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Convexity of Certain Integrals of the Calculus of Variations - II

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1. – Introduction.

In this report we deal with the convexity of the integral

$$I(u) = \int_0^1 f(x, u(x), u'(x))\, dx \tag{1.1}$$

where $f: (0,1) \times \boldsymbol{R} \times \boldsymbol{R} \to \boldsymbol{R}$ is C^2. We seek conditions on f which ensure the convexity of the integral I over the space $W_0^{1,\infty}(0,1)$ (the space of Lipschitz functions vanishing at 0 and 1). We shall show the following.

Necessary condition.

If I is convex over $W_0^{1,\infty}(0,1)$, then $f(x,u,\cdot)$ is convex (i.e., f is convex with respect to the last variable).

We then study three examples

i) Let $f(x,u,\xi) = a(u)\,\xi^{2n}$, with $n \geqq 1$, an integer, then I is convex over $W_0^{1,\infty}(0,1) \Leftrightarrow a(u) = \text{constant}$.

ii) Let $f(x,u,\xi) = a(u) + b(\xi)$, then I is convex over $W_0^{1,\infty}(0,1)$

$\Leftrightarrow$ 1) $b''(\xi) \geqq 0$, for every $\xi \in \boldsymbol{R}$

2) $\pi^2 b''(\xi) + a''(u) \geqq 0$ for every $(u,\xi) \in \boldsymbol{R}^2$.

iii) Let $f(x,u,\xi) = a(x)\,\xi^2 + 2b(x)\,u\xi + c(x)\,u^2$, with $a(x) > 0$ then I is convex over $W_0^{1,\infty}(0,1)$ (or over $H_0^1(0,1)$) if and only if there exists

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$g \in C^1(0,1)$ satisfying Jacobi equation, *i.e.*

$$a(x)\,[g'(x) + c(x)] - [b(x) + g(x)]^2 = 0.$$

(This means that there is no conjugate point in $(0,1)$.)

Furthermore in the three cases (trivially in the first one) one has that the following conjecture is true.

CONJECTURE. I is convex over $W_0^{1,\infty}(0,1)$ if and only if there exists $\tilde{f}\colon (0,1) \times \boldsymbol{R} \times \boldsymbol{R} \to \boldsymbol{R}$ with $\tilde{f}(x,\cdot,\cdot)$ convex for every $x \in (0,1)$ (*i.e.* $\tilde{f}$ is convex with respect to the last two variables) satisfying

$$I(u) = \int_0^1 \tilde{f}(x, u(x), u'(x))\,dx \tag{1.2}$$

for every $u \in W_0^{1,\infty}(0,1)$ with compact support in $(0,1)$.

In the last section we discuss the higher dimensional case, *i.e.*,

$$I(u) = \int_\Omega f(x, u(x), \nabla u(x))\,dx$$

where $u\colon \Omega \subset \boldsymbol{R}^n \to \boldsymbol{R}^m$ and $f\colon \Omega \times \boldsymbol{R}^m \times \boldsymbol{R}^{nm} \to \boldsymbol{R}$. We show that if $m = 1$, the necessary condition is still true while if $m, n > 1$, both the necessary condition and the conjecture are false.

Finally one should mention a closely related problem studied by Boccardo-Dacorogna [BD] where the monotonicity of operators of the form

$$A(u) = -\big(a(x, u(x), u'(x))\big)'$$

is investigated.

2. - Main results.

We begin with the necessary condition.

THEOREM 1. Let $f\colon (0,1) \times \boldsymbol{R} \times \boldsymbol{R} \to \boldsymbol{R}$ be continuous and such that

$$|f(x,u,\xi)| \leqq a(x, |u|, |\xi|)$$

where a is increasing with respect to $|u|$ and $|\xi|$ and locally integrable in x. If I is convex over $W_0^{1,\infty}(0,1)$ then $f(x,u,\cdot)$ is convex.

PROOF. Under the above hypotheses, a direct application of Mazur lemma gives that I is in fact weak* lower semicontinuous in $W^{1,\infty}$ and thus that $f(x, u, \cdot)$ is convex. (For more details see Dacorogna [D2]). ∎

We now turn to necessary and sufficient conditions.

THEOREM 2. Let $a \in C^2(\mathbf{R})$ be such that

$$a(u) \geqq a_0 > 0, \qquad \text{for every } u \in \mathbf{R}.$$

Let

$$f(x, u, \xi) = a(u)\, \xi^{2n}$$

with $n \geqq 1$, an integer. Then I is convex over $W_0^{1,\infty}(0,1)$ if and only if a is constant.

THEOREM 3. Let $a, b \in C^2(\mathbf{R})$ and

$$f(x, u, \xi) = a(u) + b(\xi),$$

I is convex over $W_0^{1,\infty}(0,1)$ if and only if

i) $b''(\xi) \geqq 0$ for every $\xi \in \mathbf{R}$,

ii) $\pi^2 b''(\xi) + a''(u) \geqq 0$ for every $(u, \xi) \in \mathbf{R}^2$.

Furthermore let

$$a_0 = \inf\{a''(u) \colon u \in \mathbf{R}\}, \qquad b_0 = \inf\{b''(\xi) \colon \xi \in \mathbf{R}\},$$

$$g(x) = \begin{cases} 0, & \text{if } a_0 \geqq 0, \\ \dfrac{1}{2}\sqrt{-a_0 b_0}\tan\left[\sqrt{\dfrac{-a_0}{b_0}}\left(x - \dfrac{1}{2}\right)\right], & \text{if } a_0 < 0. \end{cases}$$

$$\tilde{f}(x, u, \xi) = f(x, u, \xi) + 2g(x)\, u\xi + g'(x)\, u^2 .$$

If $\pi^2 b_0 + a_0 \geqq 0$ (*i.e.* I is convex) then $\tilde{f}(x, \cdot, \cdot)$ is convex and satisfies

$$I(u) = \int_0^1 \tilde{f}(x, u(x), u'(x))\, dx$$

for every $u \in W_0^{1,\infty}(0,1)$ with compact support in $(0,1)$ (this last condition can be dropped if $\pi^2 b_0 + a_0 > 0$).

THEOREM 4. Let $a, b, c \in C([0,1])$ with $a(x) > 0$ for every $x \in [0,1]$ and

$$f(x, u, \xi) = a(x)\, \xi^2 + 2b(x)\, u\xi + c(x)\, u^2 . \tag{2.1}$$

I is convex over $W_0^{1,\infty}(0,1)$ or $H_0^1(0,1)$ if and only if there exists $g \in C^1(0,1)$ satisfying Jacobi equation, i.e.,

$$a(x)\,[g'(x) + c(x)] - [g(x) + b(x)]^2 = 0. \tag{2.2}$$

Furthermore, set

$$\tilde{f}(x,u,\xi) = f(x,u,\xi) + 2g(x)u\xi + g'(x)u^2, \tag{2.3}$$

with g satisfying (2.2). Then I convex implies that $\tilde{f}(x,\cdot,\cdot)$ is convex and that

$$I(u) = \int_0^1 \tilde{f}(x, u(x), u'(x))\,dx \tag{2.4}$$

for every $u \in W_0^{1,\infty}(0,1)$ with compact support in $(0,1)$.

REMARKS. i) Theorems 2 and 3 have been proved in [D1]. Theorem 4 can be considered as classical in the calculus of variations c.f. for example Gelfand-Fomin [GF], however it is hard to find it, in the existing litterature, as stated and proved here.

ii) Note that the condition $\pi^2 b_0 + a_0 \geqq 0$ in Theorem 3 or (2.2) in Theorem 4 ensure the existence of a global field, in the terminology of the field theories.

iii) In Theorem 4 note (c.f. the proof below) that if I is strictly convex then g satisfying (2.2) can be chosen $C^1([0,1])$, i.e. there is no conjugate point in the closed interval $[0,1]$. Similarly if I is strictly convex then (2.4) holds without the condition on the support of u.

iv) Note that in Theorem 3 as well as Theorem 4 we have

$$\tilde{f}(x,u,u') - f(x,u,u') = \frac{d}{dx}\,[g(x)u^2].$$

However, in general, there is no reason to suppose that the right hand side is always of this form. For example if one takes $f(x,u,u') = u'^4 + u^2u'^2$ and $\Phi(x,u) = \frac{1}{24}(\tan x)\,u^4$ we have that if

$$\tilde{f}(x,u,u') = f(x,u,u') + \frac{d}{dx}\,[\Phi(x,u)]$$

then

$$\tilde{f}(x,\cdot,\cdot) \text{ is convex.}$$

EXAMPLE. Let $f(x, u, \xi) = \xi^2 - \alpha u^2$ with $\alpha \geqq 0$.

i) By Theorem 3 one finds $\alpha \leqq \pi^2$, which corresponds to the best constant in Poincaré inequality.

ii) We have that $f(x) = \sqrt{\alpha} \tan \sqrt{\alpha}\left(x - \frac{1}{2}\right)$ is a solution of (2.2) i.e., $g' - \alpha - g^2 = 0$, which is globally defined in $(0,1)$ only if $\alpha \leqq \pi^2$; using Theorem 4 we also find that I is convex provided $0 \leqq \alpha \leqq \pi^2$.

PROOF OF THEOREM 4. Before proceeding with the proof, observe that the convexity of I over $W_0^{1,\infty}(0,1)$ is equivalent to $I(u) \geqq 0$ for every $u \in W_0^{1,\infty}(0,1)$. Furthermore the convexity of $\tilde{f}(x, \cdot, \cdot)$ follows from its positivity which is ensured by (2.2).

SUFFICIENCY. Let $u \in W_0^{1,\infty}(0,1)$ with compact support in $(0,1)$, then

$$I(u) = \int_0^1 \left\{ a(x) u'^2 + 2b(x) uu' + c(x) u^2 + \frac{d}{dx}[g(x) u^2] \right\} dx =$$

$$\int_0^1 \tilde{f}(x, u(x), u'(x))\, dx = \int_0^1 \left\{ a(x) u'^2 + 2(b+g) uu' + (c+g') u^2 \right\} dx \geqq 0,$$

since g satisfies (2.2).

NECESSITY. We assume without loss of generality, that the coefficients a, b, c are in $C^1([0,1])$ and that I is strictly convex. Otherwise approximating a, b, c with a_ε, b_ε, c_ε, and applying the result to

$$I_\varepsilon(u) = \int_0^1 [a_\varepsilon u'^2 + 2b_\varepsilon uu' + c_\varepsilon u^2]\, dx + \varepsilon \int_0^1 u'^2\, dx$$

we shall have the result by letting ε tend to 0. We shall now prove that there exists $f \in C^1([0,1])$ satisfying (2.2). We shall proceed by contradiction and assume the contrary. We shall then show that we can find $w \in W_0^{1,\infty}(0,1)$, $w \neq 0$ with

$$I(w) = 0. \tag{2.5}$$

(2.5) will then immediately contradict the strict convexity of I, since letting $\lambda \in (0,1)$ we should have

$$0 \leqq I(\lambda \cdot 0 + (1-\lambda) w) < \lambda I(0) + (1-\lambda) I(w) = 0.$$

We therefore assume that there is no $g \in C^1([0,1])$ satisfying (2.2). We then extend the functions a, b, c in a C^1 way from $[0,1]$ to $\boldsymbol{R}$ with $a>0$ over $\boldsymbol{R}$. Let $0 \leqq \delta \leqq 1$ and consider

$$(2.6)\qquad \begin{cases} a(x)[g'(x)+c(x)]-[g(x)+b(x)]^2=0, \\ g(\delta)=0. \end{cases}$$

By the classical existence theorem for ordinary differential equations with Cauchy data we have that there exists a maximal open interval of existence $(x_1(\delta), x_0(\delta))$ containing δ. The structure of the equation implies that $g(x_1(\delta))=-\infty$ and $g(x_0(\delta))=+\infty$. By continuity of the solutions on the initial point δ we may choose $\delta>0$ so that $x_1(\delta)=0$ (this follows from the fact that, trivially, $x_1(0)<0$ and that, by absurd assumption $x_1(1)>0$). Observe also that for this δ, $x_0(\delta)\leqq 1$, otherwise, using again the continuity on the initial point δ, we could choose $\delta>0$ so that $x_1(\delta)<0$ and $x_0(\delta)>1$, which contradicts the fact that there is no $C^1([0,1])$ solution of (2.2). We may therefore assume that $\delta>0$ has been chosen so that (2.6) admits a solution with $x_1(\delta)=0$, $x_0(\delta)=x_0\leqq 1$ and with

$$(2.7)\qquad g(0)=-\infty, \qquad g(x_0)=+\infty.$$

Looking again at the structure of the equation (2.6) we can find $\varepsilon>0$ small and $\beta=\beta(\varepsilon)>0$, $\gamma=\gamma(\varepsilon)>0$ with the properties (c.f. the fig. 1).

$$(2.8)\qquad \begin{cases} g(x)+b(x)<-\beta \text{ and } g'(x)>\gamma, & \text{for every } x\in(0,\varepsilon], \\ g(x)+b(x)>\beta \text{ and } g'(x)>\gamma, & \text{for every } x\in[x_0-\varepsilon, x_0). \end{cases}$$

We then let

$$w(x)=\begin{cases} 0, & \text{if } x=0, \\ \exp\left\{-\displaystyle\int_{\varepsilon}^{x}\frac{b(t)+g(t)}{a(t)}\,dt\right\}, & \text{if } x\in(0,x_0), \\ 0, & \text{if } x\geqq x_0, \end{cases}$$

or equivalently

$$(2.9)\qquad w'(x)=\begin{cases} -\dfrac{b(x)+g(x)}{a(x)}\,w(x), & x\in(0,x_0), \\ 0, & x\in(x_0,1). \end{cases}$$

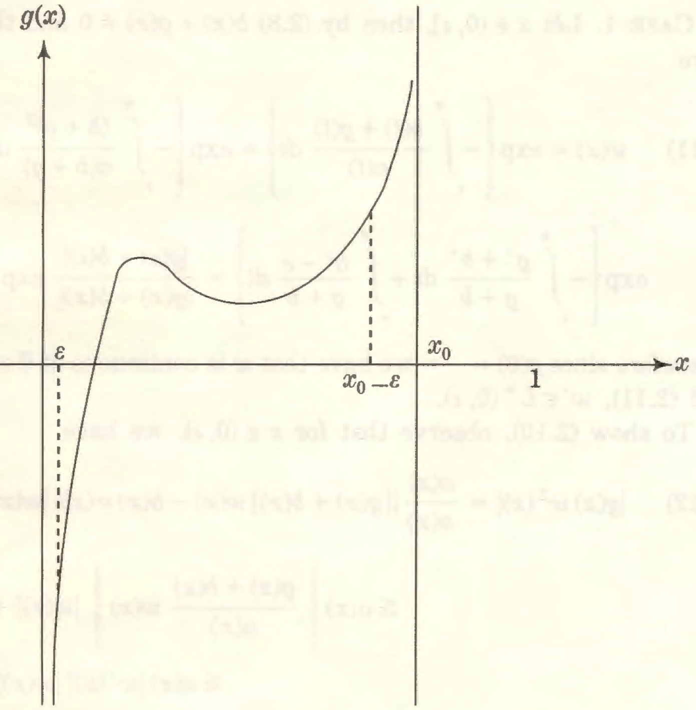

Figure 1.

We then claim that $w \in W_0^{1,\infty}(0,1)$ and satisfies (2.5) (i.e. $I(w)=0$). Observe that if we can show that

$$\lim_{x\to 0}[g(x)\,w^2(x)] = \lim_{x\to x_0}[g(x)\,w^2(x)] = 0 \tag{2.10}$$

we shall have (2.5), since using (2.6), (2.9), (2.10)

$$I(w) = \int_0^1 \{aw'^2 + 2bww' + cw^2\}\,dx =$$

$$= \int_0^{x_0}\left\{aw'^2 + 2bww' + cw^2 + \frac{d}{dx}[gw^2]\right\}dx = \int_0^{x_0} a\left[w' + \frac{b+g}{a}w\right]^2 dx = 0.$$

We therefore only need to show that $w \in W_0^{1,\infty}(0,1)$ and (2.10). We consider three cases.

CASE 1. Let $x \in (0, \varepsilon]$, then by (2.8) $b(x)+g(x) \neq 0$ and thus by (2.5) we have

$$w(x) = \exp\left\{-\int_{\varepsilon}^{x} \frac{b(t)+g(t)}{a(t)}\,dt\right\} = \exp\left\{-\int_{\varepsilon}^{x} \frac{(b+g)^2}{a(b+g)}\,dt\right\} = \tag{2.11}$$

$$\exp\left\{-\int_{\varepsilon}^{x} \frac{g'+b'}{g+b}\,dt + \int_{\varepsilon}^{x} \frac{b'-c}{g+b}\,dt\right\} = \frac{|g(\varepsilon)+b(\varepsilon)|}{|g(x)+b(x)|} \exp\left\{\int_{\varepsilon}^{x} \frac{b'-c}{g+b}\,dt\right\}.$$

Therefore since $g(0) = -\infty$ we have that w is continuous at 0 and that by (2.9) and (2.11), $w' \in L^{\infty}(0, \varepsilon)$.

To show (2.10), observe that for $x \in (0, \varepsilon)$, we have

$$|g(x)\,w^2(x)| = \frac{a(x)}{a(x)} |[g(x)+b(x)]\,w(x) - b(x)\,w(x)|\;|w(x)| \leqq \tag{2.12}$$

$$\leqq a(x) \left| \frac{g(x)+b(x)}{a(x)}\,w(x) \right| |w(x)| + |b(x)\,w^2(x)| \leqq$$

$$\leqq a(x)\,|w'(x)|\,|w(x)| + |b(x)\,w^2(x)|,$$

where we have used (2.9) in the last inequality. Since $w(0) = 0$ and $w' \in L^{\infty}(0, \varepsilon)$, we have immediately $\lim_{x\to 0} g(x)\,w^2(x) = 0$.

CASE 2. Let $x \in [\varepsilon, x_0 - \varepsilon]$, then w is well defined and even $C^1([\varepsilon, x_0 - \varepsilon])$.

CASE 3. Let $x \in [x_0 - \varepsilon, x_0)$, the using (2.8) we have $b(x)+g(x) \neq 0$. Furthermore by (2.5) as in (2.11) we have

$$w(x) = \exp\left\{-\int_{\varepsilon}^{x_0-\varepsilon} \frac{b+g}{a}\,dt\right\} \exp\left\{-\int_{x_0-\varepsilon}^{x} \frac{b+g}{a}\,dt\right\} = \tag{2.13}$$

$$\exp\left\{-\int_{\varepsilon}^{x_0-\varepsilon} \frac{b+g}{a}\,dt\right\} \cdot \frac{b(x_0-\varepsilon)+g(x_0-\varepsilon)}{b(x)+g(x)} \exp\left\{-\int_{x_0-\varepsilon}^{x} \frac{b'-c}{b+g}\,dt\right\}.$$

Using (2.7) we have indeed that w is continuous at x_0. Combining (2.9) and (2.13) we have that $w' \in L^{\infty}(x_0 - \varepsilon, x_0)$.

To conclude the proof we only need to show (2.10) at $x = x_0$ and this is done exactly in the same way as (2.12). ■

3. – Higher dimensional case.

We now consider a bounded open set $\Omega \subset \boldsymbol{R}^n$, $u: \Omega \to \boldsymbol{R}^m$ ($\nabla u \in \boldsymbol{R}^{nm}$) and a continuous function $f: \Omega \times \boldsymbol{R}^m \times \boldsymbol{R}^{nm} \to \boldsymbol{R}$. We let

$$I(u) = \int_\Omega f(x, u(x), \nabla u(x))\, dx. \tag{3.1}$$

As it is well known the result of Theorem 1 is still valid provided $m = 1$ or $n = 1$ (c.f. for example Dacorogna [D2]).

However when one considers the general vectorial case, i.e. $m, n > 1$, the necessary condition and the conjecture are both false and we give two examples below.

EXAMPLE 1. We first show that the necessary condition is false. Let $m = n = 2$ and

$$f(\xi) = \det \xi$$

then f is not convex while I is convex over $W_0^{1,\infty}(\Omega; \boldsymbol{R}^2)$ since

$$I(u) = \int_\Omega \det \nabla u(x) dx \equiv 0$$

for every $u \in W_0^{1,\infty}(\Omega; \boldsymbol{R}^2)$.

EXAMPLE 2 (c.f. Terpstra [T], Serre [S]). We now show that the conjecture is also false. The example is however more involved. We let $m = n = 3$ and denote the components of a 3×3 matrix ξ by ξ_{ij}, $1 \leqq i, j \leqq 3$. We let $\varepsilon > 0$ and

$$f(\xi) = (\xi_{11} - \xi_{32} - \xi_{23})^2 + (\xi_{12} - \xi_{31} + \xi_{13})^2 + (\xi_{21} - \xi_{31} - \xi_{13})^2 + \xi_{22}^2 + \xi_{33}^2 - \varepsilon \sum_{i,j=1}^{3} \xi_{ij}^2 .$$

One can then show (c.f. [D2] for details) that for $\varepsilon > o$ sufficiently small

i) The following holds

$$I(u) = \int_\Omega f(\nabla u(x))\, dx \geqq 0 \tag{3.2}$$

for every $u \in W_0^{1,\infty}(\Omega; \boldsymbol{R}^3)$. Since f is quadratic the positivity of I implies that I is convex over $W_0^{1,\infty}(\Omega; \boldsymbol{R}^3)$;

ii) f cannot be written as

$$f(\xi) = \tilde{f}(x, u, \xi) + \Phi(x, u, \xi) \tag{3.3}$$

with $\tilde{f}(x,\cdot,\cdot)$ convex (even with $\tilde{f}(x,u,\cdot)$ convex) and Φ satisfying

$$\int_\Omega \Phi(x, u(x), \nabla u(x))\, dx \equiv 0 \tag{3.4}$$

for every $u \in W_0^{1,\infty}(\Omega; \boldsymbol{R}^3)$.

(In the terminology of the vectorial calculus of variations, a function f satisfying (3.2) is said to be quasiconvex, (3.3) polyconvex, (3.4) quasiaffine or null Lagrangian.)

Therefore there is no $\tilde{f}: \Omega \times \boldsymbol{R}^m \times \boldsymbol{R}^{nm} \to \boldsymbol{R}$, $\tilde{f}(x,\cdot,\cdot)$ convex so that

$$I(u) = \int_\Omega f(\nabla u(x))\, dx = \int_\Omega \tilde{f}(x, u(x), \nabla u(x))\, dx$$

for every $u \in W_0^{1,\infty}(\Omega; \boldsymbol{R}^3)$, although I is convex.

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RIASSUNTO

Vedi l'introduzione.

SUMMARY

See the introduction.

REFERENCES

[BD] L. BOCCARDO - B. DACOROGNA, *Monotonicity of certain differential operators in divergence form*, Manuscripta Mathematica, 64 (1989), pp. 253-260.

[D1] B. DACOROGNA, *Convexity of certain integrals of the calculus of variations*, Proc. Roy. Soc. Edimb., 107-A (1987), pp. 15-26.

[D2] B. DACOROGNA, *Direct Methods in the Calculus of Variations*, Springer-Verlag, New-York (1989).

[GF] I. M. GELFAND - S. V. FOMIN, *Calculus of Variations*, Prentice-Hall, Englewood (1963).

[S] D. SERRE, *Formes quadratiques et calcul des variations*, J. Math. Pures Appl., 62 (1983), pp. 117-196.

[T] F. J. TERPSTRA, *Die Darstellung biquadratischer Formen als Summen von Quadraten mit Anwendung auf die Variationsrechnung*, Math. Ann., 116 (1938), pp. 166-180.