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On the minimisation of non quasiconvex integrals of the calculus of variations

1. Introduction and relaxation theorem

Let

$$F(u) = \int_{\Omega} f(\nabla u(x))dx \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set, $u: \Omega \to \mathbb{R}^m$ and thus $\nabla u = \left(\frac{\partial u^\alpha}{\partial x_i}\right)_{1\le i\le n, 1\le\alpha\le m} \in \mathbb{R}^{nm}$ and $f: \mathbb{R}^{nm} \to \mathbb{R}$ is a lower semicontinuous function.

Finally let $u_o \in W^{1,\infty}(\Omega;\mathbb{R}^m)$ (where $W^{1,\infty}$ denotes the usual Sobolev space) and

(P) $\quad \inf\{F(u): u \in u_o + W_o^{1,\infty}(\Omega;\mathbb{R}^m)\}$.

To show the existence of minima for such problems in $W^{1,p}$ $(1 < p < \infty)$, one has, in general to make a coercitivity hypothesis $\left(\text{i.e. } f(\xi) \ge \alpha|\xi|^p + \beta \text{ for } \alpha > 0\right)$ and a hypothesis of semicontinuity of F with respect to weak convergence in $W^{1,p}$. A necessary and sufficient condition to obtain such a property is the notion of quasiconvexity introduced by Morrey [1].

Definition

A lower semicontinuous function $f: \mathbb{R}^{nm} \to \mathbb{R}$, is said to be <u>quasiconvex</u> if

$$\int_{\Omega} f(\xi + \nabla\varphi(x))dx \ge f(\xi)\cdot mes\Omega \tag{1.2}$$

for every $\xi \in \mathbb{R}^{nm}$ and for every $\varphi \in W_o^{1,\infty}(\Omega;\mathbb{R}^m)$.

Remarks:

(i) Jensen inequality ensures, trivially, that every convex function is quasiconvex. If $m = 1$ or if $n = 1$, these are in fact the only quasiconvex functions.

(ii) However as soon as $m,n > 1$, there exist quasiconvex functions that are not convex. A typical example of a quasiconvex function when $m = n = 2$ is

$$f(\xi) = g(\xi, \det \xi)$$

with $g: \mathbb{R}^{2\times 2} \times \mathbb{R} \to \mathbb{R}$ convex (in particular $f(\xi) = |\det \xi|$). Or if

$$f(\xi) = |\xi|^2 \left(|\xi|^2 - 2\det \xi\right)$$

we have that f is quasiconvex but not convex. For more details one can refer to Dacorogna [2] or to Alibert-Dacorogna [1].

In this article we will discuss the existence of minima for (P) when the function f is not quasiconvex. In general problem (P) will not have a solution as soon as $n > 1$.

Remark

The case $n = 1$ is peculiar and very simple. A necessary and sufficient condition for the existence of minima is given, for example, in Dacorogna [2]. In particular if f is coercive (P) always has a solution independently of the convexity of f, this is obviously not the case as soon as $n > 1$ or even if $n = 1$ and f depends explicitly on u and not only on u'.

In the general case $n,m \geq 1$ and even if (P) has no solution one can always associate a relaxed problem and one then has

Relaxation theorem

Let Ω, f and u_o as above. Let Qf be the quasiconvex envelope of f i.e.

$$Qf(\xi) = \sup\{\varphi(\xi) : \varphi \text{ quasiconvex and } \varphi \leq f\} \tag{1.3}$$

and

(QP) $$\inf\left\{ \overline{F}(u) = \int_\Omega Qf(\nabla u(x))dx : u \in u_o + W_o^{1,\infty}(\Omega; \mathbb{R}^m) \right\}$$

then $\qquad \forall \bar{u} \in u_o + W_o^{1,\infty}(\Omega; \mathrm{IR}^m), \quad \exists\, u_\nu \in u_o + W_o^{1,\infty}(\Omega; \mathrm{IR}^m)$

$$u_\nu \xrightarrow{*W^{1,\infty}} \bar{u}$$

$$F(u_\nu) = \int_\Omega f(\nabla u_\nu(x))\, dx \to \bar{F}(\bar{u}) = \int_\Omega Qf(\nabla \bar{u}(x)) dx$$

hence in particular

$$\inf(P) = \inf(QP). \tag{1.4}$$

Remarks

(i) By $u_\nu \xrightarrow{*W^{1,\infty}} \bar{u}$, we mean that u_ν converge weak * in $W^{1,\infty}$ to $\bar{u}$.

(ii) The above theorem has been established by Dacorogna [1],[2], generalising to the vectorial case (i.e. $m > 1$) the results of L.-C. Young, Mac Shane, Ekeland... (for more details cf. Dacorogna [2]).

(iii) If, in addition, f satisfies a coercitivity hypothesis, the problem (QP) has a solution in $W^{1,p}$.

(iv) Obviously the theorem does not say anything about the existence or the non existence of solutions for problem (P) but only about the minimising sequences. However it ensures that if (P) has a solution $\bar{u} \in u_o + W_o^{1,\infty}(\Omega; \mathrm{IR}^m)$ then from (1.4) and from the fact that

$$Qf(\xi) \le f(\xi) \tag{1.5}$$

for every $\xi \in \mathrm{IR}^{nm}$, $\bar{u}$ is also a solution of (QP) and

$$Qf(\nabla \bar{u}) = f(\nabla \bar{u}) \text{ a.e..} \tag{1.6}$$

(Note that in the case $m = 1$, the equation (1.6) is an equation of Hamilton-Jacobi type).

(v) The actual computation of the quasiconvex envelope defined by (1.3) is, in general, difficult (cf. below). Note that if $n = 1$ or if $m = 1$ then the quasiconvex envelope is nothing else than the convexe envelope of f i.e.

$$Cf(\xi) = \sup\{\varphi(\xi): \varphi \text{ convex and } \varphi \le f\}. \tag{1.7}$$

In the remaining part of the article we will restrict ourselves to the case where the boundary datum is linear i.e.

$$u_o(x) = \xi_o x \tag{1.8}$$

where $\xi_o \in \mathbb{R}^{nm}$. The problem (QP) has therefore as a trivial solution u_o itself. We are then going to study the existence of solutions for problem (P). We will follow Dacorogna-Marcellini [1] and we refer to this article for a larger bibliography on this problem which has been studied by many authors.

2. SUFFICIENT CONDITION FOR THE EXISTENCE OF MINIMA

We introduce first some notations

$$\xi = \begin{pmatrix} \xi_1^1 & \cdot & \cdot & \cdot & \xi_n^1 \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \xi_1^m & \cdot & \cdot & \cdot & \xi_n^m \end{pmatrix} = \begin{pmatrix} \xi^1 \\ \cdot \\ \cdot \\ \cdot \\ \xi^m \end{pmatrix} \in \mathbb{R}^{nm}.$$

Let $\Omega \subset \mathbb{R}^n$ be bounded and open, $f: \mathbb{R}^{nm} \to \mathbb{R}$ a lower semicontinuous function and

$$K = \{\xi \in \mathbb{R}^{nm}: Qf(\xi) < f(\xi)\}. \tag{2.1}$$

We then let $\xi_o \in K$ (recall that the boundary datum is $u_o(x) = \xi_o x$). By abuse of notations, we will identify K with its connected component which contains ξ_o. We then introduce a manifold L_o whose dimension is n and is defined by

$$L_o = \{\xi = (\xi^\alpha)_{1\le\alpha\le m} \in \mathbb{R}^{nm}: \xi^\alpha = \xi_o^\alpha + \mu^\alpha(\xi_o^m - \xi^m), \alpha = 1,2,\ldots,m-1\} \tag{2.2}$$

where $\mu^\alpha \in \mathbb{R}$ (if $m = 1$ we just let $L_o = \mathbb{R}^n$). We then have the following theorem

Theorem 2.1

Let Ω, f, u_o and (P) as above. If $\xi_o \in K$ and if

(i) $K \cap L_o$ is bounded and convex

(ii) Qf is quasiaffine on $K \cap L_o$

then (P) has a solution.

Remarks :

(i) In the case $n = m = 2$, by Qf quasiaffine on $K \cap L_o$ we mean that there exists $\lambda \in \mathbb{R}^{2\times 2}$ and $\mu, \nu \in \mathbb{R}$ such that

$$Qf(\xi) = < \lambda ; \xi > + \mu \det \xi + \nu = \sum_{i,\alpha=1}^{2} \lambda_i^\alpha \xi_i^\alpha + \mu \det \xi + \nu \tag{2.3}$$

for every $\xi \in K \cap L_o$.

(ii) In the general case $n, m \geq 1$, by Qf quasiaffine we mean that Qf is the sum of an affine function and a linear combination of minors of the matrix ξ.

(iii) The hypothesis $K \cap L_o$ bounded can be considerably weakened, it suffices to assume that $K \cap L_o$ is bounded in certain directions. More precisely

$$K \cap L_o = \left\{ \xi = (\xi^\alpha)_{1 \leq \alpha \leq m} \in L_o \subset \mathbb{R}^{nm} : \left(< b_j ; \xi^m >_{\mathbb{R}^n} \right)_{1 \leq j \leq \ell} \text{bounded in } \mathbb{R}^\ell \right\} \tag{2.4}$$

where ℓ is an integer between 1 and n and $b_1, \ldots, b_\ell$ are linearly independent vectors ($<.;.>_{\mathbb{R}^n}$ denoting the scalar product).

(iv) Finally note that if $\xi_o \notin K$, then problem (P) has u_0 itself as a trivial solution.

Example

Consider the case $m = n = 2$ and

$$f(\xi) = g(\xi^1) + h(\det \xi) = g(\xi_1^1, \xi_2^1) + h(\det \xi) \tag{2.5}$$

where $g: \mathbb{R}^2 \to \mathbb{R}$ is convex (in particular $g \equiv o$) and $h: \mathbb{R} \to \mathbb{R}$ is nonconvex. It is easy to see (cf. Dacorogna-Marcellini [1]) that

$$Qf(\xi) = g(\xi^1) + Ch(\det \xi) \tag{2.6}$$

where Ch is the convex envelope of h. We also trivially have

$$K = \{\xi \in \mathbb{R}^{2\times 2}: Qf(\xi) < f(\xi)\} = \{\xi \in \mathbb{R}^{2\times 2}: Ch(\det \xi) < h(\det \xi)\}. \tag{2.7}$$

As $h: \mathbb{R} \to \mathbb{R}$, we will assume that the connected component of K which contains ξ_o is of the form

$$K = \{\xi \in \mathbb{R}^{2\times 2}: \det \xi \in (\alpha, \beta)\} \tag{2.8}$$

for $\alpha, \beta \in \mathbb{R}$ (note that the connected component of K which contains ξ_o is always of this form, with however the possibility that $\alpha = -\infty$ or $\beta = +\infty$). We then choose in (2.2) $\mu = 0$, we therefore obtain

$$L_o = \{\xi \in \mathbb{R}^{2\times 2}: \xi^1 = (\xi_o)^1\}. \tag{2.9}$$

If we then take in (2.4)

$$b = \left(-(\xi_o)_2^1, (\xi_o)_1^1\right) \in \mathbb{R}^2 \tag{2.10}$$

we indeed have that $K \cap L_o$ is bounded and convex since

$$K \cap L_o = \{\xi \in L_o \subset \mathbb{R}^{2\times 2}: <b; \xi^2> = \det \xi \in (\alpha, \beta)\}. \tag{2.11}$$

Finally observe that on $K \cap L_o$ we indeed have that Qf is affine. Since the first component ξ^1 is fixed $\left(= (\xi_o)^1\right)$ and where $Ch \neq h$ we have that Ch is affine (since $h: \mathbb{R} \to \mathbb{R}$). Theorem 2.1 then applies and we get

Corollary 2.2

Let

$$f(\xi) = g(\xi^1) + h(\det \xi)$$

and $\xi_o \in \mathbb{R}^{2\times 2}$. If g is convex and

$$\lim_{|\delta|\to\infty} \frac{h(\delta)}{|\delta|} = \infty \tag{2.12}$$

then (P) has a solution.

Remark

The hypothesis (2.12) implies that in (2.8) $\alpha \neq -\infty$ and $\beta \neq \infty$.

We next mention a second example which has been intensively studied by Kohn-Strang [1] and which finds its origins in problems of optimal design. Let

$$f(\xi) = \begin{cases} 1+|\xi|^2 & \text{if } \xi \neq 0 \\ 0 & \text{if } \xi = 0 \end{cases}. \tag{2.13}$$

One can then show

Theorem 2.3

Let $n = m = 2$ then (P) has a solution if and only if one of the following two conditions hold

(i) $\xi_o = 0$ or $|\xi_o|^2 + 2|\det \xi_o| \geq 1$

(ii) $\det \xi_o \neq 0$.

Remarks

(i) The above result, including its generalisation to $m \geq 2$, is proved in Dacorogna-Marcellini [1]. Note that part (i) of the theorem corresponds to the case where $Qf(\xi_o) = f(\xi_o)$. Indeed Kohn-Strang have shown that

$$Qf(\xi) = \begin{cases} f(\xi) = 1+|\xi|^2 & \text{if } |\xi|^2 + 2|\det \xi| \geq 1 \\ 2\left(|\xi|^2 + 2|\det \xi|\right)^{1/2} - 2|\det \xi| & \text{if } |\xi|^2 + 2|\det \xi| \leq 1 \end{cases}. \tag{2.14}$$

(ii) For the necessary part of the theorem, cf. next section.

(iii) Theorem 2.3 does not enter in the framework of Theorem 2.1. The similarity comes from the fact that if we choose

$$L = \{\xi \in \mathbb{R}^{2\times 2}: \xi = \xi^t, \text{ i.e. } \xi_2^1 = \xi_1^2 \text{ and } \det \xi > 0\} \tag{2.15}$$

and if $\det \xi_o > 0$ and trace $(\xi_o) \in (0,1)$ then

$$K \cap L = \{\xi \in \mathbb{R}^{2\times 2}: \xi_2^1 = \xi_1^2, \det \xi > 0 \text{ and trace } (\xi) = \xi_1^1 + \xi_2^2 \in (0,1)\} \tag{2.16}$$

Moreover on $K \cap L$

$$Qf(\xi) = 2(\xi_1^1 + \xi_2^2) - 2\det \xi \tag{2.17}$$

i.e. Qf is quasiaffine on $K \cap L$.

Before discussing the generalisation of these examples, we examine a necessary condition.

NECESSARY CONDITION FOR THE EXISTENCE OF MINIMA

Let us recall that the problem under consideration is

$$\text{(P)} \qquad \inf\left\{F(u) = \int_\Omega f(\nabla u(x))dx : u \in u_o + W_o^{1,\infty}(\Omega;\mathbb{R}^m)\right\}$$

where $u_o(x) = \xi_o x$ with $\xi_o \in \mathbb{R}^{nm}$. We assume that

$$\xi_o \in K = \{\xi \in \mathbb{R}^{nm}: Qf(\xi) < f(\xi)\}. \tag{3.1}$$

To express the necessary condition we must introduce the following definition

Definition

Let $h: \mathbb{R}^{nm} \to \mathbb{R}$ be convex. We say that h is strictly convex in $\xi_o = ((\xi_o)_i^\alpha)_{1\le i\le n, 1\le\alpha\le m} = (\xi_o^\alpha)_{1\le\alpha\le m} \in \mathbb{R}^{nm}$ in m directions if there exists $\lambda = (\lambda^\alpha)_{1\le\alpha\le m} \in \mathbb{R}^{nm}$ such that

$$\lambda^\alpha \neq 0 \text{ and } < \lambda^\alpha ; \xi^\alpha - \xi_o^\alpha >_{\mathbb{R}^n} = 0 \quad \forall \alpha = 1,2,...,m \tag{3.2}$$

for every $\xi = (\xi^\alpha)_{1 \leq \alpha \leq m}$ which satisfies

$$\frac{h(\xi) + h(\xi_o)}{2} = \frac{h(\xi + \xi_o)}{2}. \tag{3.3}$$

Remarks

(i) If $m = 1$ the above definition simply means that h is not affine in the neighbourhood of ξ_o, i.e. there exists at least one direction of strict convexity.

(ii) Note also that if h is strictly convex in the neighbourhood of ξ_o then it has at least m directions of strict convexity (in fact nm directions) since (3.3) holds only if $\xi = \xi_o$.

We can then state the following theorem

Theorem 3.1

Let $\xi_o \in \mathbb{R}^{nm}$ be such that

(i) $Cf(\xi_o) = Qf(\xi_o) < f(\xi_o)$

(ii) Cf is strictly convex in ξ_o in at least m directions

then (P) has no solution.

Remark

The theorem has been established by Dacorogna-Marcellini [1]. The weak point of this theorem is that we assume that $Cf(\xi_o) = Qf(\xi_o)$, which is very restrictive. Part (ii) of the theorem is however more interesting since it gives a necessary condition for the existence of a minimum; namely Cf should be affine in at least $nm - (m-1)$ directions (cf. next section for the comparison with the sufficient condition).

We can then deduce two corolaries.

Corollary 3.2

Let $m = 1$, $\xi_o \in \mathbb{R}^n$ and $Cf(\xi_o) < f(\xi_o)$. If (P) has a solution then Cf is affine in the neighbourhood of ξ_o.

Remarks :

(i) The corollary is just a restatement of the theorem since in the case $m = 1$ we always have $Cf = Qf$.

(ii) Furthermore it is interesting to compare this result with the sufficient condition. We see that if, in addition, the connected component of $K = \{Cf(\xi) < f(\xi)\}$ which contains ξ_o is bounded, then the condition is also sufficient (cf. Dacorogna-Marcellini [1] for more details).

Corollary 3.3

Let $\xi_o \in \mathbb{R}^{nm}$ and

$$f(\xi) = g(|\xi|) \tag{3.4}$$

where g is extended to be even to the whole of $\mathbb{R}$. If $n \geq 2$ and if

(i) $Cg(|\xi_o|) < g(|\xi_o|)$

(ii) Cg is strictly increasing in the neghbourhood of $|\xi_o|$

(iii) $rank\{\xi_o\} \leq 1$

then (P) has no solution.

Remark

In the case where

$$g(t)=\begin{cases}1+t^2 & \text{si} \quad t\neq 0\\ 0 & \text{si} \quad t=0\end{cases}$$

(cf. Theorem 2.3) and $m=n=2$ we see that the condition $\det \xi_o \neq 0$ (and to avoid the trivial case $Cg(|\xi_o|)\leq Qf(\xi_o)<f(\xi_o)$) is a necessary and sufficient condition for the existence of solutions.

4. FINAL REMARKS

We will now compare the necessary and sufficient conditions of the preceding sections (theorems 2.1 and 3.1). First the hypothesis $K\cap L_o$ bounded and convex appears only as a mean for solving a system of equations of de Hamilton-Jacobi type (cf.(1.6) and Dacorogna-Marcellini [1] pfor more details).

We now turn to the hypothesis $Cf(\xi_o)=Qf(\xi_o)$ in Theorem 3.1. Obviously it is too restrictive and the necessary condition is then very far from the sufficient condition (only in certain instances this is not the case, cf. corollaries 3.2 and 3.3).

However parts (ii) of theorems 2.1 and 3.1 are closer than they appear. The necessary condition requires that Qf be affine in at least $nm-(m-1)$directions, while the sufficient condition requires the (quasi) affinity of Qf on $K\cap L_o$ which is contained on the manifold L_o of dimension $nm-n(m-1)=n$. The difference between the two seems very important, however this is only an appearence since (cf. Dacorogna-Marcellini [1]).

Proposition 4.1

Let

$$L=\left\{\xi=(\xi^\alpha)_{1\leq\alpha\leq m}\in \mathrm{IR}^{nm}: \sum_{i=1}^{n} a_i^\alpha\left(\xi_i^\alpha+\mu^\alpha\xi_i^m\right)=c^\alpha;\, \alpha=1,2,\ldots,m-1\right\}$$

where $c^\alpha, \mu^\alpha \in \mathbb{R}$, $\lambda^\alpha = (a_i^\alpha)_{1\le i\le n}$ are non zero vectors of $\mathbb{R}^n$ (if m=1, we let $L = \mathbb{R}^n$). Finally let $\Omega \subset \mathbb{R}^n$, $u_o(x) = \xi_o x$ *with* $\xi_o \in L$ *and* $u \in u_o + W_o^{1,\infty}(\Omega;\mathbb{R}^m)$ then

$$\nabla u(x) \in L \text{ a.e.} \Leftrightarrow \nabla u(x) \in L_o \text{ a.e.}.$$

Remarks :

(i) Observe, this time, that the dimension of L is $nm-(m-1)$, which is the same as the one appearing in the necessary condition. Furthermore $L_o \subset L$, trivially since $\xi_o \in L$.

(ii) The proposition shows that if we fix $(m-1)$ conditions (i.e. $\nabla u(x) \in L$) and the boundary condition, then automatically $n(m-1)$ conditions are fixed (i.e. $\nabla u(x) \in L_o$).

To conclude these remarks we dicuss briefly the problem of the actual computation of the quasiconvex envelope Qf. This is a delicate problem and we refer to Dacorogna [2] for some general formulas and examples; we should also mention the recent results of Buttazzo-Dacorogna-Gangbo [1] and Dacorogna-Koshigoe [1] which allows certain simplifications in the computation of Qf when the function has certain symetries).

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