SOME RECENT RESULTS ON POLYCONVEX, QUASICONVEX AND RANK ONE CONVEX FUNCTIONS

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0. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $u : \mathbb{R}^n \to \mathbb{R}^m$, $\nabla u = \left(\frac{\partial u_i}{\partial x_j}\right)_{1 \le i \le m, 1 \le j \le n} \in \mathbb{R}^{nm}$, $f : \mathbb{R}^{nm} \to \mathbb{R}$ be a continuous function and

$$I(u) = \int_\Omega f(\nabla u(x))\, dx.$$

In order to state minimization problems involving such functionals, we need to impose some convexity hypotheses on f. We now define such conditions, starting with the well known ones.

Definition.

1) f is convex if

$$f(\lambda\xi + (1-\lambda)\eta) \le \lambda f(\xi) + (1-\lambda) f(\eta)$$

for every $\lambda \in [0,1], \xi, \eta \in \mathbb{R}^{nm}$.

2) For $\xi \in \mathbb{R}^{nm}$, let $T(\xi)$ denote the vector composed by ξ and all its $s \times s$ minors. $T(\xi) \in \mathbb{R}^{\tau(n,m)}$ where $\tau(n,m) = \sum_{s=1}^{\min\{n,m\}} \frac{m!n!}{(s!)^2(m-s)!(n-s)!}$. If $m = n = 2$, then $\tau(2,2) = 5$ and $T(\xi) = (\xi, \det\xi)$. A function is polyconvex if there exists $g : \mathbb{R}^{\tau(n,m)} \to \mathbb{R}$ convex such that

$$f(\xi) = g(T(\xi)) \text{ for every } \xi \in \mathbb{R}^{nm}.$$

3) f is said to be quasiconvex if

$$\int_\Omega f(\xi + \nabla\varphi(x))\, dx \ge f(\xi)\ \text{meas}\Omega$$

for every $\xi \in \mathbb{R}^{nm}$, every $\varphi \in W_0^{1,\infty}(\Omega; \mathbb{R}^m)$ (the set of continuous functions φ vanishing on $\partial\Omega$ and with uniformly bounded gradients).

4) f is said to be rank one convex if

$$f(\lambda\xi + (1-\lambda)\eta) \leq \lambda f(\xi) + (1-\lambda) f(\eta)$$

for every $\lambda \in [0,1], \xi, \eta \in \mathbb{R}^{nm}$, *with rank* $(\xi - \eta) \leq 1$.

Remark. As well known the preceding notions are less and less restrictive (see for general references Dacorogna [1]), i.e.

$$f \text{ convex} \Rightarrow f \text{ polyconvex} \Rightarrow f \text{ quasiconvex} \Rightarrow f \text{ rank one convex.}$$

This article is organized as follows. In the first section we discuss the example of Dacorogna-Marcellini [2] and Alibert-Dacorogna [3]. In the second one we indicate how the effect of symmetries simplifies the notions of convexity and polyconvexity. In the third section we show how these symmetries influence the computations of the different envelopes.

1. An example

Let $m = n = 2$ and let for $\xi = (\xi_{ij})_{1\leq i,j\leq 2} \in \mathbb{R}^{2\times 2}$, $|\xi|^2 = \sum_{i,j=1}^{2} \xi_{ij}^2$. Let for $\gamma \in \mathbb{R}$

$$f_\gamma(\xi) = |\xi|^2 \left(|\xi|^2 - 2\gamma \det\xi\right). \tag{1.1}$$

We have the following theorem.

Theorem 1.

1) f_γ *is convex if and only if* $|\gamma| \leq \frac{2}{3}\sqrt{2}$.

2) f_γ *is polyconvex if and only if* $|\gamma| \leq 1$. *In the case* $\gamma = 1$ *one has the following*

2^a) For every $\xi, \eta \in \mathbb{R}^{2\times 2}$

$$f_\gamma(\xi) - f_\gamma(\eta) \geq 4\left(|\eta|^2 - \det\eta\right) < \eta; \xi - \eta > -2|\eta|^2 \left(\det\xi - \det\eta\right). \tag{1.2}$$

where $< .; . >$ *stands for the usual scalar product.*

2^b) Let $g : \mathbb{R}^{2\times 2} \times \mathbb{R} \to \mathbb{R}$ *be defined by*

$$g(\xi,\delta) = \begin{cases} 0 \quad if \ |\xi|^2 + 2\det\xi \leq 4\delta \\ \left(|\xi|^2 + 2\det\xi - 4\delta\right)\left(|\xi|^2 + 2\det\xi - 2\delta\right) if \ |\xi|^2 + 2\det\xi \geq 4\delta \end{cases}$$

then g is convex and

$$f_\gamma(\xi) = g(\xi, \det\xi).$$

3) There exists $\varepsilon > 0$ *such that* f_γ *is <u>quasiconvex</u> if and only if* $|\gamma| \le 1 + \varepsilon$.

4) f_γ *is <u>rank one convex</u> if and only if* $|\gamma| \le \frac{2}{\sqrt{3}}$.

Remarks.

i) The results 1), 2), 3), 4) have been published by Dacorogna-Marcellini [1] and Alibert-Dacorogna [2]. The result 2^a), which gives immediately that f_γ is polyconvex if $\gamma = 1$, was proved by Iwaniec-Luttoborski [4]. The representation 2^b) has been found by Hartwig [5].

ii) Numerical computations to determine ε in 3) have been undertaken by Dacorogna-Douchet-Gangbo-Rappaz [6] and more recently by Gremaud [7]. They all tend to show that $1+\varepsilon = \dfrac{2}{\sqrt{3}}$. So that the function f_γ does not seem to produce a counterexample to the implication f rank one convex $\Rightarrow$ f quasiconvex. Note that such a counterexample has not yet been produced when $m = n = 2$, it has been given by Sverak [8] when $m \ge 3$ and $n \ge 2$.

iii) To understand the difficulty in computing analytically the exact value of ε giving the quasiconvexity, one should see that it is related (cf Alibert-Dacorogna [1]) to the determination of the best constant δ in

$$\int_\Omega \left(|\nabla\varphi|^2 - 2\det\nabla\varphi\right)^2 \, dx \ge \delta \int_\Omega |\nabla\varphi|^4 \, dx$$

for every $\varphi \in W_0^{1,\infty}(\Omega; \mathbb{R}^2)$. As it was pointed out to me by V. Sverak and T. iwaniec, this is related to a (still open) conjecture of Gehring (see for more details Iwaniec-Martin [9], Banuelos-Wang [10]).

2. Convexity and polyconvexity for rotationally invariant functions

We here discuss the case $m = n = 2$. We denote by $\mathbb{R}_d^{2\times 2}$ the subset of $\mathbb{R}^{2\times 2}$ consisting of diagonal matrices. We let

$$\mathcal{O} = \left\{U \in \mathbb{R}^{2\times 2} : UU^t = I\right\}$$

where U^t denotes the transpose of U and I the identity matrix. We let $\mathcal{O}^+$ be the subset of $\mathcal{O}$ consisting of those U with $\det U = 1$. We assume in this section and in the next one that f satisfies, in addition of continuity, that

$$(H^+) \qquad f(U\xi V) = f(\xi)$$

for every $\xi \in \mathbb{R}^{2\times 2}, U, V \in \mathcal{O}^+$.

Usually in applications (Ball [11], Ciarlet [12], Dacorogna [1]) one imposes that

$$(H) \qquad f(U\xi V) = f(\xi)$$

for every $\xi \in \mathbb{R}^{2\times 2}, U, V \in \mathcal{O}$. Note that the example of the previous section satisfies (H^+) but not (H).

One can then show the following result.

Theorem 2. *Let $f : \mathbb{R}^{2\times 2} \to \mathbb{R}$ satisfy (H^+). Then f is convex if and only if $f|_{\mathbb{R}^{2\times 2}_d}$ is convex .*

Remarks.

i) $f|_{\mathbb{R}^{2\times 2}_d}$ being convex, means that f is convex when restricted to diagonal matrices.

ii) in the example given above the convexity of f_γ has therefore only to be checked for

$$f_\gamma \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = (x^2 + y^2)\left(x^2 + y^2 - 2\gamma xy\right),$$

which is, at least theoretically, simpler to establish than the convexity on the whole of $\mathbb{R}^{2\times 2}$.

More interestingly Theorem 2 is also valid (c.f. the above mentioned paper) for polyconvex functions. We first define what we mean by $f|_{\mathbb{R}^{2\times 2}_d}$ is polyconvex. This just means that there exists $g : \mathbb{R}^3 \to \mathbb{R}$ convex such that

$$f\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = g(a, b, ab). \tag{2.1}$$

Theorem 3. *Let $f : \mathbb{R}^{2\times 2} \to \mathbb{R}$ satisfy (H^+). The following conditions are then equivalent:*

i) f is polyconvex.

ii) $f|_{\mathbb{R}^{2\times 2}_d}$ is polyconvex.

iii) The following holds

$$\sum_{i=1}^{4} \lambda_i f(\xi_i) \geq f\left(\sum_{i=1}^{4} \lambda_i \xi_i\right)$$

for every $\lambda_i \geq 0$ *with* $\sum_{i=1}^{4} \lambda_i = 1$ *and every* $\xi_i \in \mathbb{R}_d^{2\times 2}$ *such that*

$$\sum_{i=1}^{4} \lambda_i \det\xi_i = \det\left(\sum_{i=1}^{4} \lambda_i \xi_i\right).$$

iv) For every $\eta \in \mathbb{R}_d^{2\times 2}$, *there exist* $\alpha(\eta), \beta(\eta), \gamma(\eta) \in \mathbb{R}$ *such that*

$$f(\xi) \geq f(\eta) + < \begin{pmatrix} \alpha(\eta) & 0 \\ 0 & \beta(\eta) \end{pmatrix}; \xi - \eta > + \gamma(\eta)\left(\det\xi - \det\eta\right)$$

for every $\xi \in \mathbb{R}_d^{2\times 2}$ *and where* $< .;. >$ *denotes the scalar product in* $\mathbb{R}^{2\times 2}$.

Remark. In practice this theorem might lead to a considerable simplification in the computations. For instance in the example of the above section, Iwaniec-Luttoborski [4] have proved 2^a) essentially for diagonal matrices. In this case, i.e. $\gamma = 1$

$$f_\gamma \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = (x^2 + y^2)(x - y)^2,$$

and they have shown (lemma 10.2) that for every $x, y, a, b \in \mathbb{R}$

$$(x^2+y^2)(x-y)^2 - (a^2+b^2)(a-b)^2 \geq 4(a^2+b^2-ab)(ax+by-a^2-b^2) - 2(a^2+b^2)(xy-ab)$$

Similarly Hartwig [5] has computed its function g in 2^b) for diagonal matrices, showing that

$$g(x,0,0,y,\delta) = \begin{cases} 0 \quad if \quad (x+y)^2 \leq 4\delta \\ \left((x+y)^2 - 4\delta\right)\left((x+y)^2 - 2\delta\right) if \ (x+y)^2 \geq 4\delta \end{cases}$$

has the desired properties.

These results for convexity and polyconvexity do not extend to rank one convexity (and obviously to quasiconvexity, since quasiconvexity on diagonal matrices is meaningless). Indeed in Dacorogna-Koshigoe [13], following some computations of Dacorogna-Douchet-Gangbo-Rappaz [6], it is shown that there are functions f such that $f_{|\mathbb{R}_d^{2\times 2}}$ is rank one convex, but not rank one convex on the whole of $\mathbb{R}^{2\times 2}$. Indeed there are $\alpha \geq 2 + \sqrt{2}$ and $b \geq 0$ such that

$$f_{\alpha,b}(\xi) = |\xi|^{2\alpha} - 2^{\alpha-1} b |\det\xi|^{\alpha}$$

is rank one convex when restricted to $\mathbb{R}_d^{2\times 2}$ but not globally rank one convex.

3. The different envelopes

We assume that f satisfies (H^+) or (H) and we define

$$\begin{aligned} Cf &= \sup\{g \le f :\ g \text{ convex}\} \\ Pf &= \sup\{g \le f :\ g \text{ polyconvex}\} \\ Qf &= \sup\{g \le f :\ g \text{ quasiconvex}\} \\ Rf &= \sup\{g \le f :\ g \text{ rank one convex}\} \end{aligned}$$

Then, in general: $Cf \le Pf \le Qf \le Rf \le f$.

The first elementary result (cf Dacorogna-Koshigoe [13] and Buttazzo-Dacorogna-Gangbo [14]) is:

Theorem 4. *If f satisfies (H^+) (respectively (H)) then so do Cf, Pf, Qf, Rf.*

We quote here another result of Dacorogna-Koshigae [13].

Theorem 5. *Let $f : \mathbb{R}^{2\times 2} \to \mathbb{R}$ satisfy (H^+) then for every $\xi \in \mathbb{R}_d^{2\times 2}$*

$$Pf(\xi) = \inf\left\{ \sum_{i=1}^{4} \lambda_i f(\eta_i) :\ \eta_i \in \mathbb{R}_d^{2\times 2},\ \sum_{i=1}^{4} \lambda_i (\eta_i, \det\eta_i) = (\xi, \det\xi) \right\}.$$

Remarks

i) Compared to the theorem (cf Dacorogna [1]) for general functions f, this might be a simplification. Indeed, for general f, not satisfying (H^+), Pf is computed with 6 general matrices, instead of 4 diagonal ones.

ii) A similar result exists for convex functions.

In the more restrictive case where f satisfies (H) and hence depends only on the singular values of f (i.e. the eigenvalues of $(\xi\xi^t)^{1/2}$) one has similar simplifications for computing the convex or polyconvex envelope: cf Buttazzo-Dacorogna-Gangbo [14]. It is well known that if $f : \mathbb{R}^{n\times n} \to \mathbb{R}$ satisfies (H) then

$$f(\xi) = g(\lambda_1, ..., \lambda_n)$$

where λ_i are the singular values of ξ. We assume also that the λ_i have been ordered increasingly, i.e. $0 \le \lambda_1 \le \lambda_2 \le ..... \le \lambda_n$. We denote by $Q = \{(\lambda_1, ..., \lambda_n) \in \mathbb{R}^n : 0 \le \lambda_1 \le \lambda_2 \le ..... \le \lambda_n\}$. Then one can show that

$$Cf(\xi) = \hat{g}(\lambda_1, ..., \lambda_n)$$

where $\hat{g}$ is the greatest function less than g on Q and which is lower semicontinuous and increasing in each variable.

To conclude it is interesting to see how these results can be applied to a particular example. Several are given in Buttazzo-Dacorogna-Gangbo [14], but we would like to privilege one which was extensively studied by Kohn-Strang [15].

Let for $\xi \in \mathbb{R}^{2\times 2}$

$$f(\xi) = \begin{cases} 1 + |\xi|^2 \text{ if } \xi \neq 0 \\ 0 \quad \text{if } \xi = 0 \end{cases}$$

Then using theorem 3, 4, 5 we know that, to compute the polyconvex envelope is equivalent to compute the polyconvex envelope in $\mathbb{R}^{2\times 2}_d$ i.e. the one corresponding to

$$f\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{cases} 1 + x^2 + y^2 \text{ if } x^2 + y^2 \neq 0 \\ 0 \quad \text{if } x = y = 0 \end{cases}$$

As shown by Kohn-Strang [15] (cf also Dacorogna [1])

$$Pf\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{cases} 1 + x^2 + y^2 \text{ if } |x| + |y| \geq 1 \\ 2(|x| + |y|) - 2|x||y| \text{ if } |x| + |y| \leq 1 \end{cases}$$

Using also diagonal matrices, Dosly [16] has shown that if $s \geq 2$ and

$$f_s(\xi) = \begin{cases} 1 + |\xi|^s \text{ if } \xi \neq 0 \\ 0 \quad \text{if } \xi = 0 \end{cases}$$

then $Cf_s\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} = Pf_s\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$(i.e. $Pf_s(\xi) = Cf_s(\xi)$ if rank$\xi \leq 1$) and the difference between $Pf_s\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$ and $Cf_s\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$ is maximal when $x = y$.

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