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On rank one convex functions which are homogeneous of degree one

ABSTRACT

Let $f: \mathbb{R}^{2\times 2} \to \mathbb{R}$ be positively homogeneous of degree one. If, in addition, f is rotationally invariant and rank one convex, then it is necessarily convex.

1. INTRODUCTION

Let

$$I(u) = \int_{\Omega} f(\nabla u(x))dx \tag{1.1}$$

where :

i) $\Omega \subset \mathbb{R}^2$ is a bounded open set

ii) $u: \Omega \to \mathbb{R}^2$ and hence $\nabla u \in \mathbb{R}^{2\times 2}$

iii) $f: \mathbb{R}^{2\times 2} \to \mathbb{R}$ is continuous.

As it is well known (c.f. Morrey [8], Ball [2] or Dacorogna [3]), in order to study minimization problems involving (1.1) one has to impose some convexity hypotheses on f. Besides the usual convexity notion, one introduces the following conditions.

Definitions:

1) $f\colon \mathbb{R}^{2\times 2} \to \mathbb{R}$ is <u>rank one convex</u> if

$$f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda) f(B) \tag{1.2}$$

for every $\lambda \in [0,1]$, $A, B \in \mathbb{R}^{2\times 2}$ with $\det(A-B) = 0$.

2) $f\colon \mathbb{R}^{2\times 2} \to \mathbb{R}$ is <u>quasiconvex</u> if

$$\int_\Omega f(A + \nabla\varphi(x))dx \geq f(A)\, meas\Omega \tag{1.3}$$

for every $A \in \mathbb{R}^{2\times 2}$ and every $\varphi \in C_o^\infty(\Omega; \mathbb{R}^2)$.

3) $f\colon \mathbb{R}^{2\times 2} \to \mathbb{R}$ is <u>polyconvex</u> if there exists $g\colon \mathbb{R}^{2\times 2} \times \mathbb{R} \to \mathbb{R}$ convex such that for every $A \in \mathbb{R}^{2\times 2}$

$$f(A) = g(A, \det A). \tag{1.4}$$

In general one has

$$f \text{ convex} \Rightarrow f \text{ polyconvex} \Rightarrow f \text{ quasiconvex} \Rightarrow f \text{ rank one convex}.$$

Recently a considerable effort was made to extend the results of Goffman and Serrin [7] to quasiconvex integrands (c.f. for example Fonseca-Müller [6] and their bibliography). One then needs in particular to know if there are positively homogeneous functions of degree one that are quasivoncex but not convex. Such functions where shown to exist by Müller [9], following earlier work of Sverak [10] and Zhang [12]. Müller shows his result using weak type estimates for Cauchy-Riemann operators. So his example is in some sense implicit.

We show here that if one assumes in addition that f is rotationally invariant then Müller's result does not hold. More precisely.

Theorem 1:

Let $f: \mathbb{R}^{2\times2} \to \mathbb{R}$ be continuous and satisfying

i) $f(tA) = tf(A)$ for every $t \geq 0$ and every $A \in \mathbb{R}^{2\times2}$

ii) f is rotationally invariant, i.e. $f(RAR') = f(A)$
for every $A \in \mathbb{R}^{2\times2}$, $R, R' \in O^+ = \{RR^t = I \text{ and } \det R = 1\}$
then

$$f \text{ rank one convex} \Leftrightarrow f \text{ convex}.$$

Therefore such a result rules out all functions $f(A) = g(|A|, \det A)$ where $|A|^2 = \sum_{i,j=1}^{2} A_{ij}^2$. In terms of elasticity our theorem shows that no isotropic function can be homogeneous of degree one and quasiconvex but not convex.

We will also show that Theorem 1 can be slightly generalized.

2. THE CASE OF ROTATIONALLY INVARIANT FUNCTIONS

We start with the key lemma.

Lemma 2:

Let $g: \mathbb{R}^2 \to \mathbb{R}$ be such that

i) $g(tx, ty) = tg(x, y)$ for every $t \geq 0$ and $x, y \in \mathbb{R}$

ii) g is separately convex (i.e. $g(x,\cdot)$ and $g(\cdot,y)$ are convex for fixed x and fixed y)

then g is convex.

Remark:

For related results on separately convex functions see Tartar [11].

Proof:

The proof is of an elementary nature. We have to show that given $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 = 1$, $x_1, y_1, x_2, y_2 \in \mathbb{R}$ then

$$g(\lambda_1 x_1 + \lambda_2 x_2, \lambda_1 y_1 + \lambda_2 y_2) \leq \lambda_1 g(x_1, y_1) + \lambda_2 g(x_2, y_2). \tag{2.1}$$

In order to prove this fact we have to consider serveral cases. The idea is however essentialy included in the first case.

Case 1:

Assume that either $\underline{x_1 x_2 \geq 0 \text{ or } y_1 y_2 \geq 0}$. Since g is continuous (the fact that g is separately convex implies that g is continuous) it is enough to prove the result for $x_1 x_2 > 0$ or $y_1 y_2 > 0$. We consider the case where $x_1 x_2 > 0$, otherwise we interchange the roles of x and y. We then let

$$\sigma = \frac{\lambda_1 x_1 + \lambda_2 x_2}{|\lambda_1 x_1 + \lambda_2 x_2|} = \frac{x_1}{|x_1|} = \frac{y_1}{|y_1|}$$

(since $x_1 x_2 > 0$, then σ is well defined and equal + 1 or - 1). Using the properties of g we get

$$\begin{aligned} & g(\lambda_1 x_1 + \lambda_2 x_2, \lambda_1 y_1 + \lambda_2 y_2) \\ & = |\lambda_1 x_1 + \lambda_2 x_2| \, g\left(\sigma, \frac{\lambda_1 |x_1|}{|\lambda_1 x_1 + \lambda_2 x_2|} \frac{y_1}{|x_1|} + \frac{\lambda_2 |x_2|}{|\lambda_1 x_1 + \lambda_2 x_2|} \frac{y_2}{|x_2|}\right) \\ & \leq |\lambda_1 x_1 + \lambda_2 x_2| \left[\frac{\lambda_1 |x_1|}{|\lambda_1 x_1 + \lambda_2 x_2|} g\left(\sigma, \frac{y_1}{|x_1|}\right) + \frac{\lambda_2 |x_2|}{|\lambda_1 x_1 + \lambda_2 x_2|} g\left(\sigma, \frac{y_2}{|x_2|}\right)\right]. \end{aligned}$$

Using again the homogeneity of g we find

$$g(\lambda_1 x_1 + \lambda_2 x_2, \lambda_1 y_1 + \lambda_2 y_2) \leq \lambda_1 g(\sigma|x_1|, y_1) + \lambda_2 g(\sigma|x_2|, y_2)$$

which is nothing else than (2.1).

Case 2:

We now assume that $x_1 x_2 < 0$ and $y_1 y_2 < 0$ and that there exists $i \in \{1,2\}$ such that $x_i(\lambda_1 x_1 + \lambda_2 x_2) \geq 0$ and $y_i(\lambda_1 y_1 + \lambda_2 y_2) \leq 0$. Using again the continuity of g we may assume without loss of generality that $\lambda_1 x_1 + \lambda_2 x_2 \neq 0$ and $\lambda_1 y_1 + \lambda_2 y_2 \neq 0$.

We adopt the notations that $\lambda_3 = \lambda_1$, $x_3 = x_1$ and $y_3 = y_1$ so that $\lambda_i + \lambda_{i+1} = \lambda_1 + \lambda_2 = 1$. We then define

$$\left[\begin{array}{lll} a & = \dfrac{x_i\, y_{i+1} - x_{i+1}\, y_i}{y_{i+1} - y_i} & \quad (2.2) \\[2ex] \mu_i & = \dfrac{y_i}{\lambda_{i+1}(y_i - y_{i+1})} = \left[\lambda_{i+1}\left(1 - \dfrac{y_{i+1}}{y_i}\right)\right]^{-1}. & \quad (2.3) \\[2ex] v_{i+1} & = \dfrac{\lambda_i\, y_i + \lambda_{i+1}\, y_{i+1}}{y_{i+1}} & \quad (2.4) \end{array}\right.$$

Note that since $y_i y_{i+1} < 0$, then $\mu_i \geq 0$. Furthermore $\mu_i \leq 1$, since

$$1 - \mu_i = \frac{-(1-\lambda_{i+1})y_i - \lambda_{i+1}\, y_{i+1}}{\lambda_{i+1}(y_i - y_{i+1})} = \frac{-(\lambda_i\, y_i + \lambda_{i+1}\, y_{i+1})}{\lambda_{i+1}\, y_i\left(1 - \dfrac{y_{i+1}}{y_i}\right)} \geq 0 \qquad (2.5)$$

since $y_i(\lambda_i\, y_i + \lambda_{i+1}\, y_{i+1}) < 0$ and $y_i y_{i+1} < 0$. Therefore

$$0 \leq \mu_i \leq 1. \qquad (2.6)$$

Since $y_i y_{i+1} < 0$ and $y_i(\lambda_i y_i + \lambda_{i+1}\, y_{i+1}) < 0$, we deduce that $v_{i+1} > 0$. We now show that $v_{i+1} \leq 1$, indeed

$$1 - \nu_{i+1} = \frac{y_{i+1}(1-\lambda_{i+1}) - \lambda_i y_i}{y_{i+1}} = \lambda_i\left[1 - \frac{y_i}{y_{i+1}}\right] \geq 0. \tag{2.7}$$

Hence we deduce that

$$0 \leq \nu_{i+1} \leq 1. \tag{2.8}$$

Observe that

$$\begin{cases} \mu_i(\lambda_i x_i + \lambda_{i+1}x_{i+1}) + (1-\mu_i)x_i = a & (2.9) \\ \mu_i(\lambda_i y_i + \lambda_{i+1} y_{i+1}) + (1-\mu_i)y_i = 0. & (2.10) \end{cases}$$

Let us check (2.9),

$$\mu_i(\lambda_i x_i + \lambda_{i+1}x_{i+1}) + (1-\mu_i)x_i = x_i[1 - \mu_i(1-\lambda_i)] + \lambda_{i+1}\mu_i x_{i+1}$$
$$= x_i[1 - \lambda_{i+1}\mu_i] + \lambda_{i+1}\mu_i x_{i+1} = x_i\left[1 - \frac{y_i}{y_i - y_{i+1}}\right] + x_{i+1}\cdot\frac{y_i}{y_i - y_{i+1}}.$$
$$= \frac{x_{i+1}y_i - x_i y_{i+1}}{y_i - y_{i+1}} = a.$$

(2.10) is verified similarly.

We also note that

$$\begin{cases} \nu_{i+1}x_{i+1} + (1-\nu_{i+1})a = \lambda_i x_i + \lambda_{i+1}x_{i+1} & (2.11) \\ \nu_{i+1}y_{i+1} + (1-\nu_{i+1})0 = \nu_{i+1}y_{i+1} = \lambda_i y_i + \lambda_{i+1}y_{i+1}. & (2.12) \end{cases}$$

(2.12) is just a reformulation of (2.4). We now verify (2.11)

$$\nu_{i+1}x_{i+1} + (1-\nu_{i+1})a = \frac{\lambda_i y_i x_{i+1} + \lambda_{i+1}y_{i+1}x_{i+1} + \lambda_i(y_{i+1} - y_i)a}{y_{i+1}}$$
$$= \frac{y_{i+1}(\lambda_i x_i + \lambda_{i+1}x_{i+1})}{y_{i+1}} = \lambda_i x_i + \lambda_{i+1}x_{i+1}.$$

We are now in a position to conclude to the convexity of g.

By (2.11) and (2.12) we have

$$\begin{aligned} g(\lambda_1 x_1+\lambda_2 x_2, \lambda_1 y_1+\lambda_2 y_2) &= g(v_{i+1} x_{i+1}+(1-v_{i+1})a, v_{i+1}y_{i+1}+(1-v_{i+1})0) \\ &\le v_{i+1}\, g(x_{i+1}, y_{i+1})+(1-v_{i+1})g(a,0), \end{aligned} \tag{2.13}$$

the inequality coming from the fact that $y_{i+1}\cdot 0 \ge 0$ and from Case 1. We then use (2.9) and (2.10) to write.

$$\begin{aligned} g(a,0) &= g(\mu_i(\lambda_1 x_1+\lambda_2 x_2)+(1-\mu_i)x_i, \mu_i(\lambda_1 y_1+\lambda_2 y_2)+(1-\mu_i)y_i) \\ &\le \mu_i\, g(\lambda_1 x_1+\lambda_2 x_2, \lambda_1 y_1+\lambda_2 y_2)+(1-\mu_i)g(x_i, y_i), \end{aligned} \tag{2.14}$$

the inequality coming from Case 1 and the fact that $x_i(\lambda_1 x_1+\lambda_2 x_2)\ge 0$ (by hypothesis of Case 2).

Combining (2.13) and (2.14) we get

$$\begin{aligned} &[1-\mu_i(1-v_{i+1})]\, g(\lambda_1 x_1+\lambda_2 x_2, \lambda_1 y_1+\lambda_2 y_2) \\ &\le v_{i+1}\, g(x_{i+1}, y_{i+1})+(1-v_{i+1})(1-\mu_i)g(x_i, y_i). \end{aligned} \tag{2.15}$$

To conclude to (2.1) i.e. the convexity of g we only need to show that

$$\left[\begin{array}{ll} \dfrac{v_{i+1}}{1-\mu_i(1-v_{i+1})} = \lambda_{i+1} & (2.16) \\ \dfrac{(1-v_{i+1})(1-\mu_i)}{1-\mu_i(1-v_{i+1})} = \lambda_i . & (2.17) \end{array}\right.$$

We verify the first one (using (2.3),(2.4),(2.7))

$$\frac{v_{i+1}}{1-\mu_i(1-v_{i+1})} = \frac{\lambda_i y_i+\lambda_{i+1} y_{i+1}}{y_{i+1}-\dfrac{y_i}{\lambda_{i+1}(y_i-y_{i+1})}[\lambda_i(y_{i+1}-y_i)]} = \lambda_{i+1}$$

and similarly for (2.17) since

$$\frac{\nu_{i+1}}{1-\mu_i(1-\nu_{i+1})} + \frac{(1-\nu_{i+1})(1-\mu_i)}{1-\mu_i(1-\nu_{i+1})} = 1 = \lambda_i + \lambda_{i+1}.$$

This concludes Case 2.

Case 3:

We now assume that $\underline{x_1 x_2 < 0}$ and $\underline{y_1 y_2 < 0}$ and that there exists $i \in \{1,2\}$ with $\underline{x_i(\lambda_1 x_1 + \lambda_2 x_2) < 0}$ and $\underline{y_i(\lambda_1 y_1 + \lambda_2 y_2) < 0}$.

Note first that Cases 1, 2 and 3 cover all the possibilities, since by inverting the roles of i and $i+1$, taking into account that $x_i x_{i+1} < 0$, $y_i y_{i+1} < 0$, the convexity inequality holds by Case 2 also for $x_i(\lambda_1 x_1 + \lambda_2 x_2) \le 0$ and $y_i(\lambda_1 y_1 + \lambda_2 y_2) \ge 0$. In particular if $y_i(\lambda_1 y_1 + \lambda_2 y_2) = 0$, then the convexity inequality holds independently of the sign of $x_i(\lambda_1 x_1 + \lambda_2 x_2)$.

Therefore to establish (2.1) when Case 3 holds, we proceed exactly as in Case 2, the only point to be checked is that (2.14) holds also under the hypotheses of Case 3. Letting

$$\begin{aligned} X_1 &= \lambda_i x_i + \lambda_{i+1} x_{i+1}, & X_2 &= x_i \\ Y_1 &= \lambda_i y_i + \lambda_{i+1} y_{i+1}, & Y_2 &= y_i. \end{aligned}$$

We have by hypothesis that $X_1 X_2 < 0, Y_1 Y_2 < 0$ and by (2.10)

$$Y_1(\mu_i Y_1 + (1-\mu_i)Y_2) = 0.$$

Using the above remark we deduce from Case 2, (2.2) and (2.9), that

$$\begin{aligned} g(a,0) &= g(\mu_i X_1 + (1-\mu_i)X_2,\ \mu_i Y_1 + (1-\mu_i)Y_2) \\ &\le \mu_i g(X_1, Y_1) + (1-\mu_i) g(X_2, Y_2) \end{aligned}$$

which is nothing else than (2.14). The remaining part of the proof is then exactly the same as that of Case 2. Therefore the lemma is established. ◆

We are now in a position to prove Theorem 1.

Proof:

i) f convex $\Rightarrow f$ rank one convex. This is true for any function.

ii) f rank one convex $\Rightarrow f$ convex: Let $g: \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$g(x,y) = f\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}.$$

Since f is positively homogeneous of degree one and rank one convex, we deduce that g is positively homogeneous of degree one and separately convex. Applying Lemma 2 to g, we deduce that g is convex and so that f is convex when restricted to diagonal matrices. Using then a result of Dacorogna-Koshigoe [4], we conclude that f is convex on the whole of $\mathbb{R}^{2\times 2}$. ◆

We now turn to an example

Proposition 3:
Let

$$f(A) = \begin{cases} |A| + \alpha \dfrac{\det A}{|A|} & \text{if} \quad A \neq 0 \\ 0 & \text{if} \quad A = 0 \end{cases}$$

then

$$f \text{ rank one convex} \Leftrightarrow f \text{ convex} \Leftrightarrow |\alpha| \leq \frac{2}{3}.$$

Remark:

It is interesting to contrast this example with that of Dacorogna-Marcellini [5] (see also Alibert-Dacorogna [1])

$$f(A) = |A|^4 - 2\gamma |A|^2 \det A.$$

In this case the notions of convexity, polyconvexity and rank one convexity are different (the exact values for which these notions hold are $|\gamma| \le \frac{2}{3}\sqrt{2}, 1, \frac{2}{\sqrt{3}}$ respectively). Note also that if one looks for the simplest example of homogeneous function of degree two, one has

$$f(A) = |A|^2 - \gamma \det A.$$

In this case we have that f is rank one convex (and polyconvex) for every $\gamma \in \mathrm{IR}$, while it is convex if and only if $|\gamma| \le 2$. Therefore a natural attempt to find a rank one convex (but not convex) and homogeneous of degree one function is the one of Proposition 3. However in view of Theorem 1 this fails to give the expected example.

Proof:

The proof is elementary. Using Theorem 1 and again the result of Dacorogna-Koshigoe [4], the only thing to be checked is that

$$g(x,y) = \begin{cases} \sqrt{x^2+y^2} + \alpha \dfrac{xy}{\sqrt{x^2+y^2}} & \text{if } x^2+y^2 \neq 0 \\ 0 & \text{if } x = y = 0 \end{cases}$$

is separately convex if and only if $|\alpha| \le \frac{2}{3}$. Since g is symmetric in x and y it is enough to show that

$$\frac{\partial^2}{\partial x^2} g(x,y) \ge 0 \text{ for every } (x,y) \neq (0,0) \Leftrightarrow |\alpha| \le \frac{2}{3}.$$

This is true since (provided $(x,y) \neq (0,0)$)

$$\frac{\partial^2}{\partial x^2} g(x,y) = \frac{y^2}{(x^2+y^2)^{5/2}}(x^2 - 3\alpha xy + y^2).$$

Obviously the right hand side is positive if and only if $|\alpha| \leq \frac{2}{3}$. ◆

3. SOME GENERALIZATIONS

We now show that Theorem 1 holds also for some non rotationally invariant functions. We first fix some notations.

Notations :

We here identify $\mathbb{R}^{2\times2}$ with $\mathbb{R}^4$. For $X \in \mathbb{R}^4$, $X = (x_1, x_2, x_3, x_4)$ we let

$$|X|^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$$
$$\det X = x_1 x_4 - x_2 x_3$$
$$\operatorname{scal} X = x_1 x_2 + x_3 x_4.$$

We now assume that f satisfy the following.

Hypothesis (H) :

Let $f : \mathbb{R}^4 \cong \mathbb{R}^{2\times2} \to \mathbb{R}$ be continuous and positively homogeneous of degree one. Assume that there exists $S \in \mathbb{R}^{4\times4}$ invertible and $0 \neq A \in \mathbb{R}^4$ such that $\det A = \det(SA) = 0$. Finally let for $X \in \mathbb{R}^4$, $g: \mathbb{R}^4 \to \mathbb{R}$

$$g(X) \equiv f(SX)$$

and assume that g is rotationally invariant.

Theorem 4 :

Let f satisfy (H) then

$$f \text{ rank one convex} \Leftrightarrow f \text{ convex} \Leftrightarrow g \text{ convex} \Leftrightarrow g \text{ rank one convex}.$$

We immediately give examples of such functions.

Corollary 5 :

Let $f\colon \mathrm{IR}^4 \to \mathrm{IR}$ be positively homogeneous of degree one and having one of the following forms

1) $f(X) = \varphi(|X|, \det X)$

2) $f(X) = \varphi(|X|, \operatorname{scal} X)$

3) $f(X) = \varphi(|X|, \alpha \det X + \beta \operatorname{scal} X)$ for some $\alpha, \beta \in \mathrm{IR}\setminus\{0\}$

4) $f(X) = \varphi(\sqrt{x_1^2 + x_3^2}, \sqrt{x_2^2 + x_4^2})$

for some $\varphi\colon \mathrm{IR}^2 \to \mathrm{IR}$. Then

$$f \text{ rank one convex} \Leftrightarrow f \text{ convex}.$$

Remark :

The first result is a consequence of Theorem 1. However all the three other cases give rise to functions which are not rotationally invariant. One can also imagine several other examples where Theorem 4 apply, such as (for $\alpha, \beta \in \mathrm{IR}$)

$$f(x_1, x_2, x_3 x_4) = (x_1^2 + x_3^2 + \beta^2(x_2^2 + x_4^2))^{1/2} + \alpha \frac{x_1 x_2 + x_3 x_4}{((x_1^2 + x_3^2) + \beta^2(x_2^2 + x_4^2))^{1/2}}.$$

Proof of Corollary 5 :

1) C.f. Theorem 1, since f is then rotationally invariant.

2) Let

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

choose $A = (1,0,0,0)$. Then S and A are as in (H) and

$$g(X) = f(SX) = \varphi(|SX|, \operatorname{scal}(SX)) = \varphi(|X|, \det X).$$

Applying Theorem 4 and the first part of the Corollary we get the result.

3) Let

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{-\alpha}{\sqrt{\alpha^2+\beta^2}} & 0 & \dfrac{\beta}{\sqrt{\alpha^2+\beta^2}} \\ 0 & 0 & -1 & 0 \\ 0 & \dfrac{\beta}{\sqrt{\alpha^2+\beta^2}} & 0 & \dfrac{\alpha}{\sqrt{\alpha^2+\beta^2}} \end{pmatrix}.$$

Choose as before $A = (1,0,0,0)$ and observe that for every $X \in \mathrm{IR}^4$

$$\begin{cases} |X| = |SX| \\ \alpha \det(SX) + \beta \operatorname{scal}(SX) = \sqrt{\alpha^2+\beta^2} \det X \end{cases}$$

In view of the above identities, we find

$$g(X) = f(SX) = \varphi(|X|, \sqrt{\alpha^2+\beta^2} \det X).$$

We then apply Theorem 1 and Theorem 4 to obtain the claimed result. ◆

Remark :

If we combine Proposition 3 and Corollary 5 we find that if

$$f(X) = \begin{cases} |X| + \dfrac{\alpha \det X + \beta \operatorname{scal} X}{|X|} & \text{if} \quad |X| \neq 0 \\ 0 & \text{if} \quad X = 0 \end{cases}$$

then

$$f \text{ convex} \Leftrightarrow f \text{ rank one convex} \Leftrightarrow \alpha^2 + \beta^2 \leq \frac{4}{9}.$$

Proof of Theorem 4 :

Observe that since S is invertible we have trivially

$$f \text{ convex} \Leftrightarrow g \text{ convex}. \tag{3.1}$$

From Theorem 1 we deduce that

$$g \text{ convex} \Leftrightarrow g \text{ rank one convex}. \tag{3.2}$$

So if we show that

$$f \text{ rank one convex} \Rightarrow g \text{ rank one convex}, \tag{3.3}$$

we will have proved the theorem (since trivially f convex $\Rightarrow f$ rank one convex).

To prove that g is rank one convex, we have to show that given $X, Y \in \mathbb{R}^{2\times 2} \cong \mathbb{R}^4$ with $Y \neq 0$ and $\det Y = 0$, then the function $\varphi : \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi(t) \equiv g(X + tY)$$

is convex. (3.4)

Since g is rotationally invariant, there is no loss of generality in assuming that $Y = A$ (where A is the matrix given in (H)). Indeed there exist $R, R', Q, Q' \in O^+$ such that

$$Y = R\begin{pmatrix} |Y| & 0 \\ 0 & 0 \end{pmatrix} R'$$

$$A = Q\begin{pmatrix} |A| & 0 \\ 0 & 0 \end{pmatrix} Q'.$$

i.e.

$$Y = \frac{|Y|}{|A|} RQ' AQ''R'. \tag{3.5}$$

Therefore we in fact only need to show that given any $X \in \mathrm{IR}^4$ then the function

$$\varphi(t) = g(X + tA) \tag{3.6}$$

is convex for every $t \in \mathrm{IR}$. However, we have, by definition,

$$\varphi(t) = g(X + tA) = f(SX + tSA). \tag{3.7}$$

Since f is assumed to be rank one convex (c.f. (3.3)) and that by hypothesis $\det SA = 0$, we deduce immediately that φ is convex and thus g is rank one convex. This concludes the proof of the theorem. ◆

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