EXISTENCE AND REGULARITY OF DIFFEOMORPHISM
WITH PRESCRIBED JACOBIAN

Bernard DACOROGNA
Département de Mathématiques
Ecole Polytechnique Fédérale de Lausanne
1015 LAUSANNE, Switzerland

I. INTRODUCTION:

In this article we are concerned with the following problem, find a diffeomorphism $u\colon \bar{\Omega} \to \bar{\Omega}$, where $\Omega \subset \mathbb{R}^n$ is a bounded open set, satisfying

$$\begin{cases} g(u(x))\det\nabla u(x) = f(x) & x \in \Omega \\ u(x) = x & x \in \partial\Omega \end{cases} \tag{1.1}$$

where $f,g > 0$ in $\bar{\Omega}$ and satisfy the compatibility condition

$$\int_\Omega g(x)dx = \int_\Omega f(x)dx. \tag{1.2}$$

In geometrical terms the problem is to show that two volume elements of the some total volume (i.e.(1.2)) are diffeomorphically equivalent.

Let us state precisely the result.

Theorem 1:

Let $k \geq 0$ be an integer, $0 < \alpha < 1$ and $\Omega \subset \mathbb{R}^n$ a bounded open set with $C^{k+3,\alpha}$ boundary $\partial\Omega$ ($C^{k,\alpha}$ denoting the usual Hölder space). Let $f,g \in C^{k,\alpha}(\bar{\Omega})$, $f,g > 0$ in $\bar{\Omega}$ satisfying (1.2). Then there exists $u \in \mathrm{Diff}^{k+1,\alpha}(\bar{\Omega})$ (the set of diffeomorphisms $u\colon \bar{\Omega} \to \bar{\Omega}$ with $u,u^{-1} \in C^{k+1,\alpha}(\bar{\Omega},\bar{\Omega})$) satisfying (1.1).

<u>Remark</u>:

It is clear that (1.1) is underdetermined and uniqueness of solutions does not hold. Indeed let $f = g = 1$, N an integer then $u(x) = r(\cos(\theta+2N\pi r^2), \sin(\theta+2N\pi r^2))$ satisfy (1.1) when Ω is the unit disk of $\mathbb{R}^2$ and (r,θ) are the polar coordinates.

The above result has been obtained by Dacorogna-Moser [1]. The first result concerning (1.1) was established by Moser [1] for manifolds without boundary and for $k = \infty$. It was then extended by Banyaga [1] for manifolds with boundary and for $k = \infty$. Zehnder [1] obtained Theorem 1 for manifolds without boundary and with a smallness assumption on the $C^{0,\alpha}$ norm of $(f-g)$. Independently Tartar [1] and Dacorogna [1] proved this result for $k = \infty$ and Ω diffeomorphic to the unit ball of $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively.

In Dacorogna-Moser [1] it is also shown how to weaken the regularity of the boundary $\partial\Omega$, so as to include Lipschitz boundaries as well as isolated boundary points (as in a punctured disk)... In this article it is shown that if $k > 0$, $f,g \in C^k(\Omega)$ ($f,g > 0$ in $\bar{\Omega}$, $f + \frac{1}{f}$ and $g + \frac{1}{g}$ bounded) and satisfy (1.2), then there exists a diffeomorphism $u \in \mathrm{Diff}^k(\Omega) \cap \mathrm{Diff}^o(\bar{\Omega})$ (where $\mathrm{Diff}^o(\bar{\Omega})$ stands for the set of homemorphisms $u: \bar{\Omega} \to \bar{\Omega}$ with $u,u^{-1} \in C^o(\bar{\Omega};\bar{\Omega})$). If $k = 0$ and f,g are as above, it is proved that there exists $u \in \mathrm{Diff}^o(\bar{\Omega})$ satisfying a weak form of (1.1) namely

$$\int_E f(x)dx = \int_{u(E)} g(x)dx \tag{1.3}$$

for every open set $E \subset \Omega$ and $u(x) = x$ on $\partial\Omega$. Note that this result is weaker, from the point of view of regularity, than Theorem 1, since the expected gain in regularity is not ensured.

Theorem 1 has several applications. The first one is that we can construct a volume preserving diffeomorphism with given boundary data. More precisely let $\psi_o \in \mathrm{Diff}^{k,\alpha}(\bar{\Omega})$ with $k \geq 1$, then there exists $\psi \in \mathrm{Diff}^{k,\alpha}(\bar{\Omega})$ satisfying

$$\begin{cases} \det\nabla\psi \equiv 1 & \text{in } \Omega \\ \psi = \psi_o & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

With the above hypotheses (1.4) is equivalent to (1.1) and the existence is therefore deduced from Theorem 1.

The construction of volume preserving diffeomorphism plays a role in ergodic theory (c.f. Alpern [1], Anosov-Katok [1])and is important in incompressible elasticity (c.f. Ball [1], Dacorogna [2]). It can also be used in the calculus of variations and in problems of equilibrium of gases (c.f. Dacorogna [1],[2]). For example if one looks for solutions of

$$(P) \qquad \inf\{\int_\Omega \phi(\det\nabla u(x))dx: u = u_o \text{ on } \partial\Omega,\ u \in \mathrm{Diff}^{k,\alpha}(\bar{\Omega})\}$$

where $\phi: \mathbb{R} \to \mathbb{R}$ is convex and $u_o \in \mathrm{Diff}^{k,\alpha}(\bar{\Omega})$ with $k \geqq 1$. Indeed it follows from Jensen inequality that

$$\int_\Omega \phi(\det\nabla u(x))dx$$

$$\geqq \mathrm{meas}\Omega\ \phi(\frac{1}{\mathrm{meas}\Omega}\int_\Omega \det\nabla u(x)dx) = \mathrm{meas}\Omega\ \phi(\frac{1}{\mathrm{meas}\Omega}\int_\Omega \det\nabla u_o(x)dx)$$

$$\mathrm{meas}\Omega\ \phi(1);$$

the last indentity following from the fact that $u_o(\Omega) = \Omega$. Therefore using the fact that there exists $\bar{u} \in \mathrm{Diff}^{k,\alpha}(\bar{\Omega})$ satisfying (1.4) we find, because of the above inequality, that $\bar{u}$ is actually a minimum of (P).

Before concluding with some open problems, let us mention an interesting result for the corresponding linear problem, which will be used for obtaining Theorem 1.

Theorem 2:

Let $k \geqq 0$ be an integer, $0 < \alpha < 1$ and $\Omega \subset \mathbb{R}^n$ a bounded open set with $C^{k+3,\alpha}$ boundary $\partial\Omega$. Let $f \in C^{k,\alpha}(\bar{\Omega})$ with

$$\int_\Omega f(x)dx = 0. \tag{1.5}$$

Then there exists $u \in C^{k+1,\alpha}(\bar{\Omega};\mathbb{R}^n)$ satisfying

$$\begin{cases} \operatorname{div} u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega . \end{cases} \qquad (1.6)$$

Remark:

Several authors have proved Theorem 2 under some slightly different conditions, however as stated above it has been established by Dacorogna-Moser [1].

We now state three open problems

Problem 1:

Let Ω be C^∞ smooth, $k \geq 0$, $f \in C^k(\bar{\Omega})$, $f > 0$ in $\bar{\Omega}$ satisfying (1.2) with $g \equiv 1$. Does there exist $u \in \mathrm{Diff}^{k+1}(\bar{\Omega})$ satisfying (1.1)?

Problem 2:

Let Ω be C^∞ smooth, $k \geq 1$, $f \in C^k(\bar{\Omega})$ satisfying (1.5). Does there exist $u \in C^{k+1}(\bar{\Omega};\mathbb{R}^n)$ satisfying (1.6)?

Problem 3:

Let Ω be C^∞-smooth, $k \geq 0$, $p > 1$, $f \in W^{k,p}(\Omega)$ (the usual Sobolev space), $f(x) \geq C_0 > 0$ for almost all $x \in \Omega$ and f satisfying (1.2) with $g \equiv 1$. Does there exist $u \in W^{k+1,np}(\Omega)$ satisfying (1.1)?

Remarks

i) Note that C^k as well as $C^{k,\alpha}$ spaces have the advantage over L^p spaces to form an algebra. In that sense one has, when working with Sobolev spaces with low k and p, to ask for solution in $W^{k+1,np}$ instead of $W^{k+1,p}$ (this last case can be treated by the method developed below).

ii) The third conjecture is false when $k = 0$ and $p = 1$. Indeed Müller [1] and Coifman-Lions-Meyer-Semmes [1] have shown that given any $u \in W^{1,n}(\Omega)$ with $\det\nabla u \geq 0$ a.e. then $\det\nabla u \log(2+\det\nabla u) \in L^1_{loc}(\Omega)$.

II. PROOF OF THEOREM 1 AND 2

We now sketch the proofs of the theorems and we refer for more details to Dacorogna-Moser [1]. We first assume that Theorem 2 has been proved.

Proof of Theorem 1:

The proof is divided in four steps.

Step 1: We first show that there is no loss of generality in assuming $g \equiv 1$ in (1.1). Indeed writing

$$\begin{cases} \det\nabla\phi(x) = \dfrac{\text{meas}\,\Omega}{\int_\Omega f(x)dx} f(x) & \text{in } \Omega \\ \det\nabla\psi(x) = \dfrac{\text{meas}\,\Omega}{\int_\Omega g(x)dx} g(x) & \text{in } \Omega \\ \phi(x) = \psi(x) = x & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

we have that $u = \psi^{-1} \circ \phi$ satisfies (1.1).

Step 2: We then obtain Theorem 1 without the regularity result, i.e. we find $u \in \text{Diff}^{k,\alpha}(\bar{\Omega})$ (instead of $\text{Diff}^{k+1,\alpha}$) satisfying (1.1) with $g \equiv 1$. This is obtained by a deformation argument explained below.

Step 3: Using Theorem 2 and a smallness assumption on the $C^{0,\beta}$ norm $0 < \beta < \alpha < 1$ of $f-1$ we obtain by a fixed point argument a $\mathrm{Diff}^{k+1,\alpha}$ solution of (1.1).

Step 4: We remove the smallness assumption in Step 3 by combining two diffeomorphisms one constructed via Step 2 the other one via Step 3.

We now prove Step 2.

Step 2: Let $v \in C^{k+1,\alpha}(\bar{\Omega};\mathbb{R}^n)$ satisfying

$$\begin{cases} \operatorname{div} v = f-1 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

Such a v exists by Theorem 2 and the fact that $g \equiv 1$ in (1.2) i.e.

$$\int_\Omega (f-1)dx = 0.$$

Assume that $k \geqq 1$. Define then a vector field

$$v(t,z) = \frac{v(z)}{t+(1-t)f(z)} \cdot \tag{2.3}$$

Observe that $v(t,\cdot) \in C^{k,\alpha}$ (and not in $C^{k+1,\alpha}$). We then define the flow associated to this vector field as

$$\begin{cases} \frac{\partial}{\partial t}\Phi(t,x) = v(t,\Phi(t,x)) & t > 0 \\ \Phi(0,x) = x. \end{cases} \tag{2.4}$$

Note that because of the assumptions there exists a unique solution Φ: $[0,1] \times \bar{\Omega} \to \mathbb{R}^n$ in $\mathrm{Diff}^{k,\alpha}$ satisfying (2.4). We claim that $u(x) = \Phi(1,x)$ has all the desired properties, i.e., $u \in \mathrm{Diff}^{k,\alpha}(\bar{\Omega})$ and

$$\begin{aligned} \det\nabla u(x) &= f(x), & x \in \Omega \\ u(x) &= x & x \in \partial\Omega . \end{aligned} \tag{2.5}$$

(The aim of the next two steps will be to obtain $u \in \mathrm{Diff}^{k+1,\alpha}$ instead of $\mathrm{Diff}^{k,\alpha}$).

First observe that the fact that $u(x) = x$ on $\partial\Omega$ follows at once from the unicity of solutions of (2.4) and of the fact that $v(t,x) = 0$ if $x \in \partial\Omega$ (since $v = 0$ on $\partial\Omega$, c,f, (2.2)). It therefore remains to show that $\det \nabla u = f$ in Ω. This can be deduced from the fact that the following quantity

$$h(t,x) \equiv \det \nabla_x \Phi(t,x)[t+(1-t)f(\Phi(t,x))] \qquad (2.6)$$

is independent of t (c.f. below for a proof). The result follows then from the fact that $h(1,x) = h(0,x)$. To prove that (2.6) is independent of t is obtained from (2.2) and the fact that the solution of (2.4) satisfies (as well known)

$$\frac{\partial}{\partial t}[\det \nabla_x \Phi(t,x)] = \det \nabla_x \Phi(t,x) \operatorname{div} v(t,\Phi(t,x)).$$

We refer for more details to Dacorogna-Moser [1].

Step 3: A classical fixed point argument allows to deduce from Theorem 2 the following: let $0 < \beta \leqq \alpha < 1$, then there exists $\varepsilon = \varepsilon(\alpha,\beta,k,\Omega) > 0$ such that if $\|f-1\|_{C^{0,\beta}} \leqq \varepsilon$ then there exists $u \in \mathrm{Diff}^{k+1,\alpha}(\bar{\Omega})$ satifying (2.5).

Step 4: We now are in a position to conclude the proof of the theorem. By density of C^∞ functions in $C^{k,\alpha}$ with the $C^{0,\beta}$ norm $0 < \beta < \alpha < 1$ (in general if $k = 0$ density does not occur in $C^{0,\alpha}$ norm) we can find $\tilde{f} \in C^\infty(\bar{\Omega})$, $\tilde{f} > 0$ on $\bar{\Omega}$ such that

$$\left\| \frac{f}{\tilde{f}} - 1 \right\|_{C^{0,\beta}} \leqq \varepsilon \qquad (2.7)$$

$$\int_\Omega \frac{f(x)}{\tilde{f}(x)}\, dx = \operatorname{meas} \Omega \qquad (2.8)$$

where $\varepsilon > 0$ is as in Step 3.

We then define $b \in \mathrm{Diff}^{k+1,\alpha}(\bar{\Omega})$ to be a solution (which exists by Step 3, (2.7) and (2.8)) of

$$\begin{cases} \det \nabla b(x) = \dfrac{f(x)}{\tilde{f}(x)} & \text{in } \Omega \\ b(x) = x & \text{on } \partial\Omega. \end{cases} \tag{2.9}$$

We then define $a \in \mathrm{Diff}^{k+1,\alpha}(\bar{\Omega})$ to be a solution (which exists by Step 2, since $\tilde{f} \circ b^{-1} \in C^{k+1,\alpha}$) of

$$\begin{cases} \det \nabla a(y) = \tilde{f}(b^{-1}(y)) & y \in \Omega \\ a(y) = y & y \in \partial\Omega. \end{cases} \tag{2.10}$$

Note that the compatibility condition is satisfied since

$$\int_\Omega \tilde{f}(b^{-1}(y))dy = \int_\Omega \tilde{f}(x)\det \nabla b(x)dx = \int f(x)dx = \text{meas}\,\Omega.$$

Finally we get the desired diffeomorphism by setting $u = a \circ b$. □

Before sketching the proof of Theorem 2, we introduce some notations.

Notations:

Let $w \in C^1(\mathbb{R}^n; \mathbb{R}^{n(n-1)/2})$ with $w = (w_{ij})_{1\le i<j\le n}$. We also let $w_{ij} = - w_{ji}$ if $i \ge j$. We then define

$$\mathrm{curl}^* w = ((\mathrm{curl}^* w)_j)_{1\le j\le n} \in \mathbb{R}^n$$

by

$$(\mathrm{curl}^* w)_j = \sum_{i=1}^{n} (-1)^{i+j} \frac{\partial w_{ij}}{\partial x_i}.$$

Remarks:

i) If $n = 2$ then $\mathrm{curl}^* w = \left(\dfrac{\partial w}{\partial x_2}, -\dfrac{\partial w}{\partial x_1}\right)$

and if $n = 3$ then $\operatorname{curl}^* w = \operatorname{curl} w$ (the usual curl).

ii) In terms of differential forms $(\operatorname{curl}^* w)_j$ are just the components of $d\alpha$ where α is an $(n-2)$ form with components w_{ij}.

iii) One always have

$$\operatorname{div}(\operatorname{curl}^* w) = 0 ,$$

which corresponds to $dd\alpha = 0$.

Proof of Theorem 2:

The idea for finding u satisfying

$$\begin{cases} \operatorname{div} u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.10}$$

is to seek for solutions of the form

$$u = \operatorname{grad} v + \operatorname{curl}^* w. \tag{2.11}$$

We therefore solve (ν being the outward unit normal)

$$\begin{cases} \Delta v = f & \text{in } \Omega \\ \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

which, provided $\int f = 0$, has a unique $C^{k+2,\alpha}$ solution by the classical results. In view of (2.10) and (2.11) it therefore remains to find $w \in C^{k+2,\alpha}$ satisfying

$$\operatorname{curl}^* w = - \operatorname{grad} v \quad \text{on } \partial\Omega$$

knowing that $v \in C^{k+2,\alpha}$ and

$$\frac{\partial v}{\partial \nu} = 0.$$

Therefore Theorem 2 is reduced to finding $w \in C^{k+2,\alpha}$ satisfying

$$\operatorname{curl}^* w = g \quad \text{on } \partial\Omega \tag{2.12}$$

knowing that $g \in C^{k+1,\alpha}$ with

$$\langle g; \nu \rangle = 0 \quad \text{on } \partial\Omega \tag{2.13}$$

where $\langle \cdot ; \cdot \rangle$ stands for scalar product in $\mathbb{R}^n$.

The equations (2.12) can be further reduced to solving

$$\operatorname{grad} w_{ij} = (-1)^{i+j} (g_j \nu_i - g_i \nu_j) \quad \text{on } \partial\Omega. \tag{2.14}$$

Indeed if the above holds on $\partial\Omega$ we have, recalling that $w_{ij} = -w_{ji}$,

$$(\operatorname{curl}^* w)_j = \sum_{i=1}^{n} (-1)^{i+j} \frac{\partial w_{ij}}{\partial x_i} = \sum_{i=1}^{n} (g_j \nu_i - g_i \nu_j)\nu_i =$$

$$= g_j |\nu|^2 - \langle g; \nu \rangle \nu_j = g_j$$

where we have used (2.13) as well as the fact that $|\nu| = 1$.

We therefore have reduced the problem to finding $w_{ij} \in C^{k+2,\alpha}$ satisfying

$$\operatorname{grad} w_{ij} = g_{ij} \nu \quad \text{on } \partial\Omega \tag{2.15}$$

given $g_{ij} \in C^{k+1,\alpha}$ (here $g_{ij} = (-1)^{i+j} (g_j \nu_i - g_i \nu_j)$). We show how to solve (2.15) and find $w_{ij} \in C^{k+1,\alpha}$ (to obtain a more regular solution in $C^{k+2,\alpha}$, we refer to Dacorogna-Moser [1]). Observing that the gradient of the distance $d(x,\partial\Omega)$ is $-\nu$ whenever $x \in \partial\Omega$ we obtain a solution by setting

$$w_{ij} = -g_{ij}\, \phi(d(x,\partial\Omega))$$

where $\phi(0) = 0$, $\phi'(0) = 1$, $\phi \equiv 0$ outside a small neighbourhood of 0 and $\phi \in C^\infty$.

□

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REFERENCES

- S. Alpern [1]:
  New proofs that weak mixing is generic; Inventiones Math., Vol.32 (1976) 263-279.

- D.V. Anosov; A.B. Katok [1]:
  New examples in smooth ergodic theory, ergodic diffeomorphisms; Trans. Moscow Math. Soc., Vol.23 (1970) 1-35.

- J.M. Ball [1]:
  Convexity conditions and existence theorems in nonlinear elasticity; Arch. Rat. Mech. Anal., Vol.63 (1977) 337-403.

- A. Banyaga [1]:
  Formes volume sur les variétés à bord; Enseignement Math., Vol.20 (1974) 127-131.

- R. Coifman; P.L. Lions; Y. Meyer; S. Semmes [1]:
  Compacité par compensation et espaces de Hardy; C.R.A.S., Vol.309 (1989) 945-949.

- B. Dacorogna [1]:
  A relaxation theorem and its application to the equilibrium of gases; Arch. Rat. Mech. Anal., Vol.77 (1981) 359-385.

- B. Dacorogna [2]:
  "Direct methods in the calculus of variations", Springer, Berlin (1989).

- B. Dacorogna; J. Moser [1]:
  On a partial differential equation involving the Jacobian determinant; Ann. Inst. H. Poincaré, Anal. non-lin., Vol.7 (1990) 1-26.

- J. Moser [1]:
  On the volume element on a manifold; Trans. A.M.S. Vol.20 (1965) 286-294.

- S. Müller [1]:
  A surprising higher integrability property of mappings with positive determinant; Bull. A.M.S. Vol.21 (1989) 245-248.

- L. Tartar [1]:
  Private communication (1979).

- E. Zehnder [1]:
  Note on smoothing symplectic and volume preserving diffeomorphisms, Springer, Lect. Notes in Math., Vol.597 (1976) 828-855.