

## Existence of Solutions for a Variational Problem Associated to Models in Optimal Foraging Theory

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Realistic extensions of a model by R. Arditi and B. Dacorogna (1985, *Math. Biosci.* 76, 127–145) in optimal foraging theory are considered. Mathematically, they correspond to minimization problems (in the calculus of variations) of convex but not coercive functionals:  $E(v) = \int g(x, v') dx$  with prescribed boundary conditions and  $v' \geq \beta \geq 0$ . Existence, uniqueness, and characterization of solutions are given. The limits of the result are discussed, in particular the case with dependence on  $v$ :  $g = g(x, v, v')$ . © 1990 Academic Press, Inc.

### I. INTRODUCTION

In recent papers, Arditi and Dacorogna [1–3] have used the methods of the calculus of variations and of optimal control theory to deal with some problems of optimal foraging theory (Stephens and Krebs [15]). It is the aim of this article to generalize their results by taking into account more realistic foraging models. In particular, our mathematical result will allow us to consider more natural costs of locomotion, confusion effect, etc. We shall return to the model and to the assumptions we make in the next section. We now state the mathematical problem. We wish to find a function  $v$  which minimizes a given integral with prescribed conditions.

More precisely, the problem under consideration is

$$(P) \quad \inf \left\{ E(v) = \int_0^1 g(x, v'(x)) dx : v(0) = 0, v(1) = S, v' \geq \beta \geq 0, \right. \\ \left. \text{and } v \in C^1([0, 1]) \right\},$$

where  $C^1([0, 1])$  stands for the set of continuously differentiable functions defined in  $[0, 1]$ . In the foraging models (see next section), the problem (P) is equivalent, for example, to the maximization of the net energy gained by the animal. In the above-mentioned articles of Arditi and Dacorogna,  $g(x, v') = \rho(x)e^{-v'}$ , where  $\rho$  is the initial food distribution in the habitat.

The problem (P) is a classical problem of the calculus of variations when  $g$  is convex and coercive (i.e.,  $g(x, v')/|v'| \rightarrow +\infty$  if  $|v'| \rightarrow +\infty$ ) in the last variable (see, e.g., Cesari [4] or Dacorogna [7]). In this case, the existence of a solution is easily obtained and the problem can then be handled by using the classical necessary conditions such as the Pontryagin maximum principle (see, e.g., Cesari [4] or Clark [6]).

However, in a realistic model such as the original one or its extensions presented here,  $g$  is convex but not coercive. The classical necessary conditions still apply but their sufficiency is not any more obvious. We shall therefore prove an existence theorem which includes non-coercive integrands  $g$  and we shall give some examples showing that our hypotheses are in some sense optimal.

We shall also conclude that if the integrand  $g$  depends explicitly on  $v$  (i.e.,  $g = g(x, v, v')$ ), then in general the problem (P) does not have a solution. One such example will be  $g(x, v, v') = e^{-v} + v$ . The case where  $g$  depends explicitly on  $v$  is studied in a forthcoming paper to appear in *Journal of Differential Equations*.

## II. THE ORIGINAL BIOLOGICAL PROBLEM AND SOME EXTENSIONS

The original model of Arditi and Dacorogna [1] is a deterministic model of optimal foraging theory (McArthur and Pianka [11], Charnov [5], Pyke [12]).

Optimal foraging theory is a part of behavioural ecology (Krebs and McCleery [9]). In particular, it models the foraging behaviour of animals under constraints. It gives the strategy determined by the maximization of the fitness associated to the behaviour (see, e.g., Sibly and McFarland [14]).

In the bounded one-dimensional model of Arditi and Dacorogna [1], the animal maximizes the net energy gained and is then a "time constrained energy maximizer" (Schoener [13]). Mathematically, the problem is a minimization problem of calculus of variations:

$$(P) \quad \inf \left\{ E(v) = \int_0^1 g(x, v'(x)) dx : v \in W \right\},$$

where  $W$  is the set of admissible functions defined by

$$W = \{v \in \text{Lip}(0, 1) : v(0) = 0, v(1) = S, v'(x) \geq \beta \geq 0 \text{ a.e. in } [0, 1]\},$$

where  $\text{Lip}(0, 1)$  stands for the set of Lipschitz functions (i.e., continuous functions with almost everywhere uniformly bounded first derivative) defined in  $[0, 1]$ . It is assumed that  $S > \beta$ , since if  $S \leq \beta$ , the problem is trivial.

We refer to the above-mentioned publications of Arditi and Dacorogna for the complete derivation and the discussion of the original problem with  $g(x, v') = \rho(x)e^{-v'(x)}$ , where the initial food distribution  $\rho$  is a positive piecewise continuous function defined in the foraged habitat  $[0, 1]$  and where  $v$  represents the schedule (inverse of the trajectory) of the animal.  $S$  is the total time available to forage the whole habitat.  $1/\beta$  is the upper physiological bound of the velocity  $1/v'$  of the animal.

The existence of a solution for this problem is not a priori assured, since  $g$  is convex but not coercive with respect to  $v'$ . However, it is shown (Arditi and Dacorogna [1]) that the original problem has a solution and therefore that the necessary conditions (for example, the Pontryagin maximum principle commonly used in optimal control theory) are also sufficient.

We want to consider some realistic improvements of the original model and then solve the associated problem.

A first modification of the model, suggested by the authors themselves, is obtained by assuming that the animal does not consume any food when it travels at maximum velocity, i.e., when  $v' = \beta$  (Arditi and Dacorogna [2]). This modification yields the same mathematical problem with a new integrand:  $g(x, v') = \rho(x)e^{-v'(x) + \beta}$ .

Arditi and Dacorogna [1] also consider linear energetic costs of locomotion attached to the velocity of the animal:

$$c_l \left( \frac{dx}{dt} \right) dt, \quad \text{with } c_l > 0,$$

yielding a constant total cost  $c_l$ .

This linear dependence does not describe the overdifficulty to perform the same task more quickly. In analogy with Sibly and MacFarland [14], we can consider a quadratic dependence on the "performing activity" (here the velocity), or a more general convex super-linear function of the velocity.

Considering energetic locomotion costs of the type

$$c_l \left( \frac{dx}{dt} \right)^{p+1} dt, \quad \text{with } c_l, p > 0$$

and deducting these costs from the net energy gained by foraging, the integrand of the problem (P) becomes, after rescaling,

$$g(x, v') = \rho(x) e^{-v'(x) + \beta} + \frac{1}{[v'(x)]^p}, \quad \text{with } p > 0,$$

which is also convex but not coercive.

Another possible extension of the model takes into account the “confusion effect” (Heller and Milinski [8]). Confusion arises from the simultaneous presence in the animal’s field of vision of several prey items. Heller and Milinski associate costs to this effect in terms of loss of fitness (see also Sibly and McFarland [14]) in their model of the foraging behaviour of sticklebacks. With this extension, the model then leads to an integrand of the type

$$g(x, v') = \rho(x) \left( e^{-v'(x) + \beta} + K \frac{1}{[v'(x)]^p} \right), \quad \text{with } K, p > 0.$$

Other costs as well as combinations of them could be considered. We now give a mathematical result which allows the treatment of all the above problems.

### III. THE MAIN THEOREM

Let us introduce the assumptions made on the integrand  $g$ . The first hypothesis (regularity and strict convexity) is:

- (H1) (a) for every fixed  $y \in [\beta, +\infty[$ ,  $g(\cdot, y)$  and  $g_y(\cdot, y) = (\partial/\partial y) g(\cdot, y)$  are  $C^0([0, 1])$ ;
- (b) for every fixed  $x \in [0, 1]$ ,  $g(x, \cdot)$  is  $C^1([\beta, +\infty[)$  and strictly convex.

Instead of the standard coercivity assumption on the integrand  $g$ , we introduce a weaker assumption (H2). Let us first introduce the notations

$$\begin{aligned} \mu: [0, 1] &\rightarrow \mathbb{R} \cup \{+\infty\} \\ x \mapsto \mu(x) &= \lim_{\zeta \rightarrow +\infty} g_y(x, \zeta), \\ \mu_0 &= \inf\{\mu(x) : x \in [0, 1]\}, \\ \alpha_0 &= \inf\{g_y(x, \beta) : x \in [0, 1]\}, \\ \gamma &= \sup\{g_y(x, S) : x \in [0, 1]\}. \end{aligned}$$

By (H1),  $\mu_0 \in \mathbb{R} \cup \{+\infty\}$ ,  $\gamma \in \mathbb{R}$ , and  $\alpha_0 < \gamma$  since  $S > \beta$ .

With these notations, the second hypothesis on  $g$  is:

(H2)  $\gamma \leq \mu_0$ .

It is easy to see that all coercive functions  $g$  (i.e.,  $g(x, y)/|y| \rightarrow +\infty$  if  $|y| \rightarrow +\infty$ ) satisfy (H2). One can also verify that the original problem of Arditì and Dacorogna [1] as well as all the extensions or combinations of them presented above satisfy (H1) and (H2). For this class of problems, we

can now state the theorem that gives the existence, the uniqueness, and a complete explicit characterization of the solution of (P).

**THEOREM.** *Let  $g$  satisfy the hypotheses (H1) and (H2). Then:*

(I) (Existence and uniqueness) *There exist:*

(a) *a unique value  $\alpha \in ]\alpha_0, \gamma]$ ,*

(b) *a unique set  $\Omega_\alpha \subset [0, 1]$ ,  $\Omega_\alpha \neq \emptyset$  defined by*

$$\Omega_\alpha = \{x \in [0, 1]: \text{there exists } \zeta > \beta \text{ such that } g_y(x, \zeta) = \alpha\},$$

(c) *a unique function  $f_\alpha \in C^0(\Omega_\alpha)$ , defined by*

$$f_\alpha: \Omega_\alpha \rightarrow ]\beta, +\infty[$$

$$x \mapsto f_\alpha(x) = \zeta, \text{ solution of } g_y(x, \zeta) = \alpha,$$

*such that the following property (+) is satisfied:*

$$\int_{\Omega_\alpha} f_\alpha(x) dx = S - \beta |\Omega_\alpha^c|, \tag{+}$$

*where  $\Omega_\alpha^c = [0, 1] \setminus \Omega_\alpha$  and  $|\Omega_\alpha^c|$  stands for the measure of  $\Omega_\alpha^c$ .*

(II) (Characterization of the solution) *Let  $w: [0, 1] \rightarrow \mathbb{R}$  be defined by*

$$w(0) = 0 \quad \text{and} \quad w'(x) = \begin{cases} \beta, & \text{if } x \in \Omega_\alpha^c \\ f_\alpha(x), & \text{if } x \in \Omega_\alpha. \end{cases}$$

*Then  $w \in W$ ,  $w \in C^1([0, 1])$ , and  $w$  is the unique solution of (P).*

*Remarks.* (1)  $\Omega_\alpha$  is the set where the solution  $v$  of the Euler equation associated to (P) satisfies  $v'(x) > \beta$ .

(2) The characterization of the solution is of course equivalent to this obtained with the necessary conditions of optimal control theory (see, e.g., Lee and Markus [10]).

(3) The hypothesis of strict convexity (H1) can be weakened to convexity. In that case, the existence of a solution is still ensured, but the uniqueness no more.

(4) Note that in some cases, such as the biological models under consideration (when  $g(x, v') = \rho(x)e^{-v'}$ ,  $\rho(x)e^{-v'(x)+\beta}$ , ...), one can weaken the continuity hypothesis on  $x$  in  $g(x, v')$  by allowing a finite number of discontinuities. The solution  $w$  is then Lipschitz (Arditi and Dacorogna [1, 2]), but not  $C^1$ . The proof is unaltered since in these cases  $\mu(x) \equiv 0$  in  $[0, 1]$  and  $\gamma \leq 0$ , independently of a change in one point of the value  $\rho(x)$ .

## IV. EXAMPLES AND COUNTEREXAMPLES

Before giving the proof of the theorem, we give some examples and counterexamples of its application.

EXAMPLE 1. Let (P) be the problem with integrand

$$g(x, v'(x)) = e^{-1/x} e^{-v'(x)}, \quad \text{where } e^{-1/0} \equiv 0,$$

which satisfies (H1) and (H2), since  $\gamma = \mu_0 = 0$ . The theorem then asserts that the solution of (P) is

$$w(x) = \begin{cases} \beta x, & \text{if } x \in [0, \alpha], \\ \text{Log } \alpha - 1 + (\beta + 1/\alpha)x - \text{Log } x, & \text{if } x \in ]\alpha, 1], \end{cases}$$

where  $\alpha$  is the unique solution of

$$\text{Log } \alpha = S - \beta + 1 - 1/\alpha \quad \text{in } [0, 1].$$

EXAMPLE 2. This example shows how the loss of coercivity (or of (H2)) may imply non-existence of Lipschitz solutions. Let

$$g(x, v'(x)) = \frac{1}{2}x[v'(x)]^2,$$

which satisfies (H1) but not (H2) since  $\gamma = S > 0 = \mu_0$ . Let us show that (P) has no solution. Obviously,

$$\inf\{E(v) : v \in W\} \geq \frac{\beta^2}{4}.$$

Let us consider the following sequence  $\{v_N\} \subset W$  defined by

$$v_N(x) = \begin{cases} \beta x, & \text{if } x \in \left[0, \frac{1}{N}\right], \\ (S - \beta) + \beta x + \frac{S - \beta}{\text{Log } N} \text{Log } x, & \text{if } x \in \left]\frac{1}{N}, 1\right]. \end{cases}$$

Since we have

$$\frac{\beta^2}{4} \leq E(v_N) \leq \frac{\beta^2}{4} + \frac{2S^2}{\text{Log } N}, \quad \text{for each } N > 1,$$

we deduce that

$$\inf\{E(v) : v \in W\} = \frac{\beta^2}{4}.$$

Since  $E(w) = \beta^2/4$  and  $w' \geq \beta$  imply  $w'(x) = \beta$  a.e. in  $[0, 1]$ , we deduce that the infimum is not attained in  $W$  and thus (P) has no solution.

*Remark.* Note that even in a larger space such as  $AC(0, 1)$ , i.e., the space of continuous functions with integrable first derivative, (P) has no solution (see Cesari [4] and Dacorogna [7]).

EXAMPLE 3. *Limit of the theorem: case without constraint  $v'(x) \geq \beta$ .* We consider now the same problem as in Example 1, but without the constraint  $v' \geq \beta$ , and we show that this problem has no solution.

Let (P') denote the problem

$$(P') \quad \inf \left\{ E(v) = \int_0^1 e^{-1/x} e^{-v(x)} dx : v \in W' \right\},$$

where  $W' = \{v \in \text{Lip}(0, 1) : v(0) = 0, v(1) = 1\}$  and  $e^{-1/0} \equiv 0$ . We now show that (P') has no solution. Obviously,

$$\inf\{E(v) : v \in W'\} \geq 0.$$

Let us consider the following sequence  $\{v_N\} \subset W'$  defined by

$$v_N(x) = \begin{cases} (N + \text{Log } N)x, & \text{if } x \in \left[0, \frac{1}{N}\right], \\ 1 - \text{Log } x - (1-x) \text{Log } N, & \text{if } x \in \left]\frac{1}{N}, 1\right]. \end{cases}$$

Since we have

$$0 \leq E(v_N) \leq \frac{1}{N}, \quad \text{for each } N > 1,$$

we deduce that

$$\inf\{E(v) : v \in W'\} = 0.$$

Since  $E(w) = 0$  implies  $w'(x) = +\infty$  a.e. in  $[0, 1]$ , we deduce that the infimum is not attained in  $W'$  and thus (P') has no solution.

*Remark.* Comparing Examples 1 and 3, we see that the constraint  $v' \geq \beta$  has some "regularizing effect" on the existence of solutions.

EXAMPLE 4. *Limit of the theorem: case where  $g = g(x, v, v')$ .* The case where  $g$  depends explicitly on  $v$  is more complicated and the method developed here cannot be applied in general.

We give here a simple example which has no solution and where the dependence on  $v'$  of  $g$  is the same as in the original biological model.

Let  $(P_S)$  denote the problem

$$(P_S) \quad \inf \left\{ E(v) = \int_0^1 [e^{-v(x)} + v(x)] dx : v \in W_S \right\},$$

where for  $S > \beta \geq 0$ ,

$$W_S = \{v \in \text{Lip}(0, 1) : v(0) = 0, v(1) = S, v'(x) \geq \beta \text{ a.e. in } [0, 1]\},$$

**PROPOSITION.** For  $S > \beta + e^{-\beta}$ ,  $(P_S)$  has no solution.

*Remark.* The biological interpretation of the non-existence of solutions of  $(P_S)$  for  $S > \beta + e^{-\beta}$  would be the following.  $(P_S)$  would correspond to the foraging of an animal in a habitat with uniform food distribution ( $\rho(x) \equiv 1$ ) and with a cost  $\int_0^1 v(x) dx$  attached to the instantaneous schedule  $v(x)$  of the animal (i.e., a cost increasing with time elapsed). In view of the proof of the proposition, for a too large time  $S$  available to forage, it would be more profitable for the animal to forage with schedule  $v_0$  and to finish before the time  $S$ , in fact at time  $\beta + e^{-\beta}$ , since in comparison to profits, costs would then become too important at the end of the foraging.

*Proof of the Proposition.* Let  $v_0$  be the following function

$$v_0(x) = \begin{cases} \beta x, & \text{if } x \in [0, 1 - e^{-\beta}], \\ \beta + e^{-\beta} - (1 - x) + (1 - x) \text{Log}(1 - x), & \text{if } x \in ]1 - e^{-\beta}, 1], \end{cases}$$

where  $0 \cdot \text{Log } 0 \equiv 0$ .

Let  $S > \beta + e^{-\beta}$ . We first show that

$$\inf \{E(v) : v \in W_S\} \geq E(v_0).$$

Let

$$V = \{v \in \text{AC}(0, 1) : v(0) = 0, v'(x) \geq \beta \text{ a.e. in } [0, 1]\},$$

where  $\text{AC}(0, 1)$  stands for the set of continuous functions with integrable first derivative (note that  $\text{Lip}(0, 1) \subset \text{AC}(0, 1)$ ). Let  $p: [0, 1] \rightarrow \mathbb{R}$  be the function  $p(t) = E(v_0 + t(v - v_0))$ , where  $v \in W_S$ . Obviously,  $v, v_0 \in V$ . Since  $E$  is strictly convex over  $V$ ,  $p$  is strictly convex and then  $p(1) > p(0) + p'(0)$ . However,



$$\begin{aligned}
p'(0) &= \int_0^1 [(v(x) - v_0(x)) - (v'(x) - v'_0(x)) e^{-v'_0(x)}] dx \\
&= \int_0^{1-e^{-\beta}} [(v(x) - v_0(x)) - (v'(x) - v'_0(x)) e^{-\beta}] dx \\
&\quad + \int_{1-e^{-\beta}}^1 [(v(x) - v_0(x)) - (v'(x) - v'_0(x))(1-x)] dx \\
&= \int_0^{1-e^{-\beta}} (v(x) - v_0(x)) dx - e^{-\beta} \int_0^{1-e^{-\beta}} \frac{d}{dx} (v(x) - v_0(x)) dx \\
&\quad + \int_{1-e^{-\beta}}^1 \frac{d}{dx} [(x-1)(v(x) - v_0(x))] dx \\
&= \int_0^{1-e^{-\beta}} (v(x) - v_0(x)) dx + e^{-\beta} [v(0) - v_0(0)].
\end{aligned}$$

Since  $v'(x) \geq \beta$  in  $[0, 1]$ , then  $v(x) \geq v_0(x)$  in  $[0, 1 - e^{-\beta}]$  and since  $v(0) = v_0(0) = 0$ , we deduce that  $p'(0) \geq 0$ . Thus  $p(1) > p(0)$ , i.e.,  $E(v) > E(v_0)$ , and thus  $\inf\{E(v) : v \in W_S\} \geq E(v_0)$ . In fact, we have

$$\inf\{E(v) : v \in W_S\} = E(v_0).$$

This can be seen considering, for sufficiently large  $N$ , the following sequence  $\{v_N\} \subset W_S$ :

$$v_N(x) = \begin{cases} \beta x, & \text{if } x \in [0, 1 - e^{-\beta}], \\ \beta + e^{-\beta} - (1-x) + (1-x) \operatorname{Log}(1-x), & \\ \quad \text{if } x \in \left] 1 - e^{-\beta}, 1 - \frac{1}{N} \right], \\ S(1-N+Nx) + [N(\beta + e^{-\beta}) - (1 + \operatorname{Log} N)][1-x], & \\ \quad \text{if } x \in \left] 1 - \frac{1}{N}, 1 \right]. \end{cases}$$

By construction, we have

$$\begin{aligned}
E(v_N) &= \int_0^{1-1/N} [v_0(x) + e^{-v'_0(x)}] dx + \int_{1-1/N}^1 [v_N(x) + e^{-v'_N(x)}] dx \\
&= E(v_0) + \int_{1-1/N}^1 [v_N(x) - v_0(x) + e^{-v'_N(x)} - e^{-v'_0(x)}] dx.
\end{aligned}$$

Since  $v_N(x) \leq S$  and  $v'_N(x) \geq \beta$  in  $[1 - 1/N, 1]$ , it is then clear that  $E(v_N) \rightarrow E(v_0)$  as  $N \rightarrow \infty$ .

We are now able to show the proposition. Ab absurdo, suppose there exists  $w \in W_S$  such that  $E(w) = \inf\{E(v) : v \in W_S\}$ . From the preceding argument, we have  $E(w) = E(v_0)$ . Recall that  $v_0, w \in V$ . Let  $\tilde{w} \in V$  be defined by  $\tilde{w} = \frac{1}{2}(v_0 + w)$ . We have

$$E(\tilde{w}) = E(\frac{1}{2}v_0 + \frac{1}{2}w) < \frac{1}{2}E(v_0) + \frac{1}{2}E(w) = E(v_0).$$

by strict convexity of  $E$  in  $V$ . Since  $\tilde{w} \in W_{\tilde{S}}$ , where  $\tilde{S} = \frac{1}{2}(\beta + e^{-\beta} + S)$ ,  $E(\tilde{w}) < E(v_0)$  contradicts the fact that  $\inf\{E(v) : v \in W_{\tilde{S}}\} = E(v_0)$ . The proposition then follows. ■

V. PROOF OF THE THEOREM

The proof of the theorem is divided into five lemmas. Lemmas 1, 2, and 3 show part I and Lemmas 4 and 5 show part II of the theorem.

To simplify the notations, let us use the definitions of  $\Omega_\alpha$  and  $f_\alpha$  introduced in the theorem for any  $\alpha \in [\alpha_0, \gamma]$  and extend  $f_\alpha$  to  $[0, 1]$ , writing  $f_\alpha(x) = \beta$ , if  $x \in \Omega_\alpha^c$ .

The first lemma characterizes the  $\Omega_\alpha$ -type sets:

LEMMA 1. (i)  $\Omega_{\alpha_0} = \emptyset$ ,  $\Omega_\gamma = [0, 1]$ , and

$$\Omega_\alpha \neq \emptyset, \quad \text{if } \alpha \in ]\alpha_0, \gamma].$$

(ii) For  $\lambda \in [0, 1]$ , let

$$\gamma_\lambda = \sup\{g_\gamma(x, \lambda S + (1 - \lambda)\beta) : x \in [0, 1]\}.$$

Then, for every  $\lambda \in ]0, 1[$ ,  $\gamma_\lambda \in [\alpha_0, \gamma]$ ,

$$\gamma_\lambda \rightarrow \gamma \quad \text{when } \lambda \rightarrow 1.$$

(iii) For every  $\lambda \in ]0, 1[$ ,  $\Omega_{\gamma_\lambda} = [0, 1]$  and

$$f_{\gamma_\lambda}(x) \geq \lambda S + (1 - \lambda)\beta, \quad \text{for every } x \in [0, 1].$$

(iv) For any  $\alpha_1, \alpha_2 \in [\alpha_0, \gamma]$  with  $\alpha_1 < \alpha_2$ ,  $\Omega_{\alpha_1} \subset \Omega_{\alpha_2}$  and  $f_{\alpha_1}(x) < f_{\alpha_2}(x)$ , for every  $x \in \Omega_{\alpha_2}$ .

*Proof of Lemma 1.* Parts (i), (ii), (iii) are easily obtained using the fact that  $g_\gamma(x, \cdot)$  is strictly increasing (see (H1)). Let us show, for example, that  $\Omega_{\alpha_0} = \emptyset$ . For any fixed  $x \in [0, 1]$ ,  $\alpha_0 \leq g_\gamma(x, \beta)$ , by definition of  $\alpha_0$ . Thus the equation in  $\zeta$ :  $g_\gamma(x, \zeta) = \alpha_0$  has no solution  $\zeta > \beta$ , hence  $x \in \Omega_{\alpha_0}^c$  and then  $\Omega_{\alpha_0} = \emptyset$ .

Part (iv) is obvious if  $\alpha_1 = \alpha_0$ . For  $\alpha_1 > \alpha_0$  and for any  $x \in \Omega_{\alpha_1}$ ,

$g_y(x, \zeta_1) = \alpha_1$  for some  $\zeta_1 > \beta$ . By strict convexity of  $g(x, \cdot)$ ,  $g_y(x, \zeta_2) = \alpha_2$  for some  $\zeta_2 > \zeta_1 > \beta$ , hence  $\Omega_{\alpha_1} \subset \Omega_{\alpha_2}$ . Furthermore,  $f_{x_1}(x) < f_{x_2}(x)$  for every  $x \in \Omega_{\alpha_2}$ , since if  $\Omega_{\alpha_2} \setminus \Omega_{\alpha_1}$ , we adopted the convention that  $f_{x_1}(x) = \beta$ . ■

LEMMA 2. Let  $\alpha \in [\alpha_0, \gamma[$ . Then:

- (i)  $f_\alpha$  is bounded,
- (ii)  $f_\alpha$  is continuous.
- (iii) Furthermore, for fixed  $x \in [0, 1]$ ,  $f_\alpha(x)$  is a continuous function of  $\alpha$  in  $[\alpha_0, \gamma[$ .

*Proof of Lemma 2.* (i)  $f_\alpha$  is bounded, otherwise, using (H2),  $\alpha \geq \mu_0 \geq \gamma$  is in contradiction with  $\alpha \in [\alpha_0, \gamma[$ ,

(ii) and (iii) are easily obtained using (H1). ■

LEMMA 3. For  $\alpha \in [\alpha_0, \gamma[$ , let  $h$  be defined by

$$h(\alpha) = \int_{\Omega_\alpha} f_\alpha(x) dx - S + \beta |\Omega_\alpha^c|.$$

Then,

- (i)  $h$  is continuous in  $[\alpha_0, \gamma[$ ,
- (ii)  $h$  is strictly increasing,
- (iii)  $h(\alpha_0) < 0$ ,
- (iv)  $\lim_{\alpha \rightarrow \gamma} h(\alpha) \geq 0$ .

*Proof of Lemma 3.* (i) This follows from Lemma 2.

(ii) Let  $\alpha_1, \alpha_2 \in [\alpha_0, \gamma[$  with  $\alpha_1 < \alpha_2$ . Using Lemma 1, we have

$$\begin{aligned} h(\alpha_2) &= \int_{\Omega_{\alpha_2}} f_{\alpha_2}(x) dx - S + \beta |\Omega_{\alpha_2}^c| \\ &> \int_{\Omega_{\alpha_2}} f_{\alpha_1}(x) dx - S + \beta \{ |\Omega_{\alpha_1}^c| - |\Omega_{\alpha_2} \setminus \Omega_{\alpha_1}| \} \\ &= h(\alpha_1) + \int_{\Omega_{\alpha_2} \setminus \Omega_{\alpha_1}} [f_{\alpha_1}(x) - \beta] dx = h(\alpha_1). \end{aligned}$$

since  $f_{x_1}(x) = \beta$  for every  $x \in \Omega_{\alpha_2} \setminus \Omega_{\alpha_1}$ . Hence  $h(\alpha_2) > h(\alpha_1)$ .

(iii) By Lemma 1,  $\Omega_{\alpha_0} = \emptyset$ . Then  $h(\alpha_0) = -S + \beta < 0$ , since  $S > \beta$ .

(iv) Using Lemma 1, we have, for any  $\lambda \in ]0, 1[$ ,  $h(\gamma_\lambda) \geq -(1 - \lambda)(S - \beta)$ , and thus the result. ■

If  $\lim_{x \rightarrow \gamma} h(x) = 0$  in Lemma 3, we extend  $h$  by continuity to  $[\alpha_0, \gamma]$  by setting  $h(\gamma) = 0$ . Lemmas 1, 2, and 3 then ensure the existence and the uniqueness of  $\alpha \in ]\alpha_0, \gamma]$  such that  $h(\alpha) = 0$ , of a unique set  $\Omega_x \neq \emptyset$  and a unique function  $f_x \in C^0(\Omega_x)$ , such that the property (+) of the theorem, equivalent to  $h(\alpha) = 0$ , is satisfied. This ends up the proof of part I of the theorem.

To prove part II, we shall first show that the function  $w$  defined in the theorem is the unique solution of the problem (P) in a reduced function set  $W_1 \subset W$  (Lemma 4) and then we shall extend the result to  $W$  (Lemma 5).

LEMMA 4. *Let  $W_1 \subset W$  be defined by*

$$W_1 = \left\{ v \in W : \int_{\Omega_x} v'(x) dx = S - \beta |\Omega_x^c| \right\},$$

where  $\alpha$  and  $\Omega_x$  are given in part I of the theorem. If  $w$  is the function defined in the theorem, then:

- (i)  $w \in W_1$  and  $w \in C^1([0, 1])$ ,
- (ii)  $E(v) > E(w)$ , for every  $v \in W_1$ ,  $v \neq w$ .

*Proof of Lemma 4.* (i) This is easily verified with the property (+) of the theorem, with the boundedness and the continuity of  $f_x$  given by Lemma 2.

(ii) Let  $v \in W_1$  be fixed with  $v \neq w$  and  $p: [0, 1] \rightarrow \mathbb{R}$  be the function  $p(t) = E(w + t(v - w))$ . Its derivative at the origin is

$$\begin{aligned} p'(0) &= \int_0^1 g_y(x, w'(x)) [v'(x) - w'(x)] dx \\ &= \alpha \int_{\Omega_x} [v'(x) - w'(x)] dx + \int_{\Omega_x^c} g_y(x, \beta) [v'(x) - \beta] dx, \end{aligned}$$

by definition of  $\alpha$ ,  $\Omega_x$ , and  $w$ . The integral over  $\Omega_x^c$  is equal to zero since  $v \in W_1$ . Since  $w \in W_1$ , the integral over  $\Omega_x$  is also equal to zero because

$$\int_{\Omega_x} v'(x) dx = \int_{\Omega_x} w'(x) dx = S - \beta |\Omega_x^c|.$$

Hence  $p'(0) = 0$ . Since  $p$  is strictly convex, we deduce that  $p(1) > p(0) + p'(0) = p(0)$ , which is equivalent to  $E(v) > E(w)$ . ■

LEMMA 5. *If  $w$  is the function defined in the theorem, then*

$$E(v) > E(w), \quad \text{for every } v \in W, v \neq w.$$

*Proof of Lemma 5.* For  $v \in W$  with  $v \neq w$  being fixed, we define  $u: [0, 1] \rightarrow \mathbb{R}$  by

$$u(0) = 0 \text{ and } u'(x) = \begin{cases} \beta, & \text{if } x \in \Omega_x^c, \\ v'(x) + \frac{1}{|\Omega_x^c|} \int_{\Omega_x^c} [v'(y) - \beta] dy, & \text{if } x \in \Omega_x. \end{cases}$$

It can be easily verified that  $u \in W_1$ . As in Lemma 4, let  $p: [0, 1] \rightarrow \mathbb{R}$  be defined by  $p(t) = E(w + t(v - w))$ . Then

$$p'(0) = \alpha \int_{\Omega_x} [v'(x) - w'(x)] dx + \int_{\Omega_x^c} g_y(x, w'(x)) [v'(x) - \beta] dx,$$

which can be written

$$p'(0) = \alpha \int_{\Omega_x} [u'(x) - w'(x)] dx + \int_{\Omega_x^c} [g_y(x, \beta) - \alpha] \cdot [v'(x) - \beta] dx,$$

using the function  $u$  in  $\Omega_x$ . The first integral is zero, since  $u, w \in W_1$ . The second integral is non-negative, since  $g_y(x, \beta) - \alpha \geq 0$ , for every  $x \in \Omega_x^c$ . We then obtain  $p'(0) \geq 0$  and then  $E(v) > E(w)$ . ■

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