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On the Optimality of Certain Sobolev Exponents for the Weak Continuity of Determinants

B. DACOROGNA

Département de Mathématiques, Ecole Polytechnique Fédérale de Lausanne, CH 1015 Lausanne, Switzerland

AND

F. MURAT

Laboratoire d'Analyse Numérique, Université Paris VI, Tour 55-65, 4 place Jussieu, F 75252 Paris Cedex 05, France

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Let Ω be an open subset of $\mathbb{R}^n$ and let u_ε and u be elements of $W^{1,n}(\Omega; \mathbb{R}^n)$ such that the sequence u_ε weakly converges to u in $W^{1,p}(\Omega; \mathbb{R}^n)$. It is well known that the sequence $\det \nabla u_\varepsilon$ converges to $\det \nabla u$ in the distributional sense when $p > n^2/(n+1)$. We prove in the present paper that this weak continuity result does not hold true when $p \leqslant n^2/(n+1)$. Indeed when $p = n^2/(n+1)$ a subsequence $\det \nabla u_{\varepsilon'}$ converges in the distributional sense to $\det \nabla u + \mu$ where μ is a Radon measure which is not zero in general. When $p < n^2/(n+1)$ there exist a sequence u_ε and a test function φ in $C_0^\infty(\Omega)$ such that $\int_\Omega \det \nabla u_\varepsilon(x)\, \varphi(x)\, dx$ tends to infinity. We also prove analogous results for $\det \nabla^2 v_\varepsilon$ when v_ε weakly converges in $W^{2,p}(\Omega, \mathbb{R})$; in this setting the critical value for p is $p = n^2/(n+2)$. © 1992 Academic Press, Inc.

Soit Ω un ouvert de $\mathbb{R}^n$ et soient u_ε et u des fonctions de $W^{1,n}(\Omega; \mathbb{R}^n)$ telles que la suite u_ε converge faiblement vers u dans $W^{1,p}(\Omega, \mathbb{R}^n)$. Il est bien connu que si $p > n^2/(n+1)$, la suite $\det \nabla u_\varepsilon$ converge vers $\det \nabla u$ dans $\mathscr{D}'(\Omega)$. Nous montrons dans cet article que ce résultat de continuité faible n'est plus vrai si $p \leqslant n^2/(n+1)$. En effet, si $p = n^2/(n+1)$, une sous suite $\det \nabla u_{\varepsilon'}$ converge vers $\det \nabla u + \mu$ dans $\mathscr{D}'(\Omega)$ où μ est une mesure qui est non nulle, en général; si $p < n^2/(n+1)$, on peut trouver une suite u_ε et une fonction test φ dans $\mathscr{D}(\Omega)$ tels que $\int_\Omega \det \nabla u_\varepsilon(x)\, \varphi(x)\, dx$ tende vers l'infini. Nous établissons aussi des résultats analogues pour $\det \nabla^2 v_\varepsilon$ quand v_ε converge faiblement dans $W^{2,p}(\Omega, \mathbb{R})$; dans ce cas la valeur critique de p est $p = n^2/(n+2)$. © 1992 Academic Press, Inc.

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1. Introduction and Statement of the Results

It is now well established since the work of Reshetnyak [12] and Ball [3] that if Ω is a bounded open set of $\mathbb{R}^n$, if ε denotes a sequence which tends to 0 and if $p \geqslant n$, then

$$u_\varepsilon \rightharpoonup u \text{ weakly in } W^{1,p}(\Omega; \mathbb{R}^n) \Rightarrow \det \nabla u_\varepsilon \rightharpoonup \det \nabla u \text{ in } \mathscr{D}'(\Omega). \tag{1.1}$$

The convergence of the Jacobian determinant of course takes place in the weak topology of $L^{p/n}(\Omega)$ whenever $p > n$.

Actually if one assumes u_ε and u to belong to $W^{1,n}(\Omega; \mathbb{R}^n)$ then the above result (1.1) still holds true for $p > n^2/(n+1)$ since when v belongs to $W^{1,n}(\Omega; \mathbb{R}^n)$ one has

$$\det \nabla v = \operatorname{Det} \nabla v, \tag{1.2}$$

where $\operatorname{Det} \nabla v$ denotes the "distributional determinant" defined in formula (2.1) below (see, e.g., Ball and Murat [6, p. 249] for a summary of the properties of det and Det).

The aim of the present paper is to prove that the sequential continuity (1.1) does not hold when $1 \leqslant p \leqslant n^2/(n+1)$.

Theorem 1. *Let $n \geqslant 2$, $p \geqslant 1$, u_ε, u in $W^{1,n}(\Omega; \mathbb{R}^n)$, and $u_\varepsilon \rightharpoonup u$ weakly in $W^{1,p}(\Omega; \mathbb{R}^n)$.*

(i) *If $p > n^2/(n+1)$, then*

$$\det \nabla u_\varepsilon \rightharpoonup \det \nabla u \quad \text{in} \quad \mathscr{D}'(\Omega). \tag{1.3}$$

(ii) *If $p = n^2/(n+1)$, there exists μ in $\mathscr{D}'(\Omega)$ (with $\mu \neq 0$ in general) such that for a subsequence ε'*

$$\det \nabla u_{\varepsilon'} \rightharpoonup \det \nabla u + \mu \quad \text{in} \quad \mathscr{D}'(\Omega). \tag{1.4}$$

(iii) *If $1 \leqslant p < n^2/(n+1)$, there exist u_ε and u in $W^{1,n}(\Omega; \mathbb{R}^n)$, with $u_\varepsilon \rightharpoonup u$ weakly in $W^{1,p}(\Omega, \mathbb{R}^n)$, and φ in $\mathscr{D}(\Omega)$ such that*

$$\int_\Omega \det \nabla u_\varepsilon(x)\, \varphi(x)\, dx \to \infty. \tag{1.5}$$

Remarks 1. (i) The first part of Theorem 1 is well known (see references above).

(ii) The last part of Theorem 1 shows that in general $\det \nabla u_\varepsilon$ does not converge (to anything) in the sense of distributions when $1 \leqslant p < n^2/(n+1)$. In the case $p = n^2/(n+1)$, $\det \nabla u_\varepsilon$ does converge in the sense of distributions (and even in the so-called vague topology of

Radon measures, i.e., in the weak * topology of $\mathcal{M}(\Omega)$) but in general not to $\det \nabla u$.

(iii) The hypothesis that u_ε and u belong to $W^{1,n}(\Omega; \mathbb{R}^n)$ is assumed in order to give a meaning to $\det \nabla u$ as a function of $L^1(\Omega)$. This restriction is unimportant for what concerns the second and third points of Theorem 1 since there are negative results.

Let us mention in contrast that (1.3) continues to hold if u_ε and u only belong to $W^{1,p}(\Omega; \mathbb{R}^n)$ (with $p > n^2/(n+1)$) when $\det \nabla u_\varepsilon$ and $\det \nabla u$ are replaced by the distributional determinants $\operatorname{Det} \nabla u_\varepsilon$ and $\operatorname{Det} \nabla u$. Note also the recent result obtained by Müller [11] who proves that $\operatorname{Det} \nabla u = \det \nabla u$ if $p \geqslant n^2/(n+1)$ whenever the distribution $\operatorname{Det} \nabla u$ is assumed to belong to $L^1(\Omega)$.

We next turn out attention to results similar to those of Theorem 1 for Hessian determinants of scalar functions, i.e., for the case of a sequence of functions $v_\varepsilon \colon \Omega \subset \mathbb{R}^n \to \mathbb{R}$.

THEOREM 2. *Let $p \geqslant 1$, v_ε, v in $W^{2,n}(\Omega)$, and $v_\varepsilon \rightharpoonup v$ weakly in $W^{2,p}(\Omega)$.*

(i) *If $n \geqslant 3$ and $p > n^2/(n+2)$, or if $n = 2$ and $p \geqslant n^2/(n+2) = 1$, then*

$$\det \nabla^2 v_\varepsilon \rightharpoonup \det \nabla^2 v \qquad \text{in } \mathcal{D}'(\Omega). \tag{1.6}$$

(ii) *If $n \geqslant 3$ and $p = n^2/(n+2)$, there exists μ in $\mathcal{D}'(\Omega)$ (with $\mu \neq 0$ in general) such that for a subsequence ε'*

$$\det \nabla^2 v_{\varepsilon'} \rightharpoonup \det \nabla^2 v + \mu \qquad \text{in } \mathcal{D}'(\Omega). \tag{1.7}$$

(iii) *If $n \geqslant 3$ and $1 \leqslant p < n^2/(n+2)$, there exist v_ε and v in $W^{2,n}(\Omega)$, with $v_\varepsilon \rightharpoonup v$ weakly in $W^{2,p}(\Omega)$, and φ in $\mathcal{D}(\Omega)$ such that*

$$\int_\Omega \det \nabla^2 v_\varepsilon(x)\, \varphi(x)\, dx \to \infty. \tag{1.8}$$

Remarks 2. (i) The first part of Theorem 2 is essentially due to Ball, Currie, and Olver [5]. We shall however give below a simple proof of this fact for the sake of completeness.

(ii) The case $n = 2$, $p = n^2/(n+2) = 1$ appears as a border line case, since the Hessian determinant is weakly continuous in this case (cf. (1.6)) while it is not in the case $n \geqslant 3$ and $p = n^2/(n+2)$ (cf. (1.7)).

This is due to the fact that here $\nabla^2 v_\varepsilon$ is a sequence which is assumed to weakly converge in $(L^p(\Omega))^n$; but the weak convergence in $L^1(\Omega)$ is a much stronger assumption than the boundedness in $L^1(\Omega)$, in contrast with $L^p(\Omega)$ where the two assumptions are equivalent up to the extraction of a subsequence. We will indeed prove that for $n=2$, $p=n^2/(n+2)=1$ and for a sequence v_ε and a v in $W^{2,2}(\Omega)$ such that

$$\|v_\varepsilon\|_{W^{2,1}(\Omega)} \leqslant C, \qquad v_\varepsilon \rightharpoonup v \text{ in } \mathscr{D}'(\Omega)$$

one can extract a subsequence ε' such that

$$\det \nabla^2 v_{\varepsilon'} \rightharpoonup \det \nabla^2 v + \mu \qquad \text{in } \mathscr{D}'(\Omega), \tag{1.9}$$

where μ is a distribution which is non-zero in general (compare with (1.6)).

We finally turn our attention to the case of radial functions. Let B be the unit ball of $\mathbb{R}^n$ and define for $p \geqslant 1$

$$R_{ad}^{1,p}(B) = \left\{ u \in W^{1,p}(B; \mathbb{R}^n) \colon u(x) = f(|x|)\frac{x}{|x|} \right\}. \tag{1.10}$$

On this space the norm

$$\|u\|_{R_{ad}^{1,p}(B)} = \left(\int_0^1 \left\{ |f'(r)|^p + \left| \frac{f(r)}{r} \right|^p \right\} r^{n-1}\, dr \right)^{1/p} \tag{1.11}$$

(which is nothing but the norm of f in some weighted Sobolev space $W_{Rad}^{1,p}(0,1)$) is equivalent to the norm $\|u\|_{W^{1,p}(B;\mathbb{R}^n)}$.

In this case we have

Theorem 3. *Let $p \geqslant 1$, u_ε, u in $R_{ad}^{1,n}(B)$, and $u_\varepsilon \rightharpoonup u$ weakly in $W^{1,p}(\Omega; \mathbb{R}^n)$.*

(i) *If $n \geqslant 3$ and $p > n^2/(n+2)$, or if $n=2$ and $p \geqslant n^2/(n+2)=1$, then*

$$\det \nabla u_\varepsilon \rightharpoonup \det \nabla u \qquad \text{in } \mathscr{D}'(B). \tag{1.12a}$$

(ii) *If $n \geqslant 3$ and $p = n^2/(n+2)$, then for a subsequence ε'*

$$\det \nabla u_{\varepsilon'} \rightharpoonup \det \nabla u + c\, \Delta\delta_0 \qquad \text{in } \mathscr{D}'(B), \tag{1.12b}$$

where c is a constant (in general different from 0) and $\Delta\delta_0$ is the Laplacian of the Dirac mass, see Remark 3(ii) below.

(iii) *If $n \geqslant 3$ and $1 \leqslant p < n^2/(n+2)$ there exist u_ε and u in $R_{ad}^{1,n}(B)$, with $u_\varepsilon \rightharpoonup u$ weakly in $W^{1,p}(B; \mathbb{R}^n)$, and φ in $\mathscr{D}(B)$ such that*

$$\int_B \det \nabla u_\varepsilon(x)\, \varphi(x)\, dx \to \infty. \tag{1.13}$$

Remarks 3. (i) Except for the fact that the measure μ which appears in (1.7) is now identified as $c\,\Delta\delta_0$, Theorem 3 is a particular case of

Theorem 2. Indeed for a radial function $u_\varepsilon(x)=f_\varepsilon(|x|)x/|x|$ in $R^{1,n}_{ad}(B)$ one has

$$u_\varepsilon=\nabla U_\varepsilon,$$

where the scalar function $U_\varepsilon\colon B\to\mathbb{R}$ is defined by

$$U_\varepsilon(x)=F_\varepsilon(|x|) \qquad \text{with} \quad F_\varepsilon(r)=\int_0^r f_\varepsilon(s)\,ds.$$

With $u_\varepsilon=\nabla U_\varepsilon$ one has

$$u_\varepsilon\in W^{1,n}(B;\mathbb{R}^n)\Leftrightarrow U_\varepsilon\in W^{2,n}(B)$$

$$u_\varepsilon\rightharpoonup u \text{ weakly in } W^{1,p}(B;\mathbb{R}^n)\Leftrightarrow U_\varepsilon\rightharpoonup U \text{ weakly in } W^{2,p}(B)$$

for any $p\geqslant 1$, which proves the above assertion.

Therefore, since the second and third parts of Theorems 2 and 3 are "negative" results, we will only give examples of radial functions u_ε proving the second and third parts of Theorem 3; this provides examples U_ε proving the second and third parts of Theorem 3. Since the first part of Theorem 2 is a "positive" result, it can be deduced from the first part of Theorem 2. We will nevertheless give a separate proof.

(ii) In the second part of Theorem 3, $\Delta\delta_0$ stands for the Laplacian of the Dirac mass defined by

$$\langle\Delta\delta_0,\varphi\rangle=\operatorname{tr}(\nabla^2\varphi(0)) \qquad \forall\varphi\in\mathscr{D}(B).$$

The constant c depends not only on the limit u, but also on the subsequence $u_{\varepsilon'}$ itself; it is explicitly given by

$$c=\frac{S_{n-1}}{n^2}\lim_{\varepsilon\to 0}\int_0^1 r[(f_{\varepsilon'}(r))^n-(f(r))^n]\,dr,$$

where S_{n-1} is the area of the unit sphere of $\mathbb{R}^n$ (i.e., $S_{n-1}=\int_{|x|_{\mathbb{R}^n}=1}ds$) (cf. (2.29) below).

(iii) The case $n=2$, $p=n^2/(n+2)=1$ is a border line case (as in Theorem 2, see Remark 3(ii)). Indeed according to Theorem 3(i) one has in this case for functions u_ε and u in $R^{1,2}_{ad}(B)$

$$\det\nabla u_\varepsilon\rightharpoonup\det\nabla u \qquad \text{in } \mathscr{D}'(B)$$

when $u_\varepsilon\rightharpoonup u$ weakly in $W^{1,1}(B;\mathbb{R}^2)$, while for a sequence v_ε and a v in $R^{1,2}_{ad}(B)$ such that

$$\|v_\varepsilon\|_{W^{1,1}(B;\mathbb{R}^2)}\leqslant C, \qquad v_\varepsilon\rightharpoonup v \text{ in } \mathscr{D}'(B)$$

one has for a subsequence

$$\det \nabla v_{\varepsilon'} \rightharpoonup \det \nabla v + c\, \Delta\delta_0 \qquad \text{in } \mathscr{D}'(\Omega), \tag{1.14}$$

where c is in general non-zero.

Remarks 4. Let us conclude this introduction by recalling some other continuity (or discontinuity) properties of the determinant with respect to different topologies.

(i) There exists u_ε which belongs to $W^{1,p}(\Omega; \mathbb{R}^n)$ for any $p < n$ with

$$\begin{cases} u_\varepsilon \rightharpoonup u \text{ weakly in } W^{1,p}(\Omega; \mathbb{R}^n) & \forall p < n \\ u(x) = x, \qquad \det \nabla u_\varepsilon = 0 \text{ a.e. in } \Omega \end{cases}$$

(cf. Ball and Murat [6, Counterexample 7.4]). Note that the functions u_ε do not belong to $W^{1,n}(\Omega; \mathbb{R}^n)$; thus $\det \nabla u_\varepsilon$ is a measurable function which is only defined almost everywhere. In the quoted example, it actually differs from $\operatorname{Det} \nabla u_\varepsilon$ by a sum of Dirac masses.

(ii) There exists u_ε, u in $C^1(\bar{\Omega}; \mathbb{R}^n)$ and φ in $\mathscr{D}(\Omega)$ such that

$$\begin{cases} u_\varepsilon \to u & \text{strongly in } C^0(\bar{\Omega}; \mathbb{R}^n) \\ \det \nabla u_\varepsilon \rightharpoonup 1 & \text{weakly * in } L^\infty(\Omega) \\ \det \nabla u = 0; & \end{cases}$$

it is sufficient for that to consider (cf. Acerbi, Buttazzo, and Fusco [1]) the case

$$\begin{cases} (u_\varepsilon)_1\,(x) = \sqrt{2\varepsilon}\,\sin(x_1/\varepsilon), \\ (u_\varepsilon)_2\,(x) = x_2\,\sqrt{2\varepsilon}\,\cos(x_1/\varepsilon), \\ (u_\varepsilon)_i\,(x) = x_i \qquad \text{if} \quad 3 \leqslant i \leqslant n, \\ \quad u_1(x) = u_2(x) = 0 \\ \quad u_i(x) = x_i \qquad \text{if} \quad 3 \leqslant i \leqslant n. \end{cases}$$

(iii) If u_ε, u belong to $W^{2,n}(\Omega)$ and are convex functions such that

$$u_\varepsilon \to u \qquad \text{strongly in } C^0(\bar{\Omega})$$

then

$$\det \nabla^2 u_\varepsilon \rightharpoonup \det \nabla^2 u \qquad \text{in } \mathscr{D}'(\Omega)$$

(see Aubin [2, Proposition 8.14, p. 172]).

(iv) If u_ε belongs to $W^{1,n}(\Omega; \mathbb{R}^n)$ and if

$$\begin{cases} u_\varepsilon \rightharpoonup u & \text{weakly in } W^{1,1}(\Omega; \mathbb{R}^n) \\ \operatorname{sub}\det_\alpha(\nabla u_\varepsilon) \rightharpoonup h_\alpha & \text{weakly in } L^1(\Omega)\ \forall \alpha, \end{cases}$$

where $\operatorname{sub}\det_\alpha(M)$ denotes the determinant of any $\alpha \times \alpha$ matrix $(1 \leqslant \alpha \leqslant n)$ extracted of the $n \times n$ matrix M, then

$$h_\alpha = \operatorname{sub}\det_\alpha(\nabla u) \qquad \forall \alpha$$

(see Müller [10]). In the statement above, the assertion $u_\varepsilon \in W^{1,n}(\Omega; \mathbb{R}^n)$ can be relaxed, but some smoothness on u_ε has to be assumed.

2. Proofs of the Results

Proof of Theorem 1. *First Step.* The first part of the theorem is classical (cf. Reshetnyak [12], Ball [3]).

The second part is also the consequence of well-known results, except for what concerns the fact that in general $\mu \neq 0$. Indeed for u_ε in $W^{1,n}(\Omega; \mathbb{R}^n)$ one has

$$\det \nabla u_\varepsilon = \operatorname{Det}(\nabla u_\varepsilon) \qquad \text{in } \mathscr{D}'(\Omega),$$

where

$$\operatorname{Det} \nabla u \overset{\text{def}}{=} \frac{1}{n} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} [u_i \operatorname{adj}_{ij}(\nabla u)], \tag{2.1}$$

where $\operatorname{adj}_{ij}(M)$ denotes the (i, j) component of the adjugate matrix of M. For u_ε bounded in $W^{1,p}(\Omega; \mathbb{R}^n)$ with $p = n^2/(n+1)$, u_ε is bounded in $L^{p_*}(\Omega)$ with $1/p_* = 1/p - 1/n$, and $(u_\varepsilon)_i \operatorname{adj}_{ij}(\nabla u_\varepsilon)$ is bounded in $L^1(\Omega)$ since $1/p_* + (n-1)/p = 1$. Therefore $\det \nabla u_\varepsilon$ is the sum of first order derivatives of bounded functions of $L^1(\Omega)$. Extracting a subsequence ε', these functions converge in the weak * topology of measures and (1.4) is proved.

Note that the whole sequence $\det \nabla u_\varepsilon$ does not converge in $\mathscr{D}'(\Omega)$ in general, as it is easily seen by considering a sequence such that for $p = n^2/(n+1)$

$$u_\varepsilon \rightharpoonup u \text{ weakly in } W^{1,p}(\Omega; \mathbb{R}^n) \qquad \text{and} \qquad \det \nabla u_\varepsilon \rightharpoonup \det \nabla u + \mu \text{ in } \mathscr{D}'(\Omega)$$

with $\mu \neq 0$ and by defining a new sequence $\bar{u}_\varepsilon$ by taking sometimes $\bar{u}_\varepsilon = u_\varepsilon$ and sometimes $\bar{u}_\varepsilon = u$.

Second Step. It only remains to prove that μ can be non-zero in (1.4) and to establish the third part of Theorem 1.

To prove this we proceed as follows. Let $x=(x_1, \ldots, x_n)$, $r^2 = x_1^2 + \cdots + x_n^2$, $\Omega = B = \{x \in \mathbb{R}^n;\ |x| < 1\}$, and

$$u_\varepsilon(x) = f_\varepsilon(r) \begin{pmatrix} 1 \\ x_2/r \\ \vdots \\ x_n/r \end{pmatrix}. \tag{2.2}$$

It is easy to see that the ith row of ∇u_ε $(i \geqslant 2)$ is given by

$$\frac{f_\varepsilon(r)}{r} e_i + \left[f'_\varepsilon(r) - \frac{f_\varepsilon(r)}{r} \right] \frac{x_i}{r} \frac{x}{r}.$$

Developing $\det \nabla u_\varepsilon$ with respect to the ith row and observing that the first row is $f'_\varepsilon(r) x_1/r$ one obtains by induction

$$\det \nabla u_\varepsilon = f'_\varepsilon(r) \frac{x_1}{r} \left(\frac{f_\varepsilon(r)}{r} \right)^{n-1}. \tag{2.3}$$

We then choose for $\varepsilon > 0$ small enough and $p \geqslant 1$

$$f_\varepsilon(r) = \begin{cases} \varepsilon^{-n/p} r & \text{if } \ r \in [0, \varepsilon] \\ \varepsilon^{-n/p}(2\varepsilon - r) & \text{if } \ r \in [\varepsilon, 2\varepsilon] \\ 0 & \text{if } \ r \in [2\varepsilon, 1]. \end{cases} \tag{2.4}$$

Since

$$\left| \frac{\partial (u_\varepsilon)_i}{\partial x_j} \right| \leqslant |f'_\varepsilon(r)| + \left| \frac{f_\varepsilon(r)}{r} \right|,$$

an elementary computation shows that u_ε belongs to $W^{1,n}(B; \mathbb{R}^n)$ and satisfies

$$u_\varepsilon \rightharpoonup 0 \qquad \text{weakly in } W^{1,p}(B; \mathbb{R}^n). \tag{2.5}$$

We now claim that we can find $\varphi \in \mathscr{D}(B)$ so that

$$\int_B \det \nabla u_\varepsilon(x)\, \varphi(x)\, dx \to \begin{cases} a \neq 0 & \text{if } \ p = \dfrac{n^2}{n+1} \\ \infty & \text{if } \ 1 \leqslant p < \dfrac{n^2}{n+1}. \end{cases} \tag{2.6}$$

This is easily done, choosing $\varphi(x) = -x_1 \rho(x)$ with $\rho(x) = 1$ if $|x| < 1/2$ and $\rho \in \mathscr{D}(B)$. Indeed since $\det \nabla u_\varepsilon = 0$ if $|x| > 1/2$ we have

$$\int_B \det \nabla u_\varepsilon(x)\, \varphi(x)\, dx = -\int_{|x|<1/2} x_1 \det \nabla u_\varepsilon(x)\, dx$$
$$= -\int_{|x|<1/2} \frac{x_1^2}{r} f'_\varepsilon(r) \left(\frac{f_\varepsilon(r)}{r}\right)^{n-1} dx$$
$$= -\frac{S_{n-1}}{n} \int_0^{1/2} r f'_\varepsilon(r)(f'_\varepsilon(r))^{n-1}\, dr,$$

where S_{n-1} is the area of the unit sphere of $\mathbb{R}^n$. An elementary computation yields

$$\int_B \det \nabla u_\varepsilon(x)\, \varphi(x)\, dx = \frac{2}{n^2} \frac{S_{n-1}}{(n+1)} \varepsilon^{(n+1)-n^2/p}.$$

The result (2.6) and thus Theorem 1 follow at once. ▮

Remark 5. Note that the sequence given by (2.2) is not radial. In contrast it will result from the proof of Theorem 3 that for $\varphi(x) = -x_1 \rho(x)$ (with $\rho \in \mathscr{D}(B)$ and $\rho(x) = 1$ if $|x| < 1/2$), and for the radial function v_ε defined by $v_\varepsilon(x) = f_\varepsilon(r) x/r$ (with f_ε given by (2.4)), one has

$$\int_B \det \nabla v_\varepsilon \varphi\, dx \to 0.$$

Proof of Theorem 2. *First Step.* Although the first part of Theorem 2 is already proved in Ball, Currie, and Olver [5] we recall the proof for the convenience of the reader.

We begin with the study of the case where $n = 2$, since otherwise the notation is heavy and may make obscure the idea of the proof which in fact is very simple. Observe first that for v in $C^4(\Omega)$, $\Omega \subset \mathbb{R}^2$, one has

$$2 \det \nabla^2 v = 2\left(\frac{\partial^2 v}{\partial x^2}\frac{\partial^2 v}{\partial y^2} - \left(\frac{\partial^2 v}{\partial x\, \partial y}\right)^2\right)$$
$$= \frac{\partial^2}{\partial x^2}\left(v \frac{\partial^2 v}{\partial y^2}\right) + \frac{\partial^2}{\partial y^2}\left(v \frac{\partial^2 v}{\partial x^2}\right) - 2 \frac{\partial^2}{\partial x\, \partial y}\left(v \frac{\partial^2 v}{\partial x\, \partial y}\right). \tag{2.7}$$

Approximating a function of $W^{2,2}(\Omega)$ by a sequence of smooth functions proves that (2.7) holds true in the sense of distributions for any v in $W^{2,2}(\Omega)$.

Consider now a sequence v_ε and a v in $W^{2,2}(\Omega)$, with v_ε tending to v weakly in $W^{2,p}(\Omega)$ for some $p \geqslant 1$. From (2.7) we deduce that for any φ in $\mathscr{D}(\Omega)$

$$\iint_\Omega \det \nabla^2 v_\varepsilon \varphi\, dx\, dy$$
$$= \frac{1}{2} \iint_\Omega \left(v_\varepsilon \frac{\partial^2 v_\varepsilon}{\partial y^2}\frac{\partial^2 \varphi}{\partial x^2} + v_\varepsilon \frac{\partial^2 v_\varepsilon}{\partial x^2}\frac{\partial^2 \varphi}{\partial y^2} - 2 v_\varepsilon \frac{\partial^2 v_\varepsilon}{\partial x\, \partial y}\frac{\partial^2 \varphi}{\partial x\, \partial y}\right) dx\, dy. \tag{2.8}$$

When $p>1$, the sequence v_ε is bounded in $C^{0,\alpha}(\bar{\Omega})$ for some $\alpha>0$ and it is easy to pass to the limit in the right hand side of (2.8). This proves that $\det \nabla^2 v_\varepsilon$ tends to $\det \nabla^2 v$ in $\mathscr{D}'(\Omega)$, i.e., (1.6).

This result continues to hold for $p=1$. Indeed for $\Omega \subset \mathbb{R}^2$ the Sobolev's embedding theorem holds true in the limit case of $W^{2,1}(\Omega)$, yielding

$$\|w\|_{L^\infty(\Omega)} \leqslant C_\Omega \|w\|_{W^{2,1}(\Omega)}, \tag{2.9}$$

as it is easily seen from the formula

$$w(x, y) = w(x, b) + w(a, y) - w(a, b) + \int_a^x \int_b^y \frac{\partial^2 w}{\partial x\, \partial y}(x', y')\, dx'\, dy'.$$

On the other hand $W^{2,1}(\Omega)$ is compactly embedded in $L^q(\Omega)$ for any finite q. This proves that up to the extraction of a subsequence one has

$$\begin{cases} v_\varepsilon \to v & \text{a.e. in } \Omega,\ \|v_\varepsilon\|_{L^\infty(\Omega)} \leqslant C \\ \dfrac{\partial^2 v_\varepsilon}{\partial y^2} \rightharpoonup \dfrac{\partial^2 v}{\partial y^2} & \text{weakly in } L^1(\Omega). \end{cases} \tag{2.10}$$

Splitting Ω into a set E (where Egorov's theorem is applied to v_ε) and in its "small" remainder $\Omega \backslash E$ (where the equi-integrability of $\partial^2 v_\varepsilon/\partial y^2$ and the L^∞ bound on v_ε are used) proves that

$$\iint_\Omega v_\varepsilon \frac{\partial^2 v_\varepsilon}{\partial y^2} \frac{\partial^2 \varphi}{\partial x^2}\, dx\, dy \to \iint_\Omega v \frac{\partial^2 v}{\partial y^2} \frac{\partial^2 \varphi}{\partial x^2}\, dx\, dy.$$

The same proof holds for the other terms of the right hand side of (2.8) and proves (1.6) for $n=2$, $p=1$.

The previous proof does not continue to hold if the hypothesis that v_ε converges weakly to v in $W^{2,1}(\Omega)$ is replaced by

$$\|v_\varepsilon\|_{W^{2,1}(\Omega)} \leqslant C, \qquad v_\varepsilon \rightharpoonup v \text{ in } \mathscr{D}'(\Omega)$$

since in this case $\partial^2 v_\varepsilon/\partial y^2$ does not weakly converge to $\partial^2 v/\partial y^2$ in $L^1(\Omega)$. Nevertheless according to (2.9)

$$\left\| v_\varepsilon \frac{\partial^2 v_\varepsilon}{\partial y^2} \right\|_{L^1(\Omega)} \leqslant \|v_\varepsilon\|_{L^\infty(\Omega)} \left\| \frac{\partial^2 v_\varepsilon}{\partial y^2} \right\|_{L^1(\Omega)} \leqslant C_\Omega \|v_\varepsilon\|^2_{W^{2,1}(\Omega)},$$

and for a subsequence ε' and a Radon measure γ

$$\iint_\Omega v_{\varepsilon'} \frac{\partial^2 v_{\varepsilon'}}{\partial y^2} \frac{\partial^2 \varphi}{\partial x^2}\, dx\, dy \to \left\langle \gamma, \frac{\partial^2 \varphi}{\partial x^2} \right\rangle.$$

Passing to the limit in the other terms of the right hand side of (2.8) proves that in this setting

$$\det \nabla^2 v_{\varepsilon'} \rightharpoonup \det \nabla^2 v + \mu \qquad \text{in } \mathscr{D}'(\Omega)$$

for a certain distribution μ, i.e., (1.9). It remains to see that μ can be non-zero, but this will follow from a radial example, cf. Remark 3(i) and the second step of the proof of Theorem 3.

Second Step. We now consider the general case where $n \geqslant 3$ and $p \geqslant n^2/(n+2)$; the proof is essentially the same as the above one. The formula which replaces (2.7) is now

$$n \det \nabla^2 v = \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \, \partial x_j} (v \operatorname{adj}_{ij}(\nabla^2 v)) \qquad \text{in } \mathscr{D}'(\Omega)\ \forall v \in W^{2,n}(\Omega; \mathbb{R}^n), \tag{2.11}$$

where $\operatorname{adj}(\nabla^2 v)$ stands for the adjugate matrix of $\nabla^2 v$. Actually (2.11) is a consequence of formula (2.1). Indeed for any sufficiently smooth w one has

$$\sum_{j=1}^{n} \frac{\partial}{\partial x_j} (\operatorname{adj}_{ij}(\nabla w)) = 0 \qquad \text{for } i \text{ fixed.}$$

Since here $u = \nabla v$, using (2.1) and the symmetry of $\operatorname{adj}(\nabla^2 v)$ yields

$$\begin{aligned} \det \nabla^2 v = \operatorname{Det} \nabla^2 v &= \frac{1}{n} \sum_{ij} \frac{\partial}{\partial x_j} \left(\frac{\partial v}{\partial x_i} \operatorname{adj}_{ij}(\nabla^2 v) \right) \\ &= \frac{1}{n} \sum_{ij} \frac{\partial^2}{\partial x_i \, \partial x_j} (v \operatorname{adj}_{ij}(\nabla^2 v)). \end{aligned}$$

Let now φ be in $\mathscr{D}(\Omega)$. Integrating by parts gives

$$n \int_\Omega \det \nabla^2 v_\varepsilon \varphi \, dx = \sum_{ij} \int_\Omega v_\varepsilon (\operatorname{adj}_{ij}(\nabla^2 v_\varepsilon)) \frac{\partial^2 \varphi}{\partial x_i \, \partial x_j} \, dx. \tag{2.12}$$

The term $\operatorname{adj}_{ij}(\nabla^2 v_\varepsilon)$ is the determinant of some $(n-1) \times (n-1)$ submatrix of $\nabla^2 v_\varepsilon$. It can therefore be written as (cf. formula (2.1))

$$\operatorname{adj}_{ij}(\nabla^2 v_\varepsilon) = \frac{1}{(n-1)} \sum_{kl} \frac{\partial}{\partial x_l} \left[\frac{\partial v_\varepsilon}{\partial x_k} C^{ij}_{kl}(\nabla^2 v_\varepsilon) \right],$$

where $C^{ij}_{kl}(\nabla^2 v_\varepsilon)$ denotes the determinant of some $(n-2) \times (n-2)$ submatrix of $\nabla^2 v_\varepsilon$. From (2.12) we obtain

$$n(n-1)\int_\Omega \det \nabla^2 v_\varepsilon \varphi \, dx$$

$$= -\sum_{ijkl} \left\{ \int_\Omega \frac{\partial v_\varepsilon}{\partial x_k} \frac{\partial v_\varepsilon}{\partial x_l} C^{ij}_{kl}(\nabla^2 v_\varepsilon) \frac{\partial^2 \varphi}{\partial x_i \, \partial x_j} \, dx \right.$$

$$\left. + \int_\Omega v_\varepsilon \frac{\partial v_\varepsilon}{\partial x_k} C^{ij}_{kl}(\nabla^2 v_\varepsilon) \frac{\partial^3 \varphi}{\partial x_i \, \partial x_j \, \partial x_l} \, dx \right\}. \quad (2.13)$$

By Sobolev's embedding theorem, one has $W^{2,p}(\Omega) \subset W^{1,p_*}(\Omega)$ with $1/p_* = 1/p - 1/n$ for $p < n$. This implies that for v_ε bounded in $W^{2,p}(\Omega)$ with $p = n^2/n + 2$

$$\frac{\partial v_\varepsilon}{\partial x_k} \frac{\partial v_\varepsilon}{\partial x_l} C^{ij}_{kl}(\nabla^2 v_\varepsilon) \quad \text{and} \quad v_\varepsilon \frac{\partial v_\varepsilon}{\partial x_l} C^{ij}_{kl}(\nabla^2 v_\varepsilon) \text{ are bounded in } L^1(\Omega).$$

Extracting a subsequence ε' then proves that $\det \nabla^2 v_{\varepsilon'}$ converges in $\mathscr{D}'(\Omega)$ to some distribution which can be written as $\det \nabla v + \mu$. This establishes (1.7).

When $p > n^2/(n+2)$ the compactness of the embedding from $W^{2,p}(\Omega)$ into $W^{1,q}(\Omega)$ for any $q < p_*$ allows one to prove that

$$\frac{\partial v_\varepsilon}{\partial x_k} \frac{\partial v_\varepsilon}{\partial x_l} C^{ij}_{kl}(\nabla^2 v_\varepsilon) \rightharpoonup \frac{\partial v}{\partial x_k} \frac{\partial v}{\partial x_l} h^{ij}_{kl} \quad \text{in } \mathscr{D}'(\Omega)$$

$$v_\varepsilon \frac{\partial v_\varepsilon}{\partial x_k} C^{ij}_{kl}(\nabla^2 v^\varepsilon) \rightharpoonup v \frac{\partial v}{\partial x_k} h^{ij}_{kl} \quad \text{in } \mathscr{D}'(\Omega),$$

where h^{ij}_{kl} is the limit of $C^{ij}_{kl}(\nabla^2 v_\varepsilon)$ in the weak topology of $L^{p/(n-2)}(\Omega)$. Since $C^{ij}_{kl}(\nabla^2 v_\varepsilon)$ itself is the determinant of some $(n-2) \times (n-2)$ matrix and since $p > n^2/(n+2) \geqslant (n-2)$, a proof by induction on n (see, e.g., Dacorogna [8, Formula (6), p. 173]) implies that $h^{ij}_{kl} = C^{ij}_{kl}(\nabla^2 v)$. Using (2.13) for v implies the desired convergence $\det \nabla^2 v^\varepsilon \rightharpoonup \det \nabla^2 v$ in $\mathscr{D}'(\Omega)$, i.e., (1.6).

Third Step. It remains to prove that μ can be non-zero in (1.7) and to prove (1.8). By the observation that radial functions are in fact gradients (cf. Remark 3(i)) the examples of radial functions proving (1.12b) and (1.13) which will be exhibited below, automatically prove (1.7) and (1.8), respectively. ∎

To prove Theorem 3 we will use the following lemma:

LEMMA 4. *Let $p \geqslant 1$, u_ε, u in $R^{1,n}_{ad}(B)$, and $u_\varepsilon \rightharpoonup u$ weakly in $W^{1,p}(B; \mathbb{R}^n)$.*

(i) *If* $n \geqslant 3$ *and* $p > n^2/(n+2)$, *or if* $n = 2$ *and* $p \geqslant n^2/(n+2) = 1$, *then*

$$\lim_{\varepsilon \to 0} \int_0^1 r \, |(f_\varepsilon(r))^n - (f(r))^n| \, dr = 0. \tag{2.14}$$

(ii) *If* $n \geqslant 3$ *and* $p = n^2/(n+2)$, *then*

$$\int_0^1 r \, |f_\varepsilon(r)|^n \, dr \leqslant c, \tag{2.15}$$

where c *is a constant independent of* ε.

(iii) *If* $n \geqslant 3$ *and* $1 \leqslant p \leqslant n^2/(n+2)$, *there exists* $u_\varepsilon \in R_{ad}^{1,n}(B)$ *such that*

$$\begin{cases} u_\varepsilon \rightharpoonup 0 & \text{weakly in } W^{1,p}(B; \mathbb{R}^n) \\ f_\varepsilon(r) = 0 & \text{if} \quad r \geqslant 2\varepsilon \\ \displaystyle\int_0^1 r(f_\varepsilon(r))^n \, dr \to \begin{cases} a \neq 0 & \text{if} \quad p = n^2/(n+2) \\ \infty & \text{if} \quad 1 \leqslant p < n^2/(n+2). \end{cases} \end{cases} \tag{2.16}$$

(iv) *If* $n = 2$, $p = n^2/(n+2) = 1$ *and if the hypothesis* $u_\varepsilon \rightharpoonup u$ *weakly in* $W^{1,1}(B; \mathbb{R}^2)$ *is replaced by* $\|u_\varepsilon\|_{W^{1,1}(B;\mathbb{R}^2)} \leqslant C$ *then* (2.15) *holds true; in this setting there exists a sequence such that*

$$\begin{cases} u_\varepsilon \rightharpoonup 0 & \text{in } \mathscr{D}'(B) \\ f_\varepsilon(r) = 0 & \text{if} \quad r \geqslant 2\varepsilon \\ \displaystyle\int_0^1 r(f_\varepsilon(r))^2 \, dr \to a \neq 0. \end{cases} \tag{2.17}$$

Proof of Lemma 4. *First Step.* We first prove that for $n \geqslant 2$, $p = n^2/(n+2)$ and for any $u \in R_{ad}^{1,n}(B)$ one has

$$\int_0^1 r \, |f(r)|^n \, dr \leqslant C \left(\int_0^1 \left(|f'(r)|^p + \left| \frac{f(r)}{r} \right|^p \right) r^{n-1} \, dr \right)^{n/p}. \tag{2.18}$$

This implies (2.15) and the first part of assertion (iv).

To prove (2.18), we integrate

$$\frac{d}{dr} (r^2 \, |f(r)|^n) = 2r \, |f(r)|^n \pm r^2 \, |f(r)|^{n-1} \, |f'(r)|$$

between 0 and 1 to obtain (recalling the fact that f belongs to $C^0([0, 1])$ with $f(0) = 0$ when u belongs to $R_{ad}^{1,n}(B)$, see, e.g., Ball and Murat [6, Remark 7.2])

$$2 \int_0^1 r \, |f(r)|^n \, dr \leqslant |f(1)|^n + \int_0^1 r^2 \, |f(r)|^{n-1} \, |f'(r)| \, dr. \tag{2.19}$$

Fix some $\lambda > 0$. From

$$\|f\|_{L^\infty(\lambda,1)} \leqslant C \|f\|_{W^{1,1}(\lambda,1)}$$
$$\leqslant C_{\lambda,p}\left(\int_\lambda^1 \left(|f'(r)|^p + \left|\frac{f(r)}{r}\right|^p\right) r^{n-1}\, dr\right)^{n/p}$$

one deduces that $|f(1)|^n$ is bounded by the right-hand side of (2.18).

Consider now the case $n \geqslant 3$. Sobolev's embedding theorem implies that

$$W^{1,p}(B; \mathbb{R}^n) \subset L^{p_*}(B) \qquad \text{with} \quad p_* = n^2/2 \text{ if } p = n^2/(n+2).$$

We thus have

$$\left(\int_0^1 |f(r)|^{p_*} r^{n-1}\, dr\right)^{1/p_*} \leqslant C\left(\int_0^1 \left(|f'(r)|^p + \left|\frac{f(r)}{r}\right|^p\right) r^{n-1}\, dr\right)^{1/p}.$$

Writing

$$r^2 |f(r)|^{n-1} |f'(r)| = \left|\frac{f(r)}{r}\right|^{n-3} |f(r)|^2 |f'(r)| r^{n-1}$$

and using Hölder's inequality in the spaces $L^q(0, 1; r^{n-1}\, dr)$ with

$$\frac{n-3}{p} + \frac{2}{p_*} + \frac{1}{p} = 1$$

(which is satisfied for $p = n^2/(n+2)$) we obtain

$$\int_0^1 r^2 |f(r)|^{n-1} |f'(r)|\, dr \leqslant C\left(\int_0^1 \left(|f'(r)|^p + \left|\frac{f(r)}{r}\right|^p\right) r^{n-1}\, dr\right)^{n/p}.$$

This completes the proof of (2.18) when $n \geqslant 3$.

In the case $n = 2$, we integrate

$$\frac{d}{dr}(r\, |f(r)|) = |f(r)| \pm r\, |f'(r)|$$

between 0 and r to obtain

$$r\, |f(r)| \leqslant \int_0^r (|f(s)| + s\, |f'(s)|)\, ds$$
$$\leqslant \int_0^1 \left(|f'(r)| + \left|\frac{f(r)}{r}\right|\right) r\, dr. \tag{2.20}$$

This implies that for $n=2$

$$\int_0^1 r^2 |f(r)|^{n-1} |f'(r)| \, dr \leqslant \sup_{0 \leqslant r \leqslant 1} |rf(r)| \int_0^1 r |f'(r)| \, dr$$

$$\leqslant \left(\int_0^1 \left(|f'(r)| + \left| \frac{f(r)}{r} \right| \right) r \, dr \right)^2$$

which completes the proof of (2.18) when $n=2$.

In the case where $n \geqslant 3$, another proof of (2.18) consists in using the Bliss inequality (see Bliss [7]) which asserts that for every v with $v(0)=v(1)=0$ and every $1<\alpha<\beta$ one has

$$\left(\int_0^1 r^{(\beta/\alpha)-\beta-1} |v(r)|^\beta \, dr \right)^{1/\beta} \leqslant C \left(\int_0^1 |v'(r)|^\alpha \, dr \right)^{1/\alpha}$$

(the case $\alpha=\beta$ is just Hardy's inequality). Indeed setting

$$v(r)=r^{(n+2)(n-1)/n^2}(f(r)-f(1)), \qquad \alpha=p=n^2/(n+2), \qquad \beta=n$$

in the Bliss inequality yields

$$\left(\int_0^1 r |f(r)-f(1)|^n \, dr \right)^{1/n} \leqslant C \left(\int_0^1 \left(|f'(r)|^p + \left| \frac{f(r)-f(1)}{r} \right|^p \right) r^{n-1} \, dr \right)^{1/p}$$

and thus

$$\left(\int_0^1 r |f(r)|^n \, dr \right)^{1/n} \leqslant C \left[|f(1)| + \left(\int_0^1 \left(|f'(r)|^p + \left| \frac{f(r)}{r} \right|^p \right) r^{n-1} \, dr \right)^{1/p} \right].$$

With the help of (2.19), this proves (2.18) again.

Second Step. We now prove (2.14) for $n \geqslant 2$ and $p>n^2/(n+2)$. We first observe that for every $q>n^2/2$ one has

$$\int_0^1 r |g(r)|^n \, dr \leqslant C \left(\int_0^1 r^{n-1} |g(r)|^q \, dr \right)^{n/q}. \tag{2.21}$$

This follows from Hölder's inequality since

$$\int_0^1 r |g(r)|^n \, dt = \int_0^1 r^{n(n-1)/q} |g(r)|^n r^{1-n(n-1)/q} \, dr$$

$$\leqslant \left(\int_0^1 r^{n-1} |g(r)|^q \, dr \right)^{n/q}$$

$$\times \left(\int_0^1 (r^{1-n(n-1)/q})^{q/(q-n)} \right)^{(q-n)/q};$$

since the last integral is finite if and only if $q > n^2/2$ we have established (2.21).

On the other hand, by the compactness of Sobolev's embedding we have

$$u_\varepsilon \to u \qquad \text{strongly in } L^q(\Omega; \mathbb{R}^n) \text{ for any } q < p_* = \frac{np}{n-p}$$

which implies

$$f_\varepsilon \to f \qquad \text{strongly in } L^q(0, 1; r^{n-1}\, dr)$$

and in particular

$$\int_0^1 |(f_\varepsilon(r))^n - (f(r))^n|^{q/n}\, r^{n-1}\, dr \to 0. \tag{2.22}$$

Fixing some q with $n^2/2 < q < p_*$ and combining (2.22) and (2.21) proves (2.14) in the case where $n \geqslant 2$ *and* $p > n^2/(n+2)$.

Third Step. We now prove (2.14) in the border line case where $n = 2$ and $p = n^2/(n+2) = 1$. Note first that for any fixed $\lambda > 0$

$$\begin{aligned} &u_\varepsilon \rightharpoonup u \text{ in } W^{1,1}(B; \mathbb{R}^2) \Rightarrow f_\varepsilon \rightharpoonup f \text{ in } W^{1,1}(\lambda, 1) \Rightarrow \\ &f_\varepsilon \to f \text{ in } C^0([\lambda, 1]). \end{aligned} \tag{2.23}$$

On the other hand by a variation of (2.20) we have

$$\begin{aligned} \int_0^\lambda r\, |f_\varepsilon(r)|^2\, dr &\leqslant \sup_{0 \leqslant r \leqslant \lambda} |r f_\varepsilon(r)| \int_0^\lambda |f_\varepsilon(r)|\, dr \\ &\leqslant \left(\int_0^\lambda \left(|f'_\varepsilon(r)| + \left| \frac{f_\varepsilon(r)}{r} \right| \right) r\, dr \right)^2 \\ &\leqslant C\, \|u_\varepsilon\|^2_{W^{1,1}(B_\lambda; \mathbb{R}^2)}, \end{aligned}$$

where B_λ denotes the ball $B_\lambda = \{x \in \mathbb{R}^n;\ |x| < \lambda\}$. Since u_ε is assumed to weakly converge in $W^{1,1}(B; \mathbb{R}^2)$, the last quantity is small uniformly in ε when λ is small. Combined with (2.23) this proves (2.14) when $n = 2$ and $p = 1$.

Fourth Step. It remains to produce examples of f_ε such that (2.16) and (2.17) hold true. Consider for that the function

$$f_\varepsilon(r) = \begin{cases} \varepsilon^{-\alpha-1} r & \text{if } r \in [0, \varepsilon] \\ \varepsilon^{-\alpha-1}(2\varepsilon - r) & \text{if } r \in [\varepsilon, 2\varepsilon] \\ 0 & \text{if } r \in [2\varepsilon, 1]. \end{cases} \tag{2.24}$$

Note that $f_\varepsilon \to 0$ almost everywhere.

An elementary computation shows that

$$\|u_\varepsilon\|_{W^{1,p}(B;\,\mathbb{R}^n)} \leqslant C_0 \Leftrightarrow \alpha \leqslant (n-p)/p. \tag{2.25}$$

Similarly to (2.25) a direct computation gives that

$$\int_0^1 r\,|f_\varepsilon(r)|^n\,dr = \frac{2}{n+1}\,\varepsilon^{(n+2)-(\alpha+1)n}$$

and thus

$$\int_0^1 r\,|f_\varepsilon(r)|^n\,dr \to \begin{cases} 2/(n+1) & \text{if } \alpha = 2/n \\ \infty & \text{if } \alpha > 2/n. \end{cases} \tag{2.26}$$

Combining (2.25) and (2.26) we have found for $n=2$, $p=1$, $\alpha=1$ a radial example such that $\|u_\varepsilon\|_{W^{1,1}(B;\,\mathbb{R}^2)} \leqslant C_0$, u_ε tends to 0 in $\mathscr{D}'(\Omega;\mathbb{R}^2)$, and $\int_0^1 r(f_\varepsilon(r))^2\,dr \to 2/3$. This proves the last part of Lemma 4. Note that in this example u_ε does not converge weakly to 0 in $W^{1,1}(B;\mathbb{R}^2)$.

Similarly for $n \geqslant 3$, $1 \leqslant p \leqslant n^2/(n+2)$, $\alpha = (n-p)/p$ we have found an example satisfying (2.16). ∎

Proof of Theorem 3. The first part of Theorem 3 is a consequence of Theorem 2 (see Remark 3(i)). We nevertheless present a proof of this fact based on the same ideas which allows one to prove the second part of the theorem.

First Step. As it will appear the only difficulty in the proof is the behaviour of $\det \nabla u_\varepsilon$ at the origin. We thus write for a given function $\varphi \in \mathscr{D}(B)$

$$\varphi(x) = \varphi(0) + \nabla\varphi(0)(x/|x|)\,|x| + (1/2)\,\nabla^2\varphi(0)(x/|x|)(x/|x|)\,|x|^2 + |x|^2\,\omega(x/|x|,\,|x|),$$

where $\omega(s,r)$ is a smooth function defined on $\{|x|=1\}\times(0,1)$ with $|\omega(s,r)| \leqslant Cr$ and $|(\partial\omega/\partial r)(s,r)| \leqslant C$ for r small.

Using the symmetry we have

$$\int_{|x|=1} ds = S_{n-1}, \qquad \int_{|x|=1} \nabla\varphi(0)x\,ds = 0$$

$$\int_{|x|=1} \nabla^2\varphi(0)\,xx\,ds = \sum_{i=1}^{n} \frac{\partial^2\varphi}{\partial x_i^2}(0) \int_{|x|=1} x_i^2\,ds = \operatorname{tr}(\nabla^2\varphi(0))\,\frac{S_{n-1}}{n}.$$

Since for a radial function one has

$$\det \nabla u_\varepsilon = f'_\varepsilon(r)\left(\frac{f_\varepsilon(r)}{r}\right)^{n-1}$$

we obtain

$$\begin{aligned}\int_B \det \nabla u_\varepsilon(x)\,\varphi(x)\,dx\\ &= \varphi(0)\,S_{n-1}\int_0^1 f'_\varepsilon(r)(f_\varepsilon(r))^{n-1}\,dr + 0\\ &\quad + \operatorname{tr}(\nabla^2\varphi(0))\,\frac{S_{n-1}}{2n}\int_0^1 r^2 f'_\varepsilon(r)(f_\varepsilon(r))^{n-1}\,dr\\ &\quad + \int_0^1 r^2 f'_\varepsilon(r)(f_\varepsilon(r))^{n-1}\,\Omega(r)\,dr, \end{aligned} \tag{2.27}$$

where

$$\Omega(r) = \int_{|x|=1} \omega(s, r)\,ds.$$

Recall that for any function u_ε in $R^{1,n}_{ad}(B)$, f_ε is continuous on $[0, 1]$. Integrating the various terms of the right-hand side of (2.27) and using in particular the formula

$$r^2 f'_\varepsilon(r)(f_\varepsilon(r))^{n-1} = \frac{d}{dr}\left[\frac{r^2}{n}(f_\varepsilon(r))^n\right] - \frac{2r}{n}(f_\varepsilon(r))^n,$$

we obtain for any function u_ε in $R^{1,n}_{ad}(B)$

$$\begin{aligned}\int_\Omega \det \nabla u_\varepsilon(x)\,\varphi(x)\,dx\\ &= \left[\varphi(0)\,\frac{S_{n-1}}{n} + \operatorname{tr}(\nabla^2\varphi(0))\,\frac{S_{n-1}}{2n^2}\right](f_\varepsilon(1))^n\\ &\quad - \operatorname{tr}(\nabla^2\varphi(0))\,\frac{S_{n-1}}{n^2}\int_0^1 r(f_\varepsilon(r))^n\,dr\\ &\quad - \frac{1}{n}\int_0^1 r(f_\varepsilon(r))^n\left[2\Omega(r) + r\,\frac{d\Omega}{dr}(r)\right]dr\\ &\quad + \frac{1}{n}(f_\varepsilon(1))^n\,\Omega(1). \end{aligned} \tag{2.28}$$

Consider now a sequence of radial functions u_ε, u in $R^{1,n}_{ad}(B)$ such that $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(B; \mathbb{R}^n)$ for some $p \geqslant 1$. Since this last hypothesis ensures that $f_\varepsilon(1)$ tends to $f(1)$ (see (2.23) if necessary) we have proved that

$$\int_B [\det \nabla u_\varepsilon(x) - \det \nabla u(x)]\, \varphi(x)\, dx$$

$$+ \operatorname{tr}(\nabla^2 \varphi(0)) \frac{S_{n-1}}{n^2} \int_0^1 r[(f_\varepsilon(r))^n - (f(r))^n]\, dr$$

$$+ \frac{1}{n} \int_0^1 r[(f_\varepsilon(r))^n - (f(r))^n] \left[2\Omega(r) + r \frac{d\Omega}{dr}(r) \right] dr \to 0. \quad (2.29)$$

Second Step. Consider now the cases where $n \geqslant 3$ and $p > n^2/(n+2)$ or $n = 2$ and $p \geqslant n^2/(n+2) = 1$. The first part of Lemma 4 proves that the second and the third terms of (2.29) tend to 0 since $|2\Omega(r) + r(d\Omega/dr)(r)| \leqslant Cr$. This proves the first part of Theorem 3.

If $n \geqslant 3$ and $p = n^2/(n+2)$, the second term of (2.29) tend to $c \operatorname{tr}(\nabla^2 \varphi(0)) = c\langle \Delta\delta_0, \varphi \rangle$, where

$$c = \lim_{\varepsilon \to 0} \frac{S_{n-1}}{n^2} \int_0^1 r[(f_\varepsilon(r))^n - (f(r))^n]\, dr$$

(recall that the last term is bounded in view of (2.15)). The third term of (2.29) tends to 0, as it is seen by splitting (0, 1) into $(0, \lambda) \cup (\lambda, 1)$ for some $\lambda > 0$ small; indeed since $|2\Omega(r) + r(d\Omega/dr)(r)| \leqslant Cr$ the integral on $(0, \lambda)$ is bounded by

$$C\lambda \frac{1}{n} \int_0^1 r\, |(f_\varepsilon(r))^n - (f(r))^n|\, dr \leqslant C'\lambda$$

uniformly in ε (see (2.15)), while the integral on $(\lambda, 1)$ tends to 0 for λ fixed since

$$f_\varepsilon \rightharpoonup f \text{ weakly in } W^{1,p}(\lambda, 1) \Rightarrow f_\varepsilon \to f \text{ strongly in } C^0([\lambda, 1]).$$

When $n = 2$, $p = n/(n+2) = 1$ and when the hypothesis that u_ε tends weakly to u in $W^{1,1}(\Omega; \mathbb{R}^2)$ is replaced by

$$\|u_\varepsilon\|_{W^{1,1}(\Omega;\mathbb{R}^2)} \leqslant C, \qquad u_\varepsilon \rightharpoonup u \text{ in } \mathscr{D}'(\Omega, \mathbb{R}^2)$$

a proof similar to the proof above using now (2.20) proves that (1.14) holds true.

Third Step. It remains to construct an example satisfying (1.13), and to prove that c can be non-zero in (1.12). This is easily done by considering some φ in $\mathscr{D}(B)$ such that $\varphi(x)=x_1^2$ for $|x|<1/2$. In this case ω is identically zero in the ball $|x|<1/2$ and thus the third term of (2.29) vanishes for u_ε satisfying (2.16) or (2.17). This proves the third part of Theorem 3. ∎

Remarks 6. (i) It is worthwile to note the difference between the proof of Theorem 3 and the proof of the third part of Theorem 1. While in (2.27) the term corresponding to $\nabla\varphi(0)x$ is zero by symmetry since u_ε is a radial function, this is not the case for the function defined by (2.2). This explains why the non-radial case stops at $p=n^2/(n+1)$ while the radial case stops at $p=n^2/(n+2)$.

(ii) Recall that when u belongs to $R_{ad}^{1,p}(B)$ with $p\geqslant n$, the corresponding function f is absolutely continuous on $[0,1]$ with $f(0)=0$ (see, e.g., Ball [4] or Ball and Murat [6]); in contrast when $1\leqslant p<n$, f is no more continuous at $r=0$ and $\limsup_{r\to 0} f(r)$ can be $+\infty$ (consider, e.g., the example $f(r)=r^{-\alpha}(2+\sin r)$ with $0<\alpha<(n-p)/p$).

Consider therefore the case of radial functions in $R_{ad}^{1,p}(B)$ with $p<n$, where f_ε and f belong to $C^1([0,1])$, which allows one to define $f_\varepsilon(0)$ and $f(0)$ as well as $\det\nabla u_\varepsilon=f'_\varepsilon(r)(f_\varepsilon(r)/r)^{n-1}$ and $\det\nabla u$ as functions of $L^1(\Omega)$. If we moreover assume that

$$\begin{cases} u_\varepsilon \rightharpoonup u \text{ weakly in } W^{1,p}(B;\mathbb{R}^n) \text{ with } p>n^2/(n+2) \\ (f_\varepsilon(0))^n \text{ tends to some limit as } \varepsilon\to 0, \end{cases} \tag{2.30}$$

revisiting (2.27), (2.28), (2.29) allows one to prove that

$$\int_B (\det\nabla u_\varepsilon-\det\nabla u)\varphi\,dx \to \frac{S_{n-1}}{n}\left[(f(0))^n-\lim_{\varepsilon\to 0}(f_\varepsilon(0))^n\right]\varphi(0). \tag{2.31}$$

This corresponds to the well-known fact that for radial functions u_ε with f_ε in $C^1([0,1])$ the distributional determinant $\operatorname{Det}\nabla u_\varepsilon$ differs from $\det\nabla u_\varepsilon$ by a Dirac mass at the origin, i.e.,

$$\det\nabla u_\varepsilon=\operatorname{Det}\nabla u_\varepsilon-\frac{S_{n-1}}{n}(f_\varepsilon(0))^n\,\delta_0$$

and to the fact that (2.30) implies the convergence of $\operatorname{Det}\nabla u_\varepsilon$ to $\operatorname{Det}\nabla u$ in $\mathscr{D}'(\Omega)$.

It is worthwhile to note that (2.31) does not continue to hold when $p\leqslant n^2/(n+2)$ in (2.30); indeed for $p=n^2/(n+2)$ the second part of Theorem 3 shows that when $f_\varepsilon(0)=f(0)=0$, $\det\nabla u_\varepsilon-\det\nabla u$ tends to $c\,\Delta\delta_0$ in $\mathscr{D}'(\Omega)$ and not to zero; for $p<n^2/(n+2)$, $\det\nabla u_\varepsilon$ does not converge in the sense of distributions (see the third part of Theorem 3).

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