

A Relaxation Theorem and its Application to the Equilibrium of Gases

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I. Introduction

Many problems of the calculus of variations have no solutions in the usual sense (e.g. in a Sobolev space). The idea of relaxation is to replace the original problem by a related one possessing a solution in the usual sense, which is then interpreted as a "generalized solution" of the original one.

Relaxation finds its origins in the work of L. C. YOUNG ([11], [12]). Recent developments have led to more precise results, in particular, those of I. EKELAND [4] and I. EKELAND & R. TÉMAM [5]. In their book the latter authors consider problems of the type

$$(P_1) \inf \left\{ \int_{\Omega} g(\text{grad } u(x)) \, dx : u \in u_0 + W_0^{1,\alpha}(\Omega) \right\}$$

where Ω is a bounded open set of \mathbb{R}^n , g is a given non-convex function which satisfies some growth conditions at infinity and u_0 is a given function. When referring to Sobolev spaces we will adopt throughout the notation of ADAMS [1]. In particular $W^{1,\alpha}(\Omega)$ denotes the set of functions which are bounded in $L^\alpha(\Omega)$ and such that their derivatives (in the generalized sense) are also bounded in $L^\alpha(\Omega)$; $W_0^{1,\alpha}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $W^{1,\alpha}(\Omega)$.

The "relaxed" problem associated with (P_1) is

$$(P_1^{**}) \inf \left\{ \int_{\Omega} g^{**}(\text{grad } u(x)) \, dx : u \in u_0 + W_0^{1,\alpha}(\Omega) \right\}$$

where g^{**} is the lower convex envelope of g (i.e. g^{**} is the largest lower semi-continuous convex function which is smaller than g).

It is proved in [5] that

$$\inf (P_1) = \inf (P_1^{**}) \tag{1.1}$$

and that (P_1^{**}) possesses a solution, while, in general, (P_1) does not. EKELAND & TÉMAM gave also a relation between minimizing sequences of (P_1) and solutions of (P_1^{**}) . However, their result cannot be generalized to higher dimensions,

i.e. to the case $u: \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$. It is the aim of this article to investigate some particular cases in dimension m and to give some applications to the theory of gas equilibrium.

The article is divided into five sections:

- (1) Introduction
- (2) Preliminaries
- (3) A relaxation theorem
- (4) An existence theorem
- (5) Applications.

The third section deals with a problem of the following type:

$$(P) \inf \left\{ \int_{\Omega} f(\Phi(\nabla u(x))) \, dx : u \in u_0 + (W_0^{1,\alpha}(\Omega))^3 \right\}$$

where f satisfies some growth conditions, $\Omega \subset \mathbb{R}^3$, $u: \Omega \rightarrow \mathbb{R}^3$ and Φ is a null Lagrangian (*i.e.* the Euler-Lagrange equations for the functional $\int_{\Omega} \Phi(\nabla u(x)) \, dx$ reduce to an identity). An example of a nonlinear null Lagrangian is $\Phi(F) = \det F$. Then, following the same method as in [5], if

$$(P^{**}) \inf \left\{ \int_{\Omega} f^{**}(\Phi(\nabla u(x))) \, dx : u \in u_0 + (W_0^{1,\alpha}(\Omega))^3 \right\},$$

we prove that

$$\inf (P) = \inf (P^{**}).$$

It is interesting to note that this result differs from the aforementioned theorem of EKELAND & TÉMAM since even though our function f can be expressed as

$$f(\Phi(\nabla u(x))) = g(\nabla u(x)),$$

in general

$$f^{**}(\Phi(\nabla u(x))) \neq g^{**}(\nabla u(x)).$$

To see this, let $\Phi(\nabla u(x)) = \det \nabla u(x)$ and $f(\Phi(\nabla u(x))) = g(\nabla u(x)) = (\det \nabla u(x))^2$; then obviously we have that $f^{**}(\Phi(\nabla u(x))) = f(\Phi(\nabla u(x))) = (\det \nabla u(x))^2$, and since $\det F$ is not convex in F we get $f^{**}(\Phi(\nabla u(x))) \neq g^{**}(\nabla u(x))$.

We should emphasize, however, that, even though our result extends the theorem of EKELAND & TÉMAM to higher dimensions, it does not solve the problem for a general function g of u since null Lagrangians are of very special form. In fact the main property we will use is that Φ is a null Lagrangian if and only if Φ is rank 1 affine, *i.e.*, $\Phi(\lambda F + (1 - \lambda) G) = \lambda \Phi(F) + (1 - \lambda) \Phi(G)$ for all $0 \leq \lambda \leq 1$ and all 3×3 matrices F and G such that $F - G$ is of rank 1.

The fourth section of the article is devoted to proving that in the special case $\Phi(F) = \det F$ the minimum of (P^{**}) is attained. This result is equivalent to establishing the existence of a volume-preserving diffeomorphism \bar{u} under given boundary condition $\bar{u} = u_0$ on $\partial\Omega$. Our proof is an extension of that of L. TARTAR [10] for the case $n = 2$.

Turning now to applications (Section 5), we consider a gas which satisfies at constant entropy or temperature the equation of state

$$P = P(V)$$

where P is the pressure and V is the specific volume. One may think, for example, of the Van der Waal's equation of state. Furthermore, we assume that P satisfies appropriate growth conditions. The equilibrium equations are a special case of those of nonlinear elastostatics. We use material (Lagrangian) coordinates. Let the gas occupy the volume Ω in a given reference configuration.

We suppose that Ω is diffeomorphic to a sphere. In a deformed configuration the particle $x \in \Omega$ occupies the position $u(x) \in \mathbb{R}^3$. The energy of the deformation is then

$$E(u) = \int_{\Omega} \left(\int_1^{\det \nabla u(x)} P(V) dV \right) dx = \int_{\Omega} f(\det \nabla u(x)) dx. \tag{1.4}$$

Let

$$F(u) = \int_{\Omega} f^{**}(\det \nabla u(x)) dx. \tag{1.5}$$

We suppose that the deformation is specified on $\partial\Omega$, so that $u = u_0$ on $\partial\Omega$. Then our result is that

$$\begin{aligned} \inf(P) &= \inf \{ E(u) : u \in (C^1(\bar{\Omega}))^3; u = u_0 \text{ on } \partial\Omega \} \\ &= \inf \{ F(u) : u \in (C^1(\bar{\Omega}))^3; u = u_0 \text{ on } \partial\Omega \} = \inf(P^{**}) \end{aligned}$$

and that (P^{**}) possesses a solution \bar{u} . Moreover for any solution \bar{u} of (P^{**}) there exists a minimizing sequence $\{u_n\}$ of (P) such that

$$\begin{cases} u_n \rightarrow \bar{u} & \text{in } L^\infty, \\ \det \nabla u_n \rightharpoonup \det \nabla \bar{u} & \text{weakly in } L^\alpha \end{cases}$$

where α depends on the growth condition of the function $P(V)$ in (1.3).

Starting from the observation that to determine experimentally volume changes induced by the deformation u we actually measure some kind of ‘‘average’’ of $\det \nabla u$, we may argue, following TARTAR [9], that it is physically impossible to distinguish between a minimizing sequence $\{\det \nabla u_n\}$ and its weak limit $\det \nabla \bar{u}$. Consequently, \bar{u} should be interpreted as a ‘‘generalized equilibrium solution’’ of problem (P).

II. Null Lagrangians and Lower Convex Envelopes

Let Ω be a nonempty bounded open set of \mathbb{R}^3 . We let $M^{3 \times 3}$ denote the set of 3×3 matrices.

Definition. $\Phi : M^{3 \times 3} \rightarrow \mathbb{R}$ is said to be a null Lagrangian if

- (1) Φ is continuous,
- (2) for all $u \in (C^1(\bar{\Omega}))^3, \phi \in (C_0^\infty(\Omega))^3$

$$\int_{\Omega} \Phi(\nabla u(x) + \nabla \phi(x)) dx = \int_{\Omega} \Phi(\nabla u(x)) dx.$$

Null Lagrangians have the following remarkable properties:

Theorem 1. *The following conditions are equivalent:*

- (1) Φ is a null Lagrangian,
- (2) $\int_{\Omega} \Phi(F + \nabla\phi(x)) \, dx = \Phi(F) \times \text{volume}(\Omega)$

for all $F \in M^{3 \times 3}$ and $\phi \in (C_0^\infty(\Omega))^3$.

- (3) Φ is rank 1 affine, i.e.

$$\Phi(F + (1 - \lambda) a \otimes b) = \lambda\Phi(F) + (1 - \lambda) \Phi(F + a \otimes b)$$

for all $F \in M^{3 \times 3}$, $\lambda \in [0, 1]$, $a, b \in \mathbb{R}^3$.

- (4) There exist constants A, B_{ij}, C_{ij} and D , $1 \leq i, j \leq 3$ such that for all $F \in M^{3 \times 3}$, $F = (F_{ij})$ we have

$$\Phi(F) = A + B_{ij}F_{ij} + C_{ij}(\text{adj } F)_{ij} + D \det F$$

where $\text{adj } F$ denotes the matrix of cofactors of F . Here and throughout the article repeated indices are summed from 1 to 3.

- (5) Φ is C^1 and

$$\int_{\Omega} \frac{\partial \Phi}{\partial F_{ij}}(\nabla u(x)) \frac{\partial \phi_i}{\partial x_j} \, dx = 0$$

for all $u \in (C^1(\Omega))^3$, $\phi \in (C_0^\infty(\Omega))^3$.

Another important property of null Lagrangian is given in the following.

Theorem 2. *If $p > 3$ and $|\Phi(F)| \leq c(1 + |F|^p)$ for some constant $c > 0$ and all $F \in M^{3 \times 3}$, then the map*

$$\Phi(\nabla u(\cdot)) : (W^{1,p}(\Omega))^3 \rightarrow L^{p/3}(\Omega)$$

is sequentially weakly continuous (i.e. if $u_n \rightharpoonup u$ weakly in $(W^{1,p}(\Omega))^3$, then $\Phi(\nabla_n u(\cdot)) \rightarrow \Phi(\nabla u(\cdot))$ in $L^{p/3}(\Omega)$) if and only if Φ is a null Lagrangian.

The importance of Theorem 2 is that it asserts that

$$G(u) = \int_{\Omega} f^{**}(x, \Phi(\nabla u(x))) \, dx$$

is lower semi-continuous (i.e. if $u_n \rightharpoonup u$ in $(W^{1,p}(\Omega))^3$ then $G(u) \leq \liminf_{n \rightarrow \infty} G(u_n)$) provided f satisfies some growth conditions.

Remarks. (1) It is not difficult to see that the definitions as well as the theorems on null Lagrangians can be extended straightforwardly to the case where $\Omega \subset \mathbb{R}^n$ and u is a map from Ω into \mathbb{R}^m , for any n and m .

- (2) Observe that if $f^{**} \in C^2$ and if

$$g(F) = f^{**}(\Phi(F)),$$

then g satisfies the Legendre-Hadamard condition, namely:

$$\frac{\partial^2 g(F)}{\partial F_{i\alpha} \partial F_{j\beta}} \lambda_i \lambda_j \mu_\alpha \mu_\beta \geq 0 \quad \text{for all } F \in M^{3 \times 3} \text{ and for all } \lambda, \mu \in \mathbb{R}^3.$$

For a proof of Theorems 1 and 2 and related references see BALL [2]. For more details on necessary and sufficient conditions for weak continuity see TARTAR [9].

Definition. The lower convex envelope $f^{**}(x, \cdot)$ of $f(x, \cdot)$ is defined by $f^{**}(x, s) = \sup \{ \phi(x, s) : \phi(x, \cdot)$ lower semi-continuous and convex and

$$\phi(x, \cdot) \leq f(x, \cdot) \}.$$

Definition. $f: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is said to be a Carathéodory function if

- (1) $f(x, \cdot)$ is continuous for almost all $x \in \Omega$,
- (2) $f(\cdot, s)$ is measurable on Ω for all $s \in \mathbb{R}$.

We now have the following theorem for a proof of which we refer to EKELAND & TÉMAM ([5] Prop. 3.4 chap. IX).

Theorem 3. Let $f: \Omega \times \mathbb{R} \rightarrow \mathbb{R}_+$ be a Carathéodory function satisfying

$$f(x, t) \geq a(x) + b |t|^\alpha$$

where $1 < \alpha < \infty$, $a \in L^1(\Omega)$ and $b > 0$. Then for all measurable functions $p: \Omega \rightarrow \mathbb{R}$, there exist measurable functions $l: \Omega \rightarrow [0, 1]$ q_1 and q_2 such that for almost all $x \in \Omega$

$$\begin{cases} f^{**}(x, p(x)) = l(x)f(x, q_1(x)) + (1 - l(x))f(x, q_2(x)), \\ l(x)q_1(x) + (1 - l(x))q_2(x) = p(x). \end{cases}$$

III. Relaxation Theorem

The principal tool in our analysis will be the following lemma which is closely related to the work of L. C. YOUNG [11] and EKELAND & TÉMAM [5].

Lemma 4. Let Ω be a bounded open set in \mathbb{R}^3 . Let $0 \leq \alpha \leq 1$ and let $v, w: \Omega \rightarrow \mathbb{R}^3$ be piecewise affine functions such that

$$(H) \text{ rank} (\nabla v - \nabla w) \leq 1 \text{ a.e.}$$

Then for all $\varepsilon > 0$, there exist a $u \in (W^{1,\infty}(\Omega))^3$ and two subsets Ω_1 and Ω_2 of Ω such that

$$\begin{cases} |\text{meas } \Omega_1 - \alpha \text{ meas } \Omega| \leq \alpha \varepsilon, \\ |\text{meas } \Omega_2 - (1 - \alpha) \text{ meas } \Omega| \leq (1 - \alpha) \varepsilon, \end{cases} \tag{3.1}$$

$$\begin{cases} \nabla u(x) = \nabla v(x) \text{ a.e. in } \Omega_1, \\ \nabla u(x) = \nabla w(x) \text{ a.e. in } \Omega_2, \end{cases} \tag{3.2}$$

$$|u(x) - \alpha v(x) - (1 - \alpha) w(x)| \leq \varepsilon \text{ for all } x \in \Omega, \tag{3.3}$$

$$u(x) = \alpha v(x) + (1 - \alpha) w(x) \text{ for all } x \in \partial\Omega. \tag{3.4}$$

Proof. We suppose, without loss of generality, that v and w are affine in Ω ; otherwise we may decompose Ω into open sets on which v and w are affine and then, using (3.4), construct u on the whole of Ω .

We define

$$\begin{aligned} \bar{v} &= v - (\alpha v + (1 - \alpha) w) = (1 - \alpha)(v - w), \\ \bar{w} &= w - (\alpha v + (1 - \alpha) w) = -\alpha(v - w) \end{aligned} \tag{3.5}$$

and observe that since hypothesis (H) holds, changing, if necessary, coordinates

$$\begin{aligned} \bar{v} &= gx_1 + c, \\ \bar{w} &= hx_1 + d, \end{aligned} \tag{3.6}$$

where $g, h, c, d \in \mathbb{R}^3$ and $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Observe that since $\alpha\bar{v} + (1 - \alpha)\bar{w} = 0$, we have that

$$\alpha g + (1 - \alpha)h = 0.$$

We may also suppose that $\alpha \neq 0, 1$, since if $\alpha = 0$ or $\alpha = 1$, the lemma would be trivial.

For $i \in \mathbb{N}$ (the set of positive integers), let

$$\mathcal{K}_i = \left\{ K \subset \Omega : K = \prod_{j=1}^3 [m_j 2^{-i}, (m_j + 1) 2^{-i}] ; m_j \in \mathbb{N} \right\}$$

and

$$A_i = \bigcup_{K \in \mathcal{K}_i} K. \tag{3.7}$$

Choose i big enough so that

$$\text{meas}(\Omega - A_i) \leq \frac{\varepsilon}{2}. \tag{3.8}$$

Let N be the number of K in \mathcal{K}_i . For $K \in \mathcal{K}_i$ of the form $K = \prod_{j=1}^3 [a_j, b_j]$, and for $\eta \in (0, 2^{-(i+1)})$ let

$$\tilde{K}_\eta = (a_2 + \eta, b_2 - \eta) \times (a_3 + \eta, b_3 - \eta).$$

Choose η small enough so that

$$\left\{ \begin{aligned} \text{meas}(K - [a_1, b_1] \times \tilde{K}_\eta) &\leq \frac{\varepsilon}{4N}, \\ \eta + \alpha |\nabla v| + (1 - \alpha) |\nabla w| &\leq 3 \max\{|\nabla v|, |\nabla w|\}. \end{aligned} \right. \tag{3.9}$$

Given $m \in \mathbb{N}$, subdivide $[a_1, b_1]$ into intervals of length $(b_1 - a_1) 2^{-m}$ and each one of these intervals into two open subintervals of lengths $\alpha(b_1 - a_1) 2^{-m}$ and $(1 - \alpha)(b_1 - a_1) 2^{-m}$, respectively. Denote by I_m the union of the first subintervals and by J_m the union of the second ones. Then

$$[a_1, b_1] = \bar{I}_m \cup \bar{J}_m, \tag{3.10}$$

$$\text{meas}(I_m) = \alpha(b_1 - a_1), \tag{3.11}$$

$$\text{meas}(J_m) = (1 - \alpha)(b_1 - a_1). \tag{3.12}$$

Let

$$\phi_m : [a_1, b_1] \rightarrow \mathbb{R}^3$$

be defined by

$$\phi_m(a_1) = 0, \tag{3.13}$$

$$\phi'_m(x_1) = g \text{ on } I_m, \tag{3.14}$$

$$\phi'_m(x_1) = h \text{ on } J_m. \tag{3.15}$$

Then $\phi_m(b_1) = 0$ and $\phi_m \rightarrow 0$ uniformly in $[a_1, b_1]$. Choose m sufficiently large so that

$$|\phi_m(x_1)| \leq \min \left\{ \frac{\varepsilon}{2}, \eta^2 \right\}, \text{ for all } x_1 \in [a_1, b_1]. \tag{3.16}$$

Having fixed m in this way, we write

$$\Omega_1^K = I_m \times \tilde{K}_\eta, \tag{3.17}$$

$$\Omega_2^K = J_m \times \tilde{K}_\eta, \tag{3.18}$$

$$\tilde{u}(x) = \phi_m(x_1) + \alpha v(x) + (1 - \alpha) w(x) \text{ for all } x \in \Omega_1^K \cup \Omega_2^K. \tag{3.19}$$

Then by (3.9), (3.11) and (3.12) we get

$$\begin{cases} |\text{meas } \Omega_1^K - \alpha \text{ meas } K| \leq \alpha \frac{\varepsilon}{4N}, \\ |\text{meas } \Omega_2^K - (1 - \alpha) \text{ meas } K| \leq (1 - \alpha) \frac{\varepsilon}{4N}. \end{cases} \tag{3.20}$$

From (3.16) and (3.19) we get

$$|\tilde{u}(x) - \alpha v(x) - (1 - \alpha) w(x)| \leq \min \left\{ \frac{\varepsilon}{2}, \eta^2 \right\}. \tag{3.21}$$

Differentiating (3.19) we obtain

$$\begin{aligned} \nabla \tilde{u}(x) &= \nabla v \text{ in } \Omega_1^K, \\ \nabla \tilde{u}(x) &= \nabla w \text{ in } \Omega_2^K. \end{aligned} \tag{3.22}$$

\tilde{u} is then continuous and locally Lipschitz in $\bar{\Omega}_1^K \cup \bar{\Omega}_2^K$ by (3.19) and (3.22) and therefore $\tilde{u} \in (W^{1,\infty}(\bar{\Omega}_1^K \cup \bar{\Omega}_2^K))^3$ (see Corollary 2.4 of Chapter X in [5]).

Furthermore, by the construction of ϕ_m we have

$$\tilde{u}(x) = \alpha v(x) + (1 - \alpha) w(x) \text{ if } x \in \{a_1, b_1\} \times \tilde{K}_\eta. \tag{3.23}$$

We extend \tilde{u} to the whole of K by

$$u(x) = \phi_m(x_1) \psi(x_2) \chi(x_3) + \alpha v(x) + (1 - \alpha) w(x) \tag{3.24}$$

where

$$\psi(x_2) = \begin{cases} \frac{-a_2 + x_2}{\eta} & \text{if } a_2 \leq x_2 \leq a_2 + \eta, \\ 1 & \text{if } a_2 + \eta \leq x_2 \leq b_2 - \eta, \\ \frac{b_2 - x_2}{\eta} & \text{if } b_2 - \eta \leq x_2 \leq b_2, \end{cases}$$

$$\chi(x_3) = \begin{cases} \frac{-a_3 + x_3}{\eta} & \text{if } a_3 \leq x_3 \leq a_3 + \eta, \\ 1 & \text{if } a_3 + \eta \leq x_3 \leq b_3 - \eta, \\ \frac{b_3 - x_3}{\eta} & \text{if } b_3 - \eta \leq x_3 \leq b_3. \end{cases}$$

Then u is obviously in $(W^{1,\infty}(K))^3$ and

$$\begin{cases} u(x) = \tilde{u}(x) & \text{if } x \in \Omega_1^K \cup \Omega_2^K, \\ u(x) = \alpha v(x) + (1 - \alpha) w(x) & \text{if } x \in \partial K, \\ |\nabla u(x)| \leq 3 \max \{ |\nabla v|, |\nabla w| \}. \end{cases} \tag{3.25}$$

Summarizing we get, using (3.9) and (3.16):

$$\begin{cases} |\text{meas } \Omega_1^K - \alpha \text{meas } K| \leq \alpha \frac{\varepsilon}{4N}, \\ |\text{meas } \Omega_2^K - (1 - \alpha) \text{meas } K| \leq (1 - \alpha) \frac{\varepsilon}{4N}, \end{cases} \tag{3.26}$$

$$\begin{cases} \nabla u(x) = \nabla v & \text{a.e. in } \Omega_1^K, \\ \nabla u(x) = \nabla w & \text{a.e. in } \Omega_2^K, \end{cases} \tag{3.27}$$

$$|u(x) - \alpha v(x) - (1 - \alpha) w(x)| \leq \varepsilon, \quad x \in K, \tag{3.28}$$

$$u(x) = \alpha v(x) + (1 - \alpha) w(x), \quad x \in \partial K. \tag{3.29}$$

We have thus defined u on every $K \in \mathcal{X}_i$ and hence on $A_i = \bigcup_{K \in \mathcal{X}_i} K$. By extending u outside A_i as $\alpha v + (1 - \alpha) w$, we obtain a function $u \in (W^{1,\infty}(\Omega))^3$ which satisfies (3.2), (3.3) and (3.4) where

$$\Omega_1 = \bigcup_{K \in \mathcal{X}_1} \Omega_1^K, \quad \Omega_2 = \bigcup_{K \in \mathcal{X}_2} \Omega_2^K.$$

It only remains to prove (3.1). But this is clear since

$$\begin{aligned} |\text{meas } \Omega_1 - \alpha \text{meas } \Omega| &\leq |\text{meas } \Omega_1 - \alpha \text{meas } A_1| + \alpha |\text{meas } A_1 - \text{meas } \Omega| \\ &\leq \alpha \frac{\varepsilon}{2} + \alpha \frac{\varepsilon}{2} \end{aligned} \tag{3.30}$$

and similarly for Ω_2 . \square

In future applications we shall need the following corollary whose proof will be sketched below (for more details see [13]).

Corollary 5. *If in Lemma 4*

$$\det \nabla v > 0, \quad \det \nabla w > 0,$$

then u can be chosen so that

$$\det u > 0 \text{ a.e.}$$

Moreover there exists a $\delta > 0$ sufficiently small so that

$$\text{adj}(\nabla u(x) - \nabla u(y)) \equiv 0 \text{ a.e. for } |x - y| \leq \delta.$$

Proof. Adopting the same notation as in Lemma 4, we define

$$\nabla v = (v_{ij}), \quad \nabla w = (w_{ij}), \quad i, j = 1, 2, 3. \tag{3.31}$$

Recall that by (3.6)

$$v_{ij} = w_{ij} \text{ if } j > 1. \tag{3.32}$$

By the definition of ϕ_m we have for almost all $x \in K$

$$\phi'_m(x_1) = \lambda(x_1) \begin{bmatrix} v_{11} - w_{11} \\ v_{21} - w_{21} \\ v_{31} - w_{31} \end{bmatrix} = \begin{bmatrix} \phi'_1 \\ \phi'_2 \\ \phi'_3 \end{bmatrix},$$

where

$$\lambda(x_1) = \begin{cases} 1 - \alpha & \text{if } x_1 \in I_m, \\ -\alpha & \text{if } x_1 \in J_m. \end{cases}$$

Then by (3.24), bearing in mind (3.6), we obtain

$$\det \nabla u = (\alpha + \lambda\psi\chi) \det \nabla v + [(1 - \alpha) - \lambda\psi\chi] \det \nabla w + F(\phi_m) \tag{3.33}$$

where F is a function depending on ϕ_m .

Then since $\det \nabla v > 0$, $\det \nabla w > 0$ and $F(\phi_m)$ is arbitrarily small (since ϕ_m is small), we deduce that for almost all $x \in K$

$$\det \nabla u(x) > 0. \tag{3.34}$$

Since (3.34) holds for any K , it holds also in A_i and since $u = \alpha v + (1 - \alpha) w$ outside A_i , we deduce that (3.34) is valid on the whole of Ω .

The second part of the Corollary is based on the fact that for almost all x, y

$$\phi'_i(x) \phi_j(y) - \phi'_j(x) \phi_i(y) = \phi_i(x) \phi_j(y) - \phi_j(x) \phi_i(y) = 0, \quad 1 \leq i, j \leq 3$$

provided $|x - y|$ is sufficiently small. \square

Corollary 6. *Let Φ be a null Lagrangian. There exists a constant C such that if Ω, α, v and w are as in Lemma 4, then the conclusions of Lemma 4 hold and u can be so chosen that*

$$|\Phi(\nabla u(x))| \leq C(1 + \max(|\nabla v|^3, |\nabla w|^3)) = \varrho. \tag{3.35}$$

Proof. By Theorem 1 there exist constants A, B_{ij}, C_{ij} and $D, 1 \leq i, j \leq 3$, such that for all F

$$\Phi(F) = A + B_{ij}F_{ij} + C_{ij}(\text{adj } F)_{ij} + D \det F. \tag{3.36}$$

Then applying Lemma 4 (in particular (3.25)) to (3.35) we arrive at the result. \square

We now try to approximate $\int_{\Omega} f(x, \Phi(\nabla u(x))) dx$ by a convex combination of integrals of the same type (here we follow closely [5], Prop. X. 1.3).

Lemma 7. Let Ω be a bounded open set in $\mathbb{R}^3, 0 \leq \alpha \leq 1$ and let $v, w: \Omega \rightarrow \mathbb{R}^3$ be piecewise affine functions with the property

$$(H) \quad \text{rank}(\nabla v - \nabla w) \leq 1 \quad \text{a.e.}$$

Let Φ be a null Lagrangian and ϱ as in (3.35). Finally, assume $\mathcal{F} = \{f_1, \dots, f_{\sigma}\}$ where the $f_{\nu}, 1 \leq \nu \leq \sigma$, are Carathéodory functions on $\Omega \times \mathbb{R}$, and let $c \in L^1(\Omega)$ be positive and such that

$$c(x) \geq \sup \{|f_{\nu}(x, y)| | f_{\nu} \in \mathcal{F}, 1 \leq \nu \leq \sigma, |y| \leq \varrho\} \quad \text{a.e.} \tag{3.37}$$

Then for every $\varepsilon > 0$, there exists $u \in (W^{1,\infty}(\Omega))^3$ such that

$$|\Phi(\nabla u(x))| \leq \varrho \quad \text{a.e. in } \Omega, \tag{3.38}$$

$$|u(x) - \alpha v(x) - (1 - \alpha) w(x)| \leq \varepsilon \quad \text{for all } x \text{ in } \Omega, \tag{3.39}$$

$$u(x) = \alpha v(x) + (1 - \alpha) w(x) \quad \text{for all } x \text{ on } \partial\Omega, \tag{3.40}$$

$$\left| \int_{\Omega} f_{\nu}(x, \Phi(\nabla u(x))) dx - \alpha \int_{\Omega} f_{\nu}(x, \Phi(\nabla v(x))) dx - (1 - \alpha) \int_{\Omega} f_{\nu}(x, \Phi(\nabla w(x))) dx \right| \leq \varepsilon \quad \text{for all } \nu = 1, \dots, \sigma. \tag{3.41}$$

Proof. As in Lemma 4 there is no loss of generality in supposing v and w affine in Ω . For any $f_{\nu} (1 \leq \nu \leq \sigma)$ and for every $\varepsilon > 0$ there exists a partition of Ω into a finite number of non-empty open sets $P_{\nu,i}, 1 \leq i \leq N_{\nu}$ and a set of measure zero and also functions $\tilde{f}_{1,\nu}, \tilde{f}_{2,\nu}$ constant on every $P_{\nu,i}$ such that

$$\begin{aligned} \int_{\Omega} |f_{\nu}(x, \Phi(\nabla v(x))) - \tilde{f}_{1,\nu}(x)| dx &\leq \varepsilon, \\ \int_{\Omega} |f_{\nu}(x, \Phi(\nabla w(x))) - \tilde{f}_{2,\nu}(x)| dx &\leq \varepsilon. \end{aligned} \tag{3.42}$$

We now set

$$O_{\mu} = O_{i_1, \dots, i_{\sigma}} = P_{1,i_1} \cap \dots \cap P_{\sigma, i_{\sigma}}, \quad 1 \leq i_{\nu} \leq N_{\nu}, \quad 1 \leq \nu \leq \sigma.$$

Dropping those O_{μ} 's, if any, that are empty we end up with a family of open sets $O_i, 1 \leq i \leq N$, such that

$$(i) \quad \text{meas} \left(\Omega - \bigcup_{i=1}^N O_i \right) = 0.$$

(ii) $\tilde{f}_{1,\nu}$ and $\tilde{f}_{2,\nu}$ are constant on every O_i for all $\nu = 1, \dots, \sigma$.

We let

$$L = \max_{1 \leq \nu \leq \sigma} \{ \|\tilde{f}_{1,\nu}\|_\infty, \|\tilde{f}_{2,\nu}\|_\infty \}$$

and construct $\delta > 0$ such that

$$\delta \leq \frac{\varepsilon}{N(1+L)}, \tag{3.43}$$

$$\int_B c(x) dx \leq \frac{\varepsilon}{N} \tag{3.44}$$

for every measurable $B \subset \Omega$ with $\text{meas } B \leq \delta$.

We then construct a function $u \in (W^{1,\infty}(\Omega))^3$ which equals $\alpha v + (1 - \alpha) w$ in $\Omega - \bigcup_{i=1}^N O_i$ while its restriction on each of the O_i meets the requirements spelled out in Lemma 4 and Corollary 6. In particular there exist two disjoint open sets Ω_1^i, Ω_2^i contained in O_i such that

$$\begin{cases} |\text{meas } \Omega_1^i - \alpha \text{meas } O_i| \leq \alpha \delta, \\ |\text{meas } \Omega_2^i - (1 - \alpha) \text{meas } O_i| \leq (1 - \alpha) \delta, \end{cases} \tag{3.45}$$

$$\begin{cases} \nabla u(x) = \nabla v \quad \text{a.e. in } \Omega_1^i, \\ \nabla u(x) = \nabla w \quad \text{a.e. in } \Omega_2^i, \end{cases} \tag{3.46}$$

$$|\Phi(\nabla u(x))| \leq \varrho \quad \text{a.e. in } O_i, \tag{3.47}$$

$$|u(x) - \alpha v(x) - (1 - \alpha) w(x)| \leq \varepsilon \quad \text{for all } x \in O_i, \tag{3.48}$$

$$u(x) = \alpha v(x) + (1 - \alpha) w(x) \quad \text{for all } x \in \partial O_i. \tag{3.49}$$

Thus u satisfies (3.38), (3.39) and (3.40). Hence it only remains to prove (3.41).

Using (3.46), we obtain for any $\nu \in \{1, \dots, \sigma\}$

$$\begin{aligned} \int_{O_i} f_\nu(x, \Phi(\nabla u(x))) dx - \int_{\Omega_1^i} f_\nu(x, \Phi(\nabla v(x))) dx - \int_{\Omega_2^i} f_\nu(x, \Phi(\nabla w(x))) dx \\ = \int_{O_i - (\Omega_1^i \cup \Omega_2^i)} f_\nu(x, \Phi(\nabla u(x))) dx. \end{aligned} \tag{3.50}$$

The right hand side can be bounded in the following way. From (3.47) and (3.37) we get $|f_\nu(x, \Phi(\nabla u(x)))| \leq c(x)$; from (3.45) we obtain $\text{meas}(O_i - (\Omega_1^i \cup \Omega_2^i)) \leq \delta$ and from (3.44) we find that the right hand side is bounded by $\frac{\varepsilon}{N}$. Hence

$$\left| \int_{O_i} f_\nu(x, \Phi(\nabla u(x))) dx - \int_{\Omega_1^i} f_\nu(x, \Phi(\nabla v(x))) dx - \int_{\Omega_2^i} f_\nu(x, \Phi(\nabla w(x))) dx \right| \leq \frac{\varepsilon}{N}. \tag{3.51}$$

From (3.43) and (3.45), using the fact that $\tilde{f}_{k,\nu}$ is constant in O_i , we have

$$\left| \int_{\Omega_1^i} \tilde{f}_{1,\nu} + \int_{\Omega_2^i} \tilde{f}_{2,\nu} - \alpha \int_{O_i} \tilde{f}_{1,\nu} - (1 - \alpha) \int_{O_i} \tilde{f}_{2,\nu} \right| \leq \frac{\varepsilon}{N}. \tag{3.52}$$

If we let $\Omega_k = \bigcup_{i=1}^N \Omega_k^i$ for $k = 1, 2$, then

$$\begin{aligned}
 |A| &= \left| \int_{\Omega} f_v(x, \Phi(\nabla u(x))) dx - \alpha \int_{\Omega} f_v(x, \Phi(\nabla v(x))) dx - (1-\alpha) \int_{\Omega} f_v(x, \Phi(\nabla w(x))) dx \right| \\
 &\leq \left| \int_{\Omega} f_v(x, \Phi(\nabla u(x))) dx - \int_{\Omega_1} f_v(x, \Phi(\nabla v(x))) dx - \int_{\Omega_2} f_v(x, \Phi(\nabla w(x))) dx \right| \\
 &\quad + \int_{\Omega_1} |f_v(x, \Phi(\nabla v(x))) - \tilde{f}_{1,v}(x)| dx + \int_{\Omega_2} |f_v(x, \Phi(\nabla w(x))) - \tilde{f}_{2,v}(x)| dx \\
 &\quad + \left| \int_{\Omega_1} \tilde{f}_{1,v}(x) dx - \alpha \int_{\Omega} \tilde{f}_{1,v}(x) dx + \int_{\Omega_2} \tilde{f}_{2,v}(x) dx - (1-\alpha) \int_{\Omega} \tilde{f}_{2,v}(x) dx \right| \\
 &\quad + \alpha \int_{\Omega} |\tilde{f}_{1,v}(x) - f_v(x, \Phi(\nabla v(x)))| dx + (1-\alpha) \int_{\Omega} |\tilde{f}_{2,v}(x) - f_v(x, \Phi(\nabla w(x)))| dx.
 \end{aligned} \tag{3.53}$$

Using (3.51), (3.52) and (3.42), we get $|A| \leq 4\varepsilon$. \square

Before proving the theorem on relaxation we need the following proposition, for a proof of which we refer the reader to EKELAND & TÉMAM ([5], Prop. 2.9, Chap. X).

Proposition 8. Let Ω be a bounded open set of \mathbb{R}^3 with Lipschitz boundary. Let $u, u_0 \in (W^{1,\infty}(\Omega))^3$ and $u = u_0$ on $\partial\Omega$. Then for every $\varepsilon > 0$, there exist an open set $O \subset \Omega$ and a function $w \in (W^{1,\infty}(\Omega))^3$ such that

$$\text{meas}(\Omega - O) \leq \varepsilon, \tag{3.54}$$

$$w \text{ is piecewise affine in } O, \tag{3.55}$$

$$w = u_0 \text{ on } \partial\Omega, \tag{3.56}$$

$$|u(x) - w(x)| \leq \varepsilon \text{ for all } x \in \Omega, \tag{3.57}$$

$$\|\nabla u - \nabla w\|_{L^1} \leq \varepsilon. \tag{3.58}$$

We now turn our attention to the relaxation problem, *i.e.* to proving that

$$(P) \quad \inf \left\{ \int_{\Omega} f(x, \Phi(\nabla u(x))) dx : u \in \mathcal{W} \right\}$$

and

$$(P^{**}) \quad \inf \left\{ \int_{\Omega} f^{**}(x, \Phi(\nabla u(x))) dx : u \in \mathcal{W} \right\}$$

have the same value, where

(H1) Ω is a bounded open set of \mathbb{R}^3 with Lipschitz boundary,

(H2) $u_0 \in (W^{1,\infty}(\Omega))^3$ and $\mathcal{W} = \{u \in (W^{1,\infty}(\Omega))^3, u = u_0 \text{ on } \partial\Omega\}$,

(H3) $f: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function such that

$$c(x) + d|y|^\alpha \leq f(x, y) \leq a(x) + b|y|^\alpha$$

for some

$$a, c \in L^1(\Omega), b \geq d > 0, 1 < \alpha < \infty,$$

(H4) Φ is a null Lagrangian such that (see Theorem 1)

$$\Phi(F) = A + B_{ij}F_{ij} + C_{ij}(\text{adj } F)_{ij} + D \det F,$$

where $F \in \mathbb{R}^9 \approx M^{3 \times 3}$, $F = (F_{ij})$, $i, j = 1, 2, 3$.

Then we have the fundamental lemma:

Lemma 9. *Suppose the hypotheses (H1)–(H4) hold. Then for all piecewise affine $u \in \mathcal{W}$, for every $\varepsilon > 0$ and for all neighborhoods V of O in the weak topology of L^α (denoted by $\sigma(L^\alpha, L^{\alpha'})$ where $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$), there exists a function $v \in \mathcal{W}$ such that*

$$\Phi(\nabla v) - \Phi(\nabla u) \in V, \tag{3.59}$$

$$\|v - u\|_\infty \leq \varepsilon, \tag{3.60}$$

$$\left| \int_\Omega f(x, \Phi(\nabla v(x))) dx - \int_\Omega f^{**}(x, \Phi(\nabla u(x))) dx \right| \leq \varepsilon. \tag{3.61}$$

Proof. In the proof we shall consider two cases.

Case 1. Suppose that u is such that

$$\Phi(\nabla u(x) + a \otimes b) \neq \Phi(\nabla u(x)) \tag{3.62}$$

for all $x \in \Omega$ and for some $a, b \in \mathbb{R}^3$.

Let V be such that

$$V = \left\{ p \in L^\alpha : \left| \int_\Omega h_m(x) p(x) dx \right| \leq \theta, \quad m \in M \right\} \tag{3.63}$$

where $\theta > 0$, M is a finite set and $h_m \in L^{\alpha'}(\Omega)$.

By hypothesis, there exists a partition of Ω into open sets Δ_i , $1 \leq i \leq p$, and a set N of measure zero such that ∇u is constant on every Δ_i . If we choose $\varrho \geq \max \{ |\Phi(\nabla u(x))| : x \in \Omega \}$, f^{**} is a Carathéodory function in $\Omega \times [-\varrho, \varrho]$, and for every i , there exists a compact set $K_i \subset \Delta_i$ such that

$$\int_{\Delta_i - K_i} (a(x) + b\varrho^\alpha) dx \leq \frac{\varepsilon}{3p}, \tag{3.64}$$

$$f \text{ and } f^{**}, \text{ restricted to } K_i \times [-\varrho, \varrho], \text{ are continuous.} \tag{3.65}$$

Using (3.65) and the compactness of $[-\varrho, \varrho]$, we can find for each x in K_i an open ball B_x centred at x and contained in Ω such that

$$|f^{**}(y, \Phi(\nabla u)) - f^{**}(x, \Phi(\nabla u))| \leq \frac{\varepsilon}{6 \text{ meas } \Omega} \text{ for all } y \in B_x \cap K_i, \tag{3.66}$$

$$|f(y, \xi) - f(x, \xi)| \leq \frac{\varepsilon}{6 \text{ meas } \Omega} \text{ for all } y \in B_x \cap K_i \text{ and all } \xi \in [-\varrho, \varrho]. \tag{3.67}$$

By Theorem 3, we have that for every fixed $x \in \Delta_i$ there exist $\lambda \in [0, 1]$, s and t such that

$$\lambda s + (1 - \lambda) t = \Phi(\nabla u), \tag{3.68}$$

$$\lambda f(x, s) + (1 - \lambda) f(x, t) = f^{**}(x, \Phi(\nabla u)). \tag{3.69}$$

Note that in general λ, s, t are not necessarily constant on Δ_i but only measurable functions of x ; however, approximating λ, s, t by simple functions and then decomposing Δ_i into subsets we can assume without loss of generality that λ, s, t are constant on Δ_i and satisfy

$$\lambda s + (1 - \lambda) t = \Phi(\nabla u),$$

$$|f^{**}(x, \Phi(\nabla u)) - \lambda f(x, s) - (1 - \lambda) f(x, t)| \leq \frac{\varepsilon}{6 \text{ meas } \Omega}.$$

Hence using (3.66) and (3.67) we get

$$|f^{**}(y, \Phi(\nabla u)) - \lambda f(x, s) - (1 - \lambda) f(x, t)| \leq \frac{\varepsilon}{3 \text{ meas } \Omega} \quad \text{for all } y \in B_x \cap K_i. \tag{3.70}$$

But since K_i is compact we can cover it by a finite number of the B_x , which we denote by B_j , $1 \leq j \leq l$. Thus if we define $B'_j = B_j - \bigcup_{i=1}^{j-1} \bar{B}_i$, then all the B'_j are disjoint and $\bigcup_{j=1}^l \bar{B}'_j = \bigcup_{j=1}^l \bar{B}_j$.

We now define two functions ϕ and ψ such that

$$\text{rank } (\nabla \phi - \nabla \psi) \leq 1, \tag{3.71}$$

$$\phi \text{ and } \psi \text{ are affine in } B'_j, \tag{3.72}$$

$$u = \lambda \phi + (1 - \lambda) \psi, \tag{3.73}$$

$$\Phi(\nabla \phi) = s, \quad \Phi(\nabla \psi) = t. \tag{3.74}$$

Such a construction can be done in the following way. Let $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ be in \mathbb{R}^3 , where a_j and b_j are to be determined and where

$$\nabla \phi - \nabla \psi = a \otimes b. \tag{3.75}$$

Then (3.71) and (3.72) are satisfied. Thus it remains to find a and b which satisfies (3.74) (and therefore (3.73)). Consider

$$\Phi(\nabla u + (1 - \lambda) a \otimes b) = \Phi(\nabla \phi) = s. \tag{3.76}$$

Since Φ is a null Lagrangian (cf. Theorem 1), we obtain

$$s = \Phi(\nabla u + (1 - \lambda) a \otimes b) = \lambda \Phi(\nabla u) + (1 - \lambda) \Phi(\nabla u + a \otimes b) \tag{3.77}$$

and hence

$$(1 - \lambda) \Phi(\nabla u + a \otimes b) = s - \lambda \Phi(\nabla u). \tag{3.78}$$

Three cases may occur:

- (1) If $\lambda = 0$, then take $\psi = u$ and $\phi = 0$.
- (2) If $\lambda = 1$, then take $\phi = u$ and $\psi = 0$.
- (3) If $\lambda \neq 0, 1$, then (3.78) gives

$$\Phi(\nabla u + a \otimes b) = \frac{s - \lambda\Phi(\nabla u)}{1 - \lambda}. \tag{3.79}$$

Using hypothesis (H4) and the fact that

$$\begin{aligned} (\text{adj } F)_{ij} &= \frac{1}{2} \varepsilon_{ikt} \varepsilon_{jmn} F_{km} F_{ln}, \\ \det F &= \frac{1}{6} \varepsilon_{ijk} \varepsilon_{lmn} F_{il} F_{jm} F_{kn} \end{aligned}$$

where ε_{ijk} denotes the standard permutation symbol, *i.e.*

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i, j, k) \text{ is an even permutation of } (1, 2, 3), \\ -1 & \text{if } (i, j, k) \text{ is an odd permutation of } (1, 2, 3), \\ 0 & \text{if one of the indices is repeated,} \end{cases}$$

we deduce

$$b_\xi a_\nu \left(B_{\nu\xi} + C_{ij} \varepsilon_{ik\nu} \varepsilon_{jms} u_{km} + \frac{D}{2} \varepsilon_{ijk} \varepsilon_{smn} u_{jm} u_{kn} \right) = \frac{\Phi(\nabla u) - s}{1 - \lambda}. \tag{3.80}$$

But in (3.80) one of the coefficients of $b_\xi a_\nu$ has to be non-zero (say that of $b_i a_j$), for otherwise the left hand side of (3.80) would be identically zero for all $a, b \in \mathbb{R}^3$, and thus $\Phi(\nabla u) = s$. Using (3.79), we would then have $\Phi(\nabla u + a \otimes b) = \Phi(\nabla u)$ for all a, b , in contradiction with (3.62). Thus we can choose

$$b_i = \delta_{i\bar{i}} \quad \text{and} \quad a_j = c \delta_{j\bar{j}}$$

where δ_{ij} denotes the Kronecker symbol and

$$c = \frac{\Phi(\nabla u) - s}{(1 - \lambda) \left[B_{j\bar{i}} + C_{ij} \varepsilon_{ik\bar{j}} \varepsilon_{jmi} u_{km} + \frac{D}{2} \varepsilon_{ijk} \varepsilon_{imn} u_{jm} u_{kn} \right]}. \tag{3.81}$$

We now apply Lemma 7 to the piecewise affine functions ϕ and ψ and to the set

$$\mathcal{F} = \{f \text{ and } \langle h_m(x), \xi \rangle; \quad m \in M\}.$$

Observe that the hypothesis (3.37) on \mathcal{F} is satisfied trivially for f and for $\langle h_m(x), \xi \rangle$ since for all $m \in M$

$$|\langle h_m(x), \xi \rangle| \leq \frac{1}{\alpha'} |h_m(x)|^{\alpha'} + \frac{1}{\alpha} |\xi|^\alpha \quad \text{if } \alpha > 1. \tag{3.82}$$

We therefore obtain a function $v_{ij} \in (W^{1,\infty}(B'_j))^3$ such that

$$\left| \int_{B'_j} f(x, \Phi(\nabla v_{ij}(x))) dx - \lambda \int_{B'_j} f(x, \Phi(\nabla \phi(x))) dx - (1 - \lambda) \int_{B'_j} f(x, \Phi(\nabla \psi(x))) dx \right| \leq \frac{\varepsilon}{3pl}, \tag{3.83}$$

$$\left| \int_{B'_j} h_m(x) \Phi(\nabla v_{ij}(x)) dx - \int_{B'_j} h_m(x) \Phi(\nabla u(x)) dx \right| \leq \frac{\theta}{pl} \text{ for all } m \in M, \tag{3.84}$$

$$|v_{ij}(x) - \lambda \phi(x) - (1 - \lambda) \psi(x)| = |v_{ij}(x) - u(x)| \leq \varepsilon \text{ for all } x \in B'_j, \tag{3.85}$$

$$v_{ij}(x) = u(x) \text{ for all } x \in \partial B'_j. \tag{3.86}$$

Finally let

$$v(x) = \begin{cases} v_{ij}(x) & \text{if } x \in B'_j \subset \Delta_i, \\ u(x) & \text{if } x \in N \cup \left(\Delta_i - \bigcup_{j=1}^l B'_j \right). \end{cases} \tag{3.87}$$

Then

$$v = u \text{ on } \partial\Omega \text{ and } v \in \mathcal{W}.$$

It now remains to prove that v has the claimed properties (3.59), (3.60) and (3.61). (3.59) is deduced from (3.84) and (3.87), and similarly (3.60) is a consequence of (3.85) and (3.87). It remains only to prove (3.61).

From (H3) and (3.64) we get

$$\int_{\Delta_i - K_i} |f^{**}(x, \Phi(\nabla u(x)))| dx \leq \frac{\varepsilon}{3p}, \tag{3.88}$$

$$\int_{\Delta_i - K_i} |f(x, \Phi(\nabla v(x)))| dx \leq \frac{\varepsilon}{3p}. \tag{3.89}$$

Using (3.83) and (3.70) we get

$$\left| \int_{K_i \cap B'_j} f(x, \Phi(\nabla v(x))) dx - \int_{K_j \cap B'_j} f^{**}(x, \Phi(\nabla u(x))) dx \right| \leq \frac{\varepsilon}{3pl} + \frac{\varepsilon \text{ meas } B'_j}{3 \text{ meas } \Omega}. \tag{3.90}$$

Summing (3.90) over $j = 1, \dots, l$ and using the fact that $\text{meas} \left(K_i - \bigcup_{j=1}^l B'_j \right) = 0$, we obtain

$$\left| \int_{K_i} f(x, \Phi(\nabla v(x))) dx - \int_{K_i} f^{**}(x, \Phi(\nabla u(x))) dx \right| \leq \frac{\varepsilon}{3p} + \frac{\varepsilon \text{ meas } K_i}{3 \text{ meas } \Omega}. \tag{3.91}$$

If we now sum (3.91), (3.89) and (3.88) over $i = 1, \dots, p$ we get the result.

Case 2. Now we consider the case where (3.62) fails. Then there are two possibilities:

(a) $\Phi(F) \equiv A$ for all $F \in M^{3 \times 3}$ (A being a constant), so that the lemma is trivial.

(b) $\Phi(F)$ is not constant but (3.62) fails on a set of positive measure. We can then decompose Ω into two open sets Ω_1 and Ω_2 such that in Ω_1 (3.62) fails while in Ω_2 (3.62) holds. Without loss of generality we may assume that u is affine in Ω_1 .

If we can show that for every $\varepsilon > 0$ we can construct an open set $O \subset \Omega_1$ and a function u_1 such that

$$\text{meas}(\Omega_1 - O) \leq \varepsilon, \tag{3.92}$$

$$u_1 \text{ is piecewise affine in } O \text{ and } u_1 \in (W^{1,\infty}(\Omega_1))^3, \tag{3.93}$$

$$u_1 = u \text{ on } \partial\Omega_1, \tag{3.94}$$

$$\|u_1 - u\| \leq \varepsilon \text{ in } (W^{1,\infty}(\Omega_1))^3, \tag{3.95}$$

$$\Phi(\nabla u_1 + a \otimes b) \neq \Phi(\nabla u_1) \text{ a.e. in } O \text{ for some } a, b \in \mathbb{R}^3, \tag{3.96}$$

then Case 1, above, applies to u in Ω_2 as well as u_1 in O so combining the results we will get the assertion of Lemma 9.

Let $\nabla u = (u_{ij})$ (the u_{ij} are constant since we supposed u to be affine in Ω_1). Since (3.62) fails we have

$$B_{ij} + C_{ij}\varepsilon_{ik}\varepsilon_{jm}\bar{u}_{km} + \frac{D}{2}\varepsilon_{ijk}\varepsilon_{jmn}u_{jm}u_{kn} \equiv 0 \text{ for all } \bar{i}, \bar{j} = 1, 2, 3. \tag{3.97}$$

But, since $\Phi(F) \equiv A$, one of the constants C_{ij} or D must be non-zero.

Add $\alpha \in \mathbb{R}$ to one of the u_{ij} so that (at least) one of the equations in (3.97) is no longer valid. Denote by u' the function obtained by this process. Let O be

such that $\frac{\varepsilon}{2} \leq \text{meas}(\Omega_1 - O) \leq \varepsilon$, and construct u_1 such that

$$u_1 = \begin{cases} u' & \text{in } O, \\ u & \text{in } \partial\Omega_1 \end{cases}$$

and which satisfies (3.93)–(3.96) (obviously this can be done if we choose α sufficiently small). This completes the proof. \square

With the help of Lemma 9, we are now able to prove the main theorem of this section.

Theorem 10 (relaxation). *Under the hypotheses (H1)–(H4), for all $u \in \mathcal{W}$, for every $\varepsilon > 0$ and all neighborhoods V of O in $\sigma(L^\alpha, L^{\alpha'})$, there exists $v \in \mathcal{W}$ such that*

$$\Phi(\nabla v) - \Phi(\nabla u) \in V, \tag{3.98}$$

$$\|v - u\|_\infty \leq \varepsilon, \tag{3.99}$$

$$\left| \int_\Omega f(x, \Phi(\nabla v(x))) dx - \int_\Omega f^{**}(x, \Phi(\nabla u(x))) dx \right| \leq \varepsilon. \tag{3.100}$$

Proof. Using Proposition 8, we get a function $w \in \mathcal{W}$ and an open set $O \subset \Omega$ with Lipschitz boundary such that

$$\int_{\Omega-O} (a(x) + b\varrho^\alpha) dx \leq \frac{\varepsilon}{4}, \quad (3.101)$$

$$w \text{ is piecewise affine in } O \text{ and } w = u_0 \text{ on } \partial O, \quad (3.102)$$

$$\|u - w\|_\infty \leq \frac{\varepsilon}{2}, \quad (3.103)$$

$$\Phi(\nabla w) - \Phi(\nabla u) \in \frac{1}{2} V = \{\Phi: 2\Phi \in V\}, \quad (3.104)$$

$$\int_{\Omega} |f^{**}(x, \Phi(\nabla w(x))) - f^{**}(x, \Phi(\nabla u(x)))| dx \leq \frac{\varepsilon}{4}. \quad (3.105)$$

Using now Lemma 9 for w and O we obtain a function $v \in (W^{1,\infty}(O))^3$ such that

$$v(x) = w(x) \text{ for all } x \in \partial O, \quad (3.106)$$

$$\|v - w\|_\infty \leq \frac{\varepsilon}{2}, \quad (3.107)$$

$$\Phi(\nabla v) - \Phi(\nabla w) \in \frac{1}{2} V, \quad (3.108)$$

$$\left| \int_O f(x, \Phi(\nabla v(x))) dx - \int_O f^{**}(x, \Phi(\nabla w(x))) dx \right| \leq \frac{\varepsilon}{4}. \quad (3.109)$$

Thus if we define $v = w$ outside O , we obtain the result. \square

Remarks. (1) Theorem 10 implies that

$$\inf (P) = \inf (P^{**})$$

but it does not imply that the infimum of (P^{**}) is attained; the existence of such a minimum will be discussed in the next section. However the "advantage" of the problem (P^{**}) over (P) is that

$$\int_{\Omega} f^{**}(x, \Phi(\nabla u(x))) dx$$

is lower semi-continuous with respect to the weak convergence in $W^{1,\alpha}$, while this is not the case (in general) for

$$\int_{\Omega} f(x, \Phi(\nabla u(x))) dx.$$

(2) The theorem can be extended without difficulty to higher dimensions, i.e. if $u: \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$, provided Φ is a null Lagrangian.

(3) Let $f: \Omega \times (\mathbb{R}^n \times \mathbb{R}) \rightarrow \mathbb{R}$ be a Carathéodory function such that

$$a(x) + b|z|^\alpha \leq f(x, y, z) \leq c(x) + d|y|^\beta + e|z|^\alpha$$

for some $a, c \in L^1(\Omega)$, $e \geq b > 0$, $d \geq 0$, $1 < \alpha < \infty$ and $1 \leq \beta < \infty$. Then with the help of (3.99) we conclude that the theorem still holds for

$$\int_{\Omega} f(x, u(x), \Phi(\nabla u(x))) \, dx$$

where $f^{**}(x, y, \cdot)$ is now the lower convex envelope of $f(x, y, \cdot)$.

(4) In the case where $\Phi(F) = \det F$, we have a more precise result, namely, that if in Theorem 10 we impose the additional hypotheses

(H5) $f(x, 0) = f^{**}(x, 0)$ a.e.

(H6) $u \in (C^1(\bar{\Omega}))^3$ with $\det \nabla u(x) > 0$ for all $x \in \bar{\Omega}$,

then v can be chosen so as to satisfy the conclusions of Theorem 10 and

(i) $\det \nabla v > 0$ a.e. in Ω ,

(ii) $\text{adj}(\nabla v(y) - \nabla v(z)) \equiv 0$ a.e. for $|y - z| \leq \delta$ where $\delta > 0$ is sufficiently small.

This follows from Corollary 5 and the actual construction of v (for more details see [13]).

IV. Existence Theorem

We now turn our attention to the existence of a minimizer for the problem (P^{**}). We obtain here a partial result for a particular null-Lagrangian $\Phi(F) = \det F$, which is particularly interesting from the point of view of applications.

For this purpose we need the following lemma, which is an extension of a result of L. TARTAR [10] in dimension 2.

Lemma 11. *Let $S = \{(x, y, z) : x^2 + y^2 + z^2 < 1\}$ and let $f \in C^1(\bar{S})$ be positive and such that*

$$\int_S f(x, y, z) \, dx \, dy \, dz = \text{volume}(S) = \frac{4}{3} \pi. \tag{4.1}$$

Then the problem

$$\begin{cases} \det \nabla u(x, y, z) = f(x, y, z) & \text{in } S, \\ u(x, y, z) = \text{id}(x, y, z) = (x, y, z) & \text{on } \partial S \end{cases} \tag{4.2}$$

has a C^1 solution.

Proof. The proof proceeds in three steps:

(1) We show that there is no loss of generality if the given function f is taken to be 1 near ∂S .

(2) Similarly we prove that f can be taken to be 1 in $C \cap S$, where C is a cylinder of small radius centred at the origin.

(3) Finally we solve explicitly (4.2) subject to (4.1) when f is 1 near ∂S and in $C \cap S$.

Step 1. Consider the system

$$\begin{cases} \det \nabla v(x, y, z) = \tilde{f}(x, y, z) & \text{in } S, \\ v = \text{id} & \text{on } \partial S, \end{cases} \quad (4.3)$$

where $\tilde{f} > 0$ is a C^1 function such that

$$\tilde{f}(x, y, z) = \begin{cases} f(x, y, z) & \text{for } (x, y, z) \text{ near } \partial S, \\ 1 & \text{for } (x, y, z) \text{ near the origin} \end{cases} \quad (4.4)$$

and

$$\int_0^1 3r^2 \tilde{f}(r \cos \theta \sin \phi, r \sin \theta \sin \phi, r \cos \phi) dr = 1. \quad (4.5)$$

Such an \tilde{f} can be constructed, for example, in the following way: for $0 < r_0 < R_0 < 1$ define a C^2 function g such that

$$g(r, \theta, \phi) = \begin{cases} r^3 & \text{if } 0 \leq r \leq r_0, \\ 1 - \int_r^1 3t^2 f(t, \theta, \phi) dt & \text{if } R_0 \leq r \leq 1. \end{cases}$$

in which case

$$\frac{\partial}{\partial r} g(r, \theta, \phi) = 3r^2 \tilde{f}(r, \theta, \phi)$$

and observe that

$$\int_0^1 3r^2 \tilde{f}(r, \theta, \phi) dr = g(1, \theta, \phi) - g(0, \theta, \phi) = 1.$$

In order to solve (4.3) we look for solutions that preserve the radii, i.e.,

$$(x, y, z) \mapsto v(x, y, z) = (x\lambda(x, y, z), y\lambda(x, y, z), z\lambda(x, y, z)).$$

Then

$$\det \nabla v = \lambda^3 + \lambda^2 \left[x \frac{\partial \lambda}{\partial x} + y \frac{\partial \lambda}{\partial y} + z \frac{\partial \lambda}{\partial z} \right]. \quad (4.6)$$

Using polar coordinates

$$x = r \cos \theta \sin \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \phi,$$

we get

$$\det \nabla v = \lambda^3 + r\lambda^2 \frac{\partial \lambda}{\partial r}. \quad (4.7)$$

Then

$$r^3 \lambda^3(x, y, z) = 1 - \int_r^1 3\varrho^2 \tilde{f}(\varrho \cos \theta \sin \phi, \varrho \sin \theta \sin \phi, \varrho \cos \phi) d\varrho \quad (4.8)$$

is a solution of (4.3). Since $\tilde{f} > 0$ in \bar{S} , we can invert $v(x)$ to get $x(v)$ (this is of course possible locally and it can be done on the whole of $v(S)$ by using either degree theory or a theorem of MEISTERS & OLECH [7]). Hence, if we set $u = w \circ v$,

the system (4.2) becomes

$$\begin{cases} \det \nabla w(v) = \frac{f(v)}{\tilde{f}(v)} & \text{in } v(S) \\ w = id & \text{on } \partial v(S) = \partial S. \end{cases} \tag{4.9}$$

Thus there is no loss of generality (since \tilde{f} satisfies (4.4)) if we suppose that f in (4.2) is 1 in a neighborhood of ∂S . Observe also that the compatibility condition (4.1) is preserved in (4.9).

Step 2. Our aim now is to prove that there is no loss of generality if we take f to be 1 not only in a neighborhood of ∂S (Step-1) but also in $C \cap S$, where C is a cylinder of small radius centred at the origin. We consider the following transformation conserving the radii of the cylinder:

$$(x, y, z) \mapsto v(x, y, z) = (x\lambda(x, y, z), y\lambda(x, y, z), z).$$

Then

$$\det \nabla v = \lambda^2 + \lambda \left[x \frac{\partial \lambda}{\partial x} + y \frac{\partial \lambda}{\partial y} \right]. \tag{4.10}$$

In cylindrical coordinates

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = z.$$

Equation (4.10) becomes

$$\det \nabla v = \lambda^2 + r\lambda \frac{\partial \lambda}{\partial r}. \tag{4.11}$$

Now, as in Step 1, we can construct a new function $\tilde{f} \in C^1(\bar{S})$, $\tilde{f} > 0$ such that

$$\tilde{f}(x, y, z) = \begin{cases} f(x, y, z) & \text{if } x^2 + y^2 \text{ is near } 0, \\ 1 & \text{if } x^2 + y^2 + z^2 \text{ is near } 1 \end{cases} \tag{4.12}$$

and

$$\int_0^{\sqrt{1-z^2}} 2t\tilde{f}(t \cos \theta, t \sin \theta, z) dt = 1 - z^2 \quad \text{for all } 0 \leq \theta \leq 2\pi \tag{4.13}$$

and $0 \leq z \leq 1$.

Then the problem

$$\begin{cases} \det \nabla v = \tilde{f} & \text{in } S, \\ v = id & \text{on } \partial S \end{cases} \tag{4.14}$$

has the solution

$$r^2 \lambda^2 = \int_0^r 2t\tilde{f}(t \cos \theta, t \sin \theta, z) dt \quad \text{when } v = (x\lambda, y\lambda, z). \tag{4.15}$$

As in Step 1, by taking $u = w \circ v$, we may suppose that f is 1 near ∂S and in $C \cap S$ where C is a cylinder of small radius centred at the origin. (Observe that (4.1) is preserved.)

Step 3. We are now led to solve

$$\begin{cases} \det \nabla u = f & \text{in } S, \\ u = \text{id} & \text{on } \partial S \end{cases} \quad (4.2)$$

where $f \in C^1(\bar{S})$ is 1 near ∂S and in $C \cap S$, with C a cylinder of small radius centred at the origin and f satisfying

$$\int_S f(x, y, z) \, dx \, dy \, dz = \frac{4}{3} \pi. \quad (4.1)$$

To solve this problem we consider a transformation

$$\begin{aligned} (x, y, z) &\mapsto u(x, y, z) \\ &= (F(\sqrt{x^2 + y^2 + z^2}) \cos G(x, y, z) \sin H(x, y, z), F \sin G \sin H, F \cos H) \end{aligned}$$

under the following assumptions:

$$G(x, y, z) = g(r, \theta, \phi), \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi \quad (4.16)$$

and

$$g(r, 0, \phi) = 0, \quad g(r, 2\pi, \phi) = 2\pi,$$

$$H(x, y, z) = h(r, \phi) \quad \left(\text{so } \frac{\partial h}{\partial \theta} \equiv 0 \right) \quad (4.17)$$

and

$$h(r, 0) = 0, \quad h(r, \pi) = \pi.$$

Slices of the sphere normal to the z -axis are invariant under this transformation.

The problem therefore becomes:

$$\begin{cases} \det \nabla u = \frac{F'(r) F^2(r)}{r^2 \sin \phi} \frac{\partial g}{\partial \theta} \sin h \frac{\partial h}{\partial \phi} = f & \text{in } 0 \leq r < 1, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi \\ F(1) = 1, \quad g(1, \theta, \phi) = \theta, \quad h(1, \phi) = \phi. \end{cases} \quad (4.18)$$

We then want to construct C^1 functions F , g and h satisfying (4.18), (4.16) and (4.17).

The first equation in (4.18) gives

$$F'(r) F^2(r) \frac{\partial g}{\partial \theta} \sin h \frac{\partial h}{\partial \phi} = fr^2 \sin \phi. \quad (4.19)$$

Integrating (4.19) first with respect to θ from 0 to 2π , then with respect to ϕ from 0 to π , we obtain

$$F'(r) F^2(r) = \frac{r^2}{4\pi} \int_0^\pi \int_0^{2\pi} \sin \phi f(r \cos \theta \sin \phi, r \sin \theta \sin \phi, r \cos \phi) \, d\phi \, d\theta. \quad (4.20)$$

Finally we integrate with respect to r :

$$\begin{aligned}
 F^3(r) &= \frac{3}{4\pi} \int_0^r \int_0^\pi \int_0^{2\pi} t^2 \sin \phi f(t \cos \theta \sin \phi, t \sin \theta \sin \phi, t \cos \theta) dt d\phi d\theta \\
 &= \frac{3}{4\pi} \iiint_{x^2+y^2+z^2 \leq r^2} f(x, y, z) dx dy dz.
 \end{aligned}
 \tag{4.21}$$

Using the compatibility condition (4.1) we get

$$F(1) = 1 \quad \text{and} \quad F \in C^1([0, 1])
 \tag{4.22}$$

since $f \in C^1(\bar{S})$.

Now we can determine h (and hence H). Return to (4.19) and integrate only with respect to θ . Using (4.20), we get

$$\sin h \frac{\partial h}{\partial \phi} = \frac{2 \sin \phi \int_0^{2\pi} f(r \cos \theta \sin \phi, r \sin \theta \sin \phi, r \cos \phi) d\theta}{\int_0^\pi \int_0^{2\pi} \sin \phi f d\phi d\theta}.
 \tag{4.23}$$

Integrate now with respect to ϕ to obtain

$$\cos h = 1 - 2 \frac{\int_0^\phi \int_0^{2\pi} \sin \psi f(r \cos \theta \sin \psi, r \sin \theta \sin \psi, r \cos \psi) d\psi d\theta}{\int_0^\pi \int_0^{2\pi} \sin \phi f d\phi d\theta}.
 \tag{4.24}$$

Since f is 1 near ∂S we conclude that

$$h(1, \phi) = \phi \quad \text{and} \quad H \in C^1(\bar{S}).
 \tag{4.25}$$

Hence it only remains to find g such that $g(1, \theta, \phi) = \theta$. To this end, using (4.19), (4.20) and (4.23), we obtain

$$\frac{\partial g}{\partial \theta} = 2\pi \frac{f(r \cos \theta \sin \phi, r \sin \theta \sin \phi, r \cos \phi)}{\int_0^\pi \int_0^{2\pi} f(r \cos \theta \sin \phi, r \sin \theta \sin \phi, r \cos \phi) d\theta}.
 \tag{4.26}$$

Then by integrating (4.26) we deduce

$$g(r, \theta, \phi) = 2\pi \frac{\int_0^\theta f(r \cos \lambda \sin \phi, r \sin \lambda \sin \phi, r \cos \phi) d\lambda}{\int_0^\pi \int_0^{2\pi} f(r \cos \theta \sin \phi, r \sin \theta \sin \phi, r \cos \phi) d\theta}.
 \tag{4.27}$$

and since $f = 1$ near ∂S we get

$$g(1, \theta, \phi) = \theta \quad \text{and} \quad G \in C^1(\bar{S}).
 \tag{4.28}$$

This concludes the proof. \square

Remarks. (1) Lemma 11 implies that if u_0 is a diffeomorphism satisfying

$$(*) \quad \int_S \det \nabla u_0(x) = \text{vol}(S)$$

then there exists a volume preserving diffeomorphism u with $u = u_0$ on ∂S . This shows that when applying the direct method of the calculus of variations to incompressible elasticity, condition (*) ensures that the set of admissible functions is nonempty, thus answering a question raised in BALL [2], for the case when S is diffeomorphic to a sphere.

(2) The actual construction of Lemma 11 suggests that in general there is no unique solution of the system (4.2).

With the help of the above result we are now able to prove an existence theorem.

Consider the problems:

$$(P) \quad \inf \left\{ \int_{\bar{\Omega}} f(\det \nabla u(x)) \, dx : u \in (C^1(\bar{\Omega}))^3; u = u_0 \text{ on } \partial\Omega \right\}$$

and

$$(P^{**}) \quad \inf \left\{ \int_{\bar{\Omega}} f^{**}(\det \nabla u(x)) \, dx : u \in (C^1(\bar{\Omega}))^3; u = u_0 \text{ on } \partial\Omega \right\}$$

where

(Ha) Ω is a bounded open set of \mathbb{R}^3 , diffeomorphic to the unit sphere,

(Hb) $f: \mathbb{R}_+ \rightarrow \mathbb{R}$ is continuous, and there exist $a, c \in \mathbb{R}$ $b \geq d > 0$, $1 < \alpha < \infty$ such that

$$c + d |t|^\alpha \leq f(t) \leq a + b |t|^\alpha,$$

(Hc) $u_0 \in (C^2(\bar{\Omega}))^3$ is a homeomorphism with $\det \nabla u_0(x) > 0$ for all $x \in \bar{\Omega}$.

Theorem 12. *Under the above hypotheses*

$$\inf (P) = \inf (P^{**})$$

and (P^{**}) possesses a solution \bar{u} with $\det \nabla \bar{u} > 0$ in $\bar{\Omega}$ (by a solution \bar{u} we mean a C^1 minimizer of (P^{**})).

Furthermore if \bar{u} is a solution of (P^{**}), then there exists a minimizing sequence $\{u_n\}$ of (P) with $\det \nabla u_n > 0$ in $\bar{\Omega}$ such that

$$\begin{aligned} u_n &\rightarrow \bar{u} \text{ in } (L^\infty(\Omega))^3, \\ \det \nabla u_n &\rightarrow \det \nabla \bar{u} \text{ in } \sigma(L^\alpha, L^\alpha). \end{aligned}$$

Proof. Let us first prove that (P^{**}) possesses a solution. For this purpose observe that, since hypotheses (Ha) and (Hc) hold, the system

$$\begin{cases} \det \nabla u(x) = (\text{vol}(\Omega))^{-1} \int_{\bar{\Omega}} \det \nabla u(x) \, dx & \text{in } \Omega, \\ u = u_0 & \text{on } \partial\Omega \end{cases} \quad (4.31)$$

has a solution by Lemma 11. Since f^{**} is convex it follows from Jensen's inequality (see Lemma 1.8.2 in MORREY [8]) that

$$f^{**} \left((\text{vol}(\Omega))^{-1} \int_{\Omega} \det \nabla u_0(x) \, dx \right) \leq (\text{vol}(\Omega))^{-1} \int_{\Omega} f(\det \nabla v(x)) \, dx \quad (4.32)$$

for all v such that $v = u_0$ on $\partial\Omega$. Since \bar{u} is a solution of (4.31) we deduce that

$$f^{**}(\det \nabla \bar{u}) \leq (\text{vol}(\Omega))^{-1} \int_{\Omega} f^{**}(\det \nabla v(x)) \, dx \quad (4.33)$$

and hence

$$\int_{\Omega} f^{**}(\det \nabla \bar{u}(x)) \, dx \leq \int_{\Omega} f^{**}(\det \nabla v(x)) \, dx \quad (4.34)$$

for all v such that $v = u_0$ on $\partial\Omega$. Thus \bar{u} is a solution of (P**).

Finally extend f on the whole of \mathbb{R} by $f(0) = f^{**}(0)$; then taking into account Remark 4 following Theorem 10, we get

$$\inf(P) = \inf(P^{**}).$$

It now only remains to show that in Theorem 10 we may select u_n to be in $(C^1(\bar{\Omega}))^3$ with $\det \nabla u_n(x) > 0$ for all x in $\bar{\Omega}$. Here we only sketch the proof; for more details see [13].

Observe first that since $\bar{u} \in (C^1(\bar{\Omega}))^3$ with $\det \nabla \bar{u}(x) > 0$ for all x in $\bar{\Omega}$, we have by Remark 4 (following Theorem 10) that u_n can be so selected that

- (i) $\det \nabla u_n > 0$ a.e. in Ω ,
- (ii) $\text{adj}(\nabla u_n(x) - \nabla u_n(y)) \equiv 0$ a.e., whenever $|x - y| \leq \delta$ (where $\delta > 0$ is small enough).

We then mollify u_n (see Chapter II of [1]), i.e. define

$$v_{n,\varepsilon}(x) = \int_{\mathbb{R}^3} J_{\varepsilon}(x - y) u_n(y) \, dy. \quad (4.35)$$

Then

$$v_{n,\varepsilon} \in (C^{\infty}(\bar{\Omega}))^3$$

$$v_{n,\varepsilon} \rightarrow u_n \text{ in } (L^{\infty}(\Omega))^3, \quad (4.36)$$

$$\nabla v_{n,\varepsilon} \rightarrow \nabla u_n \text{ in } (L^p(\Omega))^9, \quad p \geq 1. \quad (4.37)$$

Observe that

$$\text{adj} \nabla v_{n,\varepsilon}(x) = \int_{\mathbb{R}^3} J_{\varepsilon}(x - y) \text{adj} \nabla u_n(y) \, dy$$

$$- \frac{1}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} J_{\varepsilon}(x - y) J_{\varepsilon}(x - z) \text{adj}(\nabla u_n(y) - \nabla u_n(z)) \, dy \, dz. \quad (4.38)$$

Choosing $\varepsilon < \delta$ and using (ii), we get

$$\text{adj} \nabla v_{n,\varepsilon}(x) = \int_{\mathbb{R}^3} J_{\varepsilon}(x - y) \text{adj} \nabla u_n(y) \, dy. \quad (4.39)$$

Similarly, as in (4.38) and (4.39), we deduce

$$\det \nabla v_{n,\varepsilon}(x) = \int_{\mathbb{R}^3} J_{\varepsilon}(x - y) \det \nabla u_n(y) \, dy. \quad (4.40)$$

Hence $\det \nabla v_{n,\varepsilon}(x) > 0$ for all $x \in \bar{\Omega}$ and this concludes the proof. \square

Remarks. (1) Obviously in general one cannot expect problem (P**) to possess a unique solution.

(2) Lemma 11 and Theorem 12 show that the regularity of solutions depends on the regularity of u_0 .

V. Application

We are now going to apply the preceding theorem to a problem in gas equilibrium. As mentioned in the introduction we use material (Lagrangian) coordinates.

Let a gas occupy the volume Ω in a given reference configuration. In a deformed configuration the particle $x \in \Omega$ occupies the position $u(x) \in \mathbb{R}^3$. We suppose, further, that the deformation is specified on $\partial\Omega$ (namely $u = u_0$ on $\partial\Omega$). The energy of the deformation is then

$$E(u) = \int_{\Omega} \int_1^{\det \nabla u(x)} P(V) dV dx \tag{5.1}$$

where P is the pressure and V the specific volume. The relation $P = P(V)$ between P and V at constant temperature or entropy is given by an equation of state. In what follows one only need to know that $P \in C((0, \infty))$ and satisfies

$$c + \lambda t^\alpha \leq P\left(\frac{1}{t}\right) \leq d + \mu t^\alpha \quad \text{for all } t > 0, \tag{5.2}$$

for some $\alpha > 1$, $c, d, \lambda, \mu \in \mathbb{R}_+$. This requirement is fulfilled by the equation of state of most gases.

As an example consider the Van der Waals equation which, at constant temperature, has the form

$$P = \frac{nRT}{V - nb} - \frac{an^2}{V^2} \tag{5.3}$$

while at constant entropy it is given by (as a first approximation)

$$P = \frac{nRT_0}{(V - nb)^{\frac{5}{3}}} - \frac{an^2}{V^2} \tag{5.4}$$

where T denotes the temperature (in (5.4) T_0 is the temperature corresponding to a fixed specific volume V_0), n is the number of moles, a, b and R are constants.

For P given by the equations (5.3) and (5.4) to satisfy the growth condition (5.2) we need to take $nb = 0$, but this is only a matter of convenience (the case $nb \neq 0$ can be treated similarly). For more details on the equations of state or about the different constants in (5.3) and (5.4) the reader is referred to any book on thermodynamics, in particular to LANDAU, AKHIEZER & LIFSHITZ [6] or to CALLEN [3].

If we set

$$e(t) = \int_1^t P(V) dV, \tag{5.5}$$

(5.1) becomes

$$E(u) = \int_{\Omega} e(\det \nabla u(x)) dx \tag{5.6}$$

and our problem can be formulated as

$$(P) \quad \inf \{E(u): u \in (C^1(\bar{\Omega}))^3; \quad u = u_0 \text{ on } \partial\Omega\}.$$

We also let

$$(P^{**}) \quad \inf \left\{ \int_{\bar{\Omega}} e^{**}(\det \nabla u(x)) \, dx : u \in (C^1(\bar{\Omega}))^3; \quad u = u_0 \text{ on } \partial\Omega \right\}$$

and state the following

Theorem 13. *Let Ω be diffeomorphic to a sphere. Let $u_0 \in (C^2(\bar{\Omega}))^3$ be a homeomorphism. Then*

$$\inf(P) = \inf(P^{**})$$

and (P^{**}) possesses a solution \bar{u} with $\det \nabla \bar{u}(x) > 0$ for all $x \in \bar{\Omega}$.

Moreover, if \bar{u} is a solution of (P^{**}) with $\det \nabla \bar{u}(x) > 0$, then there exists $\{u_n\}$ a minimizing sequence of (P) with $\det \nabla u_n(x) > 0$ for all $x \in \bar{\Omega}$, such that

$$\begin{cases} u_n \rightarrow \bar{u} & \text{in } (L^\infty(\Omega))^3, \\ \det \nabla u_n \rightarrow \det \nabla \bar{u} & \text{in } \sigma(L^\alpha, L^{\alpha'}) \end{cases}$$

where α is defined by (5.2) and $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$.

Proof. Observe first that we cannot apply Theorem 12 since e does not satisfy the required growth condition. The trick for removing the difficulty is to observe that if u is a homeomorphism with $\det \nabla u(x) > 0$ in $\bar{\Omega}$, then

$$\int_{\bar{\Omega}} e(\det \nabla u(x)) \, dx = \int_{u_0(\Omega)} e\left(\frac{1}{\det \nabla x(u)}\right) \det \nabla x(u) \, du, \tag{5.7}$$

and that if we let

$$f(t) = e\left(\frac{1}{t}\right) t, \quad \text{for } t > 0, \tag{5.8}$$

then f satisfies the required growth condition. Observe also that we have, trivially, that

$$f^{**}(t) = e^{**}\left(\frac{1}{t}\right) t. \tag{5.9}$$

We can now apply Theorem 12 to

$$(P) \quad \inf \left\{ \int_{u_0(\Omega)} f(\det \nabla x(u)) \, du : x \in (C^1(u_0(\Omega)))^3; \quad x = u_0^{-1} \text{ on } \partial u_0(\Omega) \right\}$$

and to

$$(P^{**}) \quad \inf \left\{ \int_{u_0(\Omega)} f^{**}(\det \nabla x(u)) \, du : x \in (C^1(u_0(\Omega)))^3; \quad x = u_0^{-1} \text{ on } \partial u_0(\Omega) \right\}$$

to infer that

$$\inf(P) = \inf(P^{**})$$

and to get a solution \bar{x} with $\det \nabla \bar{x}(u) > 0$ for all $u \in u_0(\Omega)$. Finally, we invert \bar{x} to get \bar{u} (as in Lemma 11, by using degree theory or the theorem of MEISTERS & OLECH [7]).

The second part of the proof is then a direct consequence of Theorem 12 (inverting also the minimizing sequences). \square

Thus one can say that in the set of deformations u with prescribed boundary value u_0 , there is a minimizing sequence $\{u_n\}$ of the energy E which is such that

$$u_n \rightarrow \bar{u} \text{ in } L^\infty, \\ \det \nabla u_n \rightarrow \det \nabla \bar{u} \text{ in } \sigma(L^\alpha, L^{\alpha'}).$$

Therefore, \bar{u} can be considered as a "generalized equilibrium solution" of the problem (P). As mentioned earlier, one cannot expect, in general, \bar{u} (and hence the sequence $\{u_n\}$) to be unique.

Note added in proof: I have recently extended Theorem 10 to include more general integrands than those considered here. This result will appear in the Journal of Functional Analysis.

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