

where μ is a finite, Radon measure in $\mathbb{R}^N$. By (3.5), for all $V \in \mathcal{A}(\Omega)$, $\varepsilon > 0$, we may find $C \subset\subset V$ such that

$$\mathcal{F}_\infty(u; V \setminus \overline{V''}) < \varepsilon.$$

First we show that

$$\mathcal{F}_\infty(u; V) \le \mu(V) \ \text{ for all } \ V \in \mathcal{A}(\Omega). \tag{3.7}$$

In fact, if $V' \in \mathcal{A}(\Omega)$ then

$$\begin{aligned}
\mathcal{F}_\infty(u; V') &\le \liminf_{n\to+\infty} \int_{V'} f(\nabla u_n)\, dx \\
&\le \mu(\overline{V'}),
\end{aligned}$$

thus, using (3.5) and given $\varepsilon > 0$ choose $V'' \subset\subset V' \subset\subset V$ such that

$$\mathcal{F}_\infty(u; V \setminus \overline{V''}) < \varepsilon,$$

so that by (3.6) we have

$$\begin{aligned}\mathcal{F}_\infty(u; V) &\le \varepsilon + \mathcal{F}_\infty(u; V') \\ &\le \varepsilon + \mu(\overline{V'}) \\ &\le \varepsilon + \mu(V)\end{aligned}$$

and (3.7) follows by letting $\varepsilon \to 0^+$.

Conversely, fixing $V \in \mathcal{A}(\Omega)$, $\varepsilon > 0$, choose $V_\varepsilon \subset\subset V$ open such that

$$\mu(V \setminus \overline{V}_\varepsilon) < \varepsilon.$$

Then

$$\begin{aligned}\mu(V) &\le \varepsilon + \mu(\overline{V}_\varepsilon) \\ &= \varepsilon + \mu(\Omega) - \mu(\Omega \setminus \overline{V}_\varepsilon) \\ &\le \varepsilon + \mathcal{F}_\infty(u; \Omega) - \mathcal{F}_\infty(u; \Omega \setminus \overline{V}_\varepsilon) \\ &\le \varepsilon + \mathcal{F}_\infty(u; V),\end{aligned}$$

where we have used (3.7) and (3.6). □

Proof of Theorem 3.1. By Proposition 2.5 we have

$$\mathcal{F}(u) \ge \int_\Omega Q_T f(u, \nabla u)\, dx,$$

and since, clearly, $\mathcal{F}_\infty(u; \Omega) \ge \mathcal{F}(u)$, it remains to show that

$$\mathcal{F}_\infty(u; \Omega) \le \int_\Omega Q_T f(u, \nabla u)\, dx,$$

or, equivalently, and in view of Proposition 3.4,

$$\frac{d\mathcal{F}_\infty(u; \cdot)}{d\mathcal{L}^N}(x_0) \le Q_T f(u(x_0), \nabla u(x_0)) \text{ for } \mathcal{L}^N \text{ a.e. } x_0 \in \Omega.$$

Let $x_0 \in \Omega$ be a Lebesgue point for $u, \nabla u$, and such that

$$\frac{d\mathcal{F}_\infty}{d\mathcal{L}^N}(u; \Omega)(x_0) \text{ exists and is finite.}$$

Fix $\varepsilon > 0$ and let $\varphi \in W_0^{1,\infty}(\mathrm{Q}; T_{y_0}(\mathcal{M}))$ be extended periodically to $\mathbb{R}^N$, where $y_0 := u(x_0)$ and (see Definition 2.1)

$$Q_T f(u(x_0), \nabla u(x_0)) + \varepsilon \ge \int_{\mathrm{Q}} f(\nabla u(x_0) + \nabla\varphi(x))\, dx. \tag{3.8}$$

Let $\Pi : U(y_0) \to U(y_0) \cap \mathcal{M}$ be a C^1 projection of an open neighborhood $U(y_0)$ of y_0 onto $U(y_0) \cap \mathcal{M}$ (see (2.2)). Since f is locally uniformly continuous, given $\varepsilon > 0$ we may find $B(y_0, \delta_0) \subset\subset U(y_0)$ and $0 < \rho < 1$ such that

$$|\xi_1|, |\xi_2| \le C_2, \quad |\xi_1 - \xi_2| < \rho \Longrightarrow |f(\xi_1) - f(\xi_0)| < \varepsilon, \tag{3.9}$$

where

$$C_2 := \left(\|\Pi\|_{L^\infty(B(y_0,\delta_0))} + 1\right)\left(2 + 2|\nabla u(x_0)| + \|\nabla\varphi\|_{L^\infty(\mathrm{Q})}\right). \tag{3.10}$$

Choose $\delta_0 > \delta > 0$ such that

$$y, y' \in B(y_0, \delta) \cap \mathcal{M} \Longrightarrow \left|\nabla\Pi(y) - \nabla\Pi(y')\right| < \theta, \tag{3.11}$$

where $0 < \theta < 1$ has been selected so that

$$\begin{aligned} &2\theta\left(\|\nabla\Pi\|_{L^\infty(B(y_0,\delta_0))} + 1\right)(1 + |\nabla u(x_0)|)\left(1 + \|\nabla\varphi\|_{L^\infty(\mathrm{Q})}\right) \\ &\quad < \frac{\rho}{2}. \end{aligned} \tag{3.12}$$

Let $\eta \in C^\infty(\mathbb{R}^d; [0,1])$ be a cut-off function such that

$$\eta = \begin{cases} 1 & \text{in} \quad B(0, \delta/4), \\ 0 & \text{outside} \quad B(0, \delta/2), \end{cases}$$

with $\|\nabla\eta\|_\infty \le \frac{C}{\delta}$. We define

$$u_n(x) := \begin{cases} u(x) & \text{if} \quad |u(x) - y_0| \ge \frac{\delta}{2}, \\ \Pi(w_n) & \text{if} \quad |u(x) - y_0| < \frac{\delta}{2}, \end{cases}$$

where

$$w_n(x) := u(x) + \eta(u(x) - y_0)\frac{1}{n}\varphi(nx).$$

Note that u_n is well defined for

$$n > 2\frac{\|\varphi\|_{L^\infty(\mathrm{Q})}}{\delta},$$

because if $|u(x) - y_0| < \frac{\delta}{2}$ then

$$|w_n(x) - y_0| < \frac{\delta}{2} + \frac{1}{n}\|\varphi\|_{L^\infty(\mathrm{Q})} < \delta,$$

and so $w_n(x) \in U(y_0)$. Notice that

$$\begin{aligned}\|u_n - u\|_{L^\infty(\Omega)} &= \|u_n - u\|_{L^\infty(\{|u-y_0|\le\frac{\delta}{2}\})}\\ &\le \|\Pi\|_{W^{1,\infty}(B_m(y_0,\delta_0))}\frac{1}{n}\|\varphi\|_\infty\\ &\to 0\end{aligned}$$

as $n \to +\infty$. Therefore,

$$\|u_n - u\|_{L^\infty(\Omega)} \to 0 \quad \text{as} \quad n \to +\infty.$$

In addition, if

$$|u(x) - y_0| \le \frac{\delta}{2}$$

then

$$\begin{aligned}\nabla u_n(x) = \nabla\Pi(w_n)\Big[&\nabla u(x) + \eta(u(x)-y_0)\nabla\varphi(nx)\\ &+\frac{1}{n}\varphi(nx)\otimes\nabla\eta(u(x)-y_0)\nabla u(x)\Big],\end{aligned}$$

so that

$$\begin{aligned}|\nabla u_n(x)| \le\ &\|\nabla\Pi\|_{L^\infty(B_m(y_0,\delta_0))}\\ &\times\Big[|\nabla u(x_0)| + |\nabla u(x) - \nabla u(x_0)|\\ &+\|\nabla\varphi\|_{L^\infty(\mathrm{Q})} + \frac{1}{n}\|\varphi\|_{L^\infty(\mathrm{Q})}\|\nabla\eta\|_{L^\infty}\,|\nabla u(x)|\Big]\\ \le\ &\|\nabla\Pi\|_{L^\infty(B_m(y_0,\delta_0))}\,[2\,|\nabla u(x) - \nabla u(x_0)|\\ &+2\,|\nabla u(x_0)| + \|\nabla\varphi\|_{L^\infty(\mathrm{Q})}\Big]\end{aligned}\tag{3.13}$$

provided that

$$n > \max\left\{2\frac{\|\varphi\|_{L^\infty}}{\delta}, \quad \|\varphi\|_{L^\infty(\mathrm{Q})}\|\nabla\eta\|_{L^\infty}\right\}.$$

We conclude that

$$|\nabla u_n(x)| \le C(1 + |\nabla u(x) - \nabla u(x_0)|) \quad \text{a. e. } x \in \Omega,\tag{3.14}$$

and, as a consequence,

$$\left\{\|\nabla u_n\|_{L^p(\Omega)}\right\} \quad \text{is bounded.}$$

Setting

$$z_n := \nabla u(x_0)x + \frac{1}{n}\varphi(nx),$$

then

$$|\nabla z_n(x)| \leq |\nabla u(x_0)| + \|\nabla \varphi\|_{L^\infty(\mathrm{Q})}.$$

It follows from (3.10) and (3.13) that if $|\nabla u(x) - \nabla u(x_0)| < 1$ then

$$|\nabla u_n(x)|\,, |\nabla w_n(x)| \leq C_2. \tag{3.15}$$

Let

$$\gamma := \min\left\{1, \quad (|\nabla \Pi(y_0)| + 1)^{-1}\,\frac{\rho}{2}\right\},$$

and assume that

$$|u(x) - y_0| < \frac{\delta}{4}, \quad |\nabla u(x) - \nabla u(x_0)| < \gamma.$$

Then $\eta(u(x) - y_0) = 1$ and

$$\begin{aligned} |\nabla u_n(x) - \nabla z_n(x)| \ &\leq \ |\nabla \Pi(w_n)\nabla u(x) - \nabla u(x_0)| \\ &\quad + |\nabla \Pi(w_n)\nabla\varphi(nx) - \nabla\varphi(nx)| \\ &\leq \ |\nabla \Pi(w_n) - \nabla \Pi(y_0)|\,|\nabla u(x)| \\ &\quad + |\nabla \Pi(y_0)|\,|\nabla u(x) - \nabla u(x_0)| \\ &\quad + |\nabla \Pi(w_n) - \nabla \Pi(y_0)|\,\|\nabla\varphi\|_{L^\infty(\mathrm{Q})} \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

By (3.11)

$$I_1 \leq \theta(1 + |\nabla u(x_0)|),$$

while, by definition of γ,

$$I_2 \leq \frac{\rho}{2},$$

and once again by (3.11) we get

$$I_3 \leq \theta\|\nabla\varphi\|_{L^\infty(\mathrm{Q})}.$$

In view of (3.12) we deduce that

$$|\nabla u_n(x) - \nabla z_n(x)| < \rho. \tag{3.16}$$

We have

$$
\begin{aligned}
&\frac{d\mathcal{F}_\infty}{d\mathcal{L}^N}(u,\cdot)(x_0) = \lim_{r\to 0^+} \frac{\mathcal{F}_\infty(u;B(x_0,r))}{\mathcal{L}^N(B(x_0,r))} \\
&\leq \liminf_{r\to 0^+} \liminf_{n\to+\infty} \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} f(\nabla u_n)\,dx \\
&\leq \liminf_{r\to 0^+} \liminf_{n\to+\infty} \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)\cap\left\{|u(x)-y_0|\geq \frac{\delta}{4}\right\}} f(\nabla u_n)\,dx \qquad (3.17)\\
&+ \limsup_{r\to 0^+} \limsup_{n\to+\infty} \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)\cap\left\{|u(x)-y_0|<\frac{\delta}{4}\right\}\cap\{|\nabla u(x)-\nabla u(x_0)|<\gamma\}} f(\nabla u_n)\,dx \\
&+ \limsup_{r\to 0^+} \limsup_{n\to+\infty} \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)\cap\left\{|u(x)-y_0|<\frac{\delta}{4}\right\}\cap\{|\nabla u(x)-\nabla u(x_0)|\geq\gamma\}} f(\nabla u_n)\,dx \\
&=: J_1 + J_2 + J_3.
\end{aligned}
$$

By virtue of (3.15), (3.16) and (3.9), we obtain

$$
\begin{aligned}
J_2 &\leq \limsup_{r\to 0^+} \limsup_{n\to+\infty} \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} f(\nabla z_n)\,dx + \varepsilon \\
&= \limsup_{r\to 0^+} \limsup_{n\to+\infty} \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} f(\nabla u(x_0) + \nabla\varphi(nx))\,dx + \varepsilon \\
&= \int_Q f(\nabla u(x_0) + \nabla\varphi(y))\,dy + \varepsilon \\
&\leq Q_T f(u(x_0), \nabla u(x_0)) + 2\varepsilon, \qquad (3.18)
\end{aligned}
$$

where we have used (3.8). By (3.14) we have

$$
\begin{aligned}
J_1 &\leq \limsup_{r\to 0^+} \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)\cap\left\{|u(x)-y_0|\geq\frac{\delta}{4}\right\}} C(1 + |\nabla u(x) - \nabla u(x_0)|^p)\,dx \\
&\leq \limsup_{r\to 0^+} \frac{C}{|B(x_0,r)|} \int_{B(x_0,r)} |\nabla u(x) - \nabla u(x_0)|^p\,dx \qquad (3.19)\\
&\quad + \limsup_{r\to 0^+} \frac{4C}{\delta} \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} |u(x) - y_0|\;dx = 0,
\end{aligned}
$$

and, finally,

$$
\begin{aligned}
J_3 &\leq \limsup_{r\to 0^+} \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)\cap\{|\nabla u(x)-\nabla u(x_0)|\geq\gamma\}} C(1 + |\nabla u(x) - \nabla u(x_0)|^p)\,dx \\
&\leq \limsup_{r\to 0^+} \frac{C}{|B(x_0,r)|} \int_{B(x_0,r)} |\nabla u(x) - \nabla u(x_0)|^p\,dx \qquad (3.20)\\
&\quad + \limsup_{r\to 0^+} \frac{C}{\gamma} \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} |\nabla u(x) - \nabla u(x_0)|\;dx = 0.
\end{aligned}
$$

The conclusion follows from (3.17), (3.18), (3.19), and (3.20). □

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