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Dirichlet Problem for Nonlinear First Order Partial Differential Equations

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This article is in honor of Stefan Hildebrandt for his 60^{th} birthday.

ABSTRACT. We establish the existence of Lipschitz continuous solutions for the Dirichlet problem for nonlinear first order partial differential equations. We consider the simpler context of the scalar case for the sake of simplicity, to show the method of proof, which in fact holds in the general context of vector valued functions.

1. Introduction and statement of the main results

We consider the Dirichlet problem

$$(1.1) \qquad \begin{cases} F(Du(x)) = 0\,, & a.e.\ x \in \Omega \\ u(x) = \varphi(x), & x \in \partial\Omega \end{cases},$$

where Ω is an open set of $\mathbb{R}^n$, $u: \Omega \to \mathbb{R}$, Du denotes the gradient of u, $F: \mathbb{R}^n \to \mathbb{R}$ and $\varphi \in W^{1,\infty}(\Omega)$. We will find more convenient to rewrite it as

$$(1.2) \qquad \begin{cases} Du(x) \in E\,, & a.e.\ x \in \Omega \\ u(x) = \varphi(x)\,, & x \in \partial\Omega \end{cases},$$

where $E = \{\xi \in \mathbb{R}^n\colon F(\xi) = 0\}$.

The existence of $W^{1,\infty}(\Omega)$ solutions of (1.1) has been widely studied and we refer here only to few relevant articles: P.D. Lax [27], A. Douglis [21], S.N.

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Kruzkov [26], M.G. Crandall - P.L. Lions [12], M.G. Crandall - L.C. Evans-P.L. Lions [10], I. Capuzzo Dolcetta - L.C. Evans [4], I. Capuzzo Dolcetta - P.L. Lions [5], M.G. Crandall - H. Ishii - P.L. Lions [11]. We also quote some of the important monographies concerned with this question: S.H. Benton [3], P.L. Lions [28], W.H. Fleming - H. M. Soner [23], G. Barles [2], I. Subbotin [31] and M. Bardi - I. Capuzzo Dolcetta [1].

One of our motivations to study the Dirichlet problem (1.1), or (1.2), besides its intrinsic interest, comes from the calculus of variations (see [15]). In this context, first order partial differential equations have been intensively studied, c.f. for example the monographies of C. Carathéodory [6] and M. Giaquinta - S.Hildebrandt [25].

In most of the above references, the existence of solutions of (1.1) uses the notion of *viscosity solutions*. These solutions have usually several other properties such as explicit formulas, uniqueness or maximality properties. However, if one is only interested in the existence of solutions, the hypotheses made to obtain viscosity solutions are too strong. In [16], [17], [18], [19] we have proposed a new approach to settle this problem. It allows us to obtain existence of solutions under much weaker hypotheses and, at the same time, to treat vectorial problems (i.e., $u: \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$, with $m > 1$).

The aim of this paper is to propose an existence theorem for the Dirichlet problem (1.2) in the scalar case; we consider this simpler context for the sake of simplicity, to show the method of proof, which in fact holds in the general context of vector valued functions. We now state our main theorem.

THEOREM. – *Let $\Omega \subset \mathbb{R}^n$ be an open set. Let $\varphi \in W^{1,\infty}(\Omega)$ be such that*

$$(1.3) \qquad D\varphi(x) \text{ is compactly contained in } \operatorname{int co} E, \text{ a.e. } x \in \Omega,$$

where int co E *denotes the interior of the convex hull of E. If in addition $\varphi \in C^1(\Omega)$ and E is closed, (1.3) can be replaced by*

$$(1.4) \qquad D\varphi(x) \in E \cup \operatorname{int co} E, \; x \in \Omega.$$

Then there exist (densely many) $u \in W^{1,\infty}(\Omega)$ such that

$$(1.5) \qquad \begin{cases} Du(x) \in E, & \text{a.e. } x \in \Omega \\ u(x) = \varphi(x), & x \in \partial\Omega \end{cases}.$$

Remarks: *i)* One should note that, in terms of the function F given in (1.1), the only hypotheses that are made are on the set of zeros of F. Therefore no convexity, coercivity or continuity on F is assumed.

ii) It will be clear in the proof of the theorem that the statement on density means that the set of solutions of (1.1) or (1.2) is dense (with some extra hypotheses on E) in

$$(1.6) \qquad V = \left\{ u \in \varphi + W_0^{1,\infty}(\Omega) \colon \; Du(x) \in \operatorname{co} E \text{ a.e. } x \in \Omega \right\},$$

endowed with the L^∞ metric.

iii) In some sense our theorem is optimal and we would not generically expect to have a $W^{1,\infty}$ solution if $D\varphi(x) \notin \operatorname{int} \operatorname{co} E$ for a set of $x \in \Omega$ of positive measure (see the example below).

iv) The previous theorem has been established in [16], [17] as a by-product of a result on vectorial problems. Here we will prove it without the use of the vectorial machinery; the proof hence being simplified a great deal and, at the same time, giving more insight on the method of proof. The main tools in proving the theorem are the Baire category method (method introduced by A. Cellina [7] and developed by F. De Blasi - G. Pianigiani [20]) and the lower semicontinuity and relaxation theorems of the calculus of variations.

v) We should finally mention that our approach allows also to treat systems of equations

$$(1.7)\qquad \begin{cases} F_1(Du(x)) = F_2(Du(x)) = \dots = F_N(Du(x)) = 0\,, & a.e.\ x \in \Omega \\ u(x) = \varphi(x)\,, & x \in \partial\Omega \end{cases}.$$

It is sufficient to set

$$E = \{\xi \in \mathbb{R}^n\colon\ F_1(\xi) = F_2(\xi) = \dots = F_N(\xi) = 0\}.$$

To end this introduction we would like to give an example where the difference between our approach and the viscosity one is particularly clear.

EXAMPLE. Let $\Omega \subset \mathbb{R}^2$ be open. Let $u = u(x_1, x_2)$ and consider the problem

$$(1.8)\qquad \begin{cases} \left(\left(\dfrac{\partial u}{\partial x_1}\right)^2 - 1\right)^2 + \left(\left(\dfrac{\partial u}{\partial x_2}\right)^2 - 1\right)^2 = 0\,, & a.e.\ x \equiv (x_1, x_2) \in \Omega \\ u(x) = \varphi(x)\,, & x \in \partial\Omega \end{cases}.$$

Then we have

$$(1.9)\qquad E = \left\{\xi = (\xi_1, \xi_2) \in \mathbb{R}^2\colon\ |\xi_1| = |\xi_2| = 1\right\},$$

$$(1.10)\qquad \operatorname{co} E = \left\{\xi = (\xi_1, \xi_2) \in \mathbb{R}^2\colon\ |\xi_1| \le 1,\ |\xi_2| \le 1\right\}.$$

Thus, if $\varphi \in C^1(\overline{\Omega})$ and $D\varphi(x) \in \operatorname{int} \operatorname{co} E$ for every $x \in \Omega$, our theorem asserts that (1.8) has solution. In particular (1.8) has $W^{1,\infty}$ solutions if $\varphi \equiv 0$.

However in [14] it is shown that, if Ω is convex and $\varphi \equiv 0$, then (1.8) has a $W^{1,\infty}$ viscosity solution if and only if Ω is a rectangle whose faces are parallel to the vectors $(1,1)$ and $(1,-1)$.

Let us finally notice that in [17] it has been proved that, if Ω is the open square $\Omega = (0,1) \times (0,1)$ and $\varphi(x_1, x_2) = x_1 + \beta x_2$, with $|\beta| < 1$, then

$D\varphi = (1,\beta) \in \operatorname{co} E$ but $D\varphi \notin \operatorname{int}\operatorname{co} E$, and the Dirichlet problem (1.8) does not have $W^{1,\infty}$ solutions.

2. Some preliminary lemmas

We start with a slight refinement of the classical finite element approximation.

LEMMA 2.1. – *Let $\Omega \subset \mathbb{R}^n$ be a bounded set. Let $K \subset \mathbb{R}^n$ be a compact and convex set, with non empty interior. Let $u \in W^{1,\infty}(\Omega)$ be such that*

(2.1) $$Du(x) \text{ is compactly contained in } \operatorname{int} K, \text{ a.e. } x \in \Omega.$$

Then there exist $\Omega_\nu \subset \Omega$ open sets and $u_\nu \in W^{1,\infty}(\Omega)$ such that

(2.2) $$\Omega_\nu \subset \Omega_{\nu+1} \text{ and } \operatorname{meas}(\Omega - \Omega_\nu) \to 0 \text{ as } \nu \to +\infty,$$

(2.3) $$u_\nu \text{ is piecewise affine on } \Omega_\nu,$$

(2.4) $$u_\nu = u \text{ on } \partial\Omega,$$

(2.5) $$u_\nu \to u \text{ uniformly in } \Omega,$$

(2.6) $$Du_\nu \to Du \text{ a.e. in } \Omega,$$

(2.7) $$\|Du_\nu\|_{L^\infty} \le \|Du\|_{L^\infty} + c(\nu), \text{ with } c(\nu) \to 0 \text{ as } \nu \to +\infty,$$

(2.8) $$Du_\nu \text{ is compactly contained in } \operatorname{int} K, \text{ a.e. in } \Omega.$$

Lemma 2.1 is proved in [17] (c.f. Lemma 6.1). Apart from the condition (2.8), it is a totally standard result.

The next lemma is a refinement of an often used result in the calculus of variations, to prove necessary conditions (c.f. for example I. Ekeland - R. Temam [22], or [13]). In the vectorial case it has been established in [17].

LEMMA 2.2. – *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let $E \subset \mathbb{R}^n$ be compact such that $\operatorname{int}\operatorname{co} E \neq \emptyset$. Let $\varphi(x) = (\xi, x)$ for $x \in \Omega$ and for fixed $\xi \in \operatorname{int}\operatorname{co} E$. Then, for every $\epsilon > 0$, there exist an open set $\tilde{\Omega}$ contained in Ω and a function $u \in W^{1,\infty}(\Omega)$ such that*

(2.9) $$\operatorname{meas}(\Omega - \tilde{\Omega}) \le \epsilon,$$

(2.10) $$u(x) = \varphi(x), \ x \in \partial\Omega,$$

(2.11) $$|u(x) - \varphi(x)| \le \epsilon, \ \forall x \in \Omega,$$

(2.12) $$Du(x) \text{ is compactly contained in } \operatorname{int}\operatorname{co} E, \text{ a.e. } x \in \Omega,$$

$$\operatorname{dist}\big(Du(x), E\big) \le \epsilon, \quad a.e.\ x \in \widetilde{\Omega}\,, \tag{2.13}$$

Proof: We divide the proof into three steps.

Step 1: Since $\xi \in \operatorname{co} E$, by Carathéodory theorem we can write

$$\xi = \sum_{i=1}^{s} \lambda_i \xi_i, \qquad \xi_i \in E\,, \tag{2.14}$$

with $s \le n+1$. Since $\xi \in \operatorname{int}\operatorname{co} E$, we can assume that, for every $\epsilon > 0$, (2.14) can be rewritten as

$$\begin{cases} \xi = \sum_{i=1}^{s} \lambda_i \xi_i \\ \xi_i \in \operatorname{int}\operatorname{co} E, \text{ and } \operatorname{dist}(\xi_i, E) \le \epsilon, \ \forall i \end{cases} . \tag{2.15}$$

Step 2: Now we reason by induction on s, and we start with the case $s = 2$, i.e.

$$\begin{cases} \xi = \lambda \xi_1 + (1-\lambda)\xi_2 \\ \xi_1, \xi_2 \in \operatorname{int}\operatorname{co} E, \quad \operatorname{dist}(\xi_1, E), \operatorname{dist}(\xi_2, E) \le \epsilon\,, \end{cases} . \tag{2.16}$$

There is no loss of generality if we assume that

$$\xi_1 - \xi_2 = \begin{pmatrix} \alpha \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{2.17}$$

otherwise perform a rotation. Then we express Ω as a union of cubes, whose faces are parallel to the axes, and a set of small measure. We set $u(x) = \varphi(x)$ on this last set and we do the construction on each cube. So, without loss of generality, we assume that $\Omega = (0,1)^n$ is the unit cube.

We let N be a fixed integer. Let us define $\psi \in W_0^{1,\infty}((0,1))$ in the following way

$$\begin{cases} [0,1] = \overline{I}_N \cup \overline{J}_N, & I_N \cap J_N = \emptyset \\ \operatorname{meas} I_N = \lambda, & \operatorname{meas} J_N = 1-\lambda \\ \psi'(x_1) = (1-\lambda)\alpha, & \text{if } x_1 \in I_N \\ \psi'(x_1) = -\lambda\alpha, & \text{if } x_1 \in J_N \\ \psi(0) = \psi(1) = 0 \\ |\psi(x_1)| \le \delta(N), & \text{where } \delta(N) \to 0 \text{ as } N \to +\infty \end{cases} , \tag{2.18}$$

Then we denote by $\Omega_\delta = (\sqrt{\delta}, 1-\sqrt{\delta})^{n-1}$. Therefore we can write

$$\Omega = (0,1)^n = \big[(0,1)\times\Omega_\delta\big]\cup\big[(0,1)\times\Omega_\delta^C\big],$$

where $\Omega_\delta^C = \Omega - \Omega_\delta$. Moreover, we define $\eta \in W^{1,\infty}(\Omega)$ to be any function satisfying

$$\text{(2.19)}\qquad \begin{cases} \eta(x) = 1\,, & \text{if } x\in(0,1)\times\Omega_\delta \\ \eta(x) = 0\,, & \text{if } x_1\in(0,1) \text{ and } (x_2,x_3,\dots,x_n)\in\partial(0,1)^{n-1} \\ 0\le\eta(x)\le 1\,, & \text{for every } x\in\Omega \\ \big|D\eta(x)\big|\le a/\sqrt{\delta} & \text{in } \Omega \quad (\text{for a certain } a>0) \end{cases}.$$

Then we let

$$\text{(2.20)}\qquad u(x) = \psi(x_1)\,\eta(x) + \varphi(x)\,.$$

First note that $u=\varphi$ on $\partial\Omega$. Indeed, if $x_1 = 0$ or $x_1 = 1$ we have $\psi = 0$ by (2.18); while, if $(x_2,x_3,\dots,x_n)\in\partial(0,1)^{n-1}$, then $\eta = 0$ by (2.19). Furthermore

$$\text{(2.21)}\qquad \begin{cases} \dfrac{\partial u}{\partial x_1} = \psi'(x_1)\,\eta(x) + \psi(x_1)\dfrac{\partial\eta}{\partial x_1} + \dfrac{\partial\varphi}{\partial x_1} \\ \dfrac{\partial u}{\partial x_i} = \psi(x_1)\dfrac{\partial\eta}{\partial x_i} + \dfrac{\partial\varphi}{\partial x_i}\,, \qquad \text{if } i\ge 2 \end{cases}$$

Since, by (2.18), (2.19), the quantity $\psi(x_1)\,\partial\eta/\partial x_i$ can be as small as we want, we have obtained the result by setting $\tilde\Omega = (0,1)\times\Omega_\delta$. Indeed, observe that, in $\tilde\Omega$, Du is either equal to ξ_1 or ξ_2, by (2.16), (2.17), (2.18), and since $D\varphi = \xi = \lambda\,\xi_1 + (1-\lambda)\,\xi_2$.

Step 3: Now we assume that the result has been established for s and we wish to show that it holds for $s+1$. So we assume that

$$\text{(2.22)}\qquad \begin{cases} \xi = \sum\limits_{i=1}^{s+1}\lambda_i\,\xi_i = = \lambda_1\,\xi_1 + (1-\lambda_1)\,\eta \\ \text{where } \eta = \dfrac{1}{1-\lambda_1}\sum\limits_{i=2}^{s+1}\lambda_i\,\xi_i \\ \xi_i\in\operatorname{int}\operatorname{co}E, \ \text{ and } \operatorname{dist}(\xi_i,E)\le\epsilon,\ \forall i \end{cases}.$$

By step 2 we can find two disjoint open sets Ω_1, Ω_1', and $u_1\in W^{1,\infty}(\Omega)$ such that

$$\big|\operatorname{meas}\Omega_1 - \lambda_1\operatorname{meas}\Omega\big|\,,\ \big|\operatorname{meas}\Omega_1' - (1-\lambda_1)\operatorname{meas}\Omega\big| \le \frac{\epsilon}{2(s+1)}\,,$$

$$u_1(x) = \varphi(x)\,, \qquad x\in\partial\Omega\,,$$

$$\big|u_1(x)-\varphi(x)\big|\le\frac{\epsilon}{2}\,, \qquad x\in\Omega\,,$$

$Du_1(x)$ is compactly contained in $\operatorname{int}\operatorname{co}E$, a.e. $x\in\Omega$,

$$Du_1(x) = \begin{cases} \xi_1 & \text{a.e. } x \in \Omega_1 \\ \eta & \text{a.e. } x \in \Omega_1' \end{cases}.$$

We apply the hypothesis of induction to each connected component of Ω_1', with φ replaced by u_1 on each component. Hence we can find $\Omega_2, \Omega_3, \ldots, \Omega_{s+1}$ disjoint open sets and $u_1' \in W^{1,\infty}(\Omega_1')$ such that

$$\left| \operatorname{meas} \Omega_i - \frac{\lambda_i}{1-\lambda_1} \operatorname{meas} \Omega_1' \right| \le \frac{\epsilon}{2(s+1)}, \quad \forall i = 2, 3, \ldots, s+1,$$

$$u_1'(x) = u_1(x), \qquad \forall x \in \partial\Omega_1',$$

$$|u_1'(x) - u_1(x)| \le \frac{\epsilon}{2}, \qquad \forall x \in \Omega_1',$$

$Du_1(x)$ is compactly contained in $\operatorname{int}\operatorname{co} E$, a.e. $x \in \Omega_1'$,

$$Du_1'(x) = \xi_i, \quad \text{a.e. } x \in \Omega_i, \quad \forall i = 2, 3, \ldots, s+1,$$

and thus

$$\operatorname{dist}(Du_1'(x), E) \le \epsilon, \qquad \text{a.e. } x \in \bigcup_{i=2}^{s+1} \Omega_i.$$

Writing

$$u(x) = \begin{cases} u_1(x) & \text{for } x \in \Omega - \Omega_1' \\ u_1'(x) & \text{for } x \in \Omega_1' \end{cases},$$

we have the conclusion of the lemma, with $\widetilde{\Omega} = \bigcup_{i=1}^{s+1} \Omega_i$.

3. Proof of the Theorem

We will first prove the theorem under stronger hypotheses on the set E. We will then remove these new hypotheses and, at the same time, treat the case $\varphi \in C^1(\Omega)$. Before enter in the proof, let us observe that there is no loss of generality in assuming Ω bounded; otherwise, cover Ω by a countable union of bounded open sets and solve the problem on each of these sets.

LEMMA 3.1. – *Let $\Omega \subset \mathbb{R}^n$ be open and $E \subset \mathbb{R}^n$ be compact. Assume that the points of E are extreme points of the convex hull of E. Let $\varphi \in W^{1,\infty}(\Omega)$ with*

(3.1) $\quad D\varphi(x)$ *compactly contained in* $\operatorname{int}\operatorname{co} E$*, a.e.* $x \in \Omega$.

Then there exist (densely many) $u \in W^{1,\infty}(\Omega)$ such that

$$(3.2)\qquad \begin{cases} Du(x) \in E\,, & a.e.\ x \in \Omega \\ u(x) = \varphi(x)\,, & x \in \partial\Omega \end{cases}.$$

Remark: The condition on extreme points just means that no $\xi \in E$ can be expressed, other that trivially, as a convex combination of other points of E.

Proof: We divide the proof into four steps.

Step 1: Let

$$(3.3)\qquad g(\xi) = \begin{cases} -\operatorname{dist}(\xi, E) & \text{if } \xi \in \operatorname{co} E \\ +\infty & \text{if } \xi \notin \operatorname{co} E \end{cases}.$$

Then define f as the convex envelope of g, i.e.

$$(3.4)\qquad f(\xi) = Cg(\xi)\,.$$

Note that $f(\xi) = +\infty$ if $\xi \notin \operatorname{co} E$. Since $\operatorname{co} E$ is closed, then f is lower semicontinuous and convex. Furthermore

$$(3.5)\qquad f(\xi) = 0 \quad \Leftrightarrow \quad \xi \in E\,.$$

($\Rightarrow$) Indeed, $0 = f(\xi) \le g(\xi) \le 0$, thus $g(\xi) = 0$ and $\xi \in E$.

($\Leftarrow$) By Carathéodory theorem

$$(3.6)\qquad f(\xi) = Cg(\xi) = \inf\left\{ \sum_{i=1}^{n+1} \lambda_i\, g(\xi_i) \colon \ \sum_{i=1}^{n+1} \lambda_i \xi_i = \xi \right\}.$$

Since $g(\xi) = +\infty$ if $\xi \notin \operatorname{co} E$, then in the infimum in (3.6) we can impose the conditions $\xi_i \in \operatorname{co} E$ for every $i = 1, 2, \ldots, n+1$.

By assumption $\xi \in E$ and points of E are extreme points of $\operatorname{co} E$; thus ξ cannot be written as a non trivial convex combination of points $\xi_i \in \operatorname{co} E$. Therefore, we deduce that, if $\xi \in E$, the infimum in (3.6) is realized only by $\xi_i = \xi$ for every $i = 1, 2, \ldots, n+1$; hence $f(\xi) = 0$.

Step 2: We define

$$V = \left\{ u \in \varphi + W_0^{1,\infty}(\Omega) \colon \ Du(x) \in \operatorname{co} E \text{ a.e. in } \Omega \right\}$$

(equivalently, by Step 1, $Du(x) \in \operatorname{co} E$ if and only if $f(Du(x)) \le 0$ a.e. in Ω).

First let us note that $V \neq \emptyset$, since $\varphi \in V$. We endow V with the L^∞ metric and we claim that V is a complete metric space.

Indeed, let $\{u_\nu\}$ be a Cauchy sequence in V. Since $\operatorname{co} E$ is bounded, we can extract a subsequence $\{u_{\nu_i}\}$ which converges weak* in $W^{1,\infty}$ to a function u. Since $\operatorname{co} E$ is closed and convex, we deduce that $u \in V$. Since $\{u_\nu\}$ is a Cauchy sequence, the whole sequence (and not only the subsequence) converges to u in

L^∞. Thus V is complete.

We then let, for $k \in \mathbb{N}$,

$$V_k = \left\{ u \in V: \int_\Omega f(Du(x))\,dx > -\frac{1}{k} \right\}.$$

Suppose that we can show that

(i) V_k is open in V (c.f. Step 3);

(ii) V_k is dense in V (c.f. Step 4).

We will then deduce, by Baire category theorem, that $\bigcap_{k=1}^{\infty} V_k$ is dense in V and hence nonempty; observe that

$$\bigcap_{k=1}^{\infty} V_k = \left\{ u \in V: \int_\Omega f(Du(x))\,dx \geq 0 \right\}$$

$$= \left\{ u \in \varphi + W_0^{1,\infty}(\Omega): f(Du(x)) = 0 \text{ a.e. in } \Omega \right\}$$

$$= \left\{ u \in \varphi + W_0^{1,\infty}(\Omega): Du(x) \in E \text{ a.e. in } \Omega \right\},$$

where we have used the definition of f and (3.5). Therefore the proof will be complete once Step 3 and Step 4 below will be established.

Step 3: We show that $V - V_k$ is closed. Indeed let

$$\begin{cases} u_\nu \to u & \text{in } L^\infty \\ u_\nu \in V - V_k \end{cases}.$$

By step 2 we know that $u \in V$. Since f is convex and lower semicontinuous, we know that

$$\int_\Omega f(Du(x))\,dx \leq \liminf_{\nu \to +\infty} \int_\Omega f(Du_\nu(x))\,dx \leq -\frac{1}{k},$$

and thus $u \in V - V_k$.

Step 4: We show that V_k is dense in V. To this aim let $u \in V$. Since $\varphi \in V$ and satisfies (3.1), i.e.

(3.7) $D\varphi(x)$ is compactly contained in $\operatorname{int} \operatorname{co} E$, a.e. $x \in \Omega$,

then we can assume that

(3.8) $Du(x)$ is compactly contained in $\operatorname{int} \operatorname{co} E$, a.e. $x \in \Omega$;

indeed, if this were not the case, using the convexity of $\operatorname{co} E$ and (3.1), we would replace u by $(1-t)u+t\varphi$, with $t>0$ sufficiently small to get (3.8).

We use Lemma 2.1 to find, for every $\epsilon>0$, $\widetilde{\Omega}\subset\Omega$ and $u_\epsilon\in W^{1,\infty}(\Omega)$ such that

$$(3.9)\qquad \begin{cases} \operatorname{meas}(\Omega-\widetilde{\Omega})\le\epsilon \\ u_\epsilon=u \quad \text{on } \partial\Omega \\ \|u_\epsilon-u\|_{L^\infty}\le\frac{\epsilon}{2} \\ u_\epsilon \ \text{ is piecewise affine in } \widetilde{\Omega} \\ Du_\epsilon \ \text{ is compactly contained in } \operatorname{int} K, \text{ a.e. in } \Omega \end{cases}.$$

Since u_ϵ is piecewise affine in $\widetilde{\Omega}$, we can find disjoint open sets $\Omega_i\subset\widetilde{\Omega}$, $1\le i\le N$ and $\xi_i\in\operatorname{co} E$ such that

$$(3.10)\qquad \begin{cases} \operatorname{meas}(\widetilde{\Omega}-\bigcup_{i=1}^{N}\Omega_i)=0 \\ Du_\epsilon(x)=\xi_i\in\operatorname{co} E \ \text{ in } \Omega_i,\ 1\le i\le N \end{cases}.$$

Now we apply Lemma 2.2 to u_ϵ and Ω_i. Therefore, for every $i\in\{1,2,\dots,N\}$, there exist $u_{\epsilon,i}\in W^{1,\infty}(\Omega_i)$ and an open set $\widetilde{\Omega}_i\subset\Omega_i$ so that

$$(3.11)\qquad \begin{cases} \operatorname{meas}(\Omega_i-\widetilde{\Omega}_i)\le\frac{\epsilon}{2^i} \\ u_\epsilon=u_{\epsilon,i} \ \text{ on } \partial\Omega_i \\ \left|u_{\epsilon,i}(x)-u_\epsilon(x)\right|\le\frac{\epsilon}{2},\quad \forall x\in\overline{\Omega}_i \\ Du_{\epsilon,i}(x) \text{ is compactly contained in } \operatorname{int}\operatorname{co} E,\ \text{ a.e. } x\in\Omega_i \\ \operatorname{dist}\left(Du_{\epsilon,i}(x),E\right)\le\frac{\epsilon}{2^i},\qquad \text{a.e. } x\in\widetilde{\Omega}_i \end{cases}.$$

The function

$$(3.12)\qquad v_\epsilon(x)=\begin{cases} u_{\epsilon,i}(x) & \text{if } x\in\overline{\Omega}_i,\ 1\le i\le N \\ u(x) & \text{if } x\in\Omega-\widetilde{\Omega} \end{cases},$$

belongs to V, since $v_\epsilon=u=\varphi$ on $\partial\Omega$ and $Dv_\epsilon\in\operatorname{co} E$ a.e. in Ω. Furthermore

$$(3.13)\qquad \|v_\epsilon-u\|_{L^\infty}\le\epsilon.$$

Therefore, if we can show that $v_\epsilon\in V_k$, indeed we will have obtained the claimed density and the proof will be complete. So let us show that $v_\epsilon\in V_k$. In fact

$$(3.14)\qquad \int_\Omega f(Dv_\epsilon(x))\,dx=$$

$$= \int_{\Omega - \tilde{\Omega}} f(Du)\,dx + \sum_{i=1}^{N} \int_{\Omega_i - \tilde{\Omega}_i} f(Du_{\epsilon,i})\,dx + \sum_{i=1}^{N} \int_{\tilde{\Omega}_i} f(Du_{\epsilon,i})\,dx$$

$$\geq -\max\left\{|f(\xi)|:\ \xi \in \operatorname{co} E\right\}\left(\operatorname{meas}(\Omega - \tilde{\Omega}) + \sum_{i=1}^{N} \operatorname{meas}(\Omega_i - \tilde{\Omega}_i)\right)$$

$$+ \sum_{i=1}^{N} \int_{\tilde{\Omega}_i} f(Du_{\epsilon,i})\,dx\,.$$

On the other hand f is Lipschitz continuous in $\operatorname{co} E$, so that there exists $L > 0$ such that, for every $\xi \in E$ ($\Rightarrow f(\xi) = 0$)

$$|f(\eta)| = |f(\eta) - f(\xi)| \leq L|\eta - \xi|;$$

hence

$$|f(\eta)| \leq L \operatorname{dist}(\eta, E), \qquad \forall \eta \in \operatorname{co} E.$$

Thus

$$\sum_{i=1}^{N} \int_{\tilde{\Omega}_i} f(Du_{\epsilon,i})\,dx \geq -L \sum_{i=1}^{N} \int_{\tilde{\Omega}_i} \operatorname{dist}\big(Du_{\epsilon,i}(x), E\big)\,dx \geq -L\epsilon \operatorname{meas}\Omega\,.$$

Hence, returning to (3.14), we have obtained that

$$\int_{\Omega} f(Dv_\epsilon(x))\,dx \geq -\epsilon \operatorname{meas}\Omega\left(L + \max\left\{|f(\xi)|:\ \xi \in \operatorname{co} E\right\}\right).$$

Choosing ϵ sufficiently small, we have indeed that $v_\epsilon \in V_k$. This achieves the proof of Lemma 3.1.

We now turn to the proof of the main theorem stated in the introduction.

Proof of the main theorem: We divide the proof into two steps. The first one will explain how to reduce the theorem to Lemma 3.1. The second one will deal with the C^1 case.

Step 1: Since (1.3) holds, we can find a compact and convex set $L \subset \mathbb{R}^n$ such that

$$D\varphi(x) \in L \subset \operatorname{int} \operatorname{co} E\,, \qquad \text{a.e. } x \in \Omega. \tag{3.15}$$

We can then find a polytope P (c.f. the proof of Theorem 20.4 in Rockafellar [29]) with the following property

$$(3.16)\qquad \begin{cases} P = \operatorname{co}\{\eta_1,\eta_2,\dots,\eta_N\} \\ L \subset \operatorname{int} P \subset P \subset \operatorname{int}\operatorname{co} E \end{cases},$$

for some $\eta_1,\eta_2,\dots,\eta_N$ in the interior of the convex hull of E. Then we use Carathéodory theorem to write

$$(3.17)\qquad \begin{cases} \eta_k = \sum_{i=1}^{n+1} \lambda_i^k \xi_i^k, \qquad \forall\, k = 1,2,\dots,N \\ \text{where } \xi_i^k \in E, \quad \forall\, k = 1,2,\dots,N, \quad \forall\, i = 1,2,\dots,n+1 \end{cases};$$

this is possible since $\eta_k \in P \subset \operatorname{co} E$ for every $k = 1,2,\dots,N$. Combining (3.15), (3.16), (3.17) we find that, for a.e. $x \in \Omega$,

$$(3.18)\qquad D\varphi(x) \in L \subset \operatorname{int} P \subset P \subset \operatorname{co}\{\xi_1^1,\dots,\xi_{n+1}^1,\dots,\xi_1^N,\dots,\xi_{n+1}^N\},$$

with $\xi_i^k \in E$ for every k and i. Among the $\{\xi_1^1,\dots,\xi_{n+1}^1,\dots,\xi_1^N,\dots,\xi_{n+1}^N\}$ we remove all the ξ_i^k which are non trivial convex combinations of the others (i.e., we keep only those which are extreme points) and we relabel the remaining ones $\{\xi_1,\xi_2,\dots,\xi_s\}$. Therefore, summarizing what we have just obtained, we can write

$$(3.19)\qquad \begin{cases} D\varphi(x) \in L \subset \operatorname{int}\operatorname{co}\{\xi_1,\xi_2,\dots,\xi_s\} \\ \xi_i \in E \text{ and none of the } \xi_i \text{ is convex combination of the other ones} \end{cases}$$

Then we can apply Lemma 3.1 with E there equal to the finite set of points $\{\xi_1,\xi_2,\dots,\xi_s\}$. We get $u \in W^{1,\infty}(\Omega)$ such that

$$(3.20)\qquad \begin{cases} Du(x) \in \{\xi_1,\xi_2,\dots,\xi_s\} \subset E, & a.e.\ x \in \Omega \\ u(x) = \varphi(x), & x \in \partial\Omega \end{cases},$$

which is the desired result.

Step 2: Now we assume that $\varphi \in C^1(\Omega) \cap W^{1,\infty}(\Omega)$, with Ω bounded. Let us define

$$(3.21)\qquad \Omega_0 = \{x \in \Omega\colon\ D\varphi(x) \in E\}.$$

Since $\varphi \in C^1(\Omega)$ and E is closed, we deduce that $\Omega - \Omega_0$ is open. We define

$$(3.22)\qquad u(x) = \varphi(x), \qquad \forall\, x \in \Omega_0.$$

It remains to solve

$$(3.23)\qquad \begin{cases} Du(x) \in E, & a.e.\ x \in \Omega - \Omega_0 \\ u(x) = \varphi(x), & x \in \partial(\Omega - \Omega_0) \end{cases}.$$

By construction we know that

$$D\varphi(x) \in \operatorname{int} \operatorname{co} E\,, \qquad \forall x \in \Omega - \Omega_0\,. \tag{3.24}$$

For every $t > 0$ let $\Omega^t = \{x \in \Omega - \Omega_0\colon\ \operatorname{dist}(D\varphi(x), \partial(\operatorname{co} E)) = t\}$. We will show below that we can find a decreasing sequence t_k of positive numbers, converging to zero, such that

$$\operatorname{meas} \Omega^{t_k} = 0\,, \qquad \forall k \in \mathbb{N}. \tag{3.25}$$

Assume for a moment that (3.25) has been established. Let

$$\Omega_k = \{x \in \Omega - \Omega_0\colon\ t_{k+1} < \operatorname{dist}(D\varphi(x), \partial(\operatorname{co} E)) < t_k\}\,, \qquad \forall k \in \mathbb{N}.$$

Then Ω_k is open and

$$\begin{cases} \overline{\Omega - \Omega_0} = \bigcup_{k=1}^{\infty} \overline{\Omega}_k \\ \Omega - \Omega_0 = \bigcup_{k=1}^{\infty} \Omega_k \cup N, \quad \text{with } \operatorname{meas} N = 0 \\ \partial\Omega_k \subset \partial(\Omega - \Omega_0) \cup \Omega^{t_k} \cup \Omega^{t_{k+1}} \end{cases} \tag{3.26}$$

(the second statement is consequence of (3.25)). Now we use the theorem on Ω_k, since $D\varphi$ is compactly contained in $\operatorname{int} \operatorname{co} E$. Then, for every $k \in \mathbb{N}$, we find $u_k \in W^{1,\infty}(\Omega_k)$ such that

$$\begin{cases} Du_k(x) \in E\,, & a.e.\ x \in \Omega \\ u_k(x) = \varphi(x)\,, & x \in \partial\Omega \end{cases}. \tag{3.27}$$

Defining

$$u(x) = \begin{cases} u_k(x) & \text{if } x \in \overline{\Omega}_k \\ \varphi(x) & \text{if } x \in \Omega_0 \end{cases}, \tag{3.28}$$

we find that u has all the claimed properties.

Therefore, it remains only to show that (3.25) holds. To this aim we define, for every $k \in \mathbb{N}$, the set

$$T_k = \left\{ t > 0\colon\ \frac{1}{k+1} \le \frac{\operatorname{meas} \Omega^t}{\operatorname{meas}(\Omega - \Omega_0)} \le \frac{1}{k} \right\}. \tag{3.29}$$

We claim that this set T_k for every $k \in \mathbb{N}$ is finite. In fact, assume for the sake of contradiction that this is not so. Then, from the fact that

$$\Omega - \Omega_0 = \bigcup_{t>0} \Omega^t \supset \bigcup_{t \in T_k} \Omega^t\,, \tag{3.30}$$

we would get

$$\text{meas}\,(\Omega - \Omega_0) \geq \sum_{t \in T_k} \text{meas}\,\Omega^t \tag{3.31}$$

$$\geq \text{meas}\,(\Omega - \Omega_0) \sum_{t \in T_k} \frac{1}{k+1} \geq \frac{\text{meas}\,(\Omega - \Omega_0)}{k+1} \sum_{t \in T_k} 1 = +\infty,$$

which contradicts the fact that Ω is bounded.

It follows that the set

$$\{t > 0: \ \text{meas}\,\Omega^t > 0\} \subset \bigcup_{k=1}^{\infty} T_k \tag{3.32}$$

is countable. Therefore the set $\{t > 0: \ \text{meas}\,\Omega^t = 0\}$ is dense in $[0,1]$ and (3.25) is established.

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4. References

1. M. BARDI, I. CAPUZZO DOLCETTA, *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*, book in preparation.
2. G. BARLES, *Solutions de viscosite des equations de Hamilton-Jacobi*, Mathematiques and Applications, 17, Springer-Verlag, Berlin, 1994.
3. S. H. BENTON, *The Hamilton Jacobi equation. A global approach*, Academic Press, New York, 1977.
4. I. CAPUZZO DOLCETTA, L. C. EVANS, *Optimal switching for ordinary differential equations*, SIAM J. Optim. Control, **22** (1988), 1133-1148.
5. I. CAPUZZO DOLCETTA, P. L. LIONS, *Viscosity solutions of Hamilton-Jacobi equations and state-constraint problem*, Trans. Amer. Math. Soc., **318** (1990), 643-683.
6. C. CARATHEODORY, *Calculus of variations and partial differential equations of the first order*, Holden Day, San Francisco, 1965.
7. A. CELLINA, *On the differential inclusion $x' \in [-1,1]$*, Atti Accad. Naz. Lincei, Rend. Sc. Fis. Mat. Nat., **69** (1980), 1-6.
8. A. CELLINA, *On minima of a functional of the gradient: necessary conditions*, Nonlinear Analysis. Theory Meth. Appl., **20** (1993), 337-341.
9. A. CELLINA, *On minima of a functional of the gradient: sufficient conditions*, Nonlinear Analysis. Theory Meth. Appl., **20** (1993), 343-347.
10. M. G. CRANDALL, L. C. EVANS, P. L. LIONS, *Some properties of viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc., **282** (1984), 487-502.
11. M. G. CRANDALL, H. ISHII, P. L. LIONS, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc., **27** (1992), 1-67.
12. M. G. CRANDALL, P. L. LIONS, *Viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc., **277** (1983), 1-4.
13. B. DACOROGNA, *Direct methods in the calculus of variations*, Applied Math. Sciences, 78, Springer-Verlag, Berlin, 1989.

14. B. DACOROGNA, W. GANGBO, N. GEORGY, *Geometric restrictions for the existence of viscosity solutions*, to appear.
15. B. DACOROGNA, P. MARCELLINI, *Existence of minimizers for non quasiconvex integrals*, Arch. Rational Mech. Anal., **131** (1995), 359-399.
16. B. DACOROGNA, P. MARCELLINI, *Théorèmes d'existence dans les cas scalaire et vectoriel pour les équations de Hamilton-Jacobi*, C. R. Acad. Sci. Paris, **322** (1996), 237-240.
17. B. DACOROGNA, P. MARCELLINI, *General existence theorems for Hamilton-Jacobi equations in the scalar and vectorial cases*, to appear.
18. B. DACOROGNA, P. MARCELLINI, *Sur le problème de Cauchy-Dirichlet pour les systèmes d'équations non linéaires du premier ordre*, C. R. Acad. Sci. Paris, **323** (1996).
19. B. DACOROGNA, P. MARCELLINI, *Cauchy-Dirichlet problem for first order nonlinear systems*, to appear.
20. F. S. DE BLASI, G. PIANIGIANI, *Non convex valued differential inclusions in Banach spaces*, J. Math. Anal. Appl., **157** (1991), 469-494.
21. A. DOUGLIS, *The continuous dependence of generalized solutions of non linear partial differential equations upon initial data*, Comm. Pure Appl. Math., **14** (1961), 267-284.
22. I. EKELAND, R. TEMAM, *Analyse convexe et problemes variationnels*, Dunod Gauthier-Villars, Paris, 1974.
23. W. H. FLEMING, H. M. SONER, *Controllated Markov processes and viscosity solutions. Applications of Mathematics*, Springer-Verlag, Berlin, 1993.
24. G. FRIESECKE, *A necessary and sufficient condition for nonattainment and formation of microstructure almost everywhere in scalar variational problems*, Proc. Royal Soc. Edinburgh, **124** (1994), 437-471.
25. M. GIAQUINTA, S. HILDEBRANDT, *Calculus of variations I and II*, Grundlehren der Math. Wiss, 310-311, Springer, Berlin, 1966.
26. S. N. KRUZKOV, *Generalized solutions of Hamilton-Jacobi equation of eikonal type. USSR Sbornik*, **27** (1975), 406-446.
27. P. D. LAX, *Nonlinear hyperbolic systems of conservation laws*, In Nonlinear Problems, Univ. Wisconsin Press, Madison, 1963.
28. P. L. LIONS, *Generalized solutions of Hamilton-Jacobi equations*, Research Notes in Math., 69, Pitman, London, 1982.
29. R. T. ROCKAFELLAR, *Convex analysis*, Princeton Univ. Press, Princeton, 1970.
30. H. RUND, *The Hamilton-Jacobi theory in the calculus of variations*, Van Nostrand, Princeton, 1966.
31. A. I. SUBBOTIN, *Generalized solutions of first order partial differential equations: the dynamical optimization perspective*, Birkhauser, Boston, 1995.

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