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Coercivity of Integrals versus Coercivity of Integrands

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1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and

$$I(u) = \int_\Omega f(x, u(x), \nabla u(x))\, dx, \qquad (1.1)$$

where $u\colon \Omega \to \mathbb{R}$ and $f\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$. The question that we wish to investigate here is the following: does any coercivity of the integral I imply a corresponding coercivity of the integrand f? More precisely assume that there exist $\alpha > 0$, $\beta \in \mathbb{R}$, $p > 1$ such that

$$I(u) \geq \alpha \|\nabla u\|_{L^p}^p + \beta, \qquad \forall u \in W_0^{1,\infty}(\Omega). \qquad (1.2)$$

The question is then, can we find $\tilde{\alpha} > 0$, $\tilde{\beta} \in \mathbb{R}$ such that

$$f(x, u, \xi) \geq \tilde{\alpha} |\xi|^p + \tilde{\beta}, \qquad \forall (x, u, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n? \qquad (1.3)$$

We will see that in general the answer is no, except for the case where f does not depend explicitly on x and u. Before being more precise we introduce some terminology. We say that f is *p-coercive* if f satisfies (1.3), I is $W_0^{1,p}$*-coercive* if it satisfies (1.2) and I is $W^{1,p}$*-coercive* if it satisfies (1.2) for every u in $W^{1,\infty}(\Omega)$, instead of $u \in W_0^{1,\infty}(\Omega)$. Since $W_0^{1,\infty}(\Omega) \subset W^{1,\infty}(\Omega)$ we have trivially

$$f\ p\text{-coercive} \Rightarrow I \quad W^{1,p}\text{-coercive} \Rightarrow I \quad W_0^{1,p}\text{-coercive}. \qquad (1.4)$$

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Our results are then the following.

Case 1. If $f(x, u, \xi) = f(\xi)$, the following holds:

$$f\ p\text{-coercive} \Leftrightarrow I \quad W^{1,p}\text{-coercive} \Leftrightarrow I \quad W_0^{1,p}\text{-coercive}. \tag{1.5}$$

Case 2. If $f(x, u, \xi) = f(x, \xi)$ the following holds:

$$I\ W^{1,p}\text{-coercive} \not\Rightarrow f\ p\text{-coercive}; \tag{1.6}$$

thus, in view of (1.4), trivially

$$I\ W_0^{1,p}\text{-coercive} \not\Rightarrow f\ p\text{-coercive}. \tag{1.7}$$

Indeed, the example $n = 1$, $\Omega = (0, 1)$, $f(x, \xi) = x\xi^4 + 1$, which does not satisfy (1.3), leads to a $W^{1,p}$-coercive integral with $p = \frac{5}{3}$.

Case 3. If $f(x, u, \xi) = f(u, \xi)$ the following holds:

$$I\ W_0^{1,p}\text{-coercive} \not\Rightarrow f\ p\text{-coercive}. \tag{1.8}$$

Of course (1.8) is well known, the Wirtinger inequality, i.e., $f(u, \xi) = \xi^2 - \alpha^2 u^2$ with $\alpha \in (0, \pi)$, $\Omega = (0, 1)$, leads to a $W_0^{1,2}$-coercive integral while f is obviously not 2-coercive. However, under some minor restrictions on f, we can prove that

$$I\ W^{1,p}\text{-coercive} \Rightarrow f\ p\text{-coercive}. \tag{1.9}$$

We now discuss briefly the extension of these results to the vectorial case, i.e., $u: \Omega \to \mathbb{R}^m$, $m > 1$. Alibert and Dacorogna [1] have shown that even if f does not depend explicitly on (x, u), as in Case 1, then (1.5) is in general false. Of course the equivalence $\{f\ p\text{-coercive} \Leftrightarrow I\ W^{1,p}\text{-coercive}\}$ persists but the other one does not. Indeed, if $n = m = 2$, denoting by ξ a 2×2 matrix, $|\xi|^2 = \sum_{i,j=1}^{2} \xi_{ij}^2$ and $\det(\xi) = \xi_{11}\xi_{22} - \xi_{12}\xi_{21}$, then $f(\xi) = (|\xi|^2 - 2\det\xi)^2$ is $W_0^{1,4}$-coercive while f is obviously not p-coercive for any $p > 1$. In fact this counterexample has motivated our analysis.

For the quadratic case, there is classical literature on necessary conditions for coercivity of quadratic forms (Garding, Agmon, Aronszajn). However, it is difficult to distinguish between coercivity and ellipticity, both being obtained via Fourier transform, while in our (general) case, ellipticity and coercivity are two independent facts.

We also mention that some related results have been obtained by Casado-Dias [2], Francfort [4], Geymonat, Müller, Triantafyllidis [5], Le Dret [6], Trudinger [9], Marcellini-Sbordone [7, 8].

Finally, we point out that the above results have their counterparts for differential operators in divergence form. For example, if $A(u) =$

$-\text{div}(a(\nabla u))$, in a similar way as in Theorem 1 below, we can obtain that if $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^n$, then

$$\langle a(\xi); \xi\rangle \geq \tilde{\alpha}|\xi|^p + \tilde{\beta}, \qquad \forall \xi \in \mathbb{R}^n$$
$$\Leftrightarrow \int_\Omega \langle a(\nabla u(x)); \nabla u(x)\rangle\, dx \geq \alpha \|\nabla u\|_{L^p}^p + \beta, \qquad \forall u \in W_0^{1,\infty}(\Omega).$$

2. The Case $f(\xi)$

Observe that the equivalence f p-coercive $\Leftrightarrow$ I $W^{1,p}$-coercive is trivial. Indeed choosing $u(x) = \langle \xi; x\rangle$ in (1.1) and using (1.2) we get immediately (1.3). This argument generalizes to the vectorial case. It is the second equivalence which is more interesting.

THEOREM 1. *Let $f: \mathbb{R}^n \to \mathbb{R}$ be continuous, $\Omega \subset \mathbb{R}^n$ be a bounded open set, $p > 1$, $\alpha > 0$, and $\beta \in \mathbb{R}$ such that*

$$I(u) = \int_\Omega f(\nabla u(x))dx \geq \alpha \int_\Omega |\nabla u(x)|^p\, dx + \beta, \qquad \forall u \in W_0^{1,\infty}(\Omega). \tag{2.1}$$

Then there exist $\tilde{\alpha} > 0$, $\tilde{\beta} \in \mathbb{R}$ such that

$$f(\xi) \geq \tilde{\alpha}|\xi|^p + \tilde{\beta}, \qquad \forall \xi \in \mathbb{R}. \tag{2.2}$$

Remark. The result is false if $p = 1$. However, in this case we can prove (see below) that

$$f(\xi) \geq \left(2\alpha - f\left(-\frac{\xi}{|\xi|}\right) + \frac{\beta}{\text{meas }\Omega}\right)|\xi| + \frac{\beta}{\text{meas }\Omega}. \tag{2.3}$$

Before proceeding with the proof we quote a lemma that can be found in Ekeland-Témam [3] (Theorem 1.2 of Chapter X) which will play the central role in our analysis.

LEMMA 2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $\lambda \in (0, 1)$, u_1, u_2 (piecewise) affine functions in Ω. Then for every $\varepsilon > 0$ there exist Ω_1, $\Omega_2 \subset \Omega$ disjoint open sets and $u \in W^{1,\infty}(\Omega)$ such that*

$$|\text{meas }\Omega_1 - \lambda \text{ meas }\Omega| \leq \lambda\varepsilon$$
$$|\text{meas }\Omega_2 - (1-\lambda)\text{ meas }\Omega| \leq (1-\lambda)\varepsilon$$
$$\nabla u(x) = \nabla u_i(x) \qquad \text{a.e. } in\ \Omega_i$$
$$\|\nabla u\|_{L^\infty} \leq \max_{i=1,2}\{\|\nabla u_i\|_{L^\infty}\}$$
$$|u(x) - \lambda u_1(x) - (1-\lambda)u_2(x)| \leq \varepsilon \qquad \textit{for every } x \in \Omega$$
$$u(x) = \lambda u_1(x) + (1-\lambda)u_2(x) \qquad \textit{for every } x \in \partial\Omega.$$

Proof of Theorem 1. Let $\xi \neq 0$ and choose

$$\lambda = \frac{1}{|\xi| + 1}, \qquad 1 - \lambda = \frac{|\xi|}{|\xi| + 1}$$

$$u_1(x) = \langle \xi; x \rangle$$

$$u_z(x) = \frac{-1}{|\xi|} \langle \xi; x \rangle.$$

For every $\varepsilon > 0$ we find Ω_1, Ω_2 and u as in the lemma (observe that $\lambda u_1 + (1 - \lambda) u_2 = 0$). Using (2.1) we get

$$\begin{aligned}
&\int_\Omega f(\nabla u(x))\, dx \\
&\quad = f(\xi) \text{ meas } \Omega_1 + f\left(-\frac{\xi}{|\xi|}\right) \text{ meas } \Omega_2 \\
&\qquad + \int_{\Omega - (\Omega_1 \cup \Omega_2)} f(\nabla u(x))\, dx \geq \beta + \alpha \int_\Omega |\nabla u(x)|^p\, dx \\
&\quad = \alpha \left[|\xi|^p \text{ meas } \Omega_1 + \text{meas } \Omega_2 + \int_{\Omega - (\Omega_1 \cup \Omega_2)} |\nabla u(x)|^p\, dx \right] + \beta \\
&\quad \geq \alpha [|\xi|^p \text{ meas } \Omega_1 + \text{meas } \Omega_2] + \beta.
\end{aligned}$$

Dividing by meas Ω_1, letting ε tend to 0,and using the properties of u we get

$$f(\xi) \geq \alpha |\xi|^p + |\xi| \left[\alpha - f\left(-\frac{\xi}{|\xi|}\right) \right] + \frac{\beta(|\xi| + 1)}{\text{meas } \Omega},$$

which implies immediately the theorem since $p > 1$. ∎

3. The Case $f(x, \xi)$

We have the following counterexample.

THEOREM 3. *Let* $\Omega = (0, 1)$, $f(x, \xi) = x\xi^4 + 1$; *then*

$$I(u) = \int_0^1 f(x, u'(x))\, dx \geq \frac{1}{4} \int_0^1 |u'(x)|^{5/3}\, dx, \qquad \forall u \in W^{1,\infty}(0,1). \quad (3.1)$$

Proof. Observe first that

$$\int_0^1 x^{-5/7}\, dx = \frac{7}{2}.$$

To show (3.1) is thus equivalent to proving

$$I_1(u) \equiv \int_0^1 \left[xu'^4 - \frac{1}{4}|u'|^{5/3} + \frac{2}{7}x^{-5/7} \right] dx \geq 0, \qquad \forall u \in W^{1,\infty}(0, 1). \tag{3.2}$$

We then change the variables and write

$$y = x^{4/7}, \qquad u(x) = v(y). \tag{3.3}$$

An elementary computation gives $u'(x) = v'(y) \cdot \frac{4}{7} y^{-3/4}$. Thus

$$\begin{aligned} I_2(v) &\equiv I_1(u) \\ &= \int_0^1 \left[y^{7/4} \left| \frac{4}{7} y^{-3/4} v'(y) \right|^4 - \frac{1}{4} \left| \frac{4}{7} y^{-3/4} v'(y) \right|^{5/3} + \frac{2}{7} y^{-5/4} \right] \cdot \frac{7}{4} y^{3/4}\, dy \\ &= \int_0^1 y^{-1/2} \left[\left(\frac{4}{7}\right)^3 |v'|^4 - \frac{1}{7^{2/3}4^{1/3}} |v'|^{5/3} + \frac{1}{2} \right] dy. \end{aligned} \tag{3.4}$$

Observing that

$$\begin{aligned} \left(\frac{4}{7}\right)^3 z^4 - \frac{1}{7^{2/3}4^{1/3}} |z|^{5/3} + \frac{1}{2} &= \left(\frac{4}{7}\right)^3 \left[z^4 - \frac{7^{7/3}}{8^{7/3}} 2^{1/3} |z|^{5/3} + \frac{7^3}{2^7} \right] \\ &\geq \left(\frac{4}{7}\right)^3 \left[z^4 - 2^{1/3}|z|^{5/3} + \frac{7}{5} \right] \\ &\geq \left(\frac{4}{7}\right)^3 \left[z^4 - \frac{12}{5}|z|^{5/3} + \frac{7}{5} \right] \geq 0, \qquad \forall z \in \mathbb{R}. \end{aligned}$$

We conclude that (3.4), and thus (3.2), is positive for every u, thus the theorem. ▮

This example shows that, in general, we cannot infer any p-coercivity of f from the corresponding $W^{1,p}$-coercivity of I. However, in some cases we can.

Remark. The same theorem can be obtained using Hoelder inequality (e.g. Marcellini-Sbordone [7]).

THEOREM 4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, f: $\Omega \times \mathbb{R}^n \to \mathbb{R}$ be a Carathéodory function (i.e., $f(x, \cdot)$ is continuous* a.e. *in x, $f(\cdot, \xi)$ is mea-*

surable in Ω) *with* $f(x, 0) = 0$, *satisfying* $|f(x, \xi)| \leq c(x)g(\xi)$, *where* $c \in L^1(\Omega)$ *and* g *is locally bounded. Assume that for* $\alpha > 0$ *and* $p > 1$ *the following holds*:

$$I(u) = \int_\Omega f(x, \nabla u(x))\, dx \geq \alpha \int_\Omega |\nabla u(x)|^p\, dx, \qquad \forall u \in W_0^{1,\infty}(\Omega). \quad (3.5)$$

Then there exist $\tilde{\alpha} > 0$, $\tilde{\beta} \in \mathbb{R}$ *such that*

$$f(x, \xi) \geq \tilde{\alpha}|\xi|^p + \tilde{\beta} \qquad \text{a.e. } x \in \Omega,\ \xi \in \mathbb{R}^n. \quad (3.6)$$

Proof. The proof is very similar to that of Theorem 1 and we omit the details. Let $\xi \neq 0$ be fixed, $\lambda = 1/(|\xi| + 1)$, $u_1(x) = \langle \xi; x \rangle$, $u_2(x) = -\langle \xi; x \rangle / |\xi|$. Let $x_0 \in \Omega$ be a Lebesgue point of $f(\cdot, \xi)$ and $f(\cdot, \xi/|\xi|)$. Let $\eta > 0$ be sufficiently small so that $B_\eta(x_0) = \{x \in \mathbb{R}^n \colon |x - x_0| < \eta\} \subset \Omega$. Applying Lemma 2 to λ, u_1, u_2, and B_η, we get for every $\varepsilon > 0$ B_1, B_2, and $u \in W_0^{1,\infty}(B_\eta)$ as in the lemma. We can also assume (see Ekeland-Témam [3], Proposition 1.3 in Chapter X, if necessary) that

$$\left| \int_{B_\eta} f(x, \nabla u)\, dx - \lambda \int_{B_\eta} f(x, \xi)\, dx - (1 - \lambda) \int_{B_\eta} f\left(x, -\frac{\xi}{|\xi|}\right) dx \right| \leq \varepsilon,$$
$$\left| \int_{B_\eta} |\nabla u|^p\, dx - [\lambda|\xi|^p + (1 - \lambda)] \operatorname{meas} B_\eta \right| \leq \varepsilon.$$

Since $f(x, 0) = 0$, we can extend u to be 0 on $\Omega - B_\eta$. We have therefore from (3.5) that

$$\begin{aligned} \lambda \int_{B_\eta} f(x, \xi)\, dx + (1 - \lambda) \int_{B_\eta} f\left(x, -\frac{\xi}{|\xi|}\right) & \\ \geq -\varepsilon + \int_\Omega f(x, \nabla u)\, dx & \\ \geq -\varepsilon + \alpha \int_\Omega |\nabla u|^p\, dx = -\varepsilon + \alpha \int_{B_\eta} |\nabla u|^p\, dx & \\ \geq -\varepsilon + \alpha[-\varepsilon + (\lambda|\xi|^p + (1 - \lambda)) \operatorname{meas} B_\eta]. & \end{aligned}$$

We thus get

$$\begin{aligned} \frac{1}{\operatorname{meas} B_\eta} \int_{B_\eta} f(x, \xi)\, dx \geq \alpha|\xi|^p - \frac{(1 + \alpha)\varepsilon}{\operatorname{meas} B_\eta} (|\xi| + 1) & \\ + \alpha|\xi| - \frac{|\xi|}{\operatorname{meas} B_\eta} \int_{B_\eta} f\left(x, \frac{-\xi}{|\xi|}\right) dx. & \end{aligned}$$

Choosing $\varepsilon = (\text{meas } B_\eta)^2$, letting η tend to 0, and using the Lebesgue theorem we find that

$$f(x_0, \xi) \geq \alpha|\xi|^p + \alpha|\xi| - |\xi| f\left(x_0, -\frac{\xi}{|\xi|}\right).$$

Since $p > 1$ we get the result. ∎

4. The Case $f(u, \xi)$

As mentioned in the Introduction no p-coercivity of f can be inferred from the $W_0^{1,p}$-coercivity of I, since best constants in Sobolev inequalities enter into the game. However, this is not the case for $W^{1,p}$-coercivity.

Theorem 5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, f: $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be continuous. If there exist $\alpha > 0$, $\beta \in \mathbb{R}$, and $p > 1$ such that*

$$I(u) = \int_\Omega f(u(x), \nabla u(x))\, dx \geq \alpha \int_\Omega |\nabla u(x)|^p + \beta, \qquad \forall u \in W^{1,\infty}(\Omega), \tag{4.1}$$

then

$$f(s, \xi) + \frac{1-\lambda}{\lambda} f\left(s, -\frac{\lambda}{1-\lambda}\xi\right) \geq \alpha\left[1 + \left(\frac{\lambda}{1-\lambda}\right)^{p-1}\right]|\xi|^p + \frac{\beta}{\lambda \text{ meas } \Omega} \qquad \forall(\lambda, s, \xi) \in (0, 1) \times \mathbb{R} \times \mathbb{R}^n. \tag{4.2}$$

In particular if either one of the three properties

$$f(s, \xi) = f(s, -\xi) \qquad \forall(s, \xi) \in \mathbb{R} \times \mathbb{R}^n \tag{4.3}$$

$$f(s, \xi) = a(s) + b(\xi) \qquad \forall(s, \xi) \in \mathbb{R} \times \mathbb{R}^n \tag{4.4}$$

$$f(s, \xi) = a(s)b(\xi) \qquad \forall(s, \xi) \in \mathbb{R} \times \mathbb{R}^n \tag{4.5}$$

holds, then there exist $\tilde{\alpha} > 0$, $\tilde{\beta} \in \mathbb{R}$ such that

$$f(s, \xi) \geq \tilde{\alpha}|\xi|^p + \tilde{\beta} \qquad \forall(s, \xi) \in \mathbb{R} \times \mathbb{R}^n. \tag{4.6}$$

Remark. We have been unable to prove that (4.1) implies (4.6) for general f; therefore this remains an open question.

Proof. To show that (4.1) implies (4.2) is done exactly as before. Apply Lemma 2 to $u_1(x) = \langle \xi; x\rangle + s$, $u_2(x) = -\lambda \langle \xi; x\rangle/(1-\lambda) + s$, and $\lambda \in (0, 1)$. We thus deduce that for $\varepsilon > 0$, there exist Ω_1, Ω_2, and $u \in W^{1,\infty}(\Omega)$ such that

$$\begin{aligned}
&|\text{meas}\ \Omega_1 - \lambda\ \text{meas}\ \Omega| \le \lambda\varepsilon,\\
&|\text{meas}\ \Omega_2 - (1-\lambda)\ \text{meas}\ \Omega| \le (1-\lambda)\varepsilon\\
&\nabla u(x) = \nabla u_i(x) \qquad \text{a.e. in } \Omega_i\\
&\|\nabla u\|_{L^\infty} \le \max\left\{\frac{\lambda}{1-\lambda}, 1\right\} \cdot |\xi| \equiv K(\lambda, \xi)\\
&|u(x) - s| \le \varepsilon, \qquad x \in \Omega\\
&u(x) = s \qquad \text{on } \partial\Omega.
\end{aligned} \tag{4.7}$$

Since f is continuous and λ, s, and ξ are fixed we deduce that there exists $\delta = \delta(\varepsilon, K)$ such that

$$|f(u(x), \nabla u(x)) - f(s, \nabla u(x))| \le \delta, \qquad x \in \Omega. \tag{4.8}$$

We then have

$$\begin{aligned}
I(u) &= \int_\Omega f(u, \nabla u)\, dx = \int_\Omega f(s, \nabla u)\, dx + \int_\Omega [f(u, \nabla u) - f(s, \nabla u)]\, dx\\
&\ge \alpha \int_\Omega |\nabla u|^p\, dx + \beta\\
&\ge \alpha\left[|\xi|^p\ \text{meas}\ \Omega_1 + \left(\frac{\lambda}{1-\lambda}\right)^p |\xi|^p\ \text{meas}\ \Omega_2\right] + \beta.
\end{aligned}$$

Using (4.7) and (4.8) we get

$$\begin{aligned}
&f(s, \xi)\ \text{meas}\ \Omega_1 + f\left(s, -\frac{\lambda}{1-\lambda}\xi\right) \text{meas}\ \Omega_2 + \int_{\Omega - (\Omega_1 \cup \Omega_2)} f(s, \nabla u)\, dx\\
&\qquad \ge \alpha\left[\text{meas}\ \Omega_1 + \left(\frac{\lambda}{1-\lambda}\right)^p \text{meas}\ \Omega_2\right] |\xi|^p + \beta - \delta\ \text{meas}\ \Omega.
\end{aligned}$$

Dividing by meas Ω_1 and letting ε tend to 0 we obtain

$$\begin{aligned}
f(s, \xi) + \frac{1-\lambda}{\lambda} f\left(s, -\frac{1}{1-\lambda}\xi\right) &\ge \alpha\left[1 + \left(\frac{\lambda}{1-\lambda}\right)^{p-1}\right] |\xi|^p\\
&\quad + \frac{\beta}{\lambda\ \text{meas}\ \Omega},
\end{aligned}$$

which is exactly (4.2).

We now show that if f satisfies (4.3), (4.4), or (4.5) then it satisfies automatically (4.6). The first case is easy; just choose $\lambda = \frac{1}{2}$.

For the second case, first choose $\xi = 0$ and $\lambda = \frac{1}{2}$ to deduce that $a(s) \geq a_0$ for every $s \in \mathbb{R}$ and for some $a_0 \in \mathbb{R}$. Then take $s = 0$ and $\lambda = 1/(1 + |\xi|)$ to get

$$a(0) + b(\xi) + |\xi| \left(a(0) + b\left(-\frac{\xi}{|\xi|}\right)\right) \geq \alpha|\xi|^p + \alpha|\xi| + \frac{\beta(|\xi| + 1)}{\text{meas } \Omega}.$$

Since $p > 1$ we get

$$b(\xi) \geq \alpha_1|\xi|^p + \beta_1.$$

Combining this with the fact that $a(s) \geq a_0$ we get immediately the result. For the last case choose $\lambda = 1/(1 + |\xi|)$ to get

$$a(s)\left[b(\xi) + |\xi| b\left(-\frac{\xi}{|\xi|}\right)\right] \geq \alpha|\xi|^p + \alpha|\xi| + \frac{\beta(|\xi| + 1)}{\text{meas } \Omega}, \qquad \forall (s, \xi) \in \mathbb{R} \times \mathbb{R}^n. \tag{4.9}$$

Observe that $a(s)$ must then have a constant sign that we may choose, without loss of generality, to be positive. Similarly it is clear that in fact $a(s) \geq a_0 > 0$ for every s. Therefore from (4.9) we deduce that since $p > 1$

$$b(\xi) + |\xi| b\left(-\frac{\xi}{|\xi|}\right) \geq \frac{\alpha_1}{a(0)}|\xi|^p + \frac{\beta_1}{a(0)},$$

which implies

$$b(\xi) \geq \alpha_2|\xi|^p + \beta_2.$$

Combining this last result with the fact that $a(s) \geq a_0 > 0$ we get immediately the result. ∎

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