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Cauchy–Dirichlet Problem for First Order Nonlinear Systems

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Let $\Omega \subset \mathbb{R}^n$ be open, $u : \Omega \to \mathbb{R}^m$ and thus the gradient matrix $Du \in \mathbb{R}^{m \times n}$. We let $E \subset \mathbb{R}^{m \times n}$ be compact and denote by $RcoE$ and $PcoE$ the rank one convex and polyconvex hull of E, respectively. We show that if $RcoE = PcoE$ (and two other hypotheses, named the segment property and the extreme points property) and if $\varphi \in C^1(\bar{\Omega}; \mathbb{R}^m)$ is such that

$$D\varphi(x) \in E \cup int\ PcoE, \qquad x \in \Omega$$

then there exists (a dense set of) $u \in W^{1,\infty}(\Omega; \mathbb{R}^m)$ such that

$$\begin{cases} Du(x) \in E, & \text{a.e. in } \Omega \\ u(x) = \varphi(x) & \text{on } \partial\Omega. \end{cases}$$

We apply this existence theorem to some relevant examples studied in the literature, as well as to problems with (x, u) dependence. © 1998 Academic Press

1. INTRODUCTION

We consider the Dirichlet problem

$$\begin{cases} F_1(Du) = \cdots = F_N(Du) = 0, \text{ a.e. in } \Omega \\ u = \varphi, \text{ on } \partial\Omega, \end{cases} \tag{1}$$

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where $\Omega \subset \mathbb{R}^n$ is open, $u: \Omega \to \mathbb{R}^m$ and therefore $Du \in \mathbb{R}^{m \times n}$, $F_i : \mathbb{R}^{m \times n} \to \mathbb{R}$, $1 \leqslant i \leqslant N$, and $\varphi \in C^1(\bar{\Omega}; \mathbb{R}^m)$ (or piecewise C^1). Below we will also consider the case where the F_i depend continuously on (x, u) as well. By letting

$$E = \{\xi \in \mathbb{R}^{m \times n}: F_1(\xi) = F_2(\xi) = \cdots = F_N(\xi) = 0\} \tag{2}$$

the problem (1) can be rewritten as

$$\begin{cases} Du(x) \in E, \text{ a.e. in } \Omega \\ u(x) = \varphi(x), \ x \in \partial\Omega. \end{cases} \tag{3}$$

In previous work (c.f. [17, 19]) we introduced a new method to prove existence of $W^{1,\infty}(\Omega; \mathbb{R}^m)$ solutions of the Dirichlet problem (3). In particular for the scalar case (i.e. $m = 1$) (3) has a solution if E is closed and

$$D\varphi(x) \in E \cup \mathit{int\ co}\ E, \qquad x \in \Omega, \tag{4}$$

where *int co* E denotes the interior of the convex hull of E. In some sense (4) is optimal for existence of $W^{1,\infty}$ solutions of (3). We also proved some results on the vectorial case but only with restrictive assumptions on the set E which are not always compatible with the natural convexity conditions of the vectorial case (see Example 1.1 below for a discussion about this point).

In this article we obtain an existence theorem in the vectorial case under general "quasiconvexity" conditions. Important applications are then considered in Section 5 (c.f. the two examples below).

Problem (1) has been intensively studied, essentially in the case $N = m = 1$ (i.e. the scalar case with one equation) in many relevant papers such as Lax [31], Douglis [22], Kruzkov [30], Crandall and Lions [15], Crandall, Evans and Lions [13], Capuzzo Dolcetta and Evans [10], Capuzzo Dolcetta and Lions [11], Crandall, Ishii and Lions [14]; as well as in books such as Rund [36], Benton [7], Lions [32], Fleming and Soner [24], Barles [6], Subbotin [37], and Bardi and Capuzzo Dolcetta [5]. In the above literature the existence of solutions is only part of the problem, the other issues are uniqueness, explicit formulas, maximality and so on. In this context the notion of viscosity solution, introduced by Crandall and Lions [15], plays a central role. Since we are considering existence for vectorial problems, we cannot use here this concept.

With the aim to state one of the main theorems we introduce the following definitions (see Section 2 for more details). Given $E \subset \mathbb{R}^{m \times n}$ we consider the class of functions

$$F_E = \{f: \mathbb{R}^{m \times n} \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}, f|_E = 0\}$$

and the sets

$$PcoE = \{\xi \in \mathbb{R}^{m \times n} \colon f(\xi) \leqslant 0, \forall f \in F_E, f \text{ polyconvex}\}, \tag{5}$$

$$RcoE = \{\xi \in \mathbb{R}^{m \times n} \colon f(\xi) \leqslant 0, \forall f \in F_E, f \text{ rank one convex}\}, \tag{6}$$

which are respectively the *polyconvex* and *rank one convex hulls* of E. Since f convex $\Rightarrow f$ polyconvex $\Rightarrow f$ rank one convex, then

$$E \subset RcoE \subset PcoE \subset coE.$$

These hulls are in general all different.

Let us introduce two more definitions (for more precise definitions see Section 4).

Let $K \subset \mathbb{R}^{m \times n}$ be a rank one convex set (i.e. $K = RcoK$). We say that K satisfies the *segment property* if for every rank one segment contained in K there exists an arbitrarily close parallel segment contained in the interior of K. Note that if K were convex with non empty interior, this property would always hold.

Finally, let $E \subset \mathbb{R}^{m \times n}$. We say that E satisfies the (polyconvex) *extreme points property* if no points of E can be expressed as a polyconvex combination of other points of E.

One of our main existence results is the following (c.f. Corollary 4.2)

THEOREM 1.1. *Let Ω be an open set of $\mathbb{R}^n$. Let E be a compact set of $\mathbb{R}^{m \times n}$ such that*

$$PcoE = RcoE. \tag{7}$$

Assume also that E satisfies the extreme points property and that $RcoE$ satisfies the segment property. Finally let $\varphi \in C^1(\bar{\Omega}; \mathbb{R}^m)$ (or piecewise C^1) such that

$$D\varphi(x) \in E \cup int\ RcoE, \qquad \forall x \in \Omega. \tag{8}$$

Then there exists (a dense set of) $u \in W^{1,\infty}(\Omega; \mathbb{R}^m)$ such that

$$\begin{cases} Du(x) \in E, \text{ a.e. in } \Omega \\ u(x) = \varphi(x), \ x \in \partial\Omega. \end{cases}$$

Let us note that the class of admissible boundary data φ is larger than the set of piecewise C^1 function and is described in Section 3.

The proof of the theorem uses similar ideas as those of [17, 19]. The principal ingredients are: (i) Baire category method, introduced in the context of ordinary differential equations by Cellina [12] and De Blasi and Pianigiani [20, 21]; (ii) relaxation methods for integrals of the calculus of variations; (iii) semicontinuity theorems for quasiconvex integrals.

The above theorem is then applied to the following examples, discussed in Section 5.

EXAMPLE 1.1. For $\xi \in \mathbb{R}^{2\times 2}$ we denote by $\lambda_1(\xi), \lambda_2(\xi)$ $(0 \leqslant \lambda_1 \leqslant \lambda_2)$ the singular values of the matrix ξ, that is the eigenvalues of the matrix $(\xi^t \xi)^{1/2}$. Recall that, for every $\xi \in \mathbb{R}^{2\times 2}$,

$$\begin{cases} (\lambda_1(\xi))^2 + (\lambda_2(\xi))^2 = |\xi|^2 \\ \lambda_1(\xi) \cdot \lambda_2(\xi) = |\det \xi|. \end{cases}$$

Let $0 < a_1 \leqslant a_2$ and let

$$E = \{\xi \in \mathbb{R}^{2\times 2} : \lambda_1(\xi) = a_1, \lambda_2(\xi) = a_2\};$$

we prove in Section 5 that

$$coE = \{\xi \in \mathbb{R}^{2\times 2} : \lambda_2(\xi) \leqslant a_2, \lambda_1(\xi) + \lambda_2(\xi) \leqslant a_1 + a_2\},$$

$$PcoE = RcoE = \{\xi \in \mathbb{R}^{2\times 2} : \lambda_2(\xi) \leqslant a_2, \lambda_1(\xi) \cdot \lambda_2(\xi) \leqslant a_1 \cdot a_2\}.$$

Note that the three hulls are the same if and only if $a_1 = a_2$. This is the main reason we could treat the case $a_1 = a_2$ in [17, 19].

We prove in Section 5 that E satisfies all the hypotheses of the theorem and therefore, for every $\varphi \in C^1(\bar{\Omega}; \mathbb{R}^2)$ (or piecewise C^1) such that

$$\lambda_2(D\varphi(x)) < a_2, \qquad \lambda_1(D\varphi(x)) \cdot \lambda_2(D\varphi(x)) < a_1 a_2, \qquad \forall x \in \Omega,$$

there exists (a dense set of) $u \in W^{1,\infty}(\Omega; \mathbb{R}^2)$ such that

$$\begin{cases} \lambda_1(Du(x)) = a_1, & \text{a.e. } x \in \Omega \\ \lambda_2(Du(x)) = a_2, & \text{a.e. } x \in \Omega \\ u(x) = \varphi(x), & x \in \partial\Omega. \end{cases}$$

EXAMPLE 1.2. The problem of potential wells, introduced by Ball and James [3, 4], is important in nonlinear elasticity. Let us briefly discuss the case of two potential wells in $\mathbb{R}^2$.

Let $\xi, \eta \in \mathbb{R}^{2\times 2}$, with $0 < \det \xi < \det \eta$. Applying our theorem we prove that there exists (a dense set of) $u \in W^{1,\infty}(\Omega; \mathbb{R}^2)$ satisfying

$$\begin{cases} Du(x) \in E = SO(2)\,\xi \cup SO(2)\,\eta, & \text{a.e. } x \in \Omega \\ u(x) = \varphi(x), & x \in \partial\Omega, \end{cases} \tag{9}$$

where $SO(2)$ denotes the set of rotations in $\mathbb{R}^{2\times 2}$, provided that the two wells $SO(2)\,\xi$ and $SO(2)\,\eta$ are rank one connected and that $\varphi \in C^1(\bar{\Omega}; \mathbb{R}^2)$ (or piecewise C^1) satisfies

$$D\varphi(x) \in E \cup int\ RcoE, \qquad \forall x \in \Omega.$$

The explicit analytic form of $RcoE = PcoE$ is given in Section 5 and has been obtained by Sverak [38].

This result, apart from the density, has also been obtained by Müller and Sverak [34] using the method of convex integration introduced by Gromov [26].

We already mentioned that our existence result for the scalar case in [17], [19] is optimal. We now discuss in which sense our Theorem 1.1 is close to an optimal existence result also for the vectorial case.

Let us first introduce, for a set $E \subset \mathbb{R}^{m\times n}$,

$$\overline{QcoE} = \{\xi \in \mathbb{R}^{m\times n} \colon f(\xi) \leqslant 0,\ \forall f \colon \mathbb{R}^{m\times n} \to \mathbb{R},\ f \text{ quasiconvex},\ f \mid E = 0\}, \quad (10)$$

which we call (closure of the) *quasiconvex hull* of E (note that in the definition (10) f takes only finite values, contrary to the definitions of the polyconvex and rank convex envelopes given above). Since f polyconvex $\Rightarrow$ f quasiconvex $\Rightarrow f$ rank one convex, we have

$$\overline{RcoE} \subset \overline{QcoE} \subset \overline{PcoE}. \quad (11)$$

A natural guess for the optimal result in the vectorial case is the following

Conjecture. Let E be a closed subset of $\mathbb{R}^{m\times n}$. The conclusion of the theorem holds provided that $\varphi \in C^1(\bar{\Omega}; \mathbb{R}^m)$ (or piecewise C^1), with

$$D\varphi(x) \in E \cup int\ \overline{QcoE}, \qquad \forall x \in \Omega.$$

The Conjecture is true if either $m = 1$ (as already mentioned) or $N = 1$ (c.f. Theorem 4.4 or Theorem 1.2 below) in (1) with F quasiconvex and E compact (in this case $E \cup int\ \overline{QcoE} = \{\xi \in \mathbb{R}^{m\times n} \colon F(\xi) \leqslant 0\}$).

In Section 6 we will consider, in (1), functions $F_i = F_i(x, s, \xi)$, with $i = 1, ..., N$, and we will present a method of reduction of the (x, u) dependence to the case independent of (x, u). In particular this will allow us to treat the above vectorial Example 1.1 with $a_i = a_i(x, u)$, $i = 1, 2$.

In the scalar case $m = 1$ our general theorem will apply to a generalization of the classical eikonal equation (namely we will treat $|\partial u/\partial x_i| = a_i(x, u)$, $i = 1, ..., n$). In the scalar case an existence result has also been obtained by Bressan and Flores [9].

When $N=1$ we will obtain (c.f. Theorem 6.2) the following

THEOREM 1.2. *Let* $\Omega \subset \mathbb{R}^n$ *be open. Let* $F: \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$ *be continuous, quasiconvex in the last variable and such that* $\{\xi \in \mathbb{R}^{m\times n}: F(x, s, \xi) \leqslant 0\}$ *is compact for every* $(x, s) \in \Omega \times \mathbb{R}^m$. *Let* $\varphi \in C^1(\bar{\Omega}; \mathbb{R}^m)$ *(or piecewise* C^1*) satisfy*

$$F(x, \varphi(x), D\varphi(x)) \leqslant 0, \qquad \text{for every} \quad x \in \Omega.$$

Then there exists (a dense set of) $u \in W^{1,\infty}(\Omega; \mathbb{R}^m)$ *such that*

$$\begin{cases} F(x, u(x), Du(x)) = 0, & \text{a.e. } x \in \Omega \\ u(x) = \varphi(x), & x \in \partial\Omega. \end{cases}$$

In particular if $m=1$ (quasiconvexity is then equivalent to convexity) we recover the classical existence results (c.f. the literature on viscosity solutions above) with however weaker conditions on the u variable.

The results presented in this paper have been announced in [18].

2. DIFFERENT CONVEX HULLS AND THEIR PROPERTIES

We start with the following definitions.

DEFINITION 2.1. Let $E \subset \mathbb{R}^{m\times n}$ and

$$F_E = \{f\colon \mathbb{R}^{m\times n} \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\},\ f|_E = 0\}.$$

Define

$$coE = \{\xi \in \mathbb{R}^{m\times n}\colon f(\xi) \leqslant 0, \forall f \in F_E, f \text{ convex}\},$$

called the convex hull of E;

$$PcoE = \{\xi \in \mathbb{R}^{m\times n}\colon f(\xi) \leqslant 0, \forall f \in F_E, f \text{ polyconvex}\},$$

called the polyconvex hull of E;

$$RcoE = \{\xi \in \mathbb{R}^{m\times n}\colon f(\xi) \leqslant 0, \forall f \in F_E, f \text{ rank one convex}\},$$

called the rank one convex hull of E.

DEFINITION 2.2. Let $E \subset \mathbb{R}^{m \times n}$. Define

$$\overline{QcoE} = \{\xi \in \mathbb{R}^{m \times n} : f(\xi) \leqslant 0, \forall f : \mathbb{R}^{m \times n} \to \mathbb{R}, \text{ quasiconvex and } f|_E = 0\},$$

called the (closure of the) quasiconvex hull of E.

Remark 2.1. (i) We should draw the attention to the fact that in the first definition we admit functions taking the value $+\infty$, while we exclude this possibility in the second. We could have adopted in Definition 2.1 functions with finite values only, we would have obtained in (i) and (ii) the closure of the convex hull and the closure of the polyconvex hull of E (since f is continuous). On the contrary, functions f with value in $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$, seem not to be consistent with Definition 2.2, indeed if $E = \{\xi_1, \xi_2\} \subset \mathbb{R}^{m \times n}$, then $QcoE$ would always be E, independently of the fact that $\xi_1 - \xi_2$ is rank one or not.

(ii) The definition of rank one convex hull that we adopted is sometimes called lamination convex hull of E. It is important in this case to consider functions with values in $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$. Indeed if, for example, $E = \{\xi_1, \xi_2, \xi_3, \xi_4\} \subset \mathbb{R}^{2 \times 2}$ is the set given by Casadio (c.f. Example 2 page 116 in [16]), or the similar one by Tartar [39], then $RcoE = E$; however if in the definition of $RcoE$ we allowed only functions with finite values, we would have found a larger set.

The following proposition is a direct consequence of the above definition.

PROPOSITION 2.1. *Let $E \subset \mathbb{R}^{m \times n}$, then*

$$E \subset RcoE \subset PcoE \subset coE.$$

Remark 2.2. One can also easily prove that

$$\bar{E} \subset \overline{RcoE} \subset \overline{QcoE} \subset \overline{PcoE} \subset \overline{coE}.$$

Indeed to prove that $\overline{RcoE} \subset \overline{QcoE}$, we just observe that

$$RcoE \subset \{\xi \in \mathbb{R}^{m \times n} : f(\xi) \leqslant 0, \forall f : \mathbb{R}^{m \times n} \to \mathbb{R},$$
$$f \text{ rank one convex and } f|_E = 0\}.$$

Since finite valued rank one convex functions are continuous, the set in the right hand side is closed. Quasiconvex functions being rank one convex, we deduce the result. The other inclusion $\overline{QcoE} \subset \overline{PcoE}$ will follow by considering finite valued f in Definition 2.1. Since we will not use this result in the sequel, we do not enter into details.

The next proposition is a consequence of Carathéodory theorem.

PROPOSITION 2.2. *Let $E \subset \mathbb{R}^{m \times n}$, then the following representation hold*

$$coE = \left\{ \xi \in \mathbb{R}^{m\times n} : \xi = \sum_{i=1}^{mn+1} t_i \xi_i, \xi_i \in E, t_i \geqslant 0 \text{ with } \sum_{i=1}^{mn+1} t_i = 1 \right\} \tag{12}$$

$$PcoE = \left\{ \xi \in \mathbb{R}^{m\times n} : T(\xi) = \sum_{i=1}^{\tau+1} t_i T(\xi_i), \xi_i \in E, t_i \geqslant 0 \text{ with } \sum_{i=1}^{\tau+1} t_i = 1 \right\} \tag{13}$$

where

$$T(\xi) = (\xi, adj_2 \xi, ..., adj_{m \wedge n} \xi) \in \mathbb{R}^{\tau(m, n)},$$

$$\tau(m, n) = \sum_{s=1}^{m \wedge n} \binom{m}{s} \binom{n}{s}$$

and $adj_s \xi$ stands for the matrix of all $s \times s$ minors of the matrix $\xi \in \mathbb{R}^{m \times n}$ (if $m = n = 2$, then $\tau = 5$ and $T(\xi) = (\xi, \det \xi)$).

Furthermore if E is compact then coE and $PcoE$ are also compact.

Proof. (1) Our definition of coE corresponds to the standard definition of convex hull of E, i.e. the smallest convex set containing E. Carathéodory theorem implies then immediately the result.

(2) Denote by Y the set in the right hand side of (13). The fact that $Y \subset PcoE$ is just the definition of polyconvex functions. The reverse inclusion is a direct consequence of Theorem 1.1 page 201 in [16] when applied to χ_E, the characteristic function of E, i.e. $P\chi_E = \chi_Y$, where $P\chi_E$ denotes the polyconvex envelope of χ_E.

(3) The fact that $PcoE$ is bounded is trivial. We now show that it is closed (the proof for coE is similar). So let $\xi^\nu \to \xi$, $\xi^\nu \in PcoE$, we have to prove that $\xi \in PcoE$. By the preceding step this is equivalent to showing that

$$P\chi_E(\xi) = 0.$$

Since $\xi^\nu \in PcoE$ we have

$$P\chi_E(\xi^\nu) = 0 = \inf \left\{ \sum_{i=1}^{\tau+1} \lambda_i^\nu \chi_E(\xi_i^\nu) : \sum_{i=1}^{\tau+1} \lambda_i^\nu T(\xi_i^\nu) = T(\xi^\nu) \right\}.$$

E being bounded and $\lambda_i^\nu \in [0, 1]$, we deduce that up to the extraction of a subsequence, $\xi_i^\nu \to \xi_i$, $\lambda_i^\nu \to \lambda_i$. Since E is closed we deduce that $\xi_i \in E$. By continuity we also have $\sum_{i=1}^{\tau+1} \lambda_i T(\xi_i) = T(\xi)$ and thus we deduce that $P\chi_E(\xi) = 0$, which is the claimed result. ∎

The following proposition is a weaker version of the previous result when applied to rank one convex hulls.

PROPOSITION 2.3. *Let* $E \subset \mathbb{R}^{m\times n}$ *and define by induction*

$$R_o coE = E$$
$$R_{i+1} coE = \{\xi \in \mathbb{R}^{m\times n} : \xi = tA + (1-t)\, B,\ t \in [0,1],$$
$$A, B \in R_i coE,\ rank\{A - B\} = 1\}.$$

Then $RcoE = \bigcup_{i\in\mathbb{N}} R_i coE$.

Proof. By induction $R_i coE \subset RcoE$ and thus $\bigcup R_i coE \subset RcoE$. We now show the reverse inclusion. We first recall the construction of rank one convex envelope of a given function $f: \mathbb{R}^{m\times n} \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ (c.f. Kohn–Strang [29]). Define by induction

$$R_o f = f$$
$$R_{i+1} f(\xi) = \inf\{t R_i f(A) + (1-t)\, R_i f(B) : t \in [0,1],$$
$$\xi = tA + (1-t)\, B,\ rank\{A - B\} = 1\}.$$

We then get that the rank one convex envelope of f is given by

$$Rf(\xi) = \inf_{i\in\mathbb{N}} R_i f(\xi).$$

Since this infimum is equal to a limit, we deduce that Rf is the maximal rank one convex function below f.

We apply this result to χ_E, the characteristic function of E. We observe that by induction

$$R_i \chi_E = \chi_{R_i coE},$$

thus

$$R\chi_E = \chi_{\bigcup R_i coE}.$$

Since $R\chi_E$ is a rank one convex function and $R\chi_E|_E = 0$ we deduce that, for $\xi \in RcoE$, $R\chi_E(\xi) = 0$. Then the above identity implies that $\xi \in \bigcup_{i\in\mathbb{N}} R_i coE$. ∎

3. THE SPACE $A^{1,p}(\Omega; K)$

We say that $u \in W^{1,\infty}(\Omega; \mathbb{R}^m)$ is a *piecewise affine function* on an open set $\Omega \subset \mathbb{R}^n$, if there exists a partition of Ω into a countable family of disjoint open sets Ω_i such that

$$\begin{cases} meas\left(\Omega - \bigcup_{i\in\mathbb{N}} \Omega_i\right) = 0 \\ Du(x) \text{ is constant in } \Omega_i, \text{ for every } i \in \mathbb{N}. \end{cases}$$

DEFINITION 3.1. (i) Let $\Omega \subset \mathbb{R}^n$ be open and let $K \subset \mathbb{R}^{m \times n}$. Let $1 \leqslant p < \infty$. We denote by $A^{1,p}(\Omega; K)$ the closure in the $W^{1,p}$ norm of the set of piecewise affine functions on Ω, whose gradient is compactly contained in *intK* a.e.

(ii) We let $A_o^{1,p}(\Omega; K) = A^{1,p}(\Omega; K) \cap W_o^{1,p}(\Omega; \mathbb{R}^m)$.

Remark 3.1. (i) If $K = \mathbb{R}^{m \times n}$, then it is easy to see that $A^{1,p} = W^{1,p}$.

(ii) Classical finite element methods show that if $u \in C^1(\bar{\Omega}; \mathbb{R}^m)$ (or piecewise C^1) and $Du(x)$ is compactly contained in *intK*, then $u \in A^{1,p}(\Omega; K)$.

(iii) Since our definition of piecewise affine functions can involve infinitely many affine pieces, iterating our procedure in Lemma 6.1 in [19] (c.f. also the proof of Proposition 3.2 below), we can assume that for every $u \in A^{1,p}(\Omega; K)$ the approximate piecewise approximation of u has the same boundary datum as u.

PROPOSITION 3.1. *If in addition K is bounded then*

$$A^{1,p}(\Omega; K) \subset W^{1,\infty}(\Omega; \mathbb{R}^m)$$

and is independent of $p \in [1, \infty)$.

Proof. Weak differentiability of piecewise affine functions passes to the limit since the gradients are a priori uniformly bounded. ∎

PROPOSITION 3.2. *Let* $K \subset \mathbb{R}^{m \times n}$ *be compact and convex. Let* $u \in W^{1,\infty}(\Omega; \mathbb{R}^m)$ *with gradient compactly contained in intK; then* $u \in A^{1,p}(\Omega; K)$ *and the approximating sequence of piecewise affine function can be chosen to be equal to u on* $\partial\Omega$.

Proof. Let $u \in W^{1,\infty}(\Omega; \mathbb{R}^m)$ and $\delta > 0$ such that

$$dist(Du(x), \partial K) \geqslant \delta \qquad \text{a.e. } x \in \Omega.$$

Let $\varepsilon > 0$, we wish to find v piecewise affine in Ω with

$$\begin{cases} v = u & \text{on } \partial\Omega \\ \|v - u\|_{W^{1,p}} \leqslant \varepsilon & \\ Dv(x) \in K & \text{a.e. in } \Omega \\ dist(Dv(x), \partial K) \geqslant \dfrac{\delta}{2} & \text{a.e. } x \in \Omega. \end{cases}$$

We apply Lemma 6.1 of [19] and get for every $\delta_1 < \delta$, $u_1 \in W^{1,\infty}(\Omega; \mathbb{R}^m)$ and $\Omega_1 \subset \Omega$ an open set with Lipschitz boundary such that

$$\begin{cases} meas(\Omega - \bar{\Omega}_1) \leqslant \varepsilon \\ u_1 = u \text{ on } \partial\Omega \\ u_1 \text{ is piecewise affine in } \Omega_1 \\ \| u - u_1 \|_{W^{1,p}} \leqslant \varepsilon \\ Du_1(x) \in K \text{ a.e. } x \in \Omega \\ dist(Du_1(x), \partial K) \geqslant \delta_1 \text{ a.e. } x \in \Omega. \end{cases}$$

We iterate the procedure and replace Ω by $\Omega - \bar{\Omega}_1$, u by u_1, δ by δ_1 and get for every $\delta_2 < \delta_1$, $\Omega_2 \subset \Omega - \bar{\Omega}_1$ open set with Lipschitz boundary and $u_2 \in W^{1,\infty}(\Omega - \bar{\Omega}_1; \mathbb{R}^m)$ such that

$$\begin{cases} meas(\Omega - (\bar{\Omega}_1 \cup \bar{\Omega}_2)) \leqslant \dfrac{\varepsilon}{2} \\ u_2 = u_1 \text{ on } \partial(\Omega - \bar{\Omega}_1) \\ u_2 \text{ is piecewise affine in } \Omega_2 \\ \| u_2 - u_1 \|_{W^{1,p}} \leqslant \dfrac{\varepsilon}{2} \\ Du_2(x) \in K \text{ a.e. } x \in \Omega - \bar{\Omega}_1 \\ dist(Du_2(x), \partial K) \geqslant \delta_2 \text{ a.e. } x \in \Omega - \bar{\Omega}_1 \end{cases}$$

We again iterate the procedure and find for $\delta_{i+1} < \delta_i$, $\Omega_{i+1} \subset \Omega - \bigcup_{k=1}^{i} \bar{\Omega}_k$ and $u_{i+1} \in W^{1,\infty}(\Omega - \bigcup_{k=1}^{i} \bar{\Omega}_k; \mathbb{R}^m)$ such that

$$\begin{cases} meas\left(\Omega - \bigcup\limits_{k=1}^{i} \bar{\Omega}_k\right) \leqslant \dfrac{\varepsilon}{i} \\ u_{i+1} = u_i \text{ on } \partial\left(\Omega - \bigcup\limits_{k=1}^{i} \bar{\Omega}_k\right) \\ u_{i+1} \text{ is piecewise affine in } \Omega_{i+1} \\ \| u_{i+1} - u_i \|_{W^{1,p}} \leqslant \dfrac{\varepsilon}{2^i} \\ Du_{i+1}(x) \in K, \text{ a.e. } x \in \Omega - \bigcup\limits_{k=1}^{i} \bar{\Omega}_k \\ dist(Du_{i+1}(x), \partial K) \geqslant \delta_{i+1} \text{ a.e. } x \in \Omega - \bigcup\limits_{k=1}^{i} \bar{\Omega}_k. \end{cases}$$

Choosing a decreasing sequence $\delta_i \to \delta/2$ and letting $v(x) = u_i(x)$ for $x \in \Omega_i$ we get the claimed density result. ∎

4. EXISTENCE THEOREMS

Before stating our main theorems, we introduce the two following definitions.

DEFINITION 4.1. We say that a rank one convex set $K \subset \mathbb{R}^{m \times n}$ (i.e. $RcoK = K$) satisfies the segment property if for every $\xi, \eta \in K$ with $rank\{\xi - \eta\} = 1$, the following two conditions hold:

(i) $(\xi, \eta) \cap int\, K \neq \phi \Rightarrow (\xi, \eta) \subset int\, K$;

(ii) for every sequence $\{\theta_\nu\} \subset K$ convergent to a point θ of the segment (ξ, η), there exists a sequence of segments $(\xi_\nu, \eta_\nu) \subset K$ parallel to (ξ, η) such that $\theta_\nu \in (\xi_\nu, \eta_\nu)$ and $\xi_\nu \to \xi$, $\eta_\nu \to \eta$.

Remark 4.1. (i) (ξ, η) denotes the open segment $t \to (1-t)\,\xi + t\eta$, $t \in (0, 1)$ and *int K* denotes the interior of K. Moreover by (ξ_ν, η_ν) parallel to (ξ, η) we mean that there exists $s_\nu \in \mathbb{R}$, $s_\nu \neq 0$, such that $\xi_\nu - \eta_\nu = s_\nu(\xi - \eta)$; instead of requiring that the segments are parallel we could have required only that rank $\{\xi_\nu - \eta_\nu\} = 1$.

(ii) If K is convex, then the above property always holds.

(iii) If rank one convexity is replaced by separate convexity, in the above definition, then one can produce examples of sets which have not the segment property.

(iv) As mentioned in the acknowledgments, our original definition of the segment property (which was only (i) in the above definition) is presumably too weak to ensure the validity of Theorem 4.1 and Lemma 4.6.

(v) In some cases it might be more convenient to check an apparently weaker condition than (i) of the segment property (c.f. [18]), namely

$$\begin{cases} \text{If } rank\{\xi - \eta\} = 1,\ \xi, \eta \in K,\ (\xi, \eta) \cap int\, K \neq \phi \\ \text{then there exist } \xi_\nu, \eta_\nu \in (\xi, \eta) \cap int\, K \text{ with } \xi_\nu \to \xi,\ \eta_\nu \to \eta. \end{cases}$$

We prove below that these notions are equivalent.

Proof (Remark 4.1(v)). It is clear that the segment property implies the above one. We wish to show the converse; so we assume

$$rank\{\xi - \eta\} = 1, \qquad (\xi, \eta) \cap int\, K \neq \phi, \qquad \xi, \eta \in K$$

and we have to prove that

$$\theta \in (\xi, \eta) \Rightarrow \theta \in int\, K.$$

By hypothesis we can find

$$\begin{cases} \xi_\nu, \eta_\nu \in (\xi, \eta) \cap int\, K, & \xi_\nu \to \xi, \quad \eta_\nu \to \eta \\ \theta \in (\xi_\nu, \eta_\nu) \text{ i.e. } \theta = t_\nu \eta_\nu + (1 - t_\nu)\, \xi_\nu \end{cases}.$$

Fix ν. Since $\xi_\nu, \eta_\nu \in int\, K$ we can find $\varepsilon > 0$ sufficiently small so that

$$B_\varepsilon(\xi_\nu),\ B_\varepsilon(\eta_\nu) \subset int\, K,$$

where $B_\varepsilon(\xi)$ stands for the ball centered at ξ and of radius ε.

We claim that $B_\varepsilon(\theta) \subset K$ and hence the result $\theta \in int\, K$. So let $\lambda \in B_\varepsilon(\theta)$ and write

$$\lambda = \theta + (\lambda - \theta) = t_\nu(\eta_\nu + (\lambda - \theta)) + (1 - t_\nu)(\xi_\nu + (\lambda - \theta)).$$

Observe that $rank\{(\xi_\nu + (\lambda - \theta)) - (\eta_\nu + (\lambda - \theta))\} = rank\{\xi_\nu - \eta_\nu\} = 1$. Furthermore since $\lambda \in B_\varepsilon(\theta)$ we deduce that

$$\xi_\nu + \lambda - \theta \in B_\varepsilon(\xi_\nu) \subset K, \qquad \eta_\nu + \lambda - \theta \in B_\varepsilon(\eta_\nu) \subset K.$$

K being rank one convex we deduce that $\lambda \in K$ i.e. $B_\varepsilon(\theta) \subset K$ and thus $\theta \in int\, K$. ▌

We now introduce a weaker assumption than the preceding one

DEFINITION 4.2. We say that a set $E \subset \mathbb{R}^{m \times n}$ has the (rank one) approximation property if, for every $\xi \in int\, RcoE$ and for every $\delta > 0$ there exists a closed set $E_\delta \subset int\, RcoE$, with $\xi \in RcoE_\delta$ and $dist(\eta, E) < \delta$ for every $\eta \in E_\delta$.

Remark 4.2. (i) The above definition is similar to a definition adopted by Müller-Sverak in [34] and called in-approximation.

(ii) The above segment property implies the (rank one) approximation property; the converse is false if rank one convexity is replaced by separate convexity.

Proof (Remark 4.2(ii)). Let $\xi \in \text{int}\, RcoE$. Since $\xi \in RcoE$, Proposition 2.3 implies that there exists $i \in \mathbb{N}$ such that $\xi \in R_i coE$. For simplicity of notations we consider only the case $i = 2$, the others being similar.

Step 1: There exist $A_1, A_2 \in R_1 coE$, with $rank\{A_1 - A_2\} = 1$, $\lambda_1, \lambda_2 \geqslant 0$, with $\lambda_1 + \lambda_2 = 1$ such that

$$\xi = \lambda_1 A_1 + \lambda_2 A_2.$$

By the segment property (i) and Remark 4.1(v) we can find a segment $(A_1^\varepsilon, A_2^\varepsilon)$ such that $\xi \in (A_1^\varepsilon, A_2^\varepsilon) \subset (A_1, A_2) \cap int\ RcoE$ and

$$\xi = \lambda_1^\varepsilon A_1^\varepsilon + \lambda_2^\varepsilon A_2^\varepsilon$$

for some $\lambda_1^\varepsilon, \lambda_2^\varepsilon \geqslant 0$, $\lambda_1^\varepsilon + \lambda_2^\varepsilon = 1$; moreover $A_i^\varepsilon \to A_i$.

Step 2: Since $A_1, A_2 \in R_1 coE$ there exist $A_{i,j} \in E$, with

$$rank\{A_{1,1} - A_{1,2}\} = rank\{A_{2,1} - A_{2,2}\} = 1,$$

$$\begin{cases} A_1 = \lambda_{1,1} A_{1,1} + \lambda_{1,2} A_{1,2} \\ A_2 = \lambda_{2,1} A_{2,1} + \lambda_{2,2} A_{2,2} \end{cases}.$$

By segment property (ii) we can find $A_{i,j}^\varepsilon$, with $A_i^\varepsilon \in (A_{i,1}^\varepsilon, A_{i,2}^\varepsilon) \subset int\ RcoE$ (in the last inclusion we have used the segment property (i)), with $rank\{A_{i,1}^\varepsilon - A_{i,2}^\varepsilon\} = 1$; namely

$$A_i^\varepsilon = \lambda_{i,1}^\varepsilon A_{i,1}^\varepsilon + \lambda_{i,2}^\varepsilon A_{i,2}^\varepsilon.$$

Letting $E_\delta = \{A_{1,1}^\varepsilon, A_{1,2}^\varepsilon, A_{2,1}^\varepsilon, A_{2,2}^\varepsilon\}$, we get that $E_\delta \subset int\ RcoE$, with $\xi \in R_2 coE_\delta \subset RcoE_\delta$ and $dist(A_{i,j}^\varepsilon, E) < \delta$ for ε sufficiently small. ∎

DEFINITION 4.3. We say that a set $E \subset \mathbb{R}^{m \times n}$ satisfies the (polyconvex) extreme points property if

$$\left.\begin{matrix} E_1 \subset E \\ E_1 \neq E \end{matrix}\right\} \Rightarrow PcoE_1 \neq PcoE$$

Remark 4.3. (i) In view of Proposition 2.2 this definition means that, if $\xi \in E$ can be written as

$$T(\xi) = \sum_{i=1}^{\tau+1} t_i T(\xi_i), \qquad \xi_i \in E, \quad t_i > 0 \quad \text{with} \quad \sum_{i=1}^{\tau+1} t_i = 1,$$

then necessarily $\xi_i = \xi$.

(ii) In particular the (polyconvex) extreme points property is satisfied if no point of E can be expressed as a non trivial convex combination of other points of E. In other words if points of E are extreme points (in the classical sense) of coE, then automatically the (polyconvex) extreme points property is satisfied.

We now have the first existence theorem.

THEOREM 4.1. *Let $\Omega \subset \mathbb{R}^n$ be open. Let $E \subset \mathbb{R}^{m \times n}$ be compact and such that*

$$PcoE = RcoE, \tag{14}$$

RcoE has the (rank one) approximation property, (15)

E has the (polyconvex) extreme points property. (16)

Let $\varphi \in A^{1,p}(\Omega; PcoE)$, then there exists (a dense set of) $u \in A^{1,p}(\Omega; PcoE)$ such that

$$\begin{cases} Du(x) \in E, \text{ a.e. in } \Omega \\ u(x) = \varphi(x), \; x \in \partial\Omega. \end{cases} \tag{17}$$

Remark 4.4 The solution u of (17) belongs to $W^{1,\infty}(\Omega; \mathbb{R}^m)$; c.f. Proposition 3.1.

The theorem has as a consequence the following

COROLLARY 4.2. *Let $\Omega \subset \mathbb{R}^n$ be open. Let $E \subset \mathbb{R}^{m \times n}$ be compact and such that*

$$PcoE = RcoE.$$

Assume also that RcoE has the (rank one) approximation property and E has the (polyconvex) extreme points property. Finally let $\varphi \in C^1(\bar{\Omega}; \mathbb{R}^m)$ (or piecewise C^1) such that

$$D\varphi(x) \in E \cup \text{int } RcoE, \qquad \forall x \in \Omega. \tag{18}$$

Then there exists (a dense set of) $u \in W^{1,\infty}(\Omega; \mathbb{R}^m)$ satisfying

$$\begin{cases} Du(x) \in E, & \text{a.e. } x \in \Omega \\ u(x) = \varphi(x), & x \in \partial\Omega. \end{cases}$$

Remark 4.5. By Remark 4.2 we see that Theorem 1.1 is then a corollary of the above one.

A particular case of interest is when E can be expressed as the set of zeroes of some quasiconvex functions F_i, $i = 1, ..., N$, as in (1).

THEOREM 4.3. *Let $\Omega \subset \mathbb{R}^n$ be open. Let $F_i^\delta: \mathbb{R}^{m \times n} \to \mathbb{R}$, $i = 1, ..., N$, be quasiconvex and continuous with respect to $\delta \in [0, \delta_0)$, for some $\delta_0 > 0$. Let us assume that*

(i) $$\text{int } Rco\{\xi \in \mathbb{R}^{m \times n}: F_i^\delta(\xi) = 0, i = 1, ..., N\}$$
$$= \{\xi \in \mathbb{R}^{m \times n}: F_i^\delta(\xi) < 0, i = 1, ..., N\} \qquad \forall \delta \in [0, \delta_0)$$

and it is bounded;

(ii) $\{\xi \in \mathbb{R}^{m\times n}: F_i^\delta = 0, i = 1, ..., N\}$

$$\subset \{\xi \in \mathbb{R}^{m\times n}: F_i^0(\xi) < 0, i = 1, ..., N\} \qquad \forall \delta \in (0, \delta_0).$$

If $\varphi \in C^1(\bar{\Omega}; \mathbb{R}^m)$ *(or piecewise* C^1*) is such that*

$$F_i^0(D\varphi(x)) < 0, \qquad \forall x \in \Omega, \quad i = 1, ..., N$$

then there exists (a dense set of) $u \in W^{1,\infty}(\Omega; \mathbb{R}^m)$ *such that*

$$\begin{cases} F_i^0(Du(x)) = 0, \text{ a.e. } x \in \Omega, \quad i = 1, ..., N \\ u(x) = \varphi(x), x \in \partial\Omega. \end{cases}$$

Remark 4.6. We have a simple case when $F_i^\delta(\xi) = F_i(\xi) + \delta$, with F_i, $i = 1, ..., N$, quasiconvex and satisfying

$$\begin{aligned} &\mathit{int}\ \mathit{Rco}\{\xi \in \mathbb{R}^{m\times n}: F_i(\xi) = -\delta, i = 1, ..., N\} \\ &\quad = \{\xi \in \mathbb{R}^{m\times n}: F_i(\xi) < -\delta, i = 1, ..., N\} \end{aligned}$$

for every $\delta \in [0, \delta_0)$.

The result takes a simpler form when $N = 1$, i.e. when there is only one quasiconvex function in (1).

THEOREM 4.4. *Let* $\Omega \subset \mathbb{R}^n$ *be open. Let* $F: \mathbb{R}^{m\times n} \to \mathbb{R}$ *be quasiconvex. Assume that* $\{\xi \in \mathbb{R}^{m\times n}: F(\xi) \leqslant 0\}$ *is compact. If* $\varphi \in C^1(\bar{\Omega}; \mathbb{R}^m)$ *(or piecewise* C^1*) is such that*

$$F(D\varphi(x)) \leqslant 0, \qquad \forall x \in \Omega,$$

then there exists (a dense set of) $u \in W^{1,\infty}(\Omega; \mathbb{R}^m)$ *such that*

$$\begin{cases} F(Du(x)) = 0, & \text{a.e. } x \in \Omega \\ u(x) = \varphi(x), & x \in \partial\Omega. \end{cases}$$

Remark 4.7. The compactness of $\{\xi \in \mathbb{R}^{m\times n}: F(\xi) \leqslant 0\}$ can be weakened as in Theorem 4.1 of [19]. It is sufficient to have compactness in one direction of rank one. More precisely it is sufficient to assume that there exists $\lambda \in \mathbb{R}^{m\times n}$, with $\mathit{rank}\{\lambda\} = 1$, such that $F(\xi + t\lambda) \to +\infty$ as $|t| \to \infty$, for every $\xi \in \mathbb{R}^{m\times n}$.

The following lemmas will be useful in the proof of Theorem 4.1. The first one is the essential ingredient in the proof that quasiconvexity implies rank one convexity and can be found in the literature (c.f. for example [16, 33]).

LEMMA 4.5. *Let $\Omega \subset \mathbb{R}^n$ be an open set. Let $\lambda \in [0, 1]$ and $A, B \in \mathbb{R}^{m \times n}$ with $\operatorname{rank}\{A - B\} = 1$. Let*

$$\varphi(x) = (\lambda A + (1-\lambda) B)\, x, \qquad x \in \bar{\Omega}.$$

Then, for every $\varepsilon > 0$, there exist a piecewise C^1 vector valued function u and disjoint open sets $\Omega_A, \Omega_B \subset \Omega$, with Lipschitz boundary, so that

$$\begin{cases} |\text{meas}\, \Omega_A - \lambda\, \text{meas}\, \Omega|, |\text{meas}\, \Omega_B - (1-\lambda)\, \text{meas}\, \Omega| \leqslant \varepsilon \\ u(x) = \varphi(x), \qquad x \in \partial\Omega \\ |u(x) - \varphi(x)| \leqslant \varepsilon, \qquad x \in \Omega \\ Du(x) = \begin{cases} A & \text{in } \Omega_A \\ B & \text{in } \Omega_B \end{cases} \\ \text{dist}(Du(x), \text{co}\{A, B\}) \leqslant \varepsilon \qquad \text{a.e. in } \Omega \end{cases}$$

Proof. The proof is our Lemma 6.2 in [19]. It is sufficient to choose there the set K to be a convex ε-neighbourhood of the closed segment $[A, B] = \text{co}\{A, B\}$. ∎

LEMMA 4.6. *Let $E \subset \mathbb{R}^{m \times n}$ be closed and such that one of the two following conditions hold:*

(i) *$RcoE$ has the (rank one) approximation property and is compact;*

(ii) *$E \supset \partial RcoE$; moreover $RcoE$ is compact in at least one direction of rank one (i.e. there exists $\eta \in \mathbb{R}^{m \times n}$ with $\operatorname{rank}\{\eta\} = 1$ such that, for every $\xi \in \text{int}\, RcoE$, there exist $t_1 < 0 < t_2$ with $\xi + t_1\eta, \xi + t_2\eta \in \partial RcoE \subset E$).*

Let $\Omega \subset \mathbb{R}^n$ be an open set, $\xi \in \text{int}\, RcoE$ and $\varphi(x) = \xi x$, $x \in \bar{\Omega}$. Then for every $\varepsilon > 0$ there exist $u \in \varphi + A_o^{1,p}(\Omega; RcoE)$ and an open set $\tilde{\Omega} \subset \Omega$ such that

$$\begin{cases} \text{meas}(\Omega - \tilde{\Omega}) \leqslant \varepsilon \\ |u(x) - \varphi(x)| \leqslant \varepsilon \qquad \text{for every} \quad x \in \Omega \\ \text{dist}(Du(x); E) \leqslant \varepsilon \qquad \text{a.e. in } \tilde{\Omega}. \end{cases}$$

Proof. Part 1: We start by assuming that $RcoE$ has the (rank one) approximation property and is compact. Using Definition 4.2 we have that for every $\delta > 0$ there exists a set E_δ, with $\xi \in RcoE_\delta$, whose rank one convex hull is compactly contained in *int RcoE*. We will prove that, for every $\varepsilon > 0$, there exist a piecewise C^1 vector valued function u and an open set $\tilde{\Omega} \subset \Omega$, with Lipschitz boundary, so that

$$\begin{cases} meas(\Omega - \tilde{\Omega}) \leqslant \varepsilon \\ u(x) = \varphi(x),\ x \in \partial\Omega \\ |u(x) - \varphi(x)| \leqslant \varepsilon, \text{ for every } \ x \in \Omega \\ dist(Du(x);\ E_\delta) \leqslant \varepsilon, \text{ a.e. } \ x \in \tilde{\Omega} \\ dist(Du(x);\ RcoE_\delta) \leqslant \varepsilon, \text{ a.e. } \ x \in \Omega. \end{cases}$$

The fact that $RcoE_\delta \subset\subset int\ RcoE$ and the last inequality imply that $Du(x)$ is compactly contained in $int\ RcoE$; by Remark 3.1(ii), $u \in A^{1,p}(\Omega;\ RcoE)$. Finally since E_δ is close to E for δ sufficiently small we will have indeed obtained the lemma.

By Proposition 2.3, we have

$$RcoE_\delta = \bigcup_{i \in \mathbb{N}} R_i coE_\delta.$$

Since $\xi \in RcoE_\delta$, we deduce that $\xi \in R_i coE_\delta$ for some $i \in \mathbb{N}$. We proceed by induction on i.

Step 1: We start with $i = 1$. We can therefore write

$$\xi = \lambda A + (1 - \lambda)\, B, \qquad rank\{A - B\} = 1, \qquad A, B \in E_\delta.$$

We then use Lemma 4.5 and get the claimed result by setting $\tilde{\Omega} = \Omega_A \cup \Omega_B$ and since $co\{A, B\} = [A; B] \subset RcoE_\delta$ and hence for ε sufficiently small

$$dist(Du(x);\ RcoE_\delta) \leqslant \varepsilon.$$

Step 2: We now let for $i > 1$

$$\xi \in R_{i+1} coE_\delta.$$

Therefore there exist $A, B \in \mathbb{R}^{m \times n}$ such that

$$\begin{cases} \xi = \lambda A + (1 - \lambda)\, B,\ rank\{A - B\} = 1 \\ A, B \in R_i coE_\delta. \end{cases}$$

We then apply Lemma 4.5 and find that there exist a piecewise C^1 vector valued function v and Ω_A, Ω_B disjoint open sets such that

$$\begin{cases} meas(\Omega - (\Omega_A \cup \Omega_B)) \leqslant \varepsilon/2 \\ v(x) = \varphi(x),\ x \in \partial\Omega \\ |v(x) - \varphi(x)| \leqslant \varepsilon/2,\ x \in \bar{\Omega} \\ Dv(x) = \begin{cases} A \text{ in } \Omega_A \\ B \text{ in } \Omega_B \end{cases} \\ dist(Dv(x); RcoE_\delta) \leqslant \varepsilon, \text{ in } \Omega. \end{cases}$$

We now use the hypothesis of induction on Ω_A, Ω_B and A, B. We then can find $\tilde{\Omega}_A$, $\tilde{\Omega}_B$, v_A piecewise C^1 in Ω_A, v_b piecewise C^1 in Ω_B satisfying

$$\begin{cases} meas(\Omega_A - \tilde{\Omega}_A), meas(\Omega_B - \tilde{\Omega}_B) \leqslant \varepsilon/2 \\ v_A(x) = v(x) \text{ on } \partial\Omega_A,\ v_B(x) = v(x) \text{ on } \partial\Omega_B \\ |v_A(x) - v(x)| \leqslant \varepsilon/2 \text{ in } \bar{\Omega}_A,\ |v_B(x) - v(x)| \leqslant \varepsilon/2 \text{ in } \bar{\Omega}_B \\ dist(Dv_A, E_\delta) \leqslant \varepsilon, \text{ a.e. in } \tilde{\Omega}_A,\ dist(Dv_B, E_\delta) \leqslant \varepsilon, \text{ a.e. in } \tilde{\Omega}_B \\ dist(Dv_A, RcoE_\delta) \leqslant \varepsilon, \text{ a.e. in } \Omega_A,\ dist(Dv_B, RcoE_\delta) \leqslant \varepsilon, \text{ a.e. in } \Omega_B. \end{cases}$$

Letting $\tilde{\Omega} = \tilde{\Omega}_A \cup \tilde{\Omega}_B$ and

$$u(x) = \begin{cases} v(x) & \text{in } \Omega - (\Omega_A \cup \Omega_B) \\ v_A(x) & \text{in } \Omega_A \\ v_B(x) & \text{in } \Omega_B \end{cases}$$

we have indeed obtained the result.

Part 2: We now assume that $E \supset \partial RcoE$ and $RcoE$ is compact in at least one direction of rank one. The result is then obtained in one step. Indeed if $\xi \in int\ RcoE$ consider the function which for $t \in \mathbb{R}$ associates $\xi(t) = \xi + t\eta \in \mathbb{R}^{m \times n}$ where η is as in the hypothesis of the lemma. We can therefore find $t_1 < 0 < t_2$ such that $\xi(t_1), \xi(t_2) \in \partial RcoE \subset E$ and $\xi(t) \in int\ RcoE$, $\forall t \in (t_1, t_2)$. We then proceed as in Step 1 and we therefore have completed the proof of the lemma. ▮

Proof. (Theorem 4.1). Without loss of generality we assume that Ω is bounded, otherwise express Ω as a countable union of bounded sets and proceed on each one of these sets. We then let $K = PcoE$.

Step 1: Define

$$g(\xi) = \begin{cases} -dist(\xi, E) & \text{if } \xi \in K \\ +\infty & \text{otherwise.} \end{cases}$$

We choose the polyconvex envelope

$$f(\xi) = Pg(\xi).$$

Note that $f(\xi) = +\infty$ if $\xi \notin K$; since K is closed then f is lower semicontinuous. We claim that f is Lipschitz continuous on K and that

$$f(\xi) = 0 \Leftrightarrow \xi \in E.$$

Let us first prove the last sentence:
$(\Rightarrow)$ $\quad 0 = f(\xi) \leqslant g(\xi) \leqslant 0 \Rightarrow g(\xi) = 0 \Rightarrow \xi \in E.$
$(\Leftarrow)$ $\quad$ By Carathéodory theorem (c.f. Theorem 1.1 page 201 in [16])

$$f(\xi) = Pg(\xi) = \inf\left\{\sum_{i=1}^{\tau+1} t_i g(\xi_i) : \sum_{i=1}^{\tau+1} t_i T(\xi_i) = T(\xi)\right\}.$$

By assumption (16) since $\xi \in E$ there does not exist $t_i > 0$, $\sum t_i = 1$, $\xi_i \in K$, $\xi_i \neq \xi$ such that

$$T(\xi) = \sum_{i=1}^{\tau+1} t_i T(\xi_i).$$

Therefore if $\xi \in E$ the infimum is taken by $\xi = \xi_i$ and thus $Pg(\xi) = g(\xi) = 0$.

Let us now prove that f is Lipschitz continuous on K, with Lipschitz constant equal to 1. This property follows from the fact that $g(\xi) = -dist(\xi, E)$ is Lipschitz continuous on K, with Lipschitz constant equal to 1. Indeed, since E is compact, then by Proposition 2.2 $K = PcoE$ is also compact. Then, if $\xi, \xi + \eta \in K$, by Carathéodory theorem we can find t_i, and $\xi_i \in K$, with

$$\xi = \sum_{i=1}^{\tau+1} t_i \xi_i, \qquad T(\xi) = \sum_{i=1}^{\tau+1} t_i T(\xi_i),$$

such that

$$\begin{aligned} f(\xi+\eta) - f(\xi) &= f(\xi+\eta) - \sum_{i=1}^{\tau+1} t_i g(\xi_i) \\ &\leqslant \sum_{i=1}^{\tau+1} t_i [g(\xi_i + \eta) - g(\xi_i)] \leqslant |\eta|. \end{aligned}$$

By interchanging the role of ξ and $\xi + \eta$ we have indeed that f is Lipschitz continuous on K, with Lipschitz constant equal to 1.

Step 2: Let (recall that $f(\xi) \leqslant 0 \Leftrightarrow \xi \in K$)

$$V = \{u \in \varphi + A_o^{1,p}(\Omega; K) : f(Du(x)) \leqslant 0 \text{ a.e. } x \in \Omega\}$$

Then $\varphi \in V$, by hypothesis. We endow V with the L^∞ norm. It is then a complete metric space since K is compact and f is polyconvex. Indeed let $O \subset \Omega$ be an open set, then since f is polyconvex and lower semicontinuous, we have

$$I(u) = \int_O f(Du(x))\, dx \leqslant \liminf_{\nu \to \infty} I(u_\nu)$$

for $u_\nu \rightharpoonup u$ weak* in $W^{1,\infty}(O; \mathbb{R}^m)$.

Since O is arbitrary we have that $f(Du(x)) \leqslant 0$ a.e. $x \in \Omega$ and thus V is a complete metric space.

Step 3: Let $k \in \mathbb{N}$ and

$$V^k = \left\{ u \in V \colon \int_\Omega f(Du(x))\, dx > -\frac{1}{k} \right\}.$$

Let us prove that:

V^k is *open* in V: By lower semicontinuity (as Step 2, the polyconvexity of f implies that $V - V^k$ is closed in V).

V^k is *dense* in V: Let $u \in V$; by definition of $A^{1,p}$ we have that, for $\varepsilon > 0$ fixed, there exist $u_\varepsilon \in W^{1,\infty}(\Omega; \mathbb{R}^m)$ piecewise affine functions with

$$\begin{cases} Du_\varepsilon(x) \text{ compactly contained in } intK \\ \|u_\varepsilon - u\|_{L^\infty} \leqslant \varepsilon/2 \\ u_\varepsilon = u \text{ on } \partial\Omega. \end{cases}$$

Since u_ε is piecewise affine, we can find disjoint open sets Ω_i such that

$$\begin{cases} meas\left(\Omega - \bigcup_{i \in \mathbb{N}} \Omega_i\right) = 0 \\ Du_\varepsilon(x) = \xi_i \in int\, K \text{ in } \Omega_i, \text{ for every } i \in \mathbb{N}. \end{cases}$$

We now apply Lemma 4.6 to u_ε and Ω_i. Therefore there exist $u_{\varepsilon,i} \in u_\varepsilon + A_o^{1,p}(\Omega_i; K)$ and $\tilde{\Omega}_i \subset \Omega_i$ such that

$$\begin{cases} meas(\Omega_i - \tilde{\Omega}_i) \leqslant \varepsilon/2^i \\ |u_{\varepsilon,i}(x) - u_\varepsilon(x)| \leqslant \varepsilon/2, \text{ for every } x \in \Omega_i \\ dist(Du_{\varepsilon,i}, E) \leqslant \varepsilon, \text{ a.e. in } \tilde{\Omega}_i. \end{cases}$$

Then the function v_ε defined as

$$v_\varepsilon(x) = u_{\varepsilon, i}(x), \qquad x \in \bar{\Omega}_i$$

belongs to V, since $v_\varepsilon = u = \varphi$ on $\partial\Omega$ and $Dv_\varepsilon(x) \in K$ a.e. in Ω. Furthermore $\|v_\varepsilon - u\|_{L^\infty} \leqslant \varepsilon$. We now compute

$$\begin{aligned}\int_\Omega f(Dv_\varepsilon(x))\,dx &= \sum_i \int_{\Omega_i - \tilde{\Omega}_i} f(Du_{\varepsilon, i})\,dx + \sum_i \int_{\tilde{\Omega}_i} f(Du_{\varepsilon, i})\,dx \\ &\geqslant -\sum_i meas(\Omega_i - \tilde{\Omega}_i) \max\{|f(\xi)| : \xi \in K\} \\ &\quad + \sum_i \int_{\tilde{\Omega}_i} f(Du_{\varepsilon, i})\,dx.\end{aligned}$$

Recall (see Step 1) that f is Lipschitz continuous on K, with Lipschitz constant equal to 1. So, for every $\xi \in E$ ($\Rightarrow f(\xi) = 0$) and $\eta \in K$, we have

$$|f(\eta) - f(\xi)| = |f(\eta)| \leqslant |\eta - \xi|$$

and hence, for every $\eta \in K$, $|f(\eta)| \leqslant dist(\eta, E)$. Thus

$$\begin{aligned}\sum_i \int_{\tilde{\Omega}_i} f(Du_{\varepsilon, i})\,dx &\geqslant -\sum_i \int_{\tilde{\Omega}_i} dist(Du_{\varepsilon, i}, E)\,dx \\ &\geqslant -\varepsilon\, meas\, \Omega.\end{aligned}$$

We therefore obtain

$$\int_\Omega f(Dv_\varepsilon)\,dx \geqslant -\varepsilon(\max\{|f(\xi)| : \xi \in K\} + meas\,\Omega).$$

Choosing ε sufficiently small we have indeed that $v_\varepsilon \in V_k$.

Step 4: By Baire category theorem, we have that

$$\bigcap_{k \in \mathbb{N}} V^k = \left\{u \in V : \int_\Omega f(Du)\,dx \geqslant 0\right\} \text{ is dense in } V,$$

hence $\{u \in \varphi + A_o^{1,p}(\Omega; K) : f(Du) = 0\}$ is dense in V.

Since $f(\xi) = 0 \Rightarrow \xi \in E$, we have indeed found a dense set of $u \in A^{1,p}(\Omega; PcoE)$ such that

$$\begin{cases} Du(x) \in E, \text{ a.e. } x \in \Omega \\ u(x) = \varphi(x), \; x \in \partial\Omega. \end{cases} \quad \blacksquare$$

We now return to the following

Proof (Corollary 4.2.). The proof is very similar to the one of Corollary 2.2 in [19] and we therefore only sketch it here. First we observe that if φ is piecewise C^1 we can do the following construction on each set where φ is C^1 and hence obtain the result on the whole of Ω. We will therefore assume that $\varphi \in C^1(\bar{\Omega})$. We first define

$$\Omega_o = \{x \in \Omega : D\varphi(x) \in E\}. \tag{19}$$

Since φ is C^1 and E is closed, we find that the set $\Omega - \Omega_o$ is open. We therefore define $u = \varphi$ in Ω_0. It remains to solve

$$\begin{cases} Du(x) \in E, & \text{a.e. in } \Omega - \Omega_o \\ u(x) = \varphi(x), & \text{on } \partial(\Omega - \Omega_o) \end{cases} \tag{20}$$

with $D\varphi(x) \in int\ RcoE$ for every $x \in \Omega - \Omega_o$.

We now observe that if for $t > 0$ we let

$$\Omega^t = \{x \in \Omega - \Omega_o \colon dist(D\varphi(x), \partial K) = t\}$$

where $K = RcoE$, then we can find a decreasing sequence $t_k > 0$ converging to zero such that

$$meas\ \Omega^{t_k} = 0 \tag{21}$$

(c.f. Step 2 of Corollary 2.2 in [19]).

We then let $\Omega_k = \{x \in \Omega - \Omega_o \colon t_{k+1} < dist(D\varphi(x), \partial K) < t_k\}$. Observe that Ω_k is open and that

$$\begin{cases} \overline{\Omega - \Omega_o} = \bigcup_{k=1}^{\infty} \bar{\Omega}_k \\ \Omega - \Omega_o = \bigcup_{k=1}^{\infty} \Omega_k \cup N \text{ with } meas\ N = 0 \\ \partial\Omega_k \subset \partial(\Omega - \Omega_o) \cup \Omega^{t_k} \cup \Omega^{t_{k+1}} \end{cases}$$

(the second statement is a consequence of (21)). Using Theorem 4.1 on Ω_k (since $\varphi \in A^{1,p}(\Omega_k, K)$) we can then find $u_k \in W^{1,\infty}(\Omega_k; \mathbb{R}^m)$ such that

$$\begin{cases} Du_k(x) \in E & \text{a.e. } x \in \Omega_k \\ u_k(x) = \varphi(x) & x \in \partial\Omega_k. \end{cases}$$

Defining

$$u(x)=\begin{cases} u_k(x) & \text{if} \quad x\in\bar{\Omega}_k \\ \varphi(x) & \text{if} \quad x\in\Omega_o \end{cases}$$

then u has all the claimed properties. ∎

We continue with

Proof (Theorem 4.3). As in Theorem 4.1 we assume without loss of generality that Ω is bounded. It is also sufficient to prove the theorem when

$$E=\{\xi\in\mathbb{R}^{m\times n}: F_i^0(\xi)=0,\, i=1, \dots, N\},$$

$$D\varphi(x) \text{ is compactly contained in } int\ RcoE,\ \forall x\in\Omega;$$

the argument of Corollary 4.2 implies the general case. We define

$$V=\{u\in\varphi+A_o^{1,p}(\Omega;\, RcoE): F_i^0(Du(x))\leqslant 0 \\ \text{a.e. } x\in\Omega,\, i=1, \dots, N\}.$$

We have $\varphi\in V$. We then endow V with the L^∞ norm. Since the F_i^0 are quasiconvex and $RcoE$ is compact, we deduce that V is a complete metric space. We then let

$$V^k=\left\{u\in V\colon \sum_{i=1}^N \int_\Omega F_i^0(Du(x))\, dx>-\frac{1}{k}\right\}.$$

We observe that V^k is open in V since the integrals in the left hand side of the inequality are weak* (in $W^{1,\infty}$) lower semicontinuous by the quasiconvexity of the F_i^0.

We now show that V^k is dense in V. Let $u\in V$; by definition of $A^{1,p}$ we have that, for $\varepsilon>0$ fixed, there exist $u_\varepsilon\in W^{1,\infty}(\Omega;\mathbb{R}^m)$ piecewise affine functions with

$$\begin{cases} Du_\varepsilon(x) \text{ compactly contained in } int\ RcoE \\ \|u_\varepsilon-u\|_{L^\infty}\leqslant\varepsilon/2 \\ u_\varepsilon=u \text{ on } \partial\Omega. \end{cases}$$

Since u_ε is piecewise affine, we can find disjoint open sets Ω_j such that

$$\begin{cases} meas\left(\Omega-\bigcup_{j\in\mathbb{N}}\Omega_j\right)=0 \\ Du_\varepsilon(x)=\xi_j\in int\ RcoE \text{ in } \Omega_j, \qquad \text{for every} \quad j\in\mathbb{N}. \end{cases}$$

Let us define E_δ to be

$$E_\delta = \{\xi \in \mathbb{R}^{m \times n} \colon F_i^\delta(\xi) = 0,\ i = 1, ..., N\}.$$

For fixed Ω_j, $j \in \mathbb{N}$, since $\xi_j \in int\ RcoE$, by (i) and by the continuity of F_i^δ with respect to $\delta \geqslant 0$, we deduce that $F_i^{\delta_j}(\xi_j) < 0$, $i = 1, ..., N$, for some positive δ_j; without loss of generality we can choose $\delta_j \leqslant \delta$, for some $\delta \in (0, \delta_0)$ and for every $j \in \mathbb{N}$. Hence $\xi_j \in RcoE_{\delta_j}$.

By the assumption (ii) and by the rank-one convexity of the set $\{\xi \in \mathbb{R}^{m \times n} \colon F_i^0(\xi) < 0\}$ we get

$$RcoE_{\delta_j} \subset \{\xi \in \mathbb{R}^{m \times n} \colon F_i^0(\xi) < 0\} = int\ RcoE$$

and thus $RcoE_{\delta_j}$ is compactly contained in $int\ RcoE$.

As in the proof of Lemma 4.6 we find a piecewise C^1 vector valued function $u_{\varepsilon, j}$ and an open set $\tilde{\Omega}_j \subset \Omega_j$, with Lipschitz boundary, so that, for every $\varepsilon_j \in (0, \varepsilon)$,

$$\begin{cases} meas(\Omega_j - \tilde{\Omega}_j) \leqslant \varepsilon/2^j \\ u_{\varepsilon, j}(x) = u_\varepsilon(x),\ x \in \partial\Omega_j \\ |u_{\varepsilon, j}(x) - u_\varepsilon(x)| \leqslant \varepsilon/2, \text{ for every } x \in \Omega_j \\ dist(Du_{\varepsilon, j}(x);\ E_{\delta_j}) \leqslant \varepsilon, \text{ a.e. } x \in \tilde{\Omega}_j \\ dist(Du_{\varepsilon, j}(x);\ RcoE_{\delta_j}) \leqslant \varepsilon_j, \text{ a.e. } x \in \Omega_j. \end{cases}$$

The fact that $RcoE_{\delta_j} \subset\subset int\ RcoE$ and the last inequality imply that $Du_{\varepsilon, j}(x)$ is compactly contained in $int\ RcoE$, provided that ε_j is sufficiently small; by Remark 3.1(ii), $u_{\varepsilon, j} \in A^{1, p}(\Omega;\ RcoE)$.

Then the function v_ε defined as $v_\varepsilon(x) = u_{\varepsilon, j}(x)$, $x \in \bar{\Omega}_j$, belongs to V, since $v_\varepsilon = u_\varepsilon$ on $\partial\Omega$ and $Dv_\varepsilon(x) \in RcoE$ a.e. in Ω.

We now compute

$$\begin{aligned} &\sum_{i=1}^N \int_\Omega F_i^0(Dv_\varepsilon(x))\, dx \\ &\quad = \sum_{i=1}^N \sum_{j \in \mathbb{N}} \int_{\Omega_j - \tilde{\Omega}_j} F_i^0(Du_{\varepsilon, j})\, dx + \sum_{i=1}^N \sum_{j \in \mathbb{N}} \int_{\tilde{\Omega}_j} F_i^0(Du_{\varepsilon, j})\, dx \\ &\quad \geqslant -\sum_{j \in \mathbb{N}} meas(\Omega_j - \tilde{\Omega}_j) \\ &\qquad \times \max\left\{\sum_{i=1}^N |F_i^0(\xi)| : F_i^0(\xi) \leqslant 0,\ i = 1, ..., N\right\} \\ &\qquad + \sum_{i, j} \int_{\tilde{\Omega}_j} F_i^{\delta_j}(Du_{\varepsilon, j})\, dx + \sum_{i, j} \int_{\tilde{\Omega}_j} \{F_i^0(Du_{\varepsilon, j}) - F_i^{\delta_j}(Du_{\varepsilon, j})\}\, dx. \end{aligned}$$

In the right hand side the first addendum is small since

$$\sum_{j \in \mathbb{N}} meas(\Omega_j - \tilde{\Omega}_j) \leqslant \varepsilon;$$

the second addendum is small since $F_i^{\delta_j}(\xi)$ is Lipschitz continuous (with some Lipschitz constant L), uniformly with $j \in \mathbb{N}$, on the bounded set $\{\xi \in \mathbb{R}^{m \times n}: F_i^0(\xi) < 0\}$, and since $dist(Du_{\varepsilon, j}(x); E_{\delta_j}) \leqslant \varepsilon$, a.e. $x \in \tilde{\Omega}_j$. Therefore, since $F_i^{\delta_j}(E_{\delta_j}) = 0$,

$$\sum_{i, j} \int_{\tilde{\Omega}_j} F_i^{\delta_j}(Du_{\varepsilon, j})\, dx \geqslant -L \sum_{i, j} \int_{\tilde{\Omega}_j} dist(Du_{\varepsilon, j}(x); E_{\delta_j})\, dx$$
$$\geqslant -\varepsilon NL\, meas\, \Omega;$$

finally the third addendum in the right hand side is small because $Du_{\varepsilon, j}(x)$ a.e. in $\tilde{\Omega}_j$ belongs to the bounded set $\{\xi \in \mathbb{R}^{m \times n}: F_i^0(\xi) < 0\}$, and $F_i^\delta(\xi)$ is uniformly continuous as $\delta \to 0^+$ and since $\delta_j \leqslant \delta$ for $j \in \mathbb{N}$, with δ sufficiently small. Therefore, for ε sufficiently small we have indeed that $v_\varepsilon \in V^k$.

Since V^k is a sequence of open and dense sets in V, by Baire category theorem we have that the intersection of the V^k, $k \in \mathbb{N}$, is still dense in V, in particular it is not empty. Any element of this intersection is a $W^{1, \infty}(\Omega; \mathbb{R}^m)$ solution of the given Dirichlet problem. ∎

We conclude this section with

Proof (Theorem 4.4). Step 1: We first show that, if

$$E = \{\xi \in \mathbb{R}^{m \times n}: F(\xi) = 0\},$$

then

$$RcoE = \{\xi \in \mathbb{R}^{m \times n}: F(\xi) \leqslant 0\}.$$

Call $X = \{\xi \in \mathbb{R}^{m \times n}: F(\xi) \leqslant 0\}$. It is clear that, since F is quasiconvex and thus rank one convex and $E \subset X$, we must have $RcoE \subset X$. We now show that $X \subset RcoE$. Let $\xi \in X$. We can assume that $F(\xi) < 0$, otherwise $\xi \in E$ and thus $\xi \in RcoE$ which is the claimed result.

Consider the function which at every $t \in \mathbb{R}$ associates $\xi(t) = \xi + t\eta \in \mathbb{R}^{m \times n}$ where η is any rank one matrix. Since E is bounded and $\xi \in X$ with $F(\xi) < 0$, we can find $t_1 < 0 < t_2$ such that $\xi(t_1), \xi(t_2) \in E$. Since $0 \in (t_1, t_2)$ we deduce that ξ is a rank one convex combination of $\xi(t_1)$ and $\xi(t_2)$; thus $\xi \in RcoE$, which is the claimed property.

Step 2: The proof of the theorem is then similar to those of Theorem 4.1 (and Corollary 4.2) and Theorem 4.3. As in Corollary 4.2 we first assume that $D\varphi$ is compactly contained in *int RcoE*. We then define

$$V = \{u \in \varphi + A_o^{1,p}(\Omega; RcoE) : F(Du(x)) \leqslant 0 \text{ a.e. } x \in \Omega\}$$

$$V^k = \left\{u \in V \colon \int_\Omega F(Du(x))\, dx > -\frac{1}{k}\right\}.$$

As in the preceding proofs it is easy to see that the quasiconvexity of F and the compactness of $\{\xi \in \mathbb{R}^{m \times n} \colon F(\xi) = 0\}$ ensure that V is a complete metric space with L^∞ norm and that V^k is open in V in the same topology. The density of V^k in V is then proved exactly as the corresponding one in Theorem 4.1 with f replaced by F. The only difference is that the hypothesis of Lemma 4.6 is satisfied there because of the (rank one) approximation property while here it is satisfied since $E \supset \partial RcoE$ by Step 1. ∎

5. APPLICATIONS

We start with the applications concerning the singular values case. We recall that for $\xi \in \mathbb{R}^{2 \times 2}$ we denote by $0 \leqslant \lambda_1(\xi) \leqslant \lambda_2(\xi)$ the eigenvalues of $(\xi^t \xi)^{1/2}$. We first give a representation formula.

LEMMA 5.1. *Let* $0 < a_1 \leqslant a_2$ *and*

$$E = \{\xi \in \mathbb{R}^{2 \times 2} \colon \lambda_1(\xi) = a_1,\ \lambda_2(\xi) = a_2\};$$

then

$$coE = \{\xi \in \mathbb{R}^{2 \times 2} \colon \lambda_2(\xi) \leqslant a_2 \text{ and } \lambda_1(\xi) + \lambda_2(\xi) \leqslant a_1 + a_2\},$$

$$PcoE = RcoE = \{\xi \in \mathbb{R}^{2 \times 2} \colon \lambda_2(\xi) \leqslant a_2 \text{ and } \lambda_1(\xi)\, \lambda_2(\xi) \leqslant a_1 a_2\},$$

$$int\ RcoE = \{\xi \in \mathbb{R}^{2 \times 2} \colon \lambda_2(\xi) < a_2 \text{ and } \lambda_1(\xi)\, \lambda_2(\xi) < a_1 a_2\}.$$

Remark 5.1. (i) Note that if $a_1 = a_2$ then $coE = PcoE = RcoE = \{\xi \colon \lambda_2(\xi) \leqslant a_2\}$.

(ii) For related properties see Aubert–Tahraoui [2].

Proof. Preliminary step. We start by expressing matrices as appropriate convex combinations. We let

$$\xi = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

with $0 \leqslant a \leqslant b \leqslant a_2$.

Case 1: If $0 \leqslant a \leqslant a_1$ and $0 \leqslant b \leqslant a_2$, we then can write

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \frac{a_1+a}{2a_1}\begin{pmatrix} a_1 & 0 \\ 0 & b \end{pmatrix} + \frac{a_1-a}{2a_1}\begin{pmatrix} -a_1 & 0 \\ 0 & b \end{pmatrix}$$
$$= \frac{a_1+a}{2a_1}\cdot\frac{a_2+b}{2a_2}\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} + \frac{a_1+a}{2a_1}\cdot\frac{a_2-b}{2a_2}\begin{pmatrix} a_1 & 0 \\ 0 & -a_2 \end{pmatrix}$$
$$+ \frac{a_1-a}{2a_1}\cdot\frac{a_2+b}{2a_2}\begin{pmatrix} -a_1 & 0 \\ 0 & a_2 \end{pmatrix}$$
$$+ \frac{a_1-a}{2a_1}\cdot\frac{a_2-b}{2a_2}\begin{pmatrix} -a_1 & 0 \\ 0 & -a_2 \end{pmatrix}. \tag{22}$$

Case 2: $0 \leqslant a_1 \leqslant a \leqslant b \leqslant a_2$ and $a+b \leqslant a_1+a_2$. We can assume that $a_1 < a_2$, otherwise this case is just $a=b=a_1=a_2$. We can also assume that $b < a_2$ otherwise $a = a_1$ (and $b = a_2$). We have

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \frac{(a_1+a_2)(a_2-a)+(a_2-a_1)(b-a_1)}{2a_2(a_2-a_1)}\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$$
$$+ \frac{a-a_1}{a_2-a_1}\begin{pmatrix} a_2 & 0 \\ 0 & a_1 \end{pmatrix} + \frac{a_1+a_2-a-b}{2a_2}\begin{pmatrix} a_1 & 0 \\ 0 & -a_2 \end{pmatrix}. \tag{23}$$

Case 3: $0 \leqslant a_1 \leqslant a \leqslant b \leqslant a_2$ and $ab \leqslant a_1 a_2$

$$\begin{pmatrix} a_1 & 0 \\ 0 & \dfrac{ab}{a_1} \end{pmatrix} = \frac{a_1 a_2 + ab}{2a_1 a_2}\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} + \frac{a_1 a_2 - ab}{2a_1 a_2}\begin{pmatrix} a_1 & 0 \\ 0 & -a_2 \end{pmatrix} \tag{24}$$

and

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \frac{1}{2}\begin{pmatrix} a & \dfrac{\sqrt{a^2-a_1^2}\sqrt{b^2-a_1^2}}{a_1} \\ 0 & b \end{pmatrix}$$
$$+ \frac{1}{2}\begin{pmatrix} a & -\dfrac{\sqrt{a^2-a_1^2}\sqrt{b^2-a_1^2}}{a_1} \\ 0 & b \end{pmatrix}. \tag{25}$$

Step 1: We now show that if

$$K = \{\xi \in \mathbb{R}^{2\times 2} \colon \lambda_2(\xi) \leqslant a_2 \text{ and } \lambda_1(\xi)+\lambda_2(\xi) \leqslant a_1+a_2\}$$

then $K = coE$.

(i) $coE \subset K$: Since the functions $\xi \to \lambda_2(\xi)$ and $\xi \to \lambda_1(\xi) + \lambda_2(\xi)$ are convex (c.f. for example Proposition 1.2 page 254 [16]) then the set K is convex. Since $E \subset K$, we deduce immediately the claimed inclusion.

(ii) $K \subset coE$: Let $\xi \in K$. Since $\xi \to \lambda_1(\xi)$ and $\xi \to \lambda_2(\xi)$ are invariant by orthogonal transformations, then we can assume, without loss of generality, that

$$\xi = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad \text{with} \quad 0 \leqslant a \leqslant b \leqslant a_2 \quad \text{and} \quad a + b \leqslant a_1 + a_2.$$

Then either Case 1 or Case 2 of the preliminary step can happen. In both cases ξ can be expressed as a convex combination of points in E, thus $K \subset coE$ and this achieves Step 1.

Step 2: We now let

$$L = \{\xi \in \mathbb{R}^{2\times 2}: \lambda_2(\xi) \leqslant a_2 \text{ and } \lambda_1(\xi)\, \lambda_2(\xi) \leqslant a_1 a_2\}.$$

We want to show that $L = RcoE = PcoE$ (we recall that $RcoE \subset PcoE$ for every E).

(i) $PcoE \subset L$: Since the functions $\xi \to \lambda_2(\xi)$ is convex and $\xi \to |\det \xi| = \lambda_1(\xi)\, \lambda_2(\xi)$ is polyconvex, we deduce that L is polyconvex; since also $E \subset L$, we obtain the claimed inclusion.

(ii) $L \subset RcoE$: Let $\xi \in L$. Since $\xi \to \lambda_2(\xi)$ and $\xi \to \lambda_1(\xi)$ are invariant by orthogonal transformations, we can assume, without loss of generality that

$$\xi = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad \text{with} \quad 0 \leqslant a \leqslant b \leqslant a_2 \quad \text{and} \quad ab \leqslant a_1 a_2. \tag{26}$$

Then either Case 1 or Case 3 of the preliminary step can happen. We examine them separately.

Case 1: Observe that by (22)

$$\begin{pmatrix} a_1 & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} -a_1 & 0 \\ 0 & b \end{pmatrix} \in R_1\, coE$$

and thus by (22), as well,

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in R_2\, coE \subset RcoE$$

and the result is established.

Case 3: Observe that by (25)

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in R_1 co \left\{ \begin{pmatrix} a & \dfrac{\sqrt{a^2-a_1^2}\sqrt{b^2-a_1^2}}{a_1} \\ 0 & b \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} a & \dfrac{-\sqrt{a^2-a_1^2}\sqrt{b^2-a_1^2}}{a_1} \\ 0 & b \end{pmatrix} \right\}. \tag{27}$$

Note also that if we denote by A_+ and A_- respectively the two matrices involved in the right hand side of (27) we find that

$$\lambda_1(A_\pm)\,\lambda_2(A_\pm) = |\det A_\pm| = ab$$

$$\lambda_1^2(A_\pm) + \lambda_2^2(A_\pm) = a^2 + b^2 + \frac{(a^2-a_1^2)(b^2-a_1^2)}{a_1^2} = \frac{a_1^4 + a^2b^2}{a_1^2}$$

leading immediately to

$$\lambda_1(A_\pm) = a_1 \qquad \lambda_2(A_\pm) = \frac{ab}{a_1}.$$

Therefore up to orthogonal transformations $Q_\pm$, $\tilde{Q}_\pm$ we have that

$$Q_\pm A_\pm \tilde{Q}_\pm = \begin{pmatrix} a_1 & 0 \\ 0 & \dfrac{ab}{a_1} \end{pmatrix}$$

and thus by (24)

$$Q_\pm A_\pm \tilde{Q}_\pm \in R_1 coE. \tag{28}$$

Combining (27), (28) and the invariance under orthogonal transformations of E, we have indeed obtained that

$$\xi \in R_2 coE \subset RcoE$$

and the result follows.

Step 3: We now wish to show that

$$int\ RcoE = \{\xi \in \mathbb{R}^{2\times 2} : \lambda_2(\xi) < a_2 \text{ and } \lambda_1(\xi)\,\lambda_2(\xi) < a_1 a_2\}.$$

Denote by Y the right hand side of the above identity.

(i) $Y \subset int\ RcoE$, since by continuity Y is open and, by Step 2, $Y \subset RcoE$.

(ii) $int\ RcoE \subset Y$. So let $\xi \in int\ RcoE$; we can therefore find $\varepsilon > 0$ sufficiently small so that (the ball centered at ξ and of radius ε) $B_\varepsilon(\xi) \subset RcoE$. Let R and R' be orthogonal matrices so that

$$\xi = R \begin{pmatrix} \lambda_1(\xi) & 0 \\ 0 & \lambda_2(\xi) \end{pmatrix} R'.$$

Let us define

$$\eta = R \begin{pmatrix} \lambda_1(\xi) & 0 \\ 0 & \lambda_2(\xi) + \dfrac{\varepsilon}{2} \end{pmatrix} R'.$$

Since $|\eta - \xi| = \varepsilon/2 < \varepsilon$, then $\eta \in RcoE$. We then get $\lambda_2(\xi) < \lambda_2(\eta) \leqslant a_2$ and, if $\lambda_1(\xi) \neq 0$,

$$\lambda_1(\xi)\, \lambda_2(\xi) < \lambda_1(\eta)\, \lambda_2(\eta) \leqslant a_1 a_2$$

which implies that $\xi \in Y$. Finally, if $\lambda_1(\xi) = 0$ then $\lambda_1(\xi)\, \lambda_2(\xi) = 0 < a_1 a_2$, which implies again that $\xi \in Y$. ∎

THEOREM 5.2. *Let $\Omega \subset \mathbb{R}^2$ be open. Let $0 < a_1 \leqslant a_2$ and $\varphi \in C^2(\bar{\Omega}; \mathbb{R}^2)$ (or piecewise C^1) such that*

$$\lambda_2(D\varphi(x)) < a_2 \text{ and } \lambda_1(D\varphi(x))\, \lambda_2(D\varphi(x)) < a_1 a_2 \text{ in } \Omega; \tag{29}$$

then there exists $u \in W^{1,\infty}(\Omega; \mathbb{R}^2)$ such that

$$\begin{cases} \lambda_1(Du(x)) = a_1,\ \lambda_2(Du(x)) = a_2, \text{ a.e. } x \in \Omega \\ u(x) = \varphi(x) \text{ on } \partial\Omega. \end{cases} \tag{30}$$

Remark 5.2. (i) Of course we could have supposed a weaker hypothesis on φ, namely $\varphi \in A^{1,p}(\Omega; PcoE)$.

(ii) Comparing with Corollary 5.2 in [19], our hypothesis (29) is more general than the one considered there. The condition (29) seems optimal (up to the fact that the inequalities are strict).

Proof. We can apply either Theorem 4.1 (and Corollary 4.2) or Theorem 4.3 with $F_1^\delta = \lambda_1 \lambda_2 - (a_1 - \delta)(a_2 - \delta)$, $F_2^\delta = \lambda_2 - (a_2 - \delta)$, $\delta \geqslant 0$. Lemma 5.1 ensures that the hypotheses of this last theorem are satisfied (see Remark 6.2 about the choice of the functions F_1 and F_2). ∎

We now turn our attention to the problem of potential wells. It has been introduced to study microstructures observed in certain materials, where the deformation gradients lie in some *potential wells*. The reference papers on the subject are Ball and James [3, 4]; see also [8, 23, 25, 27, 28, 34, 35, 38].

The problem is the following. Given two matrices $A, B \in \mathbb{R}^{2\times 2}$ with $0 < \det A < \det B$, given an open set $\Omega \subset \mathbb{R}^2$ and a boundary datum φ, find $u \in W^{1,\infty}(\Omega; \mathbb{R})$ such that

$$\begin{cases} Du(x) \in SO(2)\, A \cup SO(2)\, B, \text{ a.e. in } \Omega \\ u(x) = \varphi(x) \text{ on } \partial\Omega \end{cases}$$

where $SO(2)$ is the set of rotations (i.e. special orthogonal matrices) in $\mathbb{R}^{2\times 2}$, namely

$$SO(2) = \left\{ R \in \mathbb{R}^{2\times 2} : R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \text{ for some } \theta \in \mathbb{R} \right\}.$$

Remark 5.3. Note that in the considered case of singular values, if we take $a_1 = a_2 = 1$, then the problem is also of potentials wells type, i.e.

$$Du(x) \in SO(2)\, I \cup SO(2)\, I_-, \qquad \text{where} \quad I_- = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};$$

or in other words $Du(x) \in O(2)$ a.e. in Ω.

Up to rotation and homothety we can assume without loss of generality that

$$A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}$$

with $0 < a_1 \leqslant a_2$, $0 < b_1 \leqslant b_2$ and $a_1 a_2 < b_1 b_2$.

We now give a representation formula for *RcoE* where

$$E = SO(2)\, A \cup SO(2)\, B.$$

The result will be a consequence of the formula obtained by Sverak [38] (see also Müller–Sverak [34]).

LEMMA 5.3. *Let* $0 < b_1 < a_1 \leqslant a_2 < b_2$ *and* $a_1 a_2 < b_1 b_2$. *Let* $E = SO(2)\, A \cup SO(2)\, B$. *Define for* $\xi \in \mathbb{R}^{2\times 2}$

$$\begin{cases} F_1(\xi) = \dfrac{[(a_2\xi_{11} - a_1\xi_{22})^2 + (a_1\xi_{12} + a_2\xi_{21})^2]^{1/2}}{a_1 b_2 - a_2 b_1} + \dfrac{a_1 a_2 - \det \xi}{b_1 b_2 - a_1 a_2} \\ F_2(\xi) = \dfrac{[(b_2\xi_{11} - b_1\xi_{22})^2 + (b_1\xi_{12} + b_2\xi_{21})^2]^{1/2}}{a_1 b_2 - a_2 b_1} + \dfrac{\det \xi - b_1 b_2}{b_1 b_2 - a_1 a_2}. \\ F_3(\xi) = (\det \xi - a_1 a_2)(\det \xi - b_1 b_2) \end{cases}$$

Then $F_i \colon \mathbb{R}^{2\times 2} \to \mathbb{R}$, $i = 1, 2, 3$ *are polyconvex and invariant under the action of* $SO(2)$ *and*

$$E = \{\xi \colon F_i(\xi) = 0,\ i = 1, 2, 3\},$$

$$PcoE = RcoE = \{\xi \colon F_i(\xi) \leqslant 0,\ i = 1, 2, 3\},$$

$$int\ RcoE = \{\xi \colon F_i(\xi) < 0,\ i = 1, 2, 3\}.$$

Furthermore matrices of the form

$$A^\delta = \begin{pmatrix} a_1 - \delta & 0 \\ 0 & a_2 + T\delta \end{pmatrix}$$

are in int RcoE, for every $\delta > 0$ *sufficiently small and for* T *satisfying*

$$\frac{a_2(b_2 - a_2)(a_1 + b_1)}{a_1(b_2 + a_2)(a_1 - b_1)} < T < \frac{b_1(b_2^2 - a_2^2)}{b_2(a_1^2 - b_1^2)}.$$

Remark 5.4. The condition $b_1/a_1 < 1 < b_2/a_2$ means that the two wells are rank one connected, i.e. $\det(RA - B) = 0$ for a certain $R \in SO(2)$.

Proof (of Lemma 5.3.). Step 1: We first observe that F_i, $i = 1, 2, 3$, are polyconvex. Indeed the two first ones are a sum of a convex function and a linear function of the determinant; while the last one is a (quadratic) convex function of the determinant. The invariance under the action of $SO(2)$ is easily checked.

Step 2: We now show that $E = \{\xi \colon F_i(\xi) = 0, i = 1, 2, 3\}$. Indeed if $F_3(\xi) = 0$ then necessarily either $\det \xi = a_1 a_2$ or $\det \xi = b_1 b_2$. We examine the first possibility, the other one being handled analogously. Then since $F_1(\xi) = F_2(\xi) = 0$ and $\det \xi = a_1 a_2$, we deduce that

$$\begin{cases} a_2\xi_{11} - a_1\xi_{22} = a_1\xi_{12} + a_2\xi_{21} = 0 \\ (b_2\xi_{11} - b_1\xi_{22})^2 + (b_1\xi_{12} + b_2\xi_{21})^2 = (a_1 b_2 - a_2 b_1)^2 \end{cases}$$

i.e.

$$\begin{cases} \xi_{22} = \dfrac{a_2}{a_1}\,\xi_{11},\ \xi_{21} = -\dfrac{a_1}{a_2}\,\xi_{12} \\ \left(b_2 - b_1\dfrac{a_2}{a_1}\right)^2 \xi_{11}^2 + \left(b_1 - b_2\dfrac{a_1}{a_2}\right)^2 \xi_{12}^2 = (a_1 b_2 - a_2 b_1)^2 \\ \qquad \Leftrightarrow \dfrac{\xi_{11}^2}{a_1^2} + \dfrac{\xi_{12}^2}{a_2^2} = 1. \end{cases}$$

This leads to

$$\xi_{11} = a_1 \cos\theta,\ \xi_{12} = a_2 \sin\theta,\ \xi_{21} = -a_1 \sin\theta,\ \xi_{22} = a_2 \cos\theta,$$

i.e.

$$\xi = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \in SO(2)\,A.$$

Step 3: We now show that $PcoE = RcoE = \{\xi\colon F_i(\xi) \leqslant 0,\ i = 1, 2, 3\}$. To prove this we use the representation formula of Sverak, i.e.

$$\begin{aligned} PcoE &= RcoE \\ &= \left\{ \xi\colon \xi = \begin{pmatrix} y_1 & -y_2 \\ y_2 & y_1 \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} + \begin{pmatrix} z_1 & -z_2 \\ z_2 & z_1 \end{pmatrix} \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} \right. \\ &\qquad \left. \text{with } \sqrt{y_1^2 + y_2^2} \leqslant \frac{b_1 b_2 - \det \xi}{b_1 b_2 - a_1 a_2} \text{ and } \sqrt{z_1^2 + z_2^2} \leqslant \frac{\det \xi - a_1 a_2}{b_1 b_2 - a_1 a_2} \right\}. \end{aligned}$$

Expressing y_1, y_2, z_1, z_2 in terms of ξ_{ij} we find

$$\begin{cases} \xi_{11} = a_1 y_1 + b_1 z_1 \\ \xi_{12} = -(a_2 y_2 + b_2 z_2) \\ \xi_{21} = a_1 y_2 + b_1 z_2 \\ \xi_{22} = a_2 y_1 + b_2 z_1 \end{cases} \Leftrightarrow \begin{cases} y_1 = \dfrac{(b_2 \xi_{11} - b_1 \xi_{22})}{a_1 b_2 - a_2 b_1} \\ y_2 = \dfrac{(b_1 \xi_{12} + b_2 \xi_{21})}{a_1 b_2 - a_2 b_1} \\ z_1 = \dfrac{-(a_2 \xi_{11} - a_1 \xi_{22})}{a_1 b_2 - a_2 b_1} \\ z_2 = \dfrac{-(a_1 \xi_{12} + a_2 \xi_{21})}{a_1 b_2 - a_2 b_1} \end{cases}$$

The inequalities involving $\sqrt{y_1^2+y_2^2}$ and $\sqrt{z_1^2+z_2^2}$ lead then immediately to $F_i(\xi)\leqslant 0$, $i=1, 2, 3$.

Step 4: The fact that *int RcoE* $=\{\xi\colon F_i(\xi)<0, i=1, 2, 3\}$ follows from the fact that in the representation formula of Sverak the interior of *RcoE* is given by strict inequalities.

Step 5: We now show that A^δ given in the statement of the lemma belongs to *int RcoE*. Indeed for $\delta>0$ sufficiently small we have

$$\begin{cases} F_1(A^\delta)=\dfrac{(a_1T+a_2)\,\delta}{a_1b_2-a_2b_1}-\dfrac{(a_1T-a_2)\,\delta-T\delta^2}{b_1b_2-a_1a_2}\\ F_2(A^\delta)=\dfrac{a_1b_2-a_2b_1-(b_1T+b_2)\,\delta}{a_1b_2-a_2b_1}-1+\dfrac{(a_1T-a_2)\,\delta-T\delta^2}{b_1b_2-a_1a_2}\\ F_3(A^\delta)=(a_1a_2+(a_1T-a_2)\,\delta-T\delta^2-b_1b_2)((a_1T-a_2)\,\delta-T\delta^2). \end{cases}$$

Neglecting the terms in δ^2, we find $F_i(A^\delta)<0$, $i=1, 2, 3$ if and only if

$$\frac{(a_1T+a_2)}{a_1b_2-a_2b_1}<\frac{(a_1T-a_2)}{b_1b_2-a_1a_2}<\frac{(b_1T+b_2)}{a_1b_2-a_2b_1}.$$

These inequalities lead to

$$\frac{a_2(b_2-a_2)(a_1+b_1)}{a_1(b_2+a_2)(a_1-b_1)}<T<\frac{b_1(b_2^2-a_2^2)}{b_2(a_1^2-b_1^2)}.$$

Note that the hypotheses on a_i and b_i ensure that the left hand side is indeed smaller than the right hand side. It is also easy to see that the left hand side is larger than a_2/a_1. ∎

THEOREM 5.4. *Let $\Omega\subset\mathbb{R}^2$ be open. Let $0<b_1<a_1\leqslant a_2<b_2$ and $a_1a_2<b_1b_2$. Let $\varphi\in C^1(\bar{\Omega};\mathbb{R}^2)$ (or piecewise C^1) such that*

$$F_i(D\varphi(x))<0, \qquad x\in\Omega, i=1, 2, 3, \tag{31}$$

where the F_i, $i=1, 2, 3$ are as in Lemma 5.3. Then there exists (a dense set of) $u\in W^{1,\infty}(\Omega;\mathbb{R}^2)$ such that

$$\begin{cases} Du(x)\in SO(2)\,A\cup SO(2)\,B \text{ a.e. in } \Omega\\ u(x)=\varphi(x) \text{ on } \partial\Omega. \end{cases} \tag{32}$$

Remark 5.5. (i) The above result has also been obtained by Müller–Sverak [34], using Gromov's method [26] called convex integration.

(ii) Examples of φ satisfying (31) are linear functions $\varphi(x)=A^\delta x$ with A^δ as in Lemma 5.3.

(iii) We could have replaced (31) by the weaker hypothesis

$$D\varphi(x) = E \cup \mathit{int}\ RcoE, \qquad x \in \Omega.$$

Proof. We will use Theorem 4.3 (although Theorem 4.1 could also be applied). We now define the functions $F_i^\delta : \mathbb{R}^{2\times 2} \to \mathbb{R}$, $i = 1, 2, 3$ for $\delta \geqslant 0$. We set $F_i^0 = F_i$. Then for every $\delta > 0$ sufficiently small, we consider A^δ as in Lemma 5.3 and symmetrically B^δ of the form

$$B^\delta = \begin{pmatrix} b_1 + S\delta & 0 \\ 0 & b_2 - \delta \end{pmatrix}$$

for an appropriate $S > 0$. Then A^δ, B^δ belong to *int RcoE* and $A^\delta \to A$, $B^\delta \to B$ as $\delta \to 0$. The functions $F_1^\delta(\xi)$, $F_2^\delta(\xi)$, $F_3^\delta(\xi)$, are defined replacing $\{a_1, a_2, b_1, b_2\}$ respectively by $\{a_1 - \delta, a_2 + T\delta, b_1 + S\delta, b_2 - \delta\}$ in the definition of $F_1(\xi)$, $F_2(\xi)$, $F_3(\xi)$.

Then by Lemma 5.3 applied to A^δ, B^δ and F_i^δ we have

$$\begin{aligned} &\mathit{int}\ RcoE\{\xi \in \mathbb{R}^{2\times 2} : F_i^\delta(\xi) = 0, i = 1, 2, 3\} \\ &= \{\xi \in \mathbb{R}^{2\times 2} : F_i^\delta(\xi) < 0, i = 1, 2, 3\} \end{aligned}$$

for every $\delta \geqslant 0$ sufficiently small, therefore condition (i) of Theorem 4.3 is satisfied. Since

$$\{\xi \in \mathbb{R}^{2\times 2} : F_i^\delta(\xi) = 0, i = 1, 2, 3\} = SO(2)\, A^\delta \cup SO(2)\, B^\delta$$

and since for $i = 1, 2, 3$,

$$\begin{cases} F_i(SO(2)\, A^\delta) = F_i(A^\delta) < 0 \\ F_i(SO(2)\, B^\delta) = F_i(B^\delta) < 0 \end{cases}$$

we get (ii) of Theorem 4.3. ∎

6. THE (x, u) DEPENDENCE

We now propose a method of reduction of the (x, u) dependence to problems independent of (x, u). To explain the method we will not consider the most general possible hypotheses.

Bressan and Flores [9] studied this problem for the scalar case (this reference has recently been pointed out to us by P. Cardaliaguet). G. Pianigiani personally communicated to us that he also considers (x, u) dependence for the scalar case.

THEOREM 6.1. *Let $\Omega \subset \mathbb{R}^n$ be open. Let $F_i^\delta(x, s, \xi)$, $i=1, ..., N$, be quasiconvex with respect to $\xi \in \mathbb{R}^{m\times n}$ and continuous with respect to $(x, s) \in \Omega \times \mathbb{R}^m$ and with respect to $\delta \in [0, \delta_0)$, for some $\delta_0 > 0$. Assume that, for every $(x, s) \in \Omega \times \mathbb{R}^m$,*

$$\text{(i)} \quad \operatorname{int} Rco\{\xi \in \mathbb{R}^{m\times n}: F_i^\delta(x, s, \xi) = 0, i=1, ..., N\}$$
$$= \{\xi \in \mathbb{R}^{m\times n}: F_i^\delta(x, s, \xi) < 0, i=1, ..., N\}, \qquad \forall \delta \in [0, \delta_0)$$

and it is bounded;

$$\text{(ii)} \quad \{\xi \in \mathbb{R}^{m\times n}: F_i^\delta(x, s, \xi) = 0, i=1, ..., N\}$$
$$\subset \{\xi \in \mathbb{R}^{m\times n}: F_i^{\delta'}(x, s, \xi) < 0, i=1, ..., N\}, \qquad \forall 0 \leqslant \delta' < \delta < \delta_0.$$

If $\varphi \in C^1(\bar{\Omega}; \mathbb{R}^m)$ (or piecewise C^1) satisfies

$$F_i^0(x, \varphi(x), D\varphi(x)) < 0, \qquad \forall x \in \Omega, i=1, ..., N,$$

then there exists (a dense set of) $u \in W^{1,\infty}(\Omega; \mathbb{R}^m)$ such that

$$\begin{cases} F_i^0(x, u(x), Du(x)) = 0, \text{ a.e. } x \in \Omega, \quad i=1, ..., N \\ u(x) = \varphi(x), \ x \in \partial\Omega. \end{cases}$$

Proof. We can, as usual, assume that Ω is bounded and $\varphi \in C^1(\bar{\Omega}; \mathbb{R}^m)$ without loss of generality. We define

$$V = \text{closure in } L^\infty \text{ of } \{v \in \varphi + W_o^{1,\infty}(\Omega; \mathbb{R}^m), v \text{ piecewise affine}$$
$$\text{and } F_i^0(x, v(x), Dv(x)) < 0, \text{ a.e. } x \in \Omega, \forall i=1, ..., N\}.$$

By hypothesis $\varphi \in V$ (c.f. Remark 3.1.iii); moreover V is a complete metric space when endowed with the L^∞ norm. Observe that by quasiconvexity of F_i^0 we also have that

$$V \subset \{v \in \varphi + W_o^{1,\infty}(\Omega; \mathbb{R}^m), F_i^0(x, v(x), Dv(x)) \leqslant 0,$$
$$\text{a.e. } x \in \Omega, \forall i=1, ..., N\}.$$

For every $k \in \mathbb{N}$ let

$$V^k = \left\{v \in V: \sum_{i=1}^N \int_\Omega F_i^0(x, v(x), Dv(x))\, dx > -\frac{1}{k}\right\}.$$

As in Theorem 4.3, for every $k \in \mathbb{N}$, V^k is open in the L^∞ norm, since F_i^0 are quasiconvex and functions in V have uniformly bounded gradients.

We now show that V^k is dense in V. So let $v \in V$; by definition we can find a sequence of piecewise affine functions $v_\varepsilon \in \varphi + W_o^{1,\infty}(\Omega; \mathbb{R}^m)$ with

$$\begin{cases} v_\varepsilon \to v \text{ in } L^\infty(\Omega) \\ F_i^0(x, v_\varepsilon(x), Dv_\varepsilon(x)) < 0, \text{ a.e. in } \Omega, \quad \forall i = 1, ..., N. \end{cases}$$

By restricting to each component where Dv_ε is constant (replacing v_ε by v and Ω by the set where Dv_ε is constant) we can reduce our problem to considering affine v in Ω such that

$$F_i^0(x, v, (x), Dv) < 0 \text{ in } \bar{\Omega}, \quad \forall i = 1, ..., N. \tag{33}$$

Let $\delta > 0$ and let us divide Ω into open subsets Ω_h, $h = 1, ..., H(\delta)$, whose diameter are less than δ. For every h we let x_h be a point in Ω_h. By (33) we have

$$F_i^0(x_h, v(x_h), Dv) < 0, \quad \forall h, \quad \forall i = 1, ..., N.$$

Then, by the continuity of F_i^δ with respect to δ, for every $h = 1, ..., H(\delta)$ we consider $\delta_h < \min\{\delta_0; \delta\}$ such that

$$F_i^{\delta_h}(x_h, v(x_h), Dv) < 0, \quad \forall i = 1, ..., N.$$

By applying Theorem 4.3 we can solve the differential problem

$$\begin{cases} F_i^{\delta_h}(x_h, v(x_h), Dw(x)) = 0, \text{ a.e. } x \in \Omega_h, \quad \forall i = 1, ..., N \\ w(x) = v(x), \; x \in \partial\Omega_h \end{cases} \tag{34}$$

and find $w_h \in W^{1,\infty}(\Omega_h; \mathbb{R}^m)$. Then the function w, defined in Ω by

$$w(x) = \begin{cases} w_h(x), \; x \in \Omega_h \\ v(x), \; x \in \Omega - \bigcup_h \Omega_h, \end{cases}$$

as $\delta \to 0$ converges in $L^\infty(\Omega)$ to v, since w has uniformly bounded gradient.

With the aim to compute the integral of F_i^0 over Ω, we observe that, by the continuity of $F_i^0(x, s, \xi)$, by the fact that w converges uniformly to v as $\delta \to 0$ and that Dw is uniformly bounded, then the difference

$$F_i^0(x, w(x), Dw(x)) - F_i^{\delta_h}(x_h, v(x_h), Dw(x))$$

converges uniformly to zero as $\delta \to 0$. Choosing δ sufficiently small and using (34), for every $i = 1, ..., N$ we obtain

$$\sum_{i=1}^{N} \int_{\Omega} F_i^0(x, w(x), Dw(x))\, dx$$
$$= \sum_{i=1}^{N} \sum_h \int_{\Omega_h} F_i^{\delta_h}(x_h, v(x_h), Dw(x))\, dx + \sum_{i=1}^{N} \sum_h \int_{\Omega_h} \{F_i^0(x, w(x), Dw(x))$$
$$- F_i^{\delta_h}(x_h, v(x_h), Dw(x))\}\, dx > -\frac{1}{k}$$

and hence $w \in V^k$. The density of V^k is therefore established.

We hence use Baire category theorem to deduce that $\cap V^k$ is dense in V and thus the theorem. ▮

If $N = 1$, we obtain the simpler following result

THEOREM 6.2. *Let $\Omega \subset \mathbb{R}^n$ be open. Let $F: \Omega \times \mathbb{R}^m \times \mathbb{R}^{m \times n} \to \mathbb{R}$ be continuous and quasiconvex in the last variable. Assume that $\{\xi \in \mathbb{R}^{m \times n}: F(x, s, \xi) \leqslant 0\}$ is compact for every $(x, s) \in \Omega \times \mathbb{R}^m$. If $\varphi \in C^1(\bar{\Omega}; \mathbb{R}^m)$ (or piecewise C^1) is such that*

$$F(x, \varphi(x), D\varphi(x)) \leqslant 0, \qquad \forall x \in \Omega, \tag{35}$$

then there exists (a dense set of) $u \in W^{1,\infty}(\Omega; \mathbb{R}^m)$ such that

$$\begin{cases} F(x, u(x), Du(x)) = 0, \text{ a.e. } x \in \Omega \\ u(x) = \varphi(x), x \in \partial\Omega. \end{cases} \tag{36}$$

Proof. We first, as usual, assume that Ω is bounded and $\varphi \in C^1(\bar{\Omega}; \mathbb{R}^m)$ with

$$F(x, \varphi(x), D\varphi(x)) \leqslant -\delta_0, \qquad \forall x \in \Omega,$$

for some $\delta_0 > 0$. This can be assumed with the same argument as the one of Corollary 4.2. We then define for $\delta \in [0, \delta_0)$

$$E^\delta = \{\xi \in \mathbb{R}^{m \times n}: F(x, s, \xi) = -\delta\}.$$

We observe that (as in Step 1 of Theorem 4.4)

$$Rco E^\delta = \{\xi \in \mathbb{R}^{m \times n}: F(x, s, \xi) \leqslant -\delta\}.$$

The proof is then identical to the one of the previous theorem with $F^\delta(x, s, \xi) = F(x, s, \xi) + \delta$, the only difference being that we use Theorem 4.4 instead of Theorem 4.3. ▮

Remark 6.1. (i) As in Remark 4.7, the required compactness can be weakened and it is sufficient to assume that there exists $\eta \in \mathbb{R}^{m \times n}$ with $rank\{\eta\} = 1$ such that $F(x, s, \xi + t\eta) \to +\infty$ as $|t| \to \infty$, for every $(x, s, \xi) \in \Omega \times \mathbb{R}^m \times \mathbb{R}^{m \times n}$.

(ii) To compare our last result with the literature (devoted to the scalar case) quoted in the introduction, we specialise the above theorem to the scalar case ($m = 1$). Thus we note that problem (36) has a $W^{1,\infty}(\Omega)$ solution if F is continuous, convex in the last variable, $F(x, s, \xi) \to +\infty$ as $|\xi| \to \infty$ and provided φ satisfies (35). We therefore recover the existence part of classical results, c.f. for example Lions [32]. Note however that we have no extra hypothesis on the variable s than continuity.

(iii) It is interesting to see that in the case of the eikonal equation the above result says that if $\varphi \in C^1(\bar{\Omega})$ (or piecewise C^1) and if

$$|D\varphi(x)| \leqslant a(x, \varphi(x)), \qquad \forall x \in \Omega,$$

where $a: \Omega \times \mathbb{R} \to \mathbb{R}_+$ is continuous, then the eikonal equation has a (dense set of) solution $u \in W^{1,\infty}(\Omega)$, i.e.

$$\begin{cases} |Du(x)| = a(x, u(x)), & \text{a.e. } x \in \Omega \\ u(x) = \varphi(x), & x \in \partial\Omega. \end{cases}$$

(iv) Observe that the vectorial problem can here be obtained from the scalar one. Indeed choosing the $(m-1)$ first components of u equal to the $(m-1)$ first components of φ, we would reduce the problem to a convex scalar one (since quasiconvexity of F implies convexity with respect to the last vector of Du).

We now give two more cases where our Theorem 6.1 applies. The first one deals with a scalar problem which is a generalization of the eikonal equation.

THEOREM 6.3. *Let $\Omega \subset \mathbb{R}^n$ be open. Let $a_i: \Omega \times \mathbb{R} \to \mathbb{R}$, $1 \leqslant i \leqslant n$, be continuous and such that $a_i(x, s) \geqslant \delta_0 > 0$, $\forall (x, s) \in \Omega \times \mathbb{R}$ and $\forall i = 1, ..., n$. Let $\varphi \in C^1(\bar{\Omega})$ (or piecewise C^1) such that*

$$\left|\frac{\partial \varphi}{\partial x_i}\right| < a_i(x, \varphi(x)), \qquad x \in \Omega, \qquad \forall i = 1, ..., n. \tag{37}$$

Then there exists (a dense set of) $u \in W^{1,\infty}(\Omega)$ such that

$$\begin{cases} \left|\dfrac{\partial u}{\partial x_i}\right| = a_i(x, u(x)), & \text{a.e. } x \in \Omega, \quad \forall i = 1, ..., n \\ u(x) = \varphi(x), & x \in \partial\Omega. \end{cases}$$

The above theorem follows immediately from Theorem 6.1 with

$$F_i^\delta(x, s, \xi) = |\xi_i| - (a_i(x, s) - \delta), \qquad \forall i = 1, ..., n, \delta \in [0, \delta_0).$$

The final example is an extension of Theorem 5.2.

THEOREM 6.4. *Let $\Omega \subset \mathbb{R}^2$ be open. Let $a_i: \Omega \times \mathbb{R}^2 \to \mathbb{R}$, $1 \leqslant i \leqslant 2$ be continuous and such that $0 < \delta_0 \leqslant a_1(x, s) \leqslant a_2(x, s)$ for every $(x, s) \in \Omega \times \mathbb{R}^2$. Let $\varphi \in C^1(\bar{\Omega}; \mathbb{R}^2)$ (or piecewise C^1) such that*

$$\begin{cases} \lambda_2(D\varphi(x)) < a_2(x, \varphi(x)), & \forall x \in \Omega \\ \lambda_1(D\varphi(x))\, \lambda_2(D\varphi(x)) < a_1(x, \varphi(x))\, a_2(x, \varphi(x)), & \forall x \in \Omega. \end{cases}$$

Then there exists (a dense set of) $u \in W^{1,\infty}(\Omega; \mathbb{R}^2)$ such that

$$\begin{cases} \lambda_1(Du(x)) = a_1(x, u(x)), \text{ a.e. } x \in \Omega \\ \lambda_2(Du(x)) = a_2(x, u(x)), \text{ a.e. } x \in \Omega \\ u(x) = \varphi(x), \ x \in \partial\Omega. \end{cases}$$

Proof. Define for every $\delta \in [0, \delta_0)$,

$$\begin{cases} F_1^\delta(x, s, \xi) = \lambda_1(\xi)\, \lambda_2(\xi) - (a_1(x, s) - \delta)(a_2(x, s) - \delta) \\ F_2^\delta(x, s, \xi) = \lambda_2(\xi) - (a_2(x, s) - \delta). \end{cases}$$

Note that F_2^δ is convex in ξ, while F_1^δ is polyconvex in ξ (since $\lambda_1(\xi)\, \lambda_2(\xi) = |\det \xi|$). The other hypotheses of Theorem 6.1 are established in Theorem 5.2. ∎

Remark 6.2. It is important to note that the choice of F_1^δ, F_2^δ above cannot be arbitrary. For example we could not have taken either of the following

$$\begin{cases} F_1^0(x, s, \xi) = \lambda_1(\xi) - a_1(x, s) \\ F_2^0(x, s, \xi) = \lambda_2(\xi) - a_2(x, s), \end{cases}$$

since $\xi \to \lambda_1(\xi)$ is not polyconvex, or

$$\begin{cases} F_1^0(x, s, \xi) = \lambda_1(\xi)\, \lambda_2(\xi) - a_1(x, s)\, a_2(x, s) \\ \qquad\qquad\quad = |\det \xi| - a_1 a_2 \\ F_2^0(x, s, \xi) = \lambda_1^2(\xi) + \lambda_2^2(\xi) - a_1^2(x, s) - a_2^2(x, s) \\ \qquad\qquad\quad = |\xi|^2 - a_1^2 - a_2^2 \end{cases}$$

since, although F_1^0 and F_2^0 are polyconvex, the set $PcoE = RcoE$ would not be equal to $\{F_1^0(x, s, \xi) \leqslant 0 \text{ and } F_2^0(x, s, \xi) \leqslant 0\}$.

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