

Existence and Non-existence Results for Non-coercive Variational Problems and Applications in Ecology

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Received November 2, 1988; revised May 30, 1989

Variational problems without coercivity assumption appear in optimal foraging theory models in behavioural ecology. Using direct methods of the calculus of variations, rearrangement techniques, and relaxation theorems, we show that they possess one or no solution in AC-spaces, depending on the prescribed boundary conditions. © 1990 Academic Press, Inc.

1. INTRODUCTION

In this paper, we consider the problem of the calculus of variations

$$\inf \left\{ E(v) = \int_0^1 [\phi(x, v'(x)) + \psi(x, v(x))] dx : v \in W_S \right\}, \quad (\mathbf{P}_S)$$

where $S \geq 0$ is fixed and

$$W_S = \{v \in W^{1,1}(0, 1) : v(0) = 0, v(1) = S, v' \geq 0 \text{ a.e.}\}.$$

$W^{1,1}(0, 1)$ (sometimes denoted by $AC(0, 1)$) stands for the usual Sobolev space of continuous functions $v: [0, 1] \rightarrow \mathbb{R}$ with integrable first derivative (in the weak sense).

The main feature of the problem is that it is non-coercive, in a sense explained later.

In the ecological models considered below, where (\mathbf{P}_S) appears, the functions ϕ and ψ are often of the form

$$\begin{aligned} \phi(x, v') &= \rho(x) e^{-v'}, \\ \psi(x, v) &= h(x) v^p, \quad \text{with } p \geq 1. \end{aligned}$$

This is in sharp contrast with problems of mechanics and elasticity, where ϕ is usually of the form

$$\phi(x, v') = \frac{1}{2}[v']^2.$$

The classical existence theorem [e.g., 11, 12] does not apply to (P_S) , since no *a priori* boundedness on minimizing sequences can be inferred for the case $e^{-v'}$, contrary to the coercive case $\frac{1}{2}[v']^2$.

Using tools of the direct methods of the calculus of variations, this paper studies the existence and non-existence of solutions of (P_S) which depend on the parameter $S \geq 0$.

The simpler case, reviewed in Section 2, where

$$E(v) = \int_0^1 \phi(x, v'(x)) dx,$$

is studied in [6]. An appropriate assumption on ϕ , which includes the case $\phi(x, v') = \rho(x) e^{-v'}$ as well as the coercive case $(\phi(x, v') \geq \alpha|v'|^p, \text{ with } \alpha > 0, p > 1)$, ensures the existence of solutions for every value $S \geq 0$. Note that in this case the constraint $v' \geq 0$, which gives an L^1 bound on minimizing sequences, has a regularizing effect on the existence of minima. It is shown in [6] that for example if $\phi(x, v') = e^{-(1/x)-v'}$, then (P_S) has a solution while the unconstrained problem has none.

In Section 3, we deal with the general case and obtain a non-existence result for large values of S .

In Section 4, we obtain an existence result for small values of S for functionals which are more closely related to ecological applications. The existence of a critical value $S_c > 0$ is shown, separating the interval $[0, S_c]$, for which (P_S) has a solution, from $]S_c, +\infty[$, where (P_S) has no solution. Using rearrangement techniques [14], we show that the minimizers of (P_S) are convex functions. This result combined with a relaxation type theorem provides a criterion of compactness in the weak topology that we need to construct a solution of (P_S) .

Problem (P_S) appears in optimization models of optimal foraging theory [e.g., 26], which is part of behavioural ecology [e.g., 19]. These models attempt to find the optimal way for an animal to behave in its foraging activity (i.e., search en acquisition of food).

Recently, Arditi and Dacorogna [2-4] and Botteron and Arditi [5] developed models (which generalize existing ones [7, 18]) giving quantitative predictions [22] for the foraging behaviour of an animal in an arbitrary habitat. In mathematical terms, the problem turns out to be (P_S) (for details see Section 5). There, $v = v(x)$ corresponds to the schedule (i.e., v' is the reciprocal of the velocity) of the animal, x denotes its position in the habitat, and S denotes the foraging period of the animal, (i.e., the

available total time it has for searching and acquiring food). The minimization of the functional E corresponds to the maximization of the energy gained in [2, 3, 27] or to the maximization of a more general currency balancing gains and risks in [5, 31], which is directly related to the fitness of the animal [28].

A related problem (P) of ecological interest is to find the optimal foraging period that the animal should allocate to the foraging activity in its whole habitat. In mathematical terms, this means that we want to find, among all values $S \geq 0$ for which (P_S) has a solution, the one which minimizes the value function $m(S)$, i.e.,

$$\begin{aligned} & \inf\{m(S): S \geq 0 \text{ such that } (P_S) \text{ has a solution}\} \\ & = \inf\{\inf[E(v): v \in W_S]: S \geq 0 \text{ such that } (P_S) \text{ has a solution}\}. \quad (\text{P}) \end{aligned}$$

It will turn out that (P) has a solution denoted S_c , which is exactly the same as the above S_c (see Theorem 4.1 for more details).

2. THE CASE $\psi = 0$

We are concerned in this section with Problems (P_S) and (P), defined in the Introduction, for the functional

$$E(v) = \int_0^1 \phi(x, v'(x)) \, dx.$$

The resolution in that case is simpler than in the more general case (Sections 3, 4), but it gives more precise results. We do not consider the standard coercivity assumption on $\phi(x, \cdot)$ ($\phi(x, v') \geq \alpha|v'|^p$, with $\alpha > 0$, $p > 1$), but a weaker assumption (A2) allowing decreasing $\phi(x, \cdot)$ in view of the applications in ecology (Section 5).

We shall make the following hypotheses on ϕ :

(A1) $\phi = \phi(x, \xi)$ is non-negative, $\phi \in C^1([0, 1] \times \mathbb{R}^+)$, and $\phi(x, \cdot)$ is a strictly convex function for every fixed $x \in [0, 1]$.

(A2) $\gamma \leq \mu_0$, where

$$\left\{ \begin{array}{l} \gamma = \sup \left\{ \frac{\partial}{\partial \xi} \phi(x, S): x \in [0, 1] \right\}, \\ \mu_0 = \inf \{ \mu(x): x \in [0, 1] \}, \\ \mu(x) = \lim_{\zeta \rightarrow +\infty} \frac{\partial}{\partial \xi} \phi(x, \zeta), \quad \text{for } x \in [0, 1]. \end{array} \right.$$

EXAMPLE. The functions $\phi_1(x, \xi) = \rho(x) e^{-\xi}$, considered in the ecological applications (Section 5), and $\phi_2(x, \xi) = c(x) \xi^p$, with $p > 1$, satisfy (A1) and (A2), provided that $c, \rho \in C^1(0, 1)$ and $c(x) > 0$ in $[0, 1]$. In fact, one could weaken the regularity of ρ to admit piecewise continuous functions in the ecological applications [6].

The following theorem states the sufficiency of the classical necessary conditions applied to (P_S) and gives the solution of (P).

THEOREM 2.1. *Under the hypotheses (A1) and (A2),*

(i) (P_S) admits a unique solution $w_S \in W_S$ for every $S \geq 0$. Furthermore, $w_S \in C^1([0, 1])$ is given by the classical necessary conditions (see Eq. (1) in the proof below);

(ii) If $\phi(x, \cdot)$ is strictly decreasing (resp. strictly increasing) for almost every $x \in [0, 1]$, then the value function $m(S) = \inf\{E(v) : v \in W_S\}$ is strictly decreasing (resp. strictly increasing) and $S_c = +\infty$ (resp. $S_c = 0$) is the solution of (P).

Proof of Theorem 2.1. (i) is proved in [6]. It is shown that, for fixed $S \geq 0$, there exist a unique constant α_S , a unique set $\Omega_{\alpha_S} \subset [0, 1]$, and a unique function $f_{\alpha_S} : \Omega_{\alpha_S} \rightarrow \mathbb{R}^+$ defined by

$$\Omega_{\alpha_S} = \left\{ x \in [0, 1] : \text{there exists } \zeta > 0 \text{ such that } \frac{\partial}{\partial \xi} \phi(x, \zeta) = \alpha_S \right\},$$

$$f_{\alpha_S}(x) = \zeta \text{ is the solution of } \frac{\partial}{\partial \xi} \phi(x, \zeta) = \alpha_S, \quad \text{for every } x \in \Omega_{\alpha_S}.$$

The desired solution $w_S : [0, 1] \rightarrow \mathbb{R}$ is then the continuous function defined by

$$w_S(0) = 0 \quad \text{and} \quad w'_S(x) = \begin{cases} 0, & \text{if } x \in \Omega_{\alpha_S}^c = [0, 1] \setminus \Omega_{\alpha_S}, \\ f_{\alpha_S}(x), & \text{if } x \in \Omega_{\alpha_S}. \end{cases} \quad (1)$$

(1) is exactly the classical necessary conditions [8]. Note also that Ω_{α_S} is the set where the Euler equation is satisfied.

(ii) Consider $0 \leq S < T < +\infty$. We have, using the results of [6],

$$\alpha_S < \alpha_T, \quad \Omega_{\alpha_S} \subset \Omega_{\alpha_T}, \quad \text{and} \quad f_{\alpha_S}(x) < f_{\alpha_T}(x), \quad \text{for every } x \in \Omega_{\alpha_S}. \quad (2)$$

From (i), we have

$$m(S) - m(T) = E(w_S) - E(w_T), \quad (3)$$

where w_S (resp. w_T) stands for the unique solution of (P_S) (resp. (P_T)), characterized by (1). But with (1) and (2), (3) becomes

$$E(w_S) - E(w_T) = \int_{\Omega_{\alpha_S}} [\phi(x, f_{\alpha_S}(x)) - \phi(x, f_{\alpha_T}(x))] dx \\ + \int_{\Omega_{\alpha_T} \setminus \Omega_{\alpha_S}} [\phi(x, 0) - \phi(x, f_{\alpha_T}(x))] dx.$$

Using the strict monotonicity of ϕ , we deduce the strict monotonicity of m and hence the theorem. ■

3. NON-EXISTENCE FOR LARGE S

In this section, we consider (P_S) , defined in the Introduction for the general functional

$$E(v) = \int_0^1 [\phi(x, v') + \psi(x, v)] dx.$$

We show that for sufficiently large S , (P_S) has no solution (Theorem 3.1), and that the value function

$$m(S) = \inf\{E(v): v \in W_S\} = \inf(P_S)$$

is constant beyond a certain value of S (Lemma 3.2).

We shall make use of the following hypotheses on ϕ and ψ :

- (B1) (i) $\phi \in C(\omega; \mathbb{R}^+)$, where $\omega = [0, 1] \times [0, +\infty[\subset \mathbb{R}^2$,
(ii) there exist $c \geq 0$ and a decreasing function $\phi_0 \in C(\mathbb{R}^+; \mathbb{R}^+)$

such that

$$|\phi(x_1, \xi_1) - \phi(x_2, \xi_2)| \leq \frac{c}{2} |x_1 - x_2| + |\phi_0(\xi_1) - \phi_0(\xi_2)|,$$

for every $(x_1, \xi_1), (x_2, \xi_2) \in \omega$;

- (B2) (i) $\psi \in C(\omega; \mathbb{R}^+)$,

(ii) there exists $\psi_1 \in C(\mathbb{R}^+; \mathbb{R}^+)$ with $\psi_1(0) = 0$ and $\psi_1(r) \rightarrow +\infty$, if $r \rightarrow +\infty$, such that

$$\psi(x, r_1) - \psi(x, r_2) \geq \psi_1(r_1) - \psi_1(r_2) \quad \text{for every } x \in]0, 1[\\ \text{and for any } 0 \leq r_2 \leq r_1 < +\infty.$$

EXAMPLE. The functions $\phi(x, \xi) = \rho(x) e^{-\xi}$ and $\psi(x, r) = h(x) r^p$, with $p \geq 1$, considered in the ecological applications (Section 5), satisfy (B1) and (B2), provided that $\rho \in W^{1,\infty}(0, 1)$ (i.e., the set of Lipschitz functions) with $\rho \not\equiv 0$ and that $h \in C([0, 1])$ and $h(x) > 0$ for every $x \in [0, 1]$.

The following theorem is a non-existence result for (P_S) for large S .

THEOREM 3.1. *Under the hypotheses (B1) and (B2), there exists $S_1 > 0$ such that (P_S) has no solution, for every $S > S_1$.*

Remark. The non-existence follows in this problem from the lack of coercivity (B1) of ϕ .

To prove Theorem 3.1, we shall use the following

LEMMA 3.2. *Under the hypotheses (B1) and (B2), there exists $S_1 > 0$ such that*

$$\inf(P_S) = \inf(P_{S/2}) \quad \text{for every } S > S_1.$$

Proof of Lemma 3.2. We divide the proof into two steps:

Step 1. We first show that there exists $S_1 > 0$ such that $\inf(P_S) \geq \inf(P_{S/2})$ for every $S > S_1$. It will be sufficient to show that for every $w \in W_S$, there exists $v \in W_{S/2}$ such that $E(w) > E(v)$.

Using (B2), there exists $S_1 > 0$ such that

$$\psi_1(r) > c + \phi_0(0), \quad \text{for every } r \geq \frac{S_1}{2}, \tag{1}$$

where c and ϕ_0 are given by (B1).

Fix $S > S_1$ and any $w \in W_S$. Let $\bar{x} \in]0, 1[$ denote any point (which may be not unique) such that

$$w(1 - \bar{x}) = \frac{S}{2}.$$

Let $v \in W_{S/2}$ be defined by

$$v(x) = \begin{cases} 0, & \text{if } x \in [0, \bar{x}], \\ w(x - \bar{x}), & \text{if } x \in]\bar{x}, 1]. \end{cases} \tag{2}$$

With this definition (see Fig. 1),

$$w(x) \geq v(x), \quad \text{for every } x \in [0, 1].$$

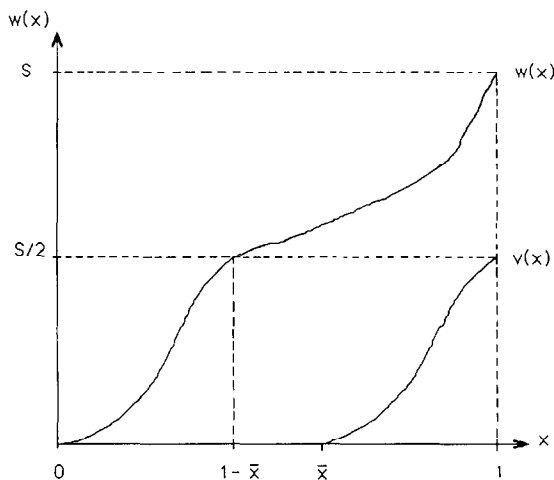


FIG. 1. Given any $w \in W_S$, example of construction of $v \in W_{S/2}$ as in (2).

Using (2), we have

$$\begin{aligned}
 & \int_0^1 [\phi(x, w') - \phi(x, v')] dx \\
 &= \int_0^{1-\bar{x}} [\phi(x, w') - \phi(x + \bar{x}, w')] dx \\
 & \quad + \int_{1-\bar{x}}^1 [\phi(x, w') - \phi(x - 1 + \bar{x}, 0)] dx \\
 & \geq -\bar{x}[c(1 - \bar{x}) + \phi_0(0)], \tag{3}
 \end{aligned}$$

since by (B1),

$$\phi(x, w') - \phi(x + \bar{x}, w') \geq -\frac{c}{2} \bar{x}, \quad \text{for every } x \in]0, 1 - \bar{x}[,$$

$$\begin{aligned}
 \phi(x, w') - \phi(x - 1 + \bar{x}, 0) & \geq -\frac{c}{2} (1 - \bar{x}) - |\phi_0(w') - \phi_0(0)| \\
 & \geq -\frac{c}{2} (1 - \bar{x}) - \phi_0(0), \quad \text{for every } x \in]1 - \bar{x}, 1[,
 \end{aligned}$$

where we used the fact that ϕ_0 is decreasing (B1) and $w' \geq 0$.

Using (B2), (2), and (1), we have

$$\int_0^1 [\psi(x, w) - \psi(x, v)] dx \geq \int_0^1 [\psi_1(w) - \psi_1(v)] dx$$

$$= \int_{1-\bar{x}}^1 \psi_1(w) dx > \bar{x}[c + \phi_0(0)], \quad (4)$$

since $S > S_1$ and $w' \geq 0$, which implies that

$$w(x) \geq \frac{S}{2}, \quad \text{for every } x \in]1 - \bar{x}, 1[.$$

Collecting (3) and (4), we get $E(w) > E(v)$. Since such a construction (2) is valid for any given $w \in W_S$, we obtain Step 1.

Step 2. We now show that in fact $\inf(P_S) = \inf(P_{S/2})$ for every $S > S_1$. It will be sufficient to show that for every $v \in W_{S/2}$, there exists a sequence $\{v_n\} \subset W_S$ such that $E(v_n) \rightarrow E(v)$ if $n \rightarrow +\infty$.

Fix $S > S_1$ and consider any given $v \in W_{S/2}$. For $n > 1$ an integer, define $v_n \in W_S$ by

$$v_n(x) = \begin{cases} v(x), & \text{if } x \in \left[0, 1 - \frac{1}{n}\right], \\ v\left(1 - \frac{1}{n}\right) + n \left[S - v\left(1 - \frac{1}{n}\right) \right] \left[x - 1 + \frac{1}{n} \right], & \text{if } x \in \left]1 - \frac{1}{n}, 1\right]. \end{cases}$$

Clearly, from the hypotheses on ϕ and ψ , we have $E(v_n) \rightarrow E(v)$, if $n \rightarrow +\infty$. This concludes the proof of Lemma 3.2. ■

Proof of Theorem 3.1. Consider $S > S_1$, where S_1 is given by Lemma 3.2. Suppose for contradiction that $w \in W_S$ is a solution of (P_S) . From the proof of Lemma 3.2, we can construct a function $v \in W_{S/2}$ such that $E(w) > E(v)$. But from Lemma 3.2 again, we have

$$E(v) < E(w) = \inf(P_S) = \inf(P_{S/2}),$$

which is absurd, since $v \in W_{S/2}$. Hence, for every fixed $S > S_1$, (P_S) has no solution. ■

4. EXISTENCE FOR SMALL S

In this section, we are concerned with Problem (P_S) ,

$$\inf\{E(v) : v \in W_S\} \tag{P_S}$$

and with problem (P),

$$\inf\{m(S): (P_S) \text{ has a solution}\}, \quad (\text{P})$$

defined in the Introduction, where

$$E(v) = \int_0^1 [\rho(x) \phi_0(v') + \psi(x, v)] dx$$

and m is the value function of (P_S) :

$$m(S) = \inf\{E(v): v \in W_S\} = \inf(P_S).$$

We consider the case where $\psi(x, \cdot)$ is increasing and ϕ_0 is decreasing and convex, which is the case related to the ecological applications (Section 5). Furthermore, we assume that ρ is increasing.

The main result (Theorem 4.1) we obtain is the existence of a critical value S_c such that (P_S) has a solution if and only if $S \in [0, S_c]$. Furthermore, S_c is the solution of (P). To prove Theorem 4.1, we proceed in several steps. The two main steps are:

Step 1. Using rearrangement techniques, we first show that minimizers of (P_S) are convex functions (Lemma 4.2).

Step 2. We then show that even if (P_S) has no solution, we can associate to (P_S) a "relaxed" problem (\tilde{P}_S) (which is the same problem as (P_S) , except that we allow functions v with $v(1) \leq S$, instead of $v(1) = S$), such that $\inf(P_S) = \inf(\tilde{P}_S)$. We show using direct methods of the calculus of variations that (\tilde{P}_S) has always a solution (Lemma 4.3).

We shall make use in this section of the following hypotheses on ρ , ϕ_0 , and ψ :

(C1) (i) $\rho \in W^{1,\infty}(0, 1)$ is a non-negative increasing function,
 (ii) $\phi_0 \in C^1([0, +\infty[)$ is a non-negative, convex, and decreasing function:

(C2) (i) $\psi = \psi(x, r)$ is a non-negative function and for every fixed $S \geq 0$, $\psi \in C^1(\omega_S)$, where $\omega_S = [0, 1] \times [0, S] \subset \mathbb{R}^2$,

(ii) there exists $\psi_1 \in C(\mathbb{R}^+; \mathbb{R}^+)$ with $\psi_1(0) = 0$ and $\psi_1(r) \rightarrow +\infty$, if $r \rightarrow +\infty$, such that

$$\psi(x, r_1) - \psi(x, r_2) \geq \psi_1(r_1) - \psi_1(r_2) \quad \text{for every } x \in]0, 1[$$

$$\text{and for any } 0 \leq r_2 \leq r_1 < +\infty;$$

(iii) for almost every fixed $x \in]0, 1[$, $\psi(x, \cdot)$ is a strictly increasing function;

(C3) for almost every fixed $x \in]0, 1[$, $\psi(x, \cdot)$ is a strictly convex function;

(C4) ϕ_0 is not identically constant.

Remarks. (1) Hypotheses (C1) and (C2) are stronger than those of the preceding section, (B1) and (B2).

(2) The regularity of ϕ_0 and ψ can be weakened to $\phi_0, \psi \in W^{1, \infty}$ (hypotheses (C1) and (C2)) for parts (i) and (ii) of Theorem 4.1.

(3) We do not need hypothesis (C4) in the main steps of the proof of Theorem 4.1. We make this hypothesis only to ensure that $S_c \neq 0$.

(4) The example given in Section 3, of ecological interest, satisfies (C1) to (C4), provided that ρ is increasing and $p > 1$. The case $p = 1$ is solved in [5] without the assumption that ρ is increasing. In that case, the solutions of (P_S) are characterized by the usual necessary conditions [e.g., 8] and the value of S_c is explicitly found.

The main result of this section is the following

THEOREM 4.1. *Under the hypotheses (C1) to (C3),*

(i) *there exists $S_c \geq 0$ such that (P_S) has a solution if and only if $S \in [0, S_c]$.*

(ii) *S_c is the solution of (P). In particular, $\inf(P_S) = \inf(P_{S_c})$ for every $S \geq S_c$.*

Furthermore, if (C4) holds, then $S_c > 0$.

Remarks. (1) For the non-existence result of Section 3, no convexity of ϕ_0 was needed, while, for an existence result, the convexity of ϕ_0 ensures the weak lower semicontinuity of the functional E [e.g., 11].

(2) In view of (C1) and (C3), the functional E is strictly convex and admits then at most one minimizer.

The proof of Theorem 4.1 is divided into several lemmas. Lemma 4.2 states that the minimizers of (P_S) are convex functions.

LEMMA 4.2. *Under the hypotheses (C1) and (C2),*

$$\inf(P_S) = \inf\{E(v) : v \in CW_S\} \quad \text{for every } S \geq 0,$$

where

$$CW_S = \{v \in W_S : v \text{ convex}\}.$$

Remark. We do not need Hypothesis (C3) for Lemma 4.2.

Proof of Lemma 4.2. To prove Lemma 4.2, we use rearrangement techniques [14]. Since $CW_S \subset W_S$, we only need to show that $\inf(P_S) \geq \inf\{E(v) : v \in CW_S\}$ to conclude. It is therefore sufficient to construct for every function $w \in W_S$ a convex function $w_c \in CW_S$ such that $E(w_c) \leq E(w)$.

Fix any $w \in W_S$. Let w'_c denote the increasing rearrangement of the derivative w' [e.g., 14, 24],

$$w'_c(x) = \begin{cases} \inf\{t \in \mathbb{R}^+ : \mu(t) > x\}, & \text{if } x \in]0, 1[, \\ \sup\{w'(y) : y \in]0, 1[\}, & \text{if } x = 1, \end{cases}$$

where $\mu : \mathbb{R}^+ \rightarrow [0, 1]$ is the distribution function of w' ,

$$\mu(t) = \text{meas}\{y \in]0, 1[: w'(y) < t\},$$

where “meas” stands for the usual Lebesgue measure in \mathbb{R} . We define w_c to be the convex rearrangement of w (see Fig. 2 for an example),

$$w_c(x) = \int_0^x w'_c(y) dy, \quad \text{for every } x \in [0, 1].$$

From the equimesurability property of rearrangements, we have

$$\int_0^1 w'_c(y) dy = \int_0^1 w'(y) dy,$$

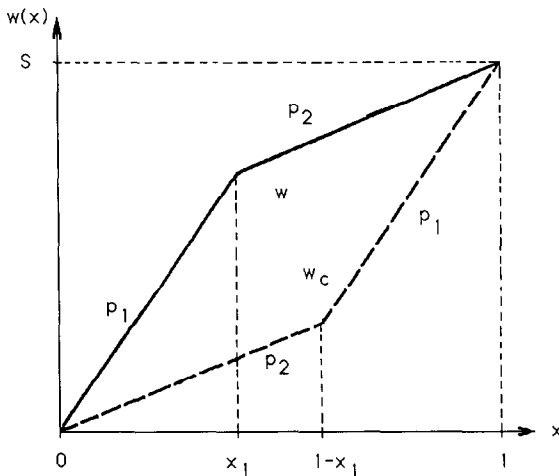


FIG. 2. Example of convex rearrangement w_c of $w \in W_S$. Let $w \in W_S$ be defined by $w'(x) = p_1$, if $x \in [0, x_1[$, and $w'(x) = p_2$, if $w \in [x_1, 1]$, with $p_1 > p_2$. The convex rearrangement $w_c \in CW_S$ of w is defined by $w'_c(x) = p_2$, if $x \in [0, 1-x_1[$, and $w'_c(x) = p_1$, if $x \in [1-x_1, 1]$.

which ensures that $w_c \in W^{1,1}(0, 1)$ and that $w_c(1) = S$. Since

$$w'_c: [0, 1] \rightarrow [\inf\{w'(y): y \in]0, 1[\}, \sup\{w'(y): y \in]0, 1[\}]$$

is increasing, w_c is convex and $w_c \in CW_S$.

The Hardy–Littlewood Theorem [24], applied to ρ and ϕ_0 , gives that

$$\int_0^1 \rho(x) \phi_0(w') dx \geq \int_0^1 \rho(x) \phi_0(w'_c) dx, \tag{1}$$

where we used (C1), i.e., that ϕ_0 is decreasing and ρ is increasing. Another property [e.g., 24, Lemma 3.1] of increasing rearrangements is that

$$w(x) \geq w_c(x), \quad \text{for every } x \in [0, 1].$$

Hence, by (C2), we then have

$$\int_0^1 \psi(x, w) dx \geq \int_0^1 \psi(x, w_c) dx. \tag{2}$$

Combining (1) and (2), we get $E(w_c) \leq E(w)$, which concludes the proof of Lemma 4.2. ■

We now introduce the “relaxed” problem (\tilde{P}_S) , which is (P_S) , except that we allow admissible functions not to reach S (i.e., $v(1) \leq S$ instead of $v(1) = S$).

More precisely, let for $S \geq 0$

$$\tilde{W}_S = \{v \in W^{1,1}(0, 1): v(0) = 0, v(1) \leq S, v' \geq 0 \text{ a.e.}\}$$

and

$$\inf\{E(v): v \in \tilde{W}_S\} = \inf(\tilde{P}_S). \tag{3}$$

We now study the relationship between (P_S) and (\tilde{P}_S) :

LEMMA 4.3 (Relaxation Lemma). *Under the hypotheses (C1) and (C2),*

- (i) $\inf(P_S) = \inf(\tilde{P}_S)$ for every $S \geq 0$;
- (ii) (\tilde{P}_S) has a suppress solution $w \in \tilde{W}_S$ for every $S \geq 0$. Furthermore, $w \in W^{1,\infty}(K)$, for every open set K such that $\bar{K} \subset]0, 1[$.

Remarks. (1) Lemma 4.3 helps characterize minimizing sequences of (P_S) , even if (P_S) has no solution.

(2) If (P_S) has a solution $v \in W_S$, then v is also a solution of (\tilde{P}_S) . Hence v' is finite in $]0, 1[$, but in general we do not have that $v'(1) < +\infty$.

Such an example would be $S=1$ and $E(v) = \int_0^1 [e^{-v'(x)} + v(x)] dx$, which has a solution $v \in W_1$ with $v'(1) = +\infty$ [6, for details].

Proof of Lemma 4.3. (i) Let $S \geq 0$ be fixed. Since $W_S \subset \tilde{W}_S$, we already have that

$$\inf(\mathbf{P}_S) \geq \inf(\tilde{\mathbf{P}}_S).$$

To get equality in this last expression, it will be sufficient to associate to every $v \in \tilde{W}_S$ a sequence $\{v_n\} \subset W_S$ such that $E(v_n) \rightarrow E(v)$ if $n \rightarrow +\infty$. The construction of such a sequence is easily done for any $v \in W_S$, proceeding as in Step 2 of the proof of Lemma 3.2.

(ii) For $N > 1$ an integer, let I_N denote the interval

$$I_N = \left] 0, 1 - \frac{1}{N} \right[.$$

We introduce the notation

$$E(v; I_N) = \int_{I_N} [\rho(x) \phi_0(v') + \psi(x, v)] dx \quad (1)$$

for every admissible function v . Let $\varepsilon > 0$ and $S \geq 0$ be fixed. Since the integrand in (1) is bounded by $\rho(1) \phi_0(0) + \max\{\psi(x, S) : x \in [0, 1]\}$, there exists $N_0 > 0$ an integer such that

$$|E(v; I_N) - E(v)| < \varepsilon \quad \text{for every } v \in \tilde{W}_S \quad \text{and} \quad \text{for every } N > N_0. \quad (2)$$

Consider a minimizing sequence $\{w_n\} \subset W_S$ of (\mathbf{P}_S) . From Lemma 4.2, we may assume that $\{w_n\} \subset CW_S$ and from (i) that $\{w_n\}$ is also a minimizing sequence of $(\tilde{\mathbf{P}}_S)$, since $W_S \subset \tilde{W}_S$. Hence, w_n is a convex function, for every n and

$$\liminf_{n \rightarrow +\infty} E(w_n) = \inf(\mathbf{P}_S) = \inf(\tilde{\mathbf{P}}_S). \quad (3)$$

We now prove that the sequence $\{w_n\}$ is in fact weak star compact in $W^{1,\infty}(I_N)$.

Fix $N > 1$ an integer. For any integer n , since w_n is convex, we have for almost every $x \in I_N$

$$S = w_n(1) \geq w_n(x) + w'_n(x)(1-x) \geq w'_n(x) \frac{1}{N},$$

since $w_n, w'_n \geq 0$. Hence,

$$\|w'_n\|_{L^\infty(I_N)} = \sup\{|w'_n(x)| : x \in I_N\} \leq NS, \quad \text{for every integer } n.$$

This last expression provides the existence of v'_N and of a subsequence if necessary such that

$$w'_n \overset{*}{\rightharpoonup} v'_N \text{ in } L^\infty(I_N) \quad (\text{weak star convergence}).$$

Letting

$$v_N(x) = \int_0^x v'_N(y) dy, \quad \text{for every } x \in \left[0, 1 - \frac{1}{N}\right],$$

we therefore have

- (i) $w_n \overset{*}{\rightharpoonup} v_N$ in $W^{1,\infty}(I_N)$
 - (ii) v_N is a convex increasing function.
- (4)

We now study the behaviour of v_N as N increases. In fact, one can easily check with (4i) that

$$v_{N+1|I_N} \equiv v_N, \quad \text{for every } N > 1.$$

This last property ensures that

$$\lim_{N \rightarrow +\infty} v_N \left(1 - \frac{1}{N}\right)$$

exists and is not greater than S . Define then

$$w(x) = \begin{cases} \lim_{N \rightarrow +\infty} v_N(x), & \text{if } x \in [0, 1[, \\ \lim_{N \rightarrow +\infty} v_N(1 - (1/N)), & \text{if } x = 1. \end{cases} \quad (5)$$

With this definition, we have that $w \in \tilde{W}_S$. Furthermore, from (4), w is convex and

$$w \in W^{1,\infty}(I_N), \quad \text{for every } N > 1.$$

We can now prove that the function w defined in (5) is a solution of (\tilde{P}_S) . Since $w \in \tilde{W}_S$, we have, using (2),

$$E(w) \leq E(w; I_N) + \varepsilon, \quad \text{if } N > N_0. \quad (6)$$

The convexity of ϕ_0 ensures the weak star lower semicontinuity of E [e.g., 11]. Hence,

$$E(w; I_N) \leq \liminf_{n \rightarrow +\infty} E(w_n; I_N).$$

Since $I_N \subset]0, 1[$, we have

$$\liminf_{n \rightarrow +\infty} E(w_n; I_N) \leq \liminf_{n \rightarrow +\infty} E(w_n). \tag{7}$$

Combining (6), (7), and (3), we get

$$E(w) \leq \inf(\tilde{P}_S) + \varepsilon.$$

Since $w \in \tilde{W}_S$ and ε is arbitrary, the function w defined by (5) is a solution of (\tilde{P}_S) , which is unique if E is strictly convex. This concludes the proof of Lemma 4.3. ■

Remarks. (1) The convexity of ϕ_0 plays a crucial role in establishing the weak star lower semicontinuity of E , but, up to now, we did not use the convexity hypothesis (C3) on $\psi(x, \cdot)$.

(2) In general, the solution $w \in \tilde{W}_S$ of (\tilde{P}_S) , defined by (5) above, is not a Lipschitz function, i.e., $w \notin W^{1,\infty}(0, 1)$ (see Remark 2 before the proof of Lemma 4.3).

With the following lemma, we characterize the non-existence of solutions of (P_S) .

LEMMA 4.4. *Let $S > 0$. Under the hypotheses (C1) to (C3), the following conditions are equivalent:*

- (i) (P_S) has no solution;
- (ii) there exists $w \in \tilde{W}_S \setminus W_S$, such that $\inf(P_T) = E(w)$, for every $T \in [w(1), S]$.

Proof of Lemma 4.4. We first show that (i) implies (ii): From Lemma 4.3, we know there exists $w \in \tilde{W}_S$ such that

$$E(w) = \inf(\tilde{P}_S) = \inf(P_S). \tag{1}$$

Since (P_S) has no solution, $w \in \tilde{W}_S \setminus W_S$ and hence $w(1) < S$. It remains to be shown that

$$\inf(P_T) = E(w), \quad \text{for every } T \in [w(1), S]. \tag{2}$$

Consider any $T \in [w(1), S]$. Since $\inf(\tilde{P}_T) = \inf(P_T)$, $w \in \tilde{W}_{w(1)}$, and

$$\tilde{W}_{w(1)} \subset \tilde{W}_T \subset \tilde{W}_S,$$

we have that

$$E(w) \geq \inf(P_{w(1)}) \geq \inf(P_T) \geq \inf(P_S). \tag{3}$$

Then (2) results from (1) and (3).

We now show that (ii) implies (i):

Suppose for contradiction that (P_S) has a solution $v \in W_S$. Suppose now that there exists w satisfying (ii); we then have $v \neq w$ and

$$E(v) = \inf(P_S) = \inf(\tilde{P}_S) = E(w),$$

by choosing $T = S$ in (ii). However, the above identity means that (\tilde{P}_S) has two minimizers, which is absurd, because of the strict convexity of E . Hence (P_S) has no solution. ■

The next lemma characterizes values S for which (P_S) has one solution or none.

LEMMA 4.5. *Let $S > 0$. Under the hypotheses (C1) to (C3),*

- (i) *if (P_S) has no solution, then neither does (P_R) , for every $R > S$;*
- (ii) *if (P_S) has a solution, then so does (P_R) , for every $R \in [0, S]$.*

Proof of Lemma 4.5. (i) From Lemma 4.4, we know there exists $w \in \tilde{W}_S \setminus W_S$ (i.e., $w(1) < S$) such that

$$\inf(P_T) = E(w), \quad \text{for every } T \in [w(1), S]. \tag{1}$$

Suppose for contradiction that $v \in W_R$ is a solution of (P_R) , for some fixed $R > S$. We then have

$$E(w) = \inf(P_S) = \inf(\tilde{P}_S) \geq \inf(\tilde{P}_R) = \inf(P_R) = E(v). \tag{2}$$

Considering any fixed $\lambda \in]0, (S - w(1))/(R - w(1))[,$ we define

$$v_\lambda(x) = \lambda v(x) + (1 - \lambda)w(x) \quad \text{for every } x \in [0, 1].$$

With this definition, $v_\lambda \in \tilde{W}_S \setminus W_S$. With (2) and the strict convexity of E , we then get

$$E(v_\lambda) < E(w) = \inf(\tilde{P}_S),$$

where we used (1) with $T = S$ and Lemma 4.3. The above expression is absurd since $v_\lambda \in \tilde{W}_S$; hence (P_R) has no solution.

(ii) follows from (i) if $R > 0$ and is trivial for $R = 0$. ■

We now define the critical value $S_c \geq 0$ separating small values S for which (P_S) has a solution from large values for which (P_S) has none:

$$S_c = \inf\{S \geq 0: (P_S) \text{ has no solution}\}.$$

The existence of S_c is ensured by Theorem 3.1. We have the following

LEMMA 4.6. *Under the hypotheses (C1) to (C3),*

- (i) (P_{S_c}) has a solution;
- (ii) (P_S) has a solution if and only if $0 \leq S \leq S_c$.

Furthermore, if (C4) holds, then

- (iii) $S_c > 0$.

Proof of Lemma 4.6. (i) Suppose for contradiction that (P_{S_c}) has no solution. We then get, using the strict convexity of E and Lemma 4.4, a value $T < S_c$ such that (P_T) has no solution. But this contradicts the definition of S_c , hence the result.

(ii) follows from (i) and Lemma 4.5.

(iii) Suppose for contradiction $S_c = 0$. Then, for every $S > 0$, (P_S) has no solution. From Lemma 4.4, we conclude that

$$\inf(P_S) = \inf(P_0) = E(w \equiv 0), \quad \text{for every } S > 0. \quad (1)$$

Choose $v(x) = Sx^n$ and define the convex function

$$f(S) = E(v) = \int_0^1 [\rho(x) \phi_0(nSx^{n-1}) + \psi(x, Sx^n)] dx, \quad \text{for } S \geq 0.$$

From (1), we have that

$$f(S) \geq f(0), \quad \text{for every } S \geq 0. \quad (2)$$

From (C1) to (C4) we deduce that for n large enough $f'(0) < 0$. But that contradicts (2) and hence $S_c > 0$. ■

The following lemma studies the properties of the value function

$$m(S) = \inf\{E(v); v \in W_S\} = \inf(P_S).$$

LEMMA 4.7. *Under the hypotheses (C1) to (C3), the value function $S \mapsto m(S)$ is*

- (i) constant in $[S_c, +\infty[$,
- (ii) strictly decreasing in $[0, S_c]$,
- (iii) convex and continuous in \mathbb{R}^+ .

Proof of Lemma 4.7. (i) Fix $S > S_c$. Since (P_S) has no solution, using Lemma 4.4, there exists $w \in \bar{W}_S \setminus W_S$ (i.e., $w(1) < S$) such that

$$m(T) = E(w), \quad \text{for every } T \in [w(1), S].$$

It is easy to check that $w(1) = S_c$; otherwise we get a contradiction with the definition of S_c . Hence

$$m(S) = m(S_c), \quad \text{for any choice of } S \geq S_c.$$

(ii) We already know from Lemma 4.3 that m is decreasing. If we suppose that m is not strictly decreasing in $[0, S_c]$, we get a contradiction with Lemma 4.4.

(iii) Using (i) and (ii), it is sufficient to show the convexity of m in $[0, S_c]$. Fix $S, T \in [0, S_c]$, with $S < T$. Let $v_S \in W_S$ (respectively $v_T \in W_T$) denote the solution of (P_S) (respectively (P_T)). For any $\lambda \in]0, 1[$, we define $S_\lambda \in [0, S_c]$ by $S_\lambda = \lambda S + (1 - \lambda)T$. Let $w_\lambda \in W_{S_\lambda}$ denote the solution of (P_{S_λ}) and let $v_\lambda \in W_{S_\lambda}$ be defined by

$$v_\lambda(x) = \lambda v_S(x) + (1 - \lambda)v_T(x), \quad \text{for every } x \in [0, 1].$$

We then have

$$\begin{aligned} m(\lambda S + (1 - \lambda)T) &= E(w_\lambda) \leq E(v_\lambda) < \lambda E(v_S) + (1 - \lambda)E(v_T) \\ &= \lambda m(S) + (1 - \lambda)m(T); \end{aligned}$$

hence m is convex (strictly in $[0, S_c]$).

The continuity of m in $\mathbb{R}^+ \setminus \{0\}$ follows as a corollary of the convexity of m . The continuity of m at $S=0$ is trivial if $S_c=0$ and is easily obtained as follows, if $S_c > 0$. Let $S \in]0, S_c]$ and $v_S \in W_S$ be the solution of (P_S) . We then have

$$\begin{aligned} 0 \leq m(0) - m(S) &= m(0) - E(v_S) \\ &= \int_0^1 \{ \rho(x)[\phi_0(0) - \phi_0(v'_S)] + [\psi(x, 0) - \psi(x, v_S)] \} dx \\ &\leq \rho(1) \left[\phi_0(0) - \int_0^1 \phi_0(v'_S) dx \right] \leq \rho(1)[\phi_0(0) - \phi_0(S)], \quad (1) \end{aligned}$$

where we used that ρ is increasing, that

$$\psi(x, 0) - \psi(x, v_S) \leq 0, \quad \text{for every } x \in]0, 1[,$$

and the Jensen inequality in the last step of (1). The continuity of m at $S=0$ follows then from the continuity of ϕ_0 . ■

The proof of Theorem 4.1 is obtained as a corollary of Lemmas 4.2 to 4.7.

5. APPLICATIONS IN ECOLOGY

The problem

$$\inf \left\{ E(v) = \int_0^1 [\rho(x) \phi_0(v') + \psi(x, v)] dx : v \in W_S \right\} \quad (P_S)$$

studied in this paper appears in optimal foraging models [20, 22, 26] of behavioural ecology [19]. These models attempt to find the optimal way for an animal to behave in its foraging activity (i.e., search and acquisition of food) and often resort to the tools of optimal control theory [9, 10].

The model of Arditi and Dacorogna [2, 3, 4, 18] deals with the situation of an animal moving through its habitat to find food. The food resource is not assumed to be uniformly distributed as in many models [e.g., 1], or to be piecewise constant (patchy) as in most studies [e.g., 7]. It is assumed that the initial food density ρ is any arbitrary piecewise continuous function, allowing therefore more general situations where patch delimitation of the food resource is impossible (e.g., an antelope grazing in the African savanna [19]).

The animal is described by the inverse function of its trajectory in the habitat (interval $[0, 1]$), i.e., by its schedule $v = v(x)$ (time as a function of position x). The animal crosses the habitat during the foraging period S ($v(0) = 0$, $v(1) = S$), which is the total time allocated to the foraging activity. The maximum velocity of the animal is constrained. This gives rise to the optimization constraint $v'(x) \geq 0$ in $[0, 1]$.

The schedule derivative $v'(x)$ is called the “foraging presence” at point x . It represents the time during which the animal consumes the resource at point x . It is assumed that the local renewal rate of food resource is slow, so that as the animal stays in the same place, the rate at which it acquires food drops [30]. This gives rise to the dependence $\phi_0(v') = e^{-v'}$, or to some other more general function ϕ_0 than the exponential satisfying hypothesis (C1) of Section 4. The energy gained G in the whole habitat is therefore

$$G(v) = \int_0^1 \rho(x) [1 - e^{-v'(x)}] dx.$$

This quantity G is directly related to the fitness that must be maximized by optimal behaviour [28]. In the first variant of the model [2], the animal is a “time constrained energy maximizer” [27]. This means that finding an optimal behaviour consists in finding an admissible v maximizing G in the function space W_S , for fixed $S \geq 0$. This is equivalent to Problem (P_S) with $\psi \equiv 0$, treated in Section 2.

In another variant of the model, Botteron and Arditi [5] introduce risk costs in terms of fitness [28]. These costs are related to the distance from the animal to a central place such as a nest or cache [17, 25]. They describe, for example, the risk of being captured while foraging [13, 16] or of losing the nest or the offspring while away [15, 23]. Other situations can be considered and this variant deals with arbitrary risk costs which alter the foraging optimal behaviour [29]. The total risk cost is usually of the form

$$C(v) = \int_0^1 h(x)[v(x)]^p dx, \quad \text{with } p \geq 1,$$

where h is related to the risk density. The case $p = 1$ is solved in [5]. The dependence v^p can of course be generalized to some other function ψ_0 such that $\psi = \psi(x, r) = h(x) \psi_0(r)$ satisfies hypotheses (C2) to (C4) of Section 4.

In this variant, the fitness to be maximized by an optimal behaviour $v \in W_S$ is $G - C$. This is equivalent to Problem (P_S) with

$$E(v) = \int_0^1 [\rho(x) e^{-v(x)} + h(x)[v(x)]^p] dx.$$

For each fixed foraging period $S \geq 0$, first (P_S) is solved (Sections 3, 4). Afterwards, Problem (P) compares the optimal behaviours for different values S and determines the finite optimal foraging period (also called "optimal duration of absence from the nest" [5]), which is S_c (Section 4). In the case $\psi \equiv 0$ (Section 2), no risk costs are introduced in the model [2]. That is why $S_c = +\infty$ is the solution of (P), corresponding to the entire consumption of the available resource ρ .

This problem is of ecological interest, because it determines the portion of the animal's time that should be allocated to foraging in order to be optimal, the remaining time being devoted to other activities [e.g., 21, 29]. The optimal behaviour in that case balances the two conflicting needs of foraging efficiently and of minimizing the risks incurred (e.g., risk of predation [31]). This conflict is translated mathematically by the decrease of $\phi(x, \cdot)$ against the growth of $\psi(x, \cdot)$.

ACKNOWLEDGMENTS

We thank R. Arditi, C. Stuart and B. Zwahlen, for interesting discussions. One of us (B.B.) received support from the Swiss National Foundation for Scientific Research (Grant 2.727-0.85).

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