On the best constant in Gaffney inequality

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August 28, 2017

Abstract

We discuss the value of the best constant in Gaffney inequality namely

$$\|\nabla\omega\|_{L^2}^2 \leq C\left(\|d\omega\|_{L^2}^2 + \|\delta\omega\|_{L^2}^2 + \|\omega\|_{L^2}^2\right)$$

when either $\nu \wedge \omega = 0$ or $\nu \lrcorner\, \omega = 0$ on $\partial\Omega$.

1 Introduction

We start by recalling Gaffney inequality for vector fields. Let $\Omega \subset \mathbb{R}^n$ be a bounded open smooth set and ν be the outward unit normal to $\partial\Omega$. Then there exists a constant $C = C(\Omega) > 0$ such that for every vector field $u \in W^{1,2}(\Omega; \mathbb{R}^n)$

$$\|\nabla u\|_{L^2}^2 \leq C\left(\|\operatorname{curl} u\|_{L^2}^2 + \|\operatorname{div} u\|_{L^2}^2 + \|u\|_{L^2}^2\right)$$

where, on $\partial\Omega$, either $\nu \wedge u = 0$ (i.e. u is parallel to ν and we write then $u \in W_T^{1,2}(\Omega; \mathbb{R}^n)$) or $\nu \lrcorner\, u = 0$ (i.e. u is orthogonal to ν and we write then $u \in W_N^{1,2}(\Omega; \mathbb{R}^n)$). In the context of differential forms (identifying $1-$forms with vector fields) this generalizes to (using the notations of [14] which are summarized in the next section) the following theorem (for references see below).

Theorem 1 (Gaffney inequality) *Let $0 \leq k \leq n$ and $\Omega \subset \mathbb{R}^n$ be a bounded open C^2 set. Then there exists a constant $C = C(\Omega, k) > 0$ such that*

$$\|\nabla\omega\|_{L^2}^2 \leq C\left(\|d\omega\|_{L^2}^2 + \|\delta\omega\|_{L^2}^2 + \|\omega\|_{L^2}^2\right)$$

for every $\omega \in W_T^{1,2}\left(\Omega; \Lambda^k\right) \cup W_N^{1,2}\left(\Omega; \Lambda^k\right)$.

The aim of this article is to study the best constant in such inequality. We therefore define

$$C_T(\Omega,k)=\sup_{\omega\in W_T^{1,2}\setminus\{0\}}\left\{\frac{\|\nabla\omega\|^2}{\|d\omega\|^2+\|\delta\omega\|^2+\|\omega\|^2}\right\} \tag{1}$$

$$C_N(\Omega,k)=\sup_{\omega\in W_N^{1,2}\setminus\{0\}}\left\{\frac{\|\nabla\omega\|^2}{\|d\omega\|^2+\|\delta\omega\|^2+\|\omega\|^2}\right\} \tag{2}$$

where $\|\cdot\|$ stands for the L^2-norm. It is easy to see (cf. Proposition 2) that $C_T(\Omega,k), C_N(\Omega,k)\geq 1$. The cases $k=0$ and $k=n$ are trivial and we always have then that $C_T=C_N=1$; so the discussion deals with the case $1\leq k\leq n-1$. In our main result, we need the concept of $k-$convexity (cf. Definition 4), which generalizes the usual notion of convexity, which corresponds to the case of $1-$convexity, while $(n-1)-$convexity means that the mean curvature of $\partial\Omega$ is non-negative. Our main result (cf. Theorem 8) shows that the following conditions are equivalent.

(i) $C_T(\Omega,k)=1$ (respectively $C_N(\Omega,k)=1$).

(ii) Ω is $(n-k)-$convex (respectively Ω is $k-$convex).

(iii) The sharper version of Gaffney inequality holds namely $\forall\,\omega\in W_T^{1,2}\left(\Omega;\Lambda^k\right)$ (respectively $\forall\,\omega\in W_N^{1,2}\left(\Omega;\Lambda^k\right)$)

$$\|\nabla\omega\|^2\leq\|d\omega\|^2+\|\delta\omega\|^2.$$

(iv) The respective supremum is not attained.

(v) C_T (respectively C_N) is scale invariant, namely, for every $t>0$

$$C_T(t\,\Omega,k)=C_T(\Omega,k).$$

The result that Ω is $(n-k)-$convex implies that $C_T(\Omega,k)=1$ was already observed by Mitrea [27].

We also show that for general non-convex domains the constants $C_T(\Omega,k)$ and $C_N(\Omega,k)$ can be arbitrarily large (see Proposition 17).

The smoothness of the domain is essential in the previous discussion. We indeed prove (see Theorem 20) that if Ω is a polytope (convex or not), or even more generally a set whose boundary is composed only of hyperplanes, then

$$\|\nabla\omega\|^2=\|d\omega\|^2+\|\delta\omega\|^2,\quad\forall\,\omega\in C_T^1\left(\overline{\Omega};\Lambda^k\right).$$

i.e. Gaffney inequality is, in fact, an equality with constant $C_T(\Omega,k)=1$ (the result is also valid with T replaced by N). This fact was already observed in [11] when $n=3$ and $k=1$ for polyhedra.

We now comment briefly on the history of Gaffney inequality. The proof goes back to Gaffney [19], [20] for manifolds without boundary and for manifolds with boundary to Friedrichs [18], Morrey [29] and Morrey-Eells [31]. Further contributions are in Bolik [5] (for a version in L^p and in Hölder spaces), Csató-Dacorogna-Kneuss [14], Iwaniec-Martin [23], Iwaniec-Scott-Stroffolini [24] (for a version in L^p), Mitrea [27], Mitrea-Mitrea [28], Morrey [30], Schwarz [33], Taylor [35] and von Wahl [36]. Specifically for the inequality for vector fields in dimension 2 and 3, we can refer to Amrouche-Bernardi-Dauge-Girault [2], Costabel [9] and Dautray-Lions [16].

A stronger version of the classical Gaffney inequality reads as

$$\|\nabla\omega\|^2\leq C\left(\|d\omega\|^2+\|\delta\omega\|^2+\|\omega\|^2\right),\quad\forall\,\omega\in W_T^{d,\delta,2}\left(\Omega;\Lambda^k\right)$$

(and similarly with T replaced by N) where

$$\omega \in W_T^{d,\delta,2}\left(\Omega;\Lambda^k\right) = \left\{\omega \in L^2\left(\Omega;\Lambda^k\right) : \left\{\begin{array}{c} d\omega \in L^2\left(\Omega;\Lambda^{k+1}\right) \\ \delta\omega \in L^2\left(\Omega;\Lambda^{k-1}\right) \\ \nu \wedge \omega = 0 \text{ on } \partial\Omega \end{array}\right\}\right.$$

and the boundary condition has to be understood in a very weak sense. Clearly $W_T^{1,2} \subset W_T^{d,\delta,2}$. This stronger inequality is, in fact, a regularity result and is valid for smooth or convex Lipschitz domains leading, a posteriori, to

$$W_T^{d,\delta,2}\left(\Omega;\Lambda^k\right) = W_T^{1,2}\left(\Omega;\Lambda^k\right).$$

However, for non-convex Lipschitz domains one has, in general, $W_T^{1,2} \neq W_T^{d,\delta,2}$. We refer to Mitrea [27] and Mitrea-Mitrea [28]; while for vector fields in dimension 2 and 3, see Amrouche-Bernardi-Dauge-Girault [2], Ben Belgacem-Bernardi-Costabel-Dauge [4], Ciarlet-Hazard-Lohrengel [7], Costabel [8], Costabel-Dauge [10] and Girault-Raviart [21].

Clearly Gaffney inequality for $k = 1$ is reminiscent of Korn inequality. The best constant in Korn inequality have been investigated by Bauer-Pauly [3] and Desvillettes-Villani [17]. Our results (cf. Corollary 32) allow us to recover the best constant found in [3].

We should end this introduction with a striking analogy with the classical Hardy inequality (cf., for example [26] and the bibliography therein). Indeed, classically the best constant μ, when the domain is convex (and in fact $(n-1)-$convex, see [25]), is independent of the dimension (in this case $\mu = 1/4$) and the best constant is not attained; while for general non-convex domains the best constant is, in general, strictly less than $1/4$. However the authors were not able to see if this connection is fortuitous or not.

2 Notations

We now fix the notations, for further details we refer to [14].

(i) A $k-$form $\omega \in \Lambda^k = \Lambda^k\left(\mathbb{R}^n\right)$ is written as

$$\omega = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \omega^{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

when convenient it is identified to a vector in $\mathbb{R}^{\binom{n}{k}}$. When necessary, we extend, in a natural way, the definition of $\omega^{i_1 \cdots i_k}$ to any $1 \leq i_1, \cdots, i_k \leq n$ (see Notations 2.5 (iii) in [14]).

(ii) The exterior product of $\nu \in \Lambda^1$ and $\omega \in \Lambda^k$ is $\nu \wedge \omega \in \Lambda^{k+1}$ and is defined as

$$\begin{aligned} \nu \wedge \omega &= \sum_{1 \leq i_1 < \cdots < i_k \leq n} \left[\sum_{j=1}^{n} \nu^j \omega^{i_1 \cdots i_k}\right] dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \\ &= \sum_{1 \leq i_1 < \cdots < i_{k+1} \leq n} \left[\sum_{\gamma=1}^{k+1} (-1)^{\gamma-1} \nu^{i_\gamma} \omega^{i_1 \cdots i_{\gamma-1} i_{\gamma+1} \cdots i_{k+1}}\right] dx^{i_1} \wedge \cdots \wedge dx^{i_{k+1}}. \end{aligned}$$

(iii) The interior product of $\nu \in \Lambda^1$ and $\omega \in \Lambda^k$ is $\nu \lrcorner \omega \in \Lambda^{k-1}$ and is defined as

$$\nu \lrcorner \omega = \sum_{1 \leq i_1 < \cdots < i_{k-1} \leq n} \left[\sum_{j=1}^{n} \nu^j \omega^{j i_1 \cdots i_{k-1}}\right] dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}}.$$

(iv) The scalar product of $\omega, \lambda \in \Lambda^k$ is defined as

$$\langle \omega ; \lambda \rangle = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \left(\omega^{i_1 \cdots i_k} \lambda^{i_1 \cdots i_k} \right) = \frac{1}{k!} \sum_{1 \leq i_1 , \cdots , i_k \leq n} \left(\omega^{i_1 \cdots i_k} \lambda^{i_1 \cdots i_k} \right)$$

the associated norm being

$$|\omega|^2 = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \left(\omega^{i_1 \cdots i_k} \right)^2 = \frac{1}{k!} \sum_{1 \leq i_1 , \cdots , i_k \leq n} \left(\omega^{i_1 \cdots i_k} \right)^2 .$$

When $k = 1$ the interior and the scalar product coincide.

(v) The Hodge $*$ operator associates to $\omega \in \Lambda^k$, $*\omega \in \Lambda^{n-k}$ via the operation

$$\omega \wedge \lambda = \langle *\omega ; \lambda \rangle \, dx^1 \wedge \cdots \wedge dx^n , \quad \text{for every } \lambda \in \Lambda^{n-k}.$$

The interior product of $\nu \in \Lambda^1$ and $\omega \in \Lambda^k$ can be then written as

$$\nu \lrcorner \omega = (-1)^{n(k-1)} * (\nu \wedge (*\omega)) .$$

We also use several times the identities

$$\nu \wedge (\nu \lrcorner \omega) + \nu \lrcorner (\nu \wedge \omega) = |\nu|^2 \omega \quad \text{and} \quad \langle \nu \wedge \alpha ; \beta \rangle = \langle \alpha ; \nu \lrcorner \beta \rangle . \tag{3}$$

(vi) The exterior derivative of $\omega \in \Lambda^k$ is $d\omega \in \Lambda^{k+1}$ and is defined as

$$\begin{aligned} d\omega &= \sum_{1 \leq i_1 < \cdots < i_k \leq n} \left[\sum_{j=1}^{n} \omega_{x_j}^{i_1 \cdots i_k} \right] dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \\ &= \sum_{1 \leq i_1 < \cdots < i_{k+1} \leq n} \left[\sum_{\gamma=1}^{k+1} (-1)^{\gamma - 1} \, \omega_{x_{i_\gamma}}^{i_1 \cdots i_{\gamma-1} i_{\gamma+1} \cdots i_{k+1}} \right] dx^{i_1} \wedge \cdots \wedge dx^{i_{k+1}} . \end{aligned}$$

When $k = 1$ we can identify $d\omega$ with $\operatorname{curl} \omega$.

(vii) The interior derivative (or codifferential) of $\omega \in \Lambda^k$ is $\delta\omega \in \Lambda^{k-1}$ and is defined as

$$\delta\omega = \sum_{1 \leq i_1 < \cdots < i_{k-1} \leq n} \left(\sum_{j=1}^{n} \omega_{x_j}^{j i_1 \cdots i_{k-1}} \right) dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}} .$$

When $k = 1$ we can identify $\delta\omega$ with $\operatorname{div} \omega$.

(viii) The spaces $W_T^{1,2}\left(\Omega; \Lambda^k\right)$ and $W_N^{1,2}\left(\Omega; \Lambda^k\right)$ are defined as

$$W_T^{1,2}\left(\Omega; \Lambda^k\right) = \left\{ \omega \in W^{1,2}\left(\Omega; \Lambda^k\right) : \nu \wedge \omega = 0 \text{ on } \partial\Omega \right\}$$

$$W_N^{1,2}\left(\Omega; \Lambda^k\right) = \left\{ \omega \in W^{1,2}\left(\Omega; \Lambda^k\right) : \nu \lrcorner \omega = 0 \text{ on } \partial\Omega \right\}$$

where ν is the outward unit normal to $\partial\Omega$.

(ix) The sets $\mathcal{H}_T\left(\Omega; \Lambda^k\right)$ and $\mathcal{H}_N\left(\Omega; \Lambda^k\right)$ are defined as

$$\mathcal{H}_T\left(\Omega; \Lambda^k\right) = \left\{ \omega \in W_T^{1,2}\left(\Omega; \Lambda^k\right) : d\omega = 0 \text{ and } \delta\omega = 0 \text{ in } \Omega \right\}$$

$$\mathcal{H}_N\left(\Omega; \Lambda^k\right) = \left\{ \omega \in W_N^{1,2}\left(\Omega; \Lambda^k\right) : d\omega = 0 \text{ and } \delta\omega = 0 \text{ in } \Omega \right\} .$$

3 Some generalities

Our first result is the following.

Proposition 2 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $0 \leq k \leq n$. Then*

$$C_T(\Omega, k), C_N(\Omega, k) \geq 1.$$

Moreover

$$C_T(\Omega, 0) = C_N(\Omega, 0) = C_T(\Omega, n) = C_N(\Omega, n) = 1 \quad \text{and} \quad C_T(\Omega, k) = C_N(\Omega, n-k).$$

Remark 3 When $k = 0$ (respectively $k = n$), $\nabla\omega$ can be identified with $d\omega$ (respectively $\delta\omega$). Therefore, for any $\omega \in W^{1,2}$,

$$\|\nabla\omega\|^2 = \|d\omega\|^2 \quad (\text{respectively } \|\nabla\omega\|^2 = \|\delta\omega\|^2).$$

Hence the statements in the proposition when $k = 0$ or n are trivial.

Proof *Step 1.* We first prove the main statement. Let $\overline{x} = (\overline{x}_1, \cdots, \overline{x}_n) \in \Omega$ and $0 < r < R$ be such that

$$B_r(\overline{x}) \subset \Omega \subset B_R(\overline{x}).$$

Choose a function $\eta \in C_0^\infty(\Omega)$ such that $\eta = 1$ in B_r and $0 \leq \eta \leq 1$ in Ω. We extend it to $\mathbb{R}^n$ by 0. Define for every $m \in \mathbb{N}$

$$\omega_m(x) = \sin(mx_1)\,\eta(x)\,dx^1 \wedge \cdots \wedge dx^k \in C_0^\infty\left(\Omega; \Lambda^k\right).$$

Since ω_m vanishes on the boundary of B_R, we have, by Theorem 5.7 in [14] (see also [13]),

$$\int_{B_R} |\nabla\omega_m|^2 = \int_{B_R} \left(|d\omega_m|^2 + |\delta\omega_m|^2\right) \quad \Rightarrow \quad \int_\Omega |\nabla\omega_m|^2 = \int_\Omega \left(|d\omega_m|^2 + |\delta\omega_m|^2\right).$$

To prove that $C_T, C_N \geq 1$ it is sufficient to show that

$$\lim_{m\to\infty} \frac{\|d\omega_m\|^2 + \|\delta\omega_m\|^2 + \|\omega_m\|^2}{\|\nabla\omega_m\|^2} = 1 + \lim_{m\to\infty} \frac{\|\omega_m\|^2}{\|\nabla\omega_m\|^2} = 1.$$

Note that

$$\int_\Omega |\omega_m|^2 \leq \operatorname{meas}\Omega.$$

On the other hand we have that

$$\int_\Omega |\nabla\omega_m|^2 \geq \int_{B_r} |\nabla\omega_m|^2 = m^2 \int_{B_r} \cos^2(mx_1)\,dx.$$

Since B_r is open, there exists m sufficiently large so that

$$\left(\overline{x}_1, \overline{x}_1 + \frac{2\pi}{m}\right) \times B_r' \subset B_r$$

where $B_r' = \left\{y = (y_2, \cdots, y_n) \in \mathbb{R}^{n-1} : |y_i - \overline{x}_i| \leq \frac{r}{n},\ i = 2, \cdots, n\right\}$ is independent of m. We thus obtain that

$$\int_\Omega |\nabla\omega_m|^2 \geq m^2 \operatorname{meas}(B_r') \int_{\overline{x}_1}^{\overline{x}_1 + 2\pi/m} \cos^2(mx_1)\,dx_1.$$

Use now the change of variables $x_1 = t/m$ to get

$$\int_\Omega |\nabla \omega_m|^2 \geq m \operatorname{meas}(B'_r) \int_{m\overline{x}_1}^{m\overline{x}_1 + 2\pi} \cos^2(t)\, dt = m\,\pi \operatorname{meas}(B'_r).$$

This shows that $\|\nabla \omega_m\|^2 \to \infty$ and thus $C_T, C_N \geq 1$ as asserted.

Step 2. The fact that $C_T(\Omega, k) = C_N(\Omega, n-k)$ is immediate through the Hodge $*$ operator. ■

4 The main theorem

4.1 Statement of the theorem

We now turn to the main theorem that gives several equivalent properties of $C_T(\Omega, k) = 1$ (or analogously $C_N(\Omega, k) = 1$). We start with a definition (cf. for a similar one [34]).

Definition 4 *Let $\Omega \subset \mathbb{R}^n$ be an open smooth set and $\Sigma = \partial\Omega$ be the associated $(n-1)-$surface. Let $\gamma_1, \cdots, \gamma_{n-1}$ be the principal curvatures of Σ. Let $1 \leq k \leq n-1$. We say that Ω is $k-$*convex *if*

$$\gamma_{i_1} + \cdots + \gamma_{i_k} \geq 0, \quad \textit{for every } 1 \leq i_1 < \cdots < i_k \leq n-1.$$

Remark 5 *(i)* When $k = 1$, it is easy to show that Ω convex implies that Ω is $1-$convex. The reverse implication is also true but deeper. The result is due to Hadamard under slightly stronger conditions and as stated to Chern-Lashof [6] (see also Alexander [1]).

(ii) When $2 \leq k \leq n-1$, the condition that Ω is $k-$convex is strictly weaker than saying that Ω is convex. In particular when $k = n-1$ the condition means that the mean curvature of $\Sigma = \partial\Omega$ is non-negative.

We will also use the following quantities.

Definition 6 *Let $\Omega \subset \mathbb{R}^n$ be open and $\nu \in C^1\left(\overline{\Omega}; \Lambda^1\right)$. We define for every $0 \leq k \leq n$ the two maps*

$$L^\nu, K^\nu : \Lambda^k(\mathbb{R}^n) \to \Lambda^k(\mathbb{R}^n)$$

by $L^\nu(\omega) = 0$ if $k = 0$ and $K^\nu(\omega) = 0$ if $k = n$, while

$$L^\nu(\omega) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \omega^{i_1 \cdots i_k}\, d\left(\nu \lrcorner \left(dx^{i_1} \wedge \cdots \wedge dx^{i_k}\right)\right), \quad \textit{if } k \geq 1$$

$$K^\nu(\omega) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \omega^{i_1 \cdots i_k}\, \delta\left(\nu \wedge \left(dx^{i_1} \wedge \cdots \wedge dx^{i_k}\right)\right), \quad \textit{if } k \leq n-1.$$

Remark 7 *(i)* If ν is the unit normal to a surface Σ and is extended to a neighborhood of Σ such that $|\nu| = 1$ everywhere, then (see [14] Lemma 5.5)

$$L^\nu(\nu \wedge \alpha) = \nu \wedge L^\nu(\alpha) \quad \text{and} \quad K^\nu(\nu \lrcorner \alpha) = \nu \lrcorner K^\nu(\alpha).$$

The right-hand sides of these expressions do not depend on the chosen extension, see [14] Theorem 3.23. We will use this frequently henceforth.

(ii) In the remaining part of the article we will always assume that ν has been extended to a neighborhood of $\Sigma = \partial\Omega$ so as to have $|\nu| = 1$.

(iii) Note that L^ν is linear in ω and ν. By definition it acts pointwise on ω. In this way, identifying $\Lambda^k(\mathbb{R}^n)$ with vectors $\mathbb{R}^{\binom{n}{k}}$, the operator L^ν can be seen as a matrix acting on ω. But in ν, on the contrary, L^ν is a local (differential) operator. The same holds true for K^ν.

We then have the main result.

Theorem 8 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open smooth set and $1 \leq k \leq n-1$. Then the following statements are equivalent.*

(i) $C_T(\Omega, k) = 1$.

(ii) *For every $\omega \in W_T^{1,2}(\Omega; \Lambda^k)$*

$$\widetilde{K}(\omega) = \langle K^\nu(\nu \lrcorner \omega); \nu \lrcorner \omega \rangle \geq 0, \quad \textit{on } \partial\Omega.$$

(iii) *The sharper version of Gaffney inequality holds, namely*

$$\|\nabla\omega\|^2 \leq \|d\omega\|^2 + \|\delta\omega\|^2, \quad \forall\, \omega \in W_T^{1,2}(\Omega; \Lambda^k).$$

(iv) Ω *is $(n-k)-$convex.*

(v) *The supremum in (1) is not attained.*

(vi) C_T *is scale invariant, namely, for every $t > 0$*

$$C_T(t\,\Omega, k) = C_T(\Omega, k).$$

Remark 9 *(i)* Using the Hodge $*$ operation, we obtain immediately, from the theorem, the following equivalent relations.

- $C_N(\Omega, k) = 1$.
- $\widetilde{L}(\omega) = \langle L^\nu(\nu \wedge \omega); \nu \wedge \omega \rangle \geq 0, \quad$ whenever $\nu \lrcorner \omega = 0$ on $\partial\Omega$.
- The sharper version of Gaffney inequality holds, namely

$$\|\nabla\omega\|^2 \leq \|d\omega\|^2 + \|\delta\omega\|^2, \quad \forall\, \omega \in W_N^{1,2}(\Omega; \Lambda^k).$$

- Ω is $k-$convex.
- The supremum in (2) is not attained.
- C_N is scale invariant.

(ii) The condition (ii) of the theorem can be equivalently rewritten (for any $\omega \in W_T^{1,2}(\Omega; \Lambda^k)$) as

$$\widetilde{K}(\omega) = \langle K^\nu(\omega); \omega \rangle \geq 0, \quad \text{on } \partial\Omega$$

since, recalling that $\nu \wedge \omega = 0$ (since $\omega \in W_T^{1,2}$),

$$\begin{aligned}\widetilde{K}(\omega) &= \langle K^\nu(\nu \lrcorner \omega); \nu \lrcorner \omega \rangle = \langle \nu \lrcorner K^\nu(\omega); \nu \lrcorner \omega \rangle = \langle K^\nu(\omega); \nu \wedge (\nu \lrcorner \omega) \rangle \\ &= \langle K^\nu(\omega); \omega - \nu \lrcorner (\nu \wedge \omega) \rangle = \langle K^\nu(\omega); \omega \rangle.\end{aligned}$$

Similar remarks hold for $\widetilde{L}$.

(iii) Note that if $C_T(\Omega, k) = 1$, then $\mathcal{H}_T(\Omega; \Lambda^k) = \{0\}$. This follows at once from (iii) of the theorem, since non-zero constant forms cannot satisfy the boundary condition. A similar remark applies to $\mathcal{H}_N$.

4.2 Some algebraic results

Lemma 10 *Let* $1 \leq k \leq n-1$ *and* $\lambda_1, \cdots, \lambda_k \in \Lambda^1$ *with*

$$\lambda_i \lrcorner \lambda_j = 0, \quad \text{if } i \neq j.$$

Then

$$\lambda_i \lrcorner (\lambda_1 \wedge \cdots \wedge \lambda_{i-1} \wedge \lambda_{i+1} \wedge \cdots \wedge \lambda_k) = 0, \quad i = 1, \cdots, k$$
$$|\lambda_1 \wedge \cdots \wedge \lambda_k| = |\lambda_1| \cdots |\lambda_k| .$$

Furthermore let

$$C^{jl} = \sum_{s_2, \cdots, s_k = 1}^{n} (\lambda_1 \wedge \cdots \wedge \lambda_k)^{j s_2 \cdots s_k} (\lambda_1 \wedge \cdots \wedge \lambda_k)^{l s_2 \cdots s_k} .$$

and $\widehat{\lambda_j} = \lambda_1 \wedge \cdots \lambda_{j-1} \wedge \lambda_{j+1} \wedge \cdots \wedge \lambda_k$, *then*

$$C^{jl} = ((k-1)!) \sum_{\gamma=1}^{k} \left| \widehat{\lambda_\gamma} \right|^2 \lambda_\gamma^j \lambda_\gamma^l .$$

Proof *Step 1.* We first establish by induction that

$$\lambda_k \lrcorner (\lambda_1 \wedge \cdots \wedge \lambda_{k-1}) = 0$$

(and similarly for all the other λ_i). Indeed if $k = 2$, this is our hypothesis; so assume that the result has been proved for k and let us prove it for $k+1$. We know from Proposition 2.16 in [14] that

$$\lambda_{k+1} \lrcorner (\lambda_1 \wedge \cdots \wedge \lambda_k) = (\lambda_{k+1} \lrcorner \lambda_1) \wedge (\lambda_2 \wedge \cdots \wedge \lambda_k) - \lambda_1 \wedge (\lambda_{k+1} \lrcorner (\lambda_2 \wedge \cdots \wedge \lambda_k))$$

applying the hypothesis of induction we have the result.

Step 2. We also proceed by induction and show that

$$|\lambda_1 \wedge \cdots \wedge \lambda_k| = |\lambda_1| \cdots |\lambda_k| .$$

When $k = 2$ we have still from Proposition 2.16 in [14] that

$$|\lambda_1 \wedge \lambda_2|^2 = |\lambda_1|^2 |\lambda_2|^2 - |\lambda_1 \lrcorner \lambda_2|^2 = |\lambda_1|^2 |\lambda_2|^2$$

as wished. So let us assume that the result has been proved for k and let us establish it for $k+1$. By the very same proposition as above we get

$$\begin{aligned} |\lambda_1|^2 |\lambda_1 \wedge \lambda_2 \wedge \cdots \wedge \lambda_k \wedge \lambda_{k+1}|^2 &= |\lambda_1 \wedge \lambda_1 \wedge \lambda_2 \wedge \cdots \wedge \lambda_{k+1}|^2 + |\lambda_1 \lrcorner (\lambda_1 \wedge \cdots \wedge \lambda_{k+1})|^2 \\ &= |\lambda_1 \lrcorner (\lambda_1 \wedge \cdots \wedge \lambda_{k+1})|^2 . \end{aligned}$$

Moreover since

$$\lambda_1 \lrcorner (\lambda_1 \wedge \cdots \wedge \lambda_{k+1}) = (\lambda_1 \lrcorner \lambda_1) \wedge (\lambda_2 \wedge \cdots \wedge \lambda_{k+1}) - \lambda_1 \wedge (\lambda_1 \lrcorner (\lambda_2 \wedge \cdots \wedge \lambda_{k+1}))$$

using Step 1 and the hypothesis of induction, we infer that

$$|\lambda_1 \lrcorner (\lambda_1 \wedge \cdots \wedge \lambda_{k+1})|^2 = |\lambda_1|^4 |\lambda_2 \wedge \cdots \wedge \lambda_{k+1}|^2 = |\lambda_1|^4 |\lambda_2|^2 \cdots |\lambda_{k+1}|^2 .$$

Combining the results we have indeed proved our claim.

Step 3. Writing

$$(\lambda_1 \wedge \cdots \wedge \lambda_k)^{js_2\cdots s_k} = \det\left[(\lambda_r^s)_{r=1,\cdots,k}^{s=j,s_2,\cdots,s_k}\right]$$

we find that

$$\begin{aligned}(\lambda_1 \wedge \cdots \wedge \lambda_k)^{js_2\cdots s_k} &= \sum_{\gamma=1}^{k} (-1)^{\gamma+1}\, \lambda_\gamma^j\, (\lambda_1 \wedge \cdots \lambda_{\gamma-1} \wedge \lambda_{\gamma+1} \wedge \cdots \wedge \lambda_k)^{s_2\cdots s_k} \\ &= \sum_{\gamma=1}^{k} (-1)^{\gamma+1}\, \lambda_\gamma^j \left(\widehat{\lambda_\gamma}\right)^{s_2\cdots s_k}\end{aligned}$$

$$\begin{aligned}(\lambda_1 \wedge \cdots \wedge \lambda_k)^{ls_2\cdots s_k} &= \sum_{\delta=1}^{k} (-1)^{\delta+1}\, \lambda_\delta^l\, (\lambda_1 \wedge \cdots \lambda_{\delta-1} \wedge \lambda_{\delta+1} \wedge \cdots \wedge \lambda_k)^{s_2\cdots s_k} \\ &= \sum_{\delta=1}^{k} (-1)^{\delta+1}\, \lambda_\delta^l \left(\widehat{\lambda_\delta}\right)^{s_2\cdots s_k}.\end{aligned}$$

Observing (using Steps 1 and 2) that

$$\sum_{s_2,\cdots,s_k=1}^{n} \left(\widehat{\lambda_\gamma}\right)^{s_2\cdots s_k} \left(\widehat{\lambda_\delta}\right)^{s_2\cdots s_k} = ((k-1)!) \left\langle \widehat{\lambda_\gamma}; \widehat{\lambda_\delta} \right\rangle = \begin{cases} (k-1)! \left|\widehat{\lambda_\gamma}\right|^2 & \text{if } \gamma = \delta \\ 0 & \text{if } \gamma \neq \delta \end{cases}$$

we find that

$$C^{jl} = ((k-1)!) \sum_{\gamma=1}^{k} \left|\widehat{\lambda_\gamma}\right|^2 \lambda_\gamma^j \lambda_\gamma^l\,.$$

as claimed. ■

We next give a way of computing the quantity K^ν.

Lemma 11 *Let $1 \leq k \leq n$, $\Sigma \subset \mathbb{R}^n$ be a smooth $(n-1)-$surface with unit normal ν and $\Omega \subset \mathbb{R}^n$ be a neighborhood of Σ. Let $\alpha, \beta \in C^1\left(\Omega; \Lambda^k\right)$ be such that, on Σ,*

$$\nu \wedge \alpha = \nu \wedge \beta = 0.$$

Then the following equation holds true, on Σ,

$$\langle K^\nu(\nu \lrcorner \alpha); \nu \lrcorner \beta\rangle + \langle K^\nu(\nu \lrcorner \beta); \nu \lrcorner \alpha\rangle = \langle \delta\alpha; \nu \lrcorner \beta\rangle + \langle \delta\beta; \nu \lrcorner \alpha\rangle - \langle \nabla(\alpha \lrcorner \beta); \nu\rangle\,. \tag{4}$$

In particular if $\alpha = \nu \wedge \lambda$ in Ω with

$$\lambda = \lambda_1 \wedge \cdots \wedge \lambda_{k-1}$$

where $\lambda_1, \cdots, \lambda_{k-1} \in C^1\left(\Omega; \Lambda^1\right)$ with, for every $i, j = 1, \cdots, k-1$ and for every $x \in \Omega$,

$$|\nu(x)| = 1, \quad \nu(x) \lrcorner \lambda_i(x) = 0 \quad \textit{and} \quad \lambda_i(x) \lrcorner \lambda_j(x) = \delta_{ij}\,,$$

then, in Ω,

$$\langle K^\nu(\lambda); \lambda\rangle = \langle \delta(\nu \wedge \lambda); \lambda\rangle\,.$$

Proof *Step 1.* Applying Lemma 5.6 in [14] we find

$$\langle K^{\nu}(\nu \lrcorner \alpha);\nu \lrcorner \beta\rangle = \langle \nu \lrcorner K^{\nu}(\alpha);\nu \lrcorner \beta\rangle = \langle \delta\alpha;\nu \lrcorner \beta\rangle - \sum_{I}\langle \nabla\alpha_I;\nu\rangle\,\beta_I$$

where $I = (i_1, \cdots, i_k) \in \mathbb{N}^k$ with $1 \leq i_1 < \cdots < i_k \leq n$. We thus deduce that

$$\langle \nu \lrcorner K^{\nu}(\alpha);\nu \lrcorner \beta\rangle + \langle \nu \lrcorner K^{\nu}(\beta);\nu \lrcorner \alpha\rangle = \langle \delta\alpha;\nu \lrcorner \beta\rangle + \langle \delta\beta;\nu \lrcorner \alpha\rangle - \sum_{I}\langle \nabla(\alpha_I\beta_I);\nu\rangle$$

or in other words

$$\langle \nu \lrcorner K^{\nu}(\alpha);\nu \lrcorner \beta\rangle + \langle \nu \lrcorner K^{\nu}(\beta);\nu \lrcorner \alpha\rangle = \langle \delta\alpha;\nu \lrcorner \beta\rangle + \langle \delta\beta;\nu \lrcorner \alpha\rangle - \langle \nabla(\alpha \lrcorner \beta);\nu\rangle$$

which is exactly (4).

Step 2. The extra statement follows from Step 1 applied to $\beta = \alpha$ and Lemma 10 since then $\nu \lrcorner \alpha = \lambda$ and

$$\langle \nabla(\alpha \lrcorner \alpha);\nu\rangle = \left\langle \nabla\left(|\lambda|^2\right);\nu\right\rangle = 0.$$

The proof is therefore complete. ■

4.3 Calculation of sums of principal curvatures

In the sequel $\Omega \subset \mathbb{R}^n$ will always be a bounded open smooth set with exterior unit normal ν. When we say that $E_1, \cdots, E_{n-1}$ is an *orthonormal frame field of principal directions* of $\partial\Omega$ with *associated principal curvatures* $\gamma_1, \cdots, \gamma_{n-1}$, we mean that $\{\nu, E_1, \cdots, E_{n-1}\}$ form an orthonormal basis of $\mathbb{R}^n$ and, for every $1 \leq i \leq n-1$,

$$\sum_{j=1}^{n}\left(E_i^j \nu_{x_j}^l\right) = \gamma_i\, E_i^l \quad \Rightarrow \quad \gamma_i = \sum_{j,l=1}^{n}\left(E_i^l E_i^j \nu_{x_j}^l\right). \tag{5}$$

Lemma 12 *Let $E_1, \cdots, E_{n-1}$ be an orthonormal frame field of principal directions of $\partial\Omega$ with associated principal curvatures $\gamma_1, \cdots, \gamma_{n-1}$. Assume that $E_1, \cdots, E_{n-1}$ are extended locally to a neighborhood of $\partial\Omega$ such that $\{\nu, E_1, \cdots, E_{n-1}\}$ form an orthonormal frame field of $\mathbb{R}^n$. Let $1 \leq k \leq n-1$,*

$$\lambda = E_{i_1 \cdots i_{k-1}} = E_{i_1} \wedge \cdots \wedge E_{i_{k-1}} \quad \textit{and} \quad \mu = E_{j_1 \cdots j_{k-1}} = E_{j_1} \wedge \cdots \wedge E_{j_{k-1}}$$

(for $k = 1$, $\lambda = \mu = 1$). Then

$$\langle \delta(\nu \wedge \lambda);\lambda\rangle = (\gamma_1 + \cdots + \gamma_{n-1}) - \left(\gamma_{i_1} + \cdots + \gamma_{i_{k-1}}\right)$$

while for $\lambda \neq \mu$

$$\langle \delta(\nu \wedge \lambda);\mu\rangle + \langle \delta(\nu \wedge \mu);\lambda\rangle = 0.$$

Proof The case $k = 1$ is immediate. Indeed the first equation reads as

$$\langle \delta(\nu \wedge \lambda);\lambda\rangle = \delta(\nu) = \operatorname{div}(\nu) = \gamma_1 + \cdots + \gamma_{n-1}.$$

While nothing is to be proved for the second equation. So we discuss now the case $k \geq 2$.

Step 1: $k \geq 2$ (first equation). We now prove that if $\lambda = E_{i_1 \cdots i_{k-1}} = E_{i_1} \wedge \cdots \wedge E_{i_{k-1}}$, then

$$\langle \delta(\nu \wedge \lambda);\lambda\rangle = (\gamma_1 + \cdots + \gamma_{n-1}) - \left(\gamma_{i_1} + \cdots + \gamma_{i_{k-1}}\right)$$

(i) We find that

$$\langle \delta\left(\nu\wedge\lambda\right);\lambda\rangle=\frac{1}{(k-1)!}\sum_{s_1,\cdots,s_{k-1}=1}^{n}\left[\sum_{j=1}^{n}\left(\nu\wedge\lambda\right)_{x_j}^{js_1\cdots s_{k-1}}\right]\lambda^{s_1\cdots s_{k-1}}=A+B$$

where (recalling that $|\lambda|=1$)

$$\begin{aligned}A&=\frac{1}{(k-1)!}\sum_{j,s_1,\cdots,s_{k-1}=1}^{n}\left[\nu^j\lambda^{s_1\cdots s_{k-1}}\right]_{x_j}\lambda^{s_1\cdots s_{k-1}}\\&=\frac{1}{(k-1)!}\sum_{j,s_1,\cdots,s_{k-1}=1}^{n}\nu_{x_j}^{j}\left(\lambda^{s_1\cdots s_{k-1}}\right)^2+\frac{1}{(k-1)!}\sum_{j=1}^{n}\nu^j\sum_{s_1,\cdots,s_{k-1}=1}^{n}\left[\left(\frac{\lambda^{s_1\cdots s_{k-1}}}{2}\right)^2\right]_{x_j}\\&=\operatorname{div}\left(\nu\right)=\gamma_1+\cdots+\gamma_{n-1}\end{aligned}$$

and

$$\begin{aligned}B&=\frac{1}{(k-1)!}\sum_{j,s_1,\cdots,s_{k-1}=1}^{n}\left[\sum_{r=1}^{k-1}(-1)^r\nu^{s_r}\lambda^{js_1\cdots s_{r-1}s_{r+1}\cdots s_{k-1}}\right]_{x_j}\lambda^{s_1\cdots s_{k-1}}\\&=\frac{1}{(k-1)!}\sum_{j,s_1,\cdots,s_{k-1}=1}^{n}\left[\sum_{r=1}^{k-1}(-1)^r\nu_{x_j}^{s_r}\lambda^{js_1\cdots s_{r-1}s_{r+1}\cdots s_{k-1}}\right]\lambda^{s_1\cdots s_{k-1}}\\&+\frac{1}{(k-1)!}\sum_{j,s_1,\cdots,s_{k-1}=1}^{n}\left[\sum_{r=1}^{k-1}(-1)^r\nu^{s_r}\lambda_{x_j}^{js_1\cdots s_{r-1}s_{r+1}\cdots s_{k-1}}\right]\lambda^{s_1\cdots s_{k-1}}\\&=B_1+B_2\,.\end{aligned}$$

The result will be established once we prove that

$$\begin{aligned}B_1&=\frac{1}{(k-1)!}\sum_{j,s_1,\cdots,s_{k-1}=1}^{n}\left[\sum_{r=1}^{k-1}(-1)^r\nu_{x_j}^{s_r}\lambda^{js_1\cdots s_{r-1}s_{r+1}\cdots s_{k-1}}\right]\lambda^{s_1\cdots s_{k-1}}\\&=-\left(\gamma_{i_1}+\cdots+\gamma_{i_{k-1}}\right)\end{aligned}$$

and

$$B_2=\frac{1}{(k-1)!}\sum_{j,s_1,\cdots,s_{k-1}=1}^{n}\left[\sum_{r=1}^{k-1}(-1)^r\nu^{s_r}\lambda_{x_j}^{js_1\cdots s_{r-1}s_{r+1}\cdots s_{k-1}}\right]\lambda^{s_1\cdots s_{k-1}}=0.$$

(ii) Let us first prove that $B_2=0$ (recalling that $\lambda=E_{i_1}\wedge\cdots\wedge E_{i_{k-1}}$). We write

$$(k-1)!\,B_2=\sum_{r=1}^{k-1}\sum_{j,s_1,\cdots s_{r-1},s_{r+1},\cdots,s_{k-1}=1}^{n}\left(\sum_{s_r=1}^{n}(-1)^r\nu^{s_r}\lambda^{s_1\cdots s_{k-1}}\right)\lambda_{x_j}^{js_1\cdots s_{r-1}s_{r+1}\cdots s_{k-1}}.$$

Observe that, for every $r=1,\cdots,k-1$,

$$\sum_{s_r=1}^{n}(-1)^r\nu^{s_r}\left(E_{i_1}\wedge\cdots\wedge E_{i_{k-1}}\right)^{s_1\cdots s_{k-1}}=-\left(\nu\lrcorner\left(E_{i_1}\wedge\cdots\wedge E_{i_{k-1}}\right)\right)^{s_1\cdots s_{r-1}s_{r+1}\cdots s_{k-1}}=0,$$

in view of Lemma 10, leading to the fact that $B_2=0$.

(iii) We finally show that $B_1 = -\left(\gamma_{i_1} + \cdots + \gamma_{i_{k-1}}\right)$. Note that, interchanging the positions of the indices s_r and $s_{r'}$, we get (recalling that $\lambda = E_{i_1} \wedge \cdots \wedge E_{i_{k-1}}$)

$$B_1 = \frac{1}{(k-1)!} \sum_{j, s_1, \cdots, s_{k-1}=1}^{n} \left[\sum_{r=1}^{k-1} (-1)^r \nu_{x_j}^{s_r} \lambda^{j s_1 \cdots s_{r-1} s_{r+1} \cdots s_{k-1}} \right] \lambda^{s_1 \cdots s_{k-1}}$$
$$= \frac{-1}{(k-2)!} \sum_{j, s_1, \cdots, s_{k-1}=1}^{n} \left[\nu_{x_j}^{s_1} \left(E_{i_1} \wedge \cdots \wedge E_{i_{k-1}}\right)^{j s_2 \cdots s_{k-1}} \left(E_{i_1} \wedge \cdots \wedge E_{i_{k-1}}\right)^{s_1 \cdots s_{k-1}} \right].$$

The result follows (cf. Lemma 10) since, for every $1 \leq j, s_1 \leq n$,

$$\sum_{r=1}^{k-1} \left(E_{i_r}^{j} E_{i_r}^{s_1} \right) = \frac{1}{(k-2)!} \sum_{s_2, \cdots, s_{k-1}=1}^{n} \left(E_{i_1} \wedge \cdots \wedge E_{i_{k-1}}\right)^{j s_2 \cdots s_{k-1}} \left(E_{i_1} \wedge \cdots \wedge E_{i_{k-1}}\right)^{s_1 \cdots s_{k-1}}$$

and thus

$$B_1 = -\sum_{r=1}^{k-1} \left(\sum_{j, s_1=1}^{n} \nu_{x_j}^{s_1} E_{i_r}^{j} E_{i_r}^{s_1} \right) = -\left(\gamma_{i_1} + \cdots + \gamma_{i_{k-1}}\right)$$

which is exactly what had to be proved.

Step 2: $k \geq 2$ (second equation). We finally establish that if

$$\lambda = E_{i_1 \cdots i_{k-1}} = E_{i_1} \wedge \cdots \wedge E_{i_{k-1}} \quad \text{and} \quad \mu = E_{j_1 \cdots j_{k-1}} = E_{j_1} \wedge \cdots \wedge E_{j_{k-1}}.$$

and $\lambda \neq \mu$, then

$$\langle \delta (\nu \wedge \lambda) ; \mu \rangle + \langle \delta (\nu \wedge \mu) ; \lambda \rangle = 0.$$

This amounts to showing that $X = 0$ where

$$X = \sum_{s_1, \cdots, s_{k-1}=1}^{n} \left[\sum_{j=1}^{n} (\nu \wedge \lambda)_{x_j}^{j s_1 \cdots s_{k-1}} \right] \mu^{s_1 \cdots s_{k-1}} + \sum_{s_1, \cdots, s_{k-1}=1}^{n} \left[\sum_{j=1}^{n} (\nu \wedge \mu)_{x_j}^{j s_1 \cdots s_{k-1}} \right] \lambda^{s_1 \cdots s_{k-1}}. \quad (6)$$

We write $X = A + B$ where

$$A = \sum_{j, s_1, \cdots, s_{k-1}=1}^{n} \left[\nu^j \lambda^{s_1 \cdots s_{k-1}} \right]_{x_j} \mu^{s_1 \cdots s_{k-1}} + \sum_{j, s_1, \cdots, s_{k-1}=1}^{n} \left[\nu^j \mu^{s_1 \cdots s_{k-1}} \right]_{x_j} \lambda^{s_1 \cdots s_{k-1}}$$
$$= 2 \sum_{j}^{n} \nu_{x_j}^{j} \sum_{s_1, \cdots, s_{k-1}=1}^{n} \lambda^{s_1 \cdots s_{k-1}} \mu^{s_1 \cdots s_{k-1}} + \sum_{j=1}^{n} \nu^j \sum_{s_1, \cdots, s_{k-1}=1}^{n} \left[\lambda^{s_1 \cdots s_{k-1}} \mu^{s_1 \cdots s_{k-1}} \right]_{x_j} = 0$$

(since $\langle \lambda ; \mu \rangle = 0$ by Lemma 10) and

$$B = \sum_{j, s_1, \cdots, s_{k-1}=1}^{n} \left[\sum_{r=1}^{k-1} (-1)^r \nu^{s_r} \lambda^{j s_1 \cdots s_{r-1} s_{r+1} \cdots s_{k-1}} \right]_{x_j} \mu^{s_1 \cdots s_{k-1}}$$
$$+ \sum_{j, s_1, \cdots, s_{k-1}=1}^{n} \left[\sum_{r=1}^{k-1} (-1)^r \nu^{s_r} \mu^{j s_1 \cdots s_{r-1} s_{r+1} \cdots s_{k-1}} \right]_{x_j} \lambda^{s_1 \cdots s_{k-1}}$$

which leads to $B = B_1 + B_2$ where

$$B_1 = \sum_{j,s_1,\cdots,s_{k-1}=1}^{n} \left[\sum_{r=1}^{k-1} (-1)^r \nu_{x_j}^{s_r} \lambda^{js_1\cdots s_{r-1}s_{r+1}\cdots s_{k-1}} \right] \mu^{s_1\cdots s_{k-1}}$$
$$+ \sum_{j,s_1,\cdots,s_{k-1}=1}^{n} \left[\sum_{r=1}^{k-1} (-1)^r \nu_{x_j}^{s_r} \mu^{js_1\cdots s_{r-1}s_{r+1}\cdots s_{k-1}} \right] \lambda^{s_1\cdots s_{k-1}}$$

$$B_2 = \sum_{j,s_1,\cdots,s_{k-1}=1}^{n} \left[\sum_{r=1}^{k-1} (-1)^r \nu^{s_r} \lambda_{x_j}^{js_1\cdots s_{r-1}s_{r+1}\cdots s_{k-1}} \right] \mu^{s_1\cdots s_{k-1}}$$
$$+ \sum_{j,s_1,\cdots,s_{k-1}=1}^{n} \left[\sum_{r=1}^{k-1} (-1)^r \nu^{s_r} \mu_{x_j}^{js_1\cdots s_{r-1}s_{r+1}\cdots s_{k-1}} \right] \lambda^{s_1\cdots s_{k-1}}.$$

It remains to prove, in order to show that $X = 0$ where X is as in (6), that $B_1 = B_2 = 0$.

(i) We start with the fact that $B_2 = 0$. We rewrite the definition as

$$B_2 = \sum_{r=1}^{k-1} \sum_{j,s_1,\cdots s_{r-1},s_{r+1},\cdots,s_{k-1}=1}^{n} \left[\sum_{s_r=1}^{n} (-1)^r \nu^{s_r} \mu^{s_1\cdots s_{k-1}} \right] \lambda_{x_j}^{js_1\cdots s_{r-1}s_{r+1}\cdots s_{k-1}}$$
$$+ \sum_{r=1}^{k-1} \sum_{j,s_1,\cdots s_{r-1},s_{r+1},\cdots,s_{k-1}=1}^{n} \left[\sum_{s_r=1}^{n} (-1)^r \nu^{s_r} \lambda^{s_1\cdots s_{k-1}} \right] \mu_{x_j}^{js_1\cdots s_{r-1}s_{r+1}\cdots s_{k-1}}.$$

Since, for every $r = 1, \cdots, k-1$,

$$\sum_{s_r=1}^{n} (-1)^r \nu^{s_r} \lambda^{s_1\cdots s_{k-1}} = -(\nu \lrcorner \lambda)^{s_1\cdots s_{r-1}s_{r+1}\cdots s_{k-1}} = 0$$

$$\sum_{s_r=1}^{n} (-1)^r \nu^{s_r} \mu^{s_1\cdots s_{k-1}} = -(\nu \lrcorner \mu)^{s_1\cdots s_{r-1}s_{r+1}\cdots s_{k-1}} = 0,$$

we find that indeed $B_2 = 0$.

(ii) We finally prove that $B_1 = 0$. Note that, interchanging the positions of the indices s_r and $s_{r'}$, we obtain

$$B_1 = -(k-1) \sum_{j,s_1,\cdots,s_{k-1}=1}^{n} \nu_{x_j}^{s_1} \left[\lambda^{js_2\cdots s_{k-1}} \mu^{s_1\cdots s_{k-1}} + \mu^{js_2\cdots s_{k-1}} \lambda^{s_1\cdots s_{k-1}} \right].$$

The result follows if we can show that, if $\lambda \neq \mu$ where

$$\lambda = \lambda_1 \wedge \cdots \wedge \lambda_{k-1} \quad \text{and} \quad \mu = \mu_1 \wedge \cdots \wedge \mu_{k-1},$$

with $\lambda_i, \mu_i \in \{E_1, \cdots, E_{n-1}\}$, then

$$\sum_{j,s_1,\cdots,s_{k-1}=1}^{n} \nu_{x_j}^{s_1} \left[\lambda^{js_2\cdots s_{k-1}} \mu^{s_1\cdots s_{k-1}} \right] = \sum_{j,s_1,\cdots,s_{k-1}=1}^{n} \nu_{x_j}^{s_1} \left[\mu^{js_2\cdots s_{k-1}} \lambda^{s_1\cdots s_{k-1}} \right] = 0.$$

Since both identities are established similarly, we prove only the first one, namely

$$\sum_{j,s_1,\cdots,s_{k-1}=1}^{n} \nu_{x_j}^{s_1}\left[(\lambda_1\wedge\cdots\wedge\lambda_{k-1})^{js_2\cdots s_{k-1}}(\mu_1\wedge\cdots\wedge\mu_{k-1})^{s_1s_2\cdots s_{k-1}}\right]=0. \tag{7}$$

Since $\lambda\neq\mu$ we assume, up to reordering, that $\lambda_1\neq\mu_1$. We claim that

$$\begin{aligned} C^{js_1} &= \sum_{s_2,\cdots,s_{k-1}=1}^{n}\left[(\lambda_1\wedge\cdots\wedge\lambda_{k-1})^{js_2\cdots s_{k-1}}(\mu_1\wedge\cdots\wedge\mu_{k-1})^{s_1s_2\cdots s_{k-1}}\right] \\ &= ((k-2)!)\ \lambda_1^j\mu_1^{s_1}\ \langle\lambda_2\wedge\cdots\wedge\lambda_{k-1};\mu_2\wedge\cdots\wedge\mu_{k-1}\rangle\,. \end{aligned} \tag{8}$$

which leads to

$$\begin{aligned} &\sum_{j,s_1,\cdots,s_{k-1}=1}^{n} \nu_{x_j}^{s_1}\left[(\lambda_1\wedge\cdots\wedge\lambda_{k-1})^{js_2\cdots s_{k-1}}(\mu_1\wedge\cdots\wedge\mu_{k-1})^{s_1s_2\cdots s_{k-1}}\right] \\ &=((k-2)!)\,\langle\lambda_2\wedge\cdots\wedge\lambda_{k-1};\mu_2\wedge\cdots\wedge\mu_{k-1}\rangle\sum_{j,s_1=1}^{n}\nu_{x_j}^{s_1}\lambda_1^j\mu_1^{s_1} \\ &=((k-2)!)\,\langle\lambda_2\wedge\cdots\wedge\lambda_{k-1};\mu_2\wedge\cdots\wedge\mu_{k-1}\rangle\,\gamma_{\lambda_1}\sum_{s_1=1}^{n}\lambda_1^{s_1}\mu_1^{s_1}=0 \end{aligned}$$

(where γ_{λ_1} is the principal curvature corresponding to λ_1) which is exactly (7). It remains to show (8). We have

$$\begin{aligned} (\lambda_1\wedge\cdots\wedge\lambda_{k-1})^{js_2\cdots s_{k-1}} &= \sum_{r=1}^{k-1}(-1)^{r+1}\,\lambda_r^j\left(\widehat{\lambda_r}\right)^{s_2\cdots s_{k-1}} \\ (\mu_1\wedge\cdots\wedge\mu_{k-1})^{s_1s_2\cdots s_{k-1}} &= \sum_{t=1}^{k-1}(-1)^{t+1}\,\mu_t^{s_1}\left(\widehat{\mu_t}\right)^{s_2\cdots s_{k-1}}\,. \end{aligned}$$

where

$$\widehat{\lambda_r}=\lambda_1\wedge\cdots\lambda_{r-1}\wedge\lambda_{r+1}\wedge\cdots\wedge\lambda_{k-1}\quad\text{and}\quad\widehat{\mu_t}=\mu_1\wedge\cdots\mu_{t-1}\wedge\mu_{t+1}\wedge\cdots\wedge\mu_{k-1}\,.$$

We therefore have that

$$\begin{aligned} C^{js_1} &= \sum_{r,t=1}^{k-1}(-1)^{r+t}\,\lambda_r^j\mu_t^{s_1}\sum_{s_2,\cdots,s_{k-1}=1}^{n}\left(\widehat{\lambda_r}\right)^{s_2\cdots s_{k-1}}\left(\widehat{\mu_t}\right)^{s_2\cdots s_{k-1}} \\ &=((k-2)!)\sum_{r,t=1}^{k-1}(-1)^{r+t}\,\lambda_r^j\mu_t^{s_1}\left\langle\widehat{\lambda_r};\widehat{\mu_t}\right\rangle. \end{aligned}$$

Invoking Lemma 10 and the fact that $\lambda_1\neq\mu_1$, we obtain that, unless $r=t=1$,

$$\left\langle\widehat{\lambda_r};\widehat{\mu_t}\right\rangle=0$$

leading to (8). The proof is therefore complete. ∎

4.4 Formulas for L^ν and K^ν in terms of principal curvatures

We first prove the symmetry of L^ν and K^ν, which essentially follows from the symmetry of the second fundamental form of a hypersurface. We use Remark 7 (i)–(iii) in the following lemma and its proof. In particular, recall that we have extended ν in a neighborhood of Σ such that for any α, β

$$[L^\nu(\nu \wedge \alpha) = \nu \wedge L^\nu(\alpha) \quad \text{and} \quad K^\nu(\nu \lrcorner \alpha) = \nu \lrcorner K^\nu(\alpha)] \quad \text{on } \Sigma.$$

Lemma 13 *Let $\Sigma \subset \mathbb{R}^n$ be a smooth $n-1$ dimensional hypersurface with unit normal ν and let $1 \leq k \leq n-1$. Then at every point x_0 of Σ and for every α, $\beta \in \Lambda^k(\mathbb{R}^n)$ the following two identities hold*

$$\langle L^\nu(\nu \wedge \alpha); \nu \wedge \beta \rangle = \langle L^\nu(\nu \wedge \beta); \nu \wedge \alpha \rangle$$

$$\langle K^\nu(\nu \lrcorner \alpha); \nu \lrcorner \beta \rangle = \langle K^\nu(\nu \lrcorner \beta); \nu \lrcorner \alpha \rangle .$$

Proof We only prove the first one, the second one follows by duality ([14] Lemma 5.3).

Step 1. Note that since $\alpha \in \Lambda^k(\mathbb{R}^n)$, i.e. has constant coefficients, $L^\nu(\alpha) = d(\nu \lrcorner \alpha)$. Let us prove that we can assume

$$\nu(x_0) = (1, 0, \cdots, 0) = e_1 .$$

Choose $A \in O(n)$ such that $A^*(\nu(x_0)) = e_1$ and set $\mu = A^*(\nu)$. It has the property that

$$\mu\left(A^{-1}x_0\right) = e_1 .$$

Set

$$\tilde{\alpha} = A^*(\alpha) \quad \text{and} \quad \tilde{\beta} = A^*(\beta) .$$

It now follows from Theorem 3.10 (note that $A^* = A^\sharp$ if $A \in O(n)$) and Proposition 2.19 in [14] that

$$\langle \nu \wedge L^\nu(\alpha); \nu \wedge \beta \rangle = \left\langle A^*(\nu) \wedge d\left(A^\sharp(\nu) \lrcorner A^*(\alpha)\right); A^*(\nu) \wedge A^*(\beta) \right\rangle = \left\langle \mu \wedge d(\mu \lrcorner \tilde{\alpha}); \mu \wedge \tilde{\beta} \right\rangle .$$

Since also $\tilde{\alpha} \in \Lambda^k(\mathbb{R}^n)$, i.e. has constant coefficients, $d(\mu \lrcorner \tilde{\alpha}) = L^\mu(\tilde{\alpha})$. This proves the claim of Step 1.

Step 2. We now assume that $\nu(x_0) = e_1$. By linearity it is sufficient to show the claim for

$$\alpha = dx^{i_1} \wedge \cdots \wedge dx^{i_k} \quad \text{and} \quad \beta = dx^{j_1} \wedge \cdots \wedge dx^{j_k}.$$

We distinguish two cases.

Case 1: $i_1 = 1$ or $j_1 = 1$. We consider only the case $i_1 = 1$, the other one being handled similarly. Then $\nu \wedge \alpha = 0$ (which implies $L^\nu(\nu \wedge \alpha) = 0$) and therefore

$$\langle L^\nu(\nu \wedge \alpha); \nu \wedge \beta \rangle = 0 = \langle L^\nu(\nu \wedge \beta); \nu \wedge \alpha \rangle$$

and the symmetry is proved.

Case 2: $i_1, j_1 > 1$. We have to show that

$$\langle \nu \wedge d(\nu \lrcorner \alpha); \nu \wedge \beta \rangle = \langle \nu \wedge d(\nu \lrcorner \beta); \nu \wedge \alpha \rangle .$$

Since $j_1 > 1$, we get at x_0 that $\beta = e_1 \lrcorner (e_1 \wedge \beta)$ and the same for α. So we have to show that

$$\langle d(\nu \lrcorner \alpha); \beta \rangle = \langle d(\nu \lrcorner \beta); \alpha \rangle . \tag{9}$$

Clearly we can assume that $(i_1, \cdots, i_k) \neq (j_1, \cdots, j_k)$. By a direct calculation one obtains

$$d(\nu \lrcorner \alpha) = \sum_{s=1}^{n} \sum_{\gamma=1}^{k} (-1)^{\gamma-1} \nu_{x_s}^{i_\gamma} dx^s \wedge dx^{i_1} \wedge \cdots \wedge \widehat{dx^{i_\gamma}} \wedge \cdots \wedge dx^{i_k},$$

where $\widehat{a}$ means that a has been omitted. Two possibilities may then happen.

Case 2.1. $\{i_1, \cdots, i_k\}$ and $\{j_1, \cdots, j_k\}$ differ in more than one index (considered as sets). Then for any $s = 1, \cdots, n$

$$\left\langle dx^s \wedge dx^{i_1} \wedge \cdots \wedge \widehat{dx^{i_\gamma}} \wedge \cdots \wedge dx^{i_k}; dx^{j_1} \wedge \cdots \wedge dx^{j_k} \right\rangle = 0.$$

It follows that $\langle d(\nu \lrcorner \alpha); \beta \rangle = 0$ and by symmetry (9) follows.

Case 2.2. There exist $\tau, \gamma \in \{1, \cdots, k\}$ such that $\left(i_1, \cdots, \widehat{i_\gamma}, \cdots, i_k\right) = \left(j_1, \cdots, \widehat{j_\tau}, \cdots, j_k\right)$. Without loss of generality $\tau \leq \gamma$ and hence

$$(j_1, \cdots, j_k) = \left(i_1, \cdots, i_{\tau-1}, j_\tau, i_\tau, \cdots, \widehat{i_\gamma}, \cdots, i_k\right).$$

We immediately find

$$\langle d(\nu \lrcorner \alpha); \beta \rangle = (-1)^{\tau+\gamma} \nu_{x_{j_\tau}}^{i_\gamma}.$$

In the same way, using that

$$(i_1, \cdots, i_k) = \left(j_1, \cdots, \widehat{j_\tau}, \cdots, j_\gamma, i_\gamma, j_{\gamma+1}, \cdots, j_k\right),$$

one obtains

$$\langle d(\nu \lrcorner \beta); \alpha \rangle = (-1)^{\tau+\gamma} \nu_{x_{i_\gamma}}^{j_\tau}.$$

So to prove the symmetry we have to show that

$$\nu_{x_{j_\tau}}^{i_\gamma} = \nu_{x_{i_\gamma}}^{j_\tau} \quad \text{or equivalently} \quad \left\langle \nabla\nu \cdot e_{j_\tau}; e_{i_\gamma} \right\rangle = \left\langle \nabla\nu \cdot e_{i_\gamma}; e_{j_\tau} \right\rangle.$$

This last equality follows from the symmetry of the second fundamental form of the hypersurface Σ at the point x_0, because e_{i_γ} and e_{j_τ} are tangent vectors. ■

We now improve Lemma 13 (for a different proof of the next result see Lemma 37).

Lemma 14 *Let $\Sigma \subset \mathbb{R}^n$ be a smooth $n-1$ dimensional hypersurface with unit normal ν and let $1 \leq k \leq n-1$. Let $E_1, \cdots, E_{n-1}$ be an orthonormal set of principal directions of Σ with associated principal curvatures $\gamma_1, \cdots, \gamma_{n-1}$. Then, at every point $x_0 \in \Sigma$ and for every α, $\beta \in \Lambda^k(\mathbb{R}^n)$, the following two identities hold*

$$\langle L^\nu(\nu \wedge \alpha); \nu \wedge \beta \rangle = \sum_{1 \leq i_1 < \cdots < i_k \leq n-1} \langle \alpha; E_{i_1 \cdots i_k} \rangle \langle \beta; E_{i_1 \cdots i_k} \rangle \sum_{j \in \{i_1, \cdots, i_k\}} \gamma_j \tag{10}$$

$$\langle K^\nu(\nu \lrcorner \alpha); \nu \lrcorner \beta \rangle = \sum_{1 \leq i_1 < \cdots < i_{k-1} \leq n-1} \left\langle \alpha; \nu \wedge E_{i_1 \cdots i_{k-1}} \right\rangle \left\langle \beta; \nu \wedge E_{i_1 \cdots i_{k-1}} \right\rangle \sum_{j \notin \{i_1, \cdots, i_{k-1}\}} \gamma_j \tag{11}$$

where $E_{i_1 \cdots i_k} = E_{i_1} \wedge \cdots \wedge E_{i_k}$.

Remark 15 When $k = n-1$, the first formula reads as

$$\langle L^{\nu}(\nu \wedge \alpha); \nu \wedge \beta\rangle = \langle \alpha; E_1 \wedge \cdots \wedge E_{n-1}\rangle \langle \beta; E_1 \wedge \cdots \wedge E_{n-1}\rangle \sum_{j=1}^{n-1} \gamma_j$$

while, when $k = 1$, the second one reads as

$$\langle K^{\nu}(\nu \lrcorner \alpha); \nu \lrcorner \beta\rangle = \langle \alpha; \nu\rangle \langle \beta; \nu\rangle \sum_{j=1}^{n-1} \gamma_j .$$

Proof We only prove the second statement. The first one can be deduced from the other one by duality, using for instance [14] Lemma 5.3. Since both sides of the equation are bilinear in (α, β) it is sufficient to show the identity for basis vectors of $\Lambda^k(\mathbb{R}^n)$. We choose basis vectors of the type

$$\text{(a)} \quad \alpha = E_{i_1} \wedge \cdots \wedge E_{i_k}, \quad \beta = E_{j_1} \wedge \cdots \wedge E_{j_k}$$

or of the type

$$\text{(b)} \quad \alpha = \nu \wedge E_{i_1} \wedge \cdots \wedge E_{i_{k-1}}, \quad \beta = \nu \wedge E_{j_1} \wedge \cdots \wedge E_{j_{k-1}} .$$

If either one of α or β is of the type (a), then one immediately obtains that both sides of (11) are zero and the equation is trivially satisfied (see Lemma 10 and (3)). So we only need to consider the case that α and β are both of type (b). We distinguish two cases. We also let $x_0 \in \Sigma$.

Case 1: $(i_1, \cdots, i_{k-1}) = (j_1, \cdots, j_{k-1})$. In that case the right hand side of (11) is equal to

$$\sum_{1 \leq l_1 < \cdots < l_{k-1} \leq n-1} \left\langle \alpha; \nu \wedge E_{l_1 \cdots l_{k-1}}\right\rangle \left\langle \beta; \nu \wedge E_{l_1 \cdots l_{k-1}}\right\rangle \sum_{j \notin \{l_1, \cdots, l_{k-1}\}} \gamma_j = \sum_{j \notin \{i_1, \cdots, i_{k-1}\}} \gamma_j .$$

We now use Lemma 11 with $\lambda = E_{i_1} \wedge \cdots \wedge E_{i_{k-1}}$. We can assume that $\{\nu, E_1, \ldots, E_{n-1}\}$ are extended to an orthonormal basis in a neighborhood of x_0. Note that from (3) and Lemma 10 we get that $\nu \lrcorner (\nu \wedge \lambda) = \lambda - \nu \wedge (\nu \lrcorner \lambda) = \lambda$. So Lemma 11 gives

$$\langle K^{\nu}(\nu \lrcorner \alpha); \nu \lrcorner \alpha\rangle = \langle K^{\nu}(\lambda); \lambda\rangle = \langle \delta(\nu \wedge \lambda); \lambda\rangle . \tag{12}$$

We get the result appealing to Lemma 12 namely

$$\langle \delta(\nu \wedge \lambda); \lambda\rangle = (\gamma_1 + \cdots + \gamma_{n-1}) - \left(\gamma_{i_1} + \cdots + \gamma_{i_{k-1}}\right) = \sum_{j \notin \{i_1, \cdots, i_{k-1}\}} \gamma_j .$$

Case 2: $(i_1, \cdots, i_{k-1}) \neq (j_1, \cdots, j_{k-1})$. The right hand side of (11) is now 0. So we have to show that

$$\langle K^{\nu}(\nu \lrcorner \alpha); \nu \lrcorner \beta\rangle = 0$$

Let $\lambda = E_{i_1} \wedge \cdots \wedge E_{i_{k-1}}$ and $\mu = E_{j_1} \wedge \cdots \wedge E_{j_{k-1}}$. It follows from Lemma 11 (using (3) as in Case 1) that

$$\langle K^{\nu}(\nu \lrcorner \alpha); \nu \lrcorner \beta\rangle + \langle K^{\nu}(\nu \lrcorner \beta); \nu \lrcorner \alpha\rangle = \langle \delta(\nu \wedge \lambda); \mu\rangle + \langle \delta(\nu \wedge \mu); \lambda\rangle - \langle \nabla(\alpha \lrcorner \beta); \nu\rangle .$$

Recall that $\{\nu, E_1, \ldots, E_{n-1}\}$ are extended to an orthonormal basis in a neighborhood of x_0 and therefore $\nabla(\alpha \lrcorner \beta) = 0$. Thus it follows from Lemmas 12 and 13 that

$$\langle K^{\nu}(\nu \lrcorner \alpha); \nu \lrcorner \beta\rangle = \frac{1}{2}\left(\langle \delta(\nu \wedge \lambda); \mu\rangle + \langle \delta(\nu \wedge \mu); \lambda\rangle\right) = 0,$$

which proves the claim of the present case. ■

4.5 Proof of the main theorem

For the equivalence (i) $\Leftrightarrow$ (iv), we give below a proof which is elementary and self-contained. A second proof can be obtained from Theorem 28 and the remark following it. Still another proof, more in the language of differential geometry, can be given using Theorem 35. These two other proofs are independent of the one given below and of the previous analysis..

Proof (Theorem 8). We know from Theorem 5.7 in [14] (see also [13] or Theorem 33 for a slightly different way of expressing the identity) that, for every $\omega \in W_T^{1,2}\left(\Omega;\Lambda^k\right) \cup W_N^{1,2}\left(\Omega;\Lambda^k\right)$,

$$\int_\Omega \left(|d\omega|^2 + |\delta\omega|^2 - |\nabla\omega|^2\right) = \int_{\partial\Omega} \left(\langle L^\nu\left(\nu \wedge \omega\right); \nu \wedge \omega\rangle + \langle K^\nu\left(\nu \lrcorner \omega\right); \nu \lrcorner \omega\rangle\right). \tag{13}$$

Step 1: (i) $\Rightarrow$ (ii). Assume that $C_T\left(\Omega, k\right) = 1$. This means that, for every $\omega \in W_T^{1,2}\left(\Omega;\Lambda^k\right)$,

$$0 \leq \|d\omega\|^2 + \|\delta\omega\|^2 - \|\nabla\omega\|^2 + \|\omega\|^2 = \|\omega\|^2 + K\left(\omega\right)$$

where

$$K\left(\omega\right) = \int_{\partial\Omega} \widetilde{K}\left(\omega\right) = \int_{\partial\Omega} \langle K^\nu\left(\nu \lrcorner \omega\right); \nu \lrcorner \omega\rangle.$$

Next let $\varphi \in W_T^{1,2}\left(\Omega;\Lambda^k\right)$ be such that $\varphi = \omega$ on $\partial\Omega$ and $\varphi \equiv 0$ in Ω outside an $\epsilon-$neighborhood of $\partial\Omega$. Note that, since $\varphi = \omega$ on $\partial\Omega$, then

$$K^\nu\left(\nu \lrcorner \varphi\right) = K^\nu\left(\nu \lrcorner \omega\right) \quad \text{on } \partial\Omega.$$

We thus have, by (13) and since $\varphi = \omega$ on $\partial\Omega$,

$$0 \leq \|\varphi\|^2 + K\left(\varphi\right) = \|\varphi\|^2 + K\left(\omega\right).$$

Since $\|\varphi\|^2$ is as small as we want, we deduce that

$$K\left(\omega\right) = \int_{\partial\Omega} \widetilde{K}\left(\omega\right) = \int_{\partial\Omega} \langle K^\nu\left(\nu \lrcorner \omega\right); \nu \lrcorner \omega\rangle \geq 0, \quad \forall\, \omega \in W_T^{1,2}\left(\Omega;\Lambda^k\right). \tag{14}$$

We now prove (ii) from the above inequality. Choose $\omega \in W_T^{1,2}\left(\Omega;\Lambda^k\right)$, $\psi \in C^\infty\left(\overline{\Omega}\right)$ and $\alpha = \psi\,\omega \in W_T^{1,2}\left(\Omega;\Lambda^k\right)$. Invoking (14) we find that

$$0 \leq K\left(\alpha\right) = \int_{\partial\Omega} \widetilde{K}\left(\alpha\right) = \int_{\partial\Omega} \psi^2\, \widetilde{K}\left(\omega\right).$$

Since ψ is arbitrary, we have the claim, i.e. $\widetilde{K}\left(\omega\right) \geq 0$.

Step 2: (ii) $\Rightarrow$ (iii). From (13) we have

$$\|d\omega\|^2 + \|\delta\omega\|^2 - \|\nabla\omega\|^2 = K\left(\omega\right) = \int_{\partial\Omega} \widetilde{K}\left(\omega\right)$$

and thus the result, since $K\left(\omega\right) \geq 0$ (because $\widetilde{K}\left(\omega\right) \geq 0$).

Step 3: (iii) $\Rightarrow$ (i). This is trivial, once coupled with Proposition 2.

Step 4: (ii) $\Rightarrow$ (iv). We choose in (ii), for $1 \leq i_1 < \cdots < i_{k-1} \leq n-1$,

$$\omega = \nu \wedge \lambda \quad \text{with} \quad \lambda = E_{i_1} \wedge \cdots \wedge E_{i_{k-1}}$$

From the assumption and the second conclusion in Lemma 14, we then obtain

$$0 \leq \langle K^{\nu} (\nu \lrcorner \omega) ; \nu \lrcorner \omega \rangle = \sum_{j \notin \{i_1, \cdots, i_{k-1}\}} \gamma_j \, .$$

Step 5: (iv) ⇒ (ii). This follows from the second conclusion of Lemma 14.

Step 6: (iii) ⇒ (v). The fact that the supremum is not attained follows from (iii), since, for every $\omega \in W_T^{1,2} (\Omega; \Lambda^k)$,

$$\|\nabla \omega\|^2 \leq \|d\omega\|^2 + \|\delta \omega\|^2 \leq \|d\omega\|^2 + \|\delta \omega\|^2 + \|\omega\|^2$$

hence the result.

Step 7: (v) ⇒ (i). In order to prove the statement, we show that if $C_T > 1$, then there exists a maximizer. We divide the proof into three substeps.

Step 7.1. Let $\omega_s \in W_T^{1,2} (\Omega; \Lambda^k) \setminus \{0\}$ be a maximizing sequence (and hence ω_s is not a constant form), i.e.

$$\lim_{s \to \infty} \frac{\|\nabla \omega_s\|^2}{\|d\omega_s\|^2 + \|\delta \omega_s\|^2 + \|\omega_s\|^2} = C_T \, .$$

Without loss of generality, up to replacing ω_s by $\omega_s / \|\nabla \omega_s\|$, we can assume that $\|\nabla \omega_s\| = 1$ and hence

$$\lim_{s \to \infty} \left[\|d\omega_s\|^2 + \|\delta \omega_s\|^2 + \|\omega_s\|^2 \right] = \frac{1}{C_T} < 1. \tag{15}$$

In particular $\|\omega_s\|$ is bounded and thus, up to a subsequence that we do not relabel, there exists $\omega \in W_T^{1,2} (\Omega; \Lambda^k)$ such that

$$\omega_s \rightharpoonup \omega \quad \text{in } W^{1,2}.$$

We prove in the next substeps that ω is a maximizer.

Step 7.2. We first show that $\omega \neq 0$. Suppose, for the sake of contradiction, that $\omega = 0$, then from (15) we get

$$\lim_{s \to \infty} \left[\|d\omega_s\|^2 + \|\delta \omega_s\|^2 \right] = \frac{1}{C_T} < 1. \tag{16}$$

From (13), we infer that there exists $c_1 = c_1 (\Omega)$ such that

$$\|d\omega_s\|^2 + \|\delta \omega_s\|^2 = 1 + \int_{\partial \Omega} \langle K^{\nu} (\nu \lrcorner \omega_s) ; \nu \lrcorner \omega_s \rangle \geq 1 - c_1 \int_{\partial \Omega} |\omega_s|^2 \, .$$

Since (cf. Proposition 5.15 in [14]) there exists $c_2 = c_2 (\Omega)$ such that for every $\epsilon > 0$

$$\int_{\partial \Omega} |\omega_s|^2 \leq \epsilon \|\nabla \omega_s\|^2 + \frac{c_2}{\epsilon} \|\omega_s\|^2 = \epsilon + \frac{c_2}{\epsilon} \|\omega_s\|^2$$

we deduce that

$$\|d\omega_s\|^2 + \|\delta \omega_s\|^2 \geq 1 - c_1 \epsilon - \frac{c_1 c_2}{\epsilon} \|\omega_s\|^2 \, .$$

Letting $s \to \infty$ we find

$$\lim_{s \to \infty} \left[\|d\omega_s\|^2 + \|\delta \omega_s\|^2 \right] \geq 1 - c_1 \epsilon$$

and, since ϵ is arbitrary, we find a contradiction with (16).

Step 7.3. We may now conclude. In the sequel we will have to pass several times to subsequences in order that all limits are true limits but, for the sake of not burdening the notations, we do not relabel these subsequences.

(i) We have, recalling that $\omega \in W_T^{1,2}\left(\Omega;\Lambda^k\right)$,

$$\begin{aligned}1 = \|\nabla\omega_s\|^2 &= \|\nabla(\omega_s-\omega)\|^2 + \|\nabla\omega\|^2 + 2\int_\Omega \langle\nabla(\omega_s-\omega);\nabla\omega\rangle \\ &\le \|\nabla(\omega_s-\omega)\|^2 + C_T\left(\|d\omega\|_{L^2}^2 + \|\delta\omega\|_{L^2}^2 + \|\omega\|_{L^2}^2\right) + 2\int_\Omega \langle\nabla(\omega_s-\omega);\nabla\omega\rangle .\end{aligned}$$

Since $\omega_s \rightharpoonup \omega$ in $W^{1,2}$, we find that

$$1 = \lim_{s\to\infty}\|\nabla(\omega_s-\omega)\|^2 + \|\nabla\omega\|^2 \le \lim_{s\to\infty}\|\nabla(\omega_s-\omega)\|^2 + C_T\left(\|d\omega\|_{L^2}^2 + \|\delta\omega\|_{L^2}^2 + \|\omega\|_{L^2}^2\right).$$

(ii) Since $\omega_s - \omega \in W_T^{1,2}\left(\Omega;\Lambda^k\right)$, we have

$$\begin{aligned}\|\nabla(\omega_s-\omega)\|^2 &\le C_T\left(\|d(\omega_s-\omega)\|_{L^2}^2 + \|\delta(\omega_s-\omega)\|_{L^2}^2 + \|(\omega_s-\omega)\|_{L^2}^2\right) \\ &= C_T\left(\|d\omega_s\|^2 + \|\delta\omega_s\|^2 + \|\omega_s\|^2 + \|d\omega\|_{L^2}^2 + \|\delta\omega\|_{L^2}^2 + \|\omega\|_{L^2}^2\right) \\ &\quad - C_T\left(\int_\Omega 2\left[\langle d\omega_s;d\omega\rangle + \langle\delta\omega_s;\delta\omega\rangle + \langle\omega_s;\omega\rangle\right]\right).\end{aligned}$$

Passing to the limit, recalling (15) and that $\omega_s \rightharpoonup \omega$ in $W^{1,2}$, we get

$$\lim_{s\to\infty}\|\nabla(\omega_s-\omega)\|^2 \le 1 - C_T\left(\|d\omega\|_{L^2}^2 + \|\delta\omega\|_{L^2}^2 + \|\omega\|_{L^2}^2\right).$$

(iii) Combining (i) and (ii) we obtain

$$\begin{aligned}1 &= \lim_{s\to\infty}\|\nabla(\omega_s-\omega)\|^2 + \|\nabla\omega\|^2 \\ &\le \lim_{s\to\infty}\|\nabla(\omega_s-\omega)\|^2 + C_T\left(\|d\omega\|_{L^2}^2 + \|\delta\omega\|_{L^2}^2 + \|\omega\|_{L^2}^2\right) \le 1\end{aligned}$$

which implies that

$$\|\nabla\omega\|^2 = C_T\left(\|d\omega\|_{L^2}^2 + \|\delta\omega\|_{L^2}^2 + \|\omega\|_{L^2}^2\right)$$

as wished.

Step 8: (i) ⇒ (vi). Since (i) (and thus (iii)) holds, we find, for $\omega \in W_T^{1,2}\left(t\,\Omega;\Lambda^k\right)$ and setting $\omega(x) = u(x/t)$,

$$\begin{aligned}\|\nabla\omega\|_{L^2(t\,\Omega)}^2 &= \int_{t\,\Omega}|\nabla\omega(y)|^2\,dy = t^{n-2}\int_\Omega|\nabla u(x)|^2\,dx = t^{n-2}\|\nabla u\|_{L^2(\Omega)}^2 \\ &\le t^{n-2}\|du\|_{L^2(\Omega)}^2 + t^{n-2}\|\delta u\|_{L^2(\Omega)}^2 = \|d\omega\|_{L^2(t\,\Omega)}^2 + \|\delta\omega\|_{L^2(t\,\Omega)}^2\end{aligned}$$

which shows that $C_T(t\,\Omega,k) = C_T(\Omega,k) = 1$.

Step 9: (vi) ⇒ (i). Without loss of generality we can assume that $t < 1$. We reason by contradiction and assume that $C_T(\Omega,k) > 1$. Invoking (v) we have that there exists $u \in W_T^{1,2}\left(\Omega;\Lambda^k\right)$ such that

$$C_T(\Omega,k) = \frac{\|\nabla u\|_{L^2(\Omega)}^2}{\|du\|_{L^2(\Omega)}^2 + \|\delta u\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2}.$$

Setting $\omega(x) = u(x/t)$, we obtain that $\omega \in W_T^{1,2}\left(t\,\Omega;\Lambda^k\right)$ and

$$C_T(\Omega,k) = \frac{\|\nabla\omega\|_{L^2(t\,\Omega)}^2}{\|d\omega\|_{L^2(t\,\Omega)}^2 + \|\delta\omega\|_{L^2(t\,\Omega)}^2 + t^{-2}\|\omega\|_{L^2(t\,\Omega)}^2}.$$

Since $t < 1$, we get

$$C_T(\Omega, k) < \frac{\|\nabla\omega\|^2_{L^2(t\,\Omega)}}{\|d\omega\|^2_{L^2(t\,\Omega)} + \|\delta\omega\|^2_{L^2(t\,\Omega)} + \|\omega\|^2_{L^2(t\,\Omega)}} \le C_T(t\,\Omega, k)$$

which is our claim. ■

Theorem 8 (combined with Remark 9) has as an immediate corollary the following.

Corollary 16 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open smooth set and $k = 1$. Then*

(i) $C_T(\Omega, 1) = 1$ *if and only if the mean curvature of $\partial\Omega$ is non-negative;*

(ii) $C_N(\Omega, 1) = 1$ *if and only if Ω is convex.*

5 Some examples

We now deal with some special cases where we can make C_T, C_N arbitrarily large.

Proposition 17 *Let $1 \le k \le n-1$. Then there exists a set $\Omega_k \subset B$ (a fixed ball of $\mathbb{R}^n$) such that $C_T(\Omega_k, k), C_N(\Omega_k, n-k)$ are arbitrarily large.*

Remark 18 Except for the case $k = 1$, the sets Ω_k that we construct are not smooth. However it is easy to modify slightly these sets so as to make them smooth, while preserving the proposition.

Proof Since $C_T(\Omega, k) = C_N(\Omega, n-k)$, it is sufficient to prove the result for $C_T(\Omega, k)$. For the sake of clarity we deal with the case $k = 1$ separately.

Step 1 ($k = 1$). We let, for $x \in \mathbb{R}^n$, $|x|$ denote the usual Euclidean norm. Let $0 < r < 1$ and $\Omega_1 = \{x \in \mathbb{R}^n : r < |x| < 1\}$. We then choose $\lambda \in C^1([r, 1])$ arbitrary and

$$\omega(x) = \lambda(|x|) \sum_{i=1}^{n} x_i\, dx^i \in W_T^{1,2}\left(\Omega_1; \Lambda^1\right).$$

Clearly

$$\omega^j_{x_i} = \lambda\,\delta^{ij} + \lambda'\,\frac{x_i x_j}{|x|}, \quad d\omega = 0 \quad \text{and} \quad \delta\omega = \operatorname{div}(x\,\lambda) = n\,\lambda + |x|\,\lambda'$$

leading to

$$|\delta\omega|^2 = (n\,\lambda + |x|\,\lambda')^2 \quad \text{and} \quad |\nabla\omega|^2 = n\,\lambda^2 + 2\,|x|\,\lambda\,\lambda' + |x|^2\,(\lambda')^2.$$

Choose $\lambda(s) = s^{-n}$ (with this choice we have $\delta\omega = 0$). We therefore have (denoting by σ_n the measure of the unit sphere of $\mathbb{R}^n$) that

$$\int_{\Omega_1} |\nabla\omega|^2 = \sigma_n \int_r^1 \left(n^2 - n\right) s^{-n-1} ds = \left.\frac{\sigma_n\left(n^2 - n\right) s^{-n}}{-n}\right|_r^1 = \sigma_n\,(n-1)\left[r^{-n} - 1\right]$$

while

$$\int_{\Omega_1} \left(|d\omega|^2 + |\delta\omega|^2 + |\omega|^2\right) = \sigma_n \int_r^1 s^{-n+1} ds = \begin{cases} \frac{\sigma_n}{n-2}\left[r^{-n+2} - 1\right] & \text{if } n > 2 \\ -\sigma_n \log r & \text{if } n = 2. \end{cases}$$

Therefore, when $r \to 0$, we find (writing $\sim$ for the asymptotic behavior)

$$\frac{\|\nabla\omega\|^2}{\|d\omega\|^2 + \|\delta\omega\|^2 + \|\omega\|^2} \sim \begin{cases} \frac{(n-2)(n-1)}{r^2} & \text{if } n > 2 \\ -\frac{1}{r^2 \log r} & \text{if } n = 2. \end{cases}$$

Thus, for r sufficiently small, we deduce that $C_T(\Omega_1, 1)$ is arbitrarily large as wished.

Step 2 ($2 \leq k \leq n-1$)*.* We divide the proof into two parts.

Step 2.1. Let us introduce some notations.

1) We write for $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$

$$|x|_k = \sqrt{x_1^2 + \cdots + x_{n-k+1}^2}\,.$$

2) Let $0 < r < 1$. The set $\Omega_k \subset \mathbb{R}^n$ is then chosen as

$$\Omega_k = \left\{x \in \mathbb{R}^n : r < |x|_k < 1 \text{ and } 0 < x_{n-k+2}, \cdots, x_n < 1\right\}.$$

3) We finally let $\lambda \in C^1([r, 1])$ to be chosen below,

$$\varphi_k(x) = \sum_{i=1}^{n-k+1} x_i\, dx^i \quad \text{and} \quad \omega_k(x) = \lambda\left(|x|_k\right)\varphi_k(x) \wedge dx^{n-k+2} \wedge \cdots \wedge dx^n \in \Lambda^k.$$

Step 2.2. Observe the following facts.

(i) If ν is the outward unit normal to Ω_k , then

$$\nu \wedge \omega_k = 0 \quad \text{on } \partial\Omega.$$

Indeed one sees that this is the case by distinguishing between the lateral boundaries $|x|_k = r, 1$ where

$$\nu = \frac{\pm 1}{|x|_k}(x_1, \cdots, x_{n-k+1}, 0, \cdots, 0) \quad \Rightarrow \quad \nu \wedge \varphi_k = 0$$

and the horizontal boundaries $x_s = 0, 1$ $(n-k+2 \leq s \leq n)$ where

$$\nu = \pm e_s \quad \Rightarrow \quad \nu \wedge dx^{n-k+2} \wedge \cdots \wedge dx^n = 0.$$

(ii) We have, for $1 \leq i_1 < \cdots < i_k \leq n$, that

$$\omega_k^{i_1 \cdots i_k}(x) = \begin{cases} \lambda\left(\sqrt{x_1^2 + \cdots + x_{n-k+1}^2}\right) x_{i_1} & \text{if} \quad \begin{array}{l} 1 \leq i_1 \leq n-k+1 \\ (i_2, \cdots, i_k) = ((n-k+2), \cdots, n) \end{array} \\ 0 & \text{otherwise.} \end{cases}$$

and thus, if $1 \leq i, j \leq n-k+1$,

$$\frac{\partial \omega_k^{i\,(n-k+2)\cdots n}}{\partial x_j} = \lambda\, \delta^{ij} + \lambda'\, \frac{x_i x_j}{|x|_k}$$

and all the other partial derivatives are 0.

(iii) This leads to $d\omega_k = 0$. Indeed if we set

$$\mu'(s) = s\,\lambda(s) \quad \text{and} \quad \eta(x) = \mu\left(|x|_k\right)$$

we see that

$$\lambda\left(|x|_k\right)\varphi_k(x) = d\eta(x) \quad \Rightarrow \quad d\omega_k = 0.$$

(iv) We now prove that

$$|\delta\omega_k|^2 = \left((n-k+1)\,\lambda + |x|_k\,\lambda'\right)^2.$$

Indeed, since

$$(\delta\omega_k)^{i_1\cdots i_{k-1}} = \sum_{\gamma=1}^{k} (-1)^{\gamma-1} \sum_{i_{\gamma-1}<j<i_\gamma} \frac{\partial \omega_k^{i_1\cdots i_{\gamma-1} j i_\gamma \cdots i_{k-1}}}{\partial x_j},$$

we have $(\delta\omega_k)^{i_1\cdots i_{k-1}} = 0$ unless $(i_1,\cdots,i_{k-1}) = ((n-k+2),\cdots,n)$; while

$$(\delta\omega_k)^{(n-k+2)\cdots n} = \sum_{j=1}^{n-k+1} \frac{\partial \omega_k^{jn-k+2\cdots n}}{\partial x_j} = \sum_{j=1}^{n-k+1} \frac{\partial\left(\lambda\left(|x|_k\right)x_j\right)}{\partial x_j} = (n-k+1)\,\lambda + |x|_k\,\lambda'.$$

(v) We next observe that

$$|\nabla\omega_k|^2 = (n-k+1)\,\lambda^2 + 2\,|x|_k\,\lambda\,\lambda' + |x|_k^2\,(\lambda')^2\,.$$

(vi) Finally choose $\lambda(s) = s^{-(n-k+1)}$ (with this choice we have $\delta\omega_k = 0$). We therefore have (σ_{n-k+1} denoting the measure of the unit sphere of $\mathbb{R}^{n-k+1}$) that

$$\begin{aligned}\int_{\Omega_k} |\nabla\omega_k|^2 &= \sigma_{n-k+1}\int_r^1 (n-k+1)\,(n-k)\,s^{-n+k-2}ds = \sigma_{n-k+1}\,(k-n)\,s^{-n+k-1}\Big|_r^1 \\ &= \sigma_{n-k+1}\,(n-k)\left[r^{-n+k-1}-1\right]\end{aligned}$$

while

$$\int_{\Omega_k}\left(|d\omega_k|^2 + |\delta\omega_k|^2 + |\omega_k|^2\right) = \sigma_{n-k+1}\int_r^1 s^{-n+k}ds = \begin{cases} \frac{\sigma_{n-k+1}}{n-k-1}\left[r^{-n+k+1}-1\right] & \text{if } n > k+1 \\ -\sigma_{n-k+1}\log r & \text{if } n = k+1. \end{cases}$$

Therefore, when $r \to 0$, we find (writing $\sim$ for the asymptotic behavior)

$$\frac{\|\nabla\omega_k\|^2}{\|d\omega_k\|^2 + \|\delta\omega_k\|^2 + \|\omega_k\|^2} \sim \begin{cases} \frac{(n-k-1)(n-k)}{r^2} & \text{if } n > k+1 \\ -\frac{1}{r^2\log r} & \text{if } n = k+1. \end{cases}$$

Thus, for r sufficiently small, we deduce that $C_T(\Omega_k, k)$ is arbitrarily large as wished. ■

6 The case of polytopes

Definition 19 *$\Omega \subset \mathbb{R}^n$ is said to be a* generalized polytope*, if there exist $\Omega_0, \Omega_1, \cdots, \Omega_M$ bounded open polytopes such that, for every $i, j = 1, \cdots, M$ with $i \neq j$,*

$$\overline{\Omega}_i \subset \Omega_0\,, \quad \overline{\Omega}_i \cap \overline{\Omega}_j = \emptyset \quad \text{and} \quad \Omega = \Omega_0 \setminus \left(\bigcup_{i=1}^{M} \overline{\Omega}_i\right).$$

In this case $\overline{\Omega}_i$, $i = 1, \cdots, M$, are called the holes.

Theorem 20 *Let $\Omega \subset \mathbb{R}^n$ be a generalized polytope. Then the following identity holds*

$$\|\nabla\omega\|^2 = \|d\omega\|^2 + \|\delta\omega\|^2\,, \quad \forall\,\omega \in C_T^1\left(\overline{\Omega};\Lambda^k\right) \cup C_N^1\left(\overline{\Omega};\Lambda^k\right).$$

Remark 21 *(i)* Note that we do not make any assumption on the topology of the domain and that holes are allowed. The identity shows that there are no non-trivial harmonic fields with vanishing tangential (or normal) component which are of class C^1. However, in presence of holes, there are non-trivial harmonic fields with weaker regularity (this is, of course, a problem only on the boundary, since harmonic fields are C^∞ in the interior).

(ii) In the case $k = 1$, see also [15].

Before proceeding with the proof, we need to introduce a few notations that would help us keep track of the signs in the proof.

Notation 22 (i) *For $1 \leqslant k \leqslant n$, we write*

$$\mathcal{T}^k = \{(i_1, \cdots, i_k) \in \mathbb{N}^k : 1 \leqslant i_1 < \cdots < i_k \leqslant n\}.$$

For $I = (i_1, \cdots, i_k) \in \mathcal{T}^k$, we write dx^I to denote $dx^{i_1} \wedge \cdots \wedge dx^{i_k}$.

(ii) *For $i \in I$, we write $I_{\widehat{i}} = (i_1, \cdots, \widehat{i}, \cdots, i_k)$, where $\widehat{i}$ denotes the absence of the named index i. Note that, $I_{\widehat{i_p}} \in \mathcal{T}^{k-1}$, for all $1 \leqslant p \leqslant k$. Similarly, for $i, j \in I$, $i < j$, we write $I_{\widehat{ij}} = (i_1, \cdots, \widehat{i}, \cdots, \widehat{j}, \cdots, i_k)$.*

(iii) *Given $I \in \mathcal{T}^k$ and $i, j \notin I$, $i \neq j$, we write $[iI]$ to denote the increasing multiindex formed by the index i and the indices in I. In other words $[iI]$ is the permutation of the indices such that $[iI] \in \mathcal{T}^{k+1}$. Furthermore, we define the sign of $[i, I]$, denoted by $\operatorname{sgn}[i, I]$, as*

$$dx^{[iI]} = \operatorname{sgn}[i, I]\, dx^i \wedge dx^I.$$

Similarly, $[ijI]$ is the permutation of the indices such that $[ijI] \in \mathcal{T}^{k+2}$ and $\operatorname{sgn}[i, j, I]$ is given by

$$dx^{[ijI]} = \operatorname{sgn}[i, j, I]\, dx^i \wedge dx^j \wedge dx^I.$$

We need a few lemmas for the theorem.

Lemma 23 *Let $n \geq 2$ and $1 \leq k \leq n-1$ be integers. Then for any $I \in \mathcal{T}^{k+1}$ and every $i, j \in I$ with $i \neq j$,*

$$\operatorname{sgn}\left[i, I_{\widehat{ij}}\right] \operatorname{sgn}\left[j, I_{\widehat{ij}}\right] = -\operatorname{sgn}\left[i, I_{\widehat{i}}\right] \operatorname{sgn}\left[j, I_{\widehat{j}}\right]$$

and for any $I \in \mathcal{T}^{k-1}$ and every $i, j \notin I$ with $i \neq j$,

$$\operatorname{sgn}[i, [jI]] \operatorname{sgn}[j, [iI]] = -\operatorname{sgn}[i, I] \operatorname{sgn}[j, I].$$

Remark 24 When $k = 1$ both equations read as

$$\operatorname{sgn}[i, j] \operatorname{sgn}[j, i] = -1.$$

Indeed elements $I \in \mathcal{T}^{k-1}$ are as if they were absent, i.e. $[jI] = j$ and $\operatorname{sgn}[i, I] = 1$.

Proof Since $I_{\widehat{i}} = \left[jI_{\widehat{ij}}\right]$ and $I_{\widehat{j}} = \left[iI_{\widehat{ij}}\right]$, we have the identity

$$\operatorname{sgn}\left[i, I_{\widehat{i}}\right] \operatorname{sgn}\left[j, I_{\widehat{ij}}\right] = \operatorname{sgn}\left[i, j, I_{\widehat{ij}}\right] = -\operatorname{sgn}\left[j, i, I_{\widehat{ij}}\right] = -\operatorname{sgn}\left[j, I_{\widehat{j}}\right] \operatorname{sgn}\left[i, I_{\widehat{ij}}\right].$$

This proves the first identity. The second one is just the first one, where $I \in \mathcal{T}^{k-1}$ plays the role of $I_{\widehat{ij}}$. This finishes the proof. ■

The next lemma gives a pointwise identity.

Lemma 25 *Let $U \subset \mathbb{R}^n$ be open and let $\omega \in C^1\left(U; \Lambda^k\right)$. Then, for any $x \in U$,*

$$|d\omega|^2 + |\delta\omega|^2 - |\nabla\omega|^2 = \sum_{I \in \mathcal{T}^{k+1}} \sum_{\substack{i,j \in I \\ i \neq j}} \operatorname{sgn}\left[i, I_{\widehat{i}}\right] \operatorname{sgn}\left[j, I_{\widehat{j}}\right] \left(\frac{\partial \omega^{I_{\widehat{i}}}}{\partial x_i} \frac{\partial \omega^{I_{\widehat{j}}}}{\partial x_j} - \frac{\partial \omega^{I_{\widehat{j}}}}{\partial x_i} \frac{\partial \omega^{I_{\widehat{i}}}}{\partial x_j} \right) \tag{17}$$

$$|d\omega|^2 + |\delta\omega|^2 - |\nabla\omega|^2 = \sum_{I \in \mathcal{T}^{k-1}} \sum_{\substack{i,j \notin I \\ i < j}} \operatorname{sgn}[i, I] \operatorname{sgn}[j, I] \left(\frac{\partial \omega^{[iI]}}{\partial x_i} \frac{\partial \omega^{[jI]}}{\partial x_j} - \frac{\partial \omega^{[jI]}}{\partial x_i} \frac{\partial \omega^{[iI]}}{\partial x_j} \right). \tag{18}$$

Remark 26 When $k = 1$ the lemma reads as

$$|d\omega|^2 + |\delta\omega|^2 - |\nabla\omega|^2 = 2\sum_{i<j}\left(\frac{\partial\omega^i}{\partial x_i}\frac{\partial\omega^j}{\partial x_j} - \frac{\partial\omega^j}{\partial x_i}\frac{\partial\omega^i}{\partial x_j}\right)$$

while for $k = 2$

$$\begin{aligned}&|d\omega|^2 + |\delta\omega|^2 - |\nabla\omega|^2\\&= 2\sum_{i<j<k}\left(\frac{\partial\omega^{ij}}{\partial x_j}\frac{\partial\omega^{ik}}{\partial x_k} - \frac{\partial\omega^{ij}}{\partial x_k}\frac{\partial\omega^{ik}}{\partial x_j} + \frac{\partial\omega^{ij}}{\partial x_k}\frac{\partial\omega^{jk}}{\partial x_i} - \frac{\partial\omega^{ij}}{\partial x_i}\frac{\partial\omega^{jk}}{\partial x_k} + \frac{\partial\omega^{ik}}{\partial x_i}\frac{\partial\omega^{jk}}{\partial x_j} - \frac{\partial\omega^{ik}}{\partial x_j}\frac{\partial\omega^{jk}}{\partial x_i}\right).\end{aligned}$$

Proof We calculate

$$d\omega = \sum_{I\in\mathcal{T}^{k+1}}\left(\sum_{i\in I}\operatorname{sgn}\left[i, I_{\widehat{i}}\right]\frac{\partial\omega^{I_{\widehat{i}}}}{\partial x_i}\right)dx^I \quad\text{and}\quad \delta\omega = \sum_{I\in\mathcal{T}^{k-1}}\left(\sum_{i\notin I}\operatorname{sgn}[i, I]\frac{\partial\omega^{[iI]}}{\partial x_i}\right)dx^I.$$

We start by evaluating $|d\omega|^2$, we find

$$|d\omega|^2 = \sum_{I\in\mathcal{T}^{k+1}}\sum_{i\in I}\left(\frac{\partial\omega^{I_{\widehat{i}}}}{\partial x_i}\right)^2 + 2\sum_{I\in\mathcal{T}^{k+1}}\sum_{\substack{i,j\in I\\ i<j}}\operatorname{sgn}\left[i, I_{\widehat{i}}\right]\operatorname{sgn}\left[j, I_{\widehat{j}}\right]\frac{\partial\omega^{I_{\widehat{i}}}}{\partial x_i}\frac{\partial\omega^{I_{\widehat{j}}}}{\partial x_j}.$$

Rewriting the terms in two different ways, we obtain,

$$|d\omega|^2 = \sum_{I\in\mathcal{T}^{k}}\sum_{i\notin I}\left(\frac{\partial\omega^{I}}{\partial x_i}\right)^2 + 2\sum_{I\in\mathcal{T}^{k+1}}\sum_{\substack{i,j\in I\\ i<j}}\operatorname{sgn}\left[i, I_{\widehat{i}}\right]\operatorname{sgn}\left[j, I_{\widehat{j}}\right]\frac{\partial\omega^{I_{\widehat{i}}}}{\partial x_i}\frac{\partial\omega^{I_{\widehat{j}}}}{\partial x_j} \tag{19}$$

$$|d\omega|^2 = \sum_{I\in\mathcal{T}^{k}}\sum_{i\notin I}\left(\frac{\partial\omega^{I}}{\partial x_i}\right)^2 + 2\sum_{I\in\mathcal{T}^{k-1}}\sum_{\substack{i,j\notin I\\ i<j}}\operatorname{sgn}[i,[jI]]\operatorname{sgn}[j,[iI]]\frac{\partial\omega^{[jI]}}{\partial x_i}\frac{\partial\omega^{[iI]}}{\partial x_j}. \tag{20}$$

We next evaluate $|\delta\omega|^2$, we get

$$|\delta\omega|^2 = \sum_{I\in\mathcal{T}^{k-1}}\sum_{i\notin I}\left(\frac{\partial\omega^{[iI]}}{\partial x_i}\right)^2 + 2\sum_{I\in\mathcal{T}^{k-1}}\sum_{\substack{i,j\notin I\\ i<j}}\operatorname{sgn}[i, I]\operatorname{sgn}[j, I]\frac{\partial\omega^{[iI]}}{\partial x_i}\frac{\partial\omega^{[jI]}}{\partial x_j}.$$

Rewriting the terms in two different ways, we find

$$|\delta\omega|^2 = \sum_{I\in\mathcal{T}^{k}}\sum_{i\in I}\left(\frac{\partial\omega^{I}}{\partial x_i}\right)^2 + 2\sum_{I\in\mathcal{T}^{k-1}}\sum_{\substack{i,j\notin I\\ i<j}}\operatorname{sgn}[i, I]\operatorname{sgn}[j, I]\frac{\partial\omega^{[iI]}}{\partial x_i}\frac{\partial\omega^{[jI]}}{\partial x_j} \tag{21}$$

$$|\delta\omega|^2 = \sum_{I\in\mathcal{T}^{k}}\sum_{i\in I}\left(\frac{\partial\omega^{I}}{\partial x_i}\right)^2 + 2\sum_{I\in\mathcal{T}^{k+1}}\sum_{\substack{i,j\in I\\ i<j}}\operatorname{sgn}\left[i, I_{\widehat{ij}}\right]\operatorname{sgn}\left[j, I_{\widehat{ij}}\right]\frac{\partial\omega^{I_{\widehat{j}}}}{\partial x_i}\frac{\partial\omega^{I_{\widehat{i}}}}{\partial x_j}. \tag{22}$$

Appealing to (19) and (22), we infer that

$$\begin{aligned}&|d\omega|^2 + |\delta\omega|^2 - |\nabla\omega|^2\\&= 2\sum_{I\in\mathcal{T}^{k+1}}\sum_{\substack{i,j\in I\\ i<j}}\left(\operatorname{sgn}\left[i, I_{\widehat{i}}\right]\operatorname{sgn}\left[j, I_{\widehat{j}}\right]\frac{\partial\omega^{I_{\widehat{i}}}}{\partial x_i}\frac{\partial\omega^{I_{\widehat{j}}}}{\partial x_j} + \operatorname{sgn}\left[i, I_{\widehat{ij}}\right]\operatorname{sgn}\left[j, I_{\widehat{ij}}\right]\frac{\partial\omega^{I_{\widehat{j}}}}{\partial x_i}\frac{\partial\omega^{I_{\widehat{i}}}}{\partial x_j}\right).\end{aligned}$$

Invoking (20) and (21), we find

$$|d\omega|^2 + |\delta\omega|^2 - |\nabla\omega|^2$$
$$= \sum_{I \in \mathcal{T}^{k-1}} \sum_{\substack{i,j \notin I \\ i<j}} \left(\operatorname{sgn}[i,I] \operatorname{sgn}[j,I] \frac{\partial \omega^{[iI]}}{\partial x_i} \frac{\partial \omega^{[jI]}}{\partial x_j} + \operatorname{sgn}[i,[jI]] \operatorname{sgn}[j,[iI]] \frac{\partial \omega^{[jI]}}{\partial x_i} \frac{\partial \omega^{[iI]}}{\partial x_j} \right).$$

Using Lemma 23, the last two identities establish (17) and (18), respectively. ■

Lemma 27 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open Lipschitz set and $\omega \in C^1\left(\overline{\Omega}; \Lambda^k\right)$. Then*

$$\|d\omega\|^2 + \|\delta\omega\|^2 - \|\nabla\omega\|^2 = \int_{\partial\Omega} \langle \nu \wedge \omega; d\omega \rangle - \int_{\partial\Omega} \sum_{I \in \mathcal{T}^{k+1}} \sum_{i,j \in I} \operatorname{sgn}\left[i, I_{\widehat{i}}\right] \operatorname{sgn}\left[j, I_{\widehat{j}}\right] \omega^{I_{\widehat{j}}} \nu^i \frac{\partial \omega^{I_{\widehat{i}}}}{\partial x_j} \quad (23)$$

$$\|d\omega\|^2 + \|\delta\omega\|^2 - \|\nabla\omega\|^2 = \int_{\partial\Omega} \langle \nu \lrcorner \omega; \delta\omega \rangle - \int_{\partial\Omega} \sum_{I \in \mathcal{T}^{k-1}} \sum_{i,j \notin I} \operatorname{sgn}[i,I] \operatorname{sgn}[j,I] \omega^{[iI]} \nu^j \frac{\partial \omega^{[jI]}}{\partial x_i}. \quad (24)$$

Proof We divide the proof in three steps.

Step 1. Integrate the equations of Lemma 25 to get

$$\|d\omega\|^2 + \|\delta\omega\|^2 - \|\nabla\omega\|^2 = \int_{\Omega} \sum_{I \in \mathcal{T}^{k+1}} \sum_{\substack{i,j \in I \\ i \neq j}} \operatorname{sgn}\left[i, I_{\widehat{i}}\right] \operatorname{sgn}\left[j, I_{\widehat{j}}\right] \left(\frac{\partial \omega^{I_{\widehat{i}}}}{\partial x_i} \frac{\partial \omega^{I_{\widehat{j}}}}{\partial x_j} - \frac{\partial \omega^{I_{\widehat{j}}}}{\partial x_i} \frac{\partial \omega^{I_{\widehat{i}}}}{\partial x_j} \right) \quad (25)$$

$$\|d\omega\|^2 + \|\delta\omega\|^2 - \|\nabla\omega\|^2 = \int_{\Omega} \sum_{I \in \mathcal{T}^{k-1}} \sum_{\substack{i,j \notin I \\ i<j}} \operatorname{sgn}[i,I] \operatorname{sgn}[j,I] \left(\frac{\partial \omega^{[iI]}}{\partial x_i} \frac{\partial \omega^{[jI]}}{\partial x_j} - \frac{\partial \omega^{[jI]}}{\partial x_i} \frac{\partial \omega^{[iI]}}{\partial x_j} \right). \quad (26)$$

Step 2. Noting that (the following argument uses the fact that $\omega \in C^2$ but by density the identity (27) is valid for $\omega \in C^1$)

$$\frac{\partial \omega^{I_{\widehat{i}}}}{\partial x_i} \frac{\partial \omega^{I_{\widehat{j}}}}{\partial x_j} - \frac{\partial \omega^{I_{\widehat{j}}}}{\partial x_i} \frac{\partial \omega^{I_{\widehat{i}}}}{\partial x_j} = \frac{\partial}{\partial x_j} \left(\omega^{I_{\widehat{j}}} \frac{\partial \omega^{I_{\widehat{i}}}}{\partial x_i} \right) - \frac{\partial}{\partial x_i} \left(\omega^{I_{\widehat{j}}} \frac{\partial \omega^{I_{\widehat{i}}}}{\partial x_j} \right)$$

we integrate by parts (25), bearing in mind that Ω is Lipschitz, to obtain

$$\|d\omega\|^2 + \|\delta\omega\|^2 - \|\nabla\omega\|^2 = \int_{\partial\Omega} \sum_{I \in \mathcal{T}^{k+1}} \sum_{\substack{i,j \in I \\ i \neq j}} \operatorname{sgn}\left[i, I_{\widehat{i}}\right] \operatorname{sgn}\left[j, I_{\widehat{j}}\right] \left(\omega^{I_{\widehat{j}}} \nu^j \frac{\partial \omega^{I_{\widehat{i}}}}{\partial x_i} - \omega^{I_{\widehat{j}}} \nu^i \frac{\partial \omega^{I_{\widehat{i}}}}{\partial x_j} \right). \quad (27)$$

Step 3. Since Ω is Lipschitz, ν is defined for a.e. $x \in \partial\Omega$. Thus, for a.e. $x \in \partial\Omega$, we find

$$\nu \wedge \omega = \sum_{I \in \mathcal{T}^{k+1}} \left(\sum_{j \in I} \operatorname{sgn}\left[j, I_{\widehat{j}}\right] \nu^j \omega^{I_{\widehat{j}}} \right) dx^I.$$

But, since $\omega \in C^1\left(\overline{\Omega}; \Lambda^k\right)$, for every $x \in \overline{\Omega}$, we have

$$d\omega = \sum_{I \in \mathcal{T}^{k+1}} \left(\sum_{i \in I} \operatorname{sgn}\left[i, I_{\widehat{i}}\right] \frac{\partial \omega^{I_{\widehat{i}}}}{\partial x_i} \right) dx^I.$$

We therefore deduce the pointwise identity, for a.e. $x \in \partial\Omega$,

$$\begin{aligned}\langle \nu \wedge \omega ; d\omega \rangle &= \sum_{I \in \mathcal{T}^{k+1}} \sum_{i,j \in I} \operatorname{sgn}\left[i, I_{\widehat{i}}\right] \operatorname{sgn}\left[j, I_{\widehat{j}}\right] \omega^{I_{\widehat{j}}} \, \nu^j \, \frac{\partial \omega^{I_{\widehat{i}}}}{\partial x_i} \\ &= \sum_{I \in \mathcal{T}^{k+1}} \sum_{i \in I} \omega^{I_{\widehat{i}}} \, \nu^i \, \frac{\partial \omega^{I_{\widehat{i}}}}{\partial x_i} + \sum_{I \in \mathcal{T}^{k+1}} \sum_{\substack{i,j \in I \\ i \neq j}} \operatorname{sgn}\left[i, I_{\widehat{i}}\right] \operatorname{sgn}\left[j, I_{\widehat{j}}\right] \omega^{I_{\widehat{j}}} \, \nu^j \, \frac{\partial \omega^{I_{\widehat{i}}}}{\partial x_i} .\end{aligned}$$

We then have

$$\sum_{I \in \mathcal{T}^{k+1}} \sum_{\substack{i,j \in I \\ i \neq j}} \operatorname{sgn}\left[i, I_{\widehat{i}}\right] \operatorname{sgn}\left[j, I_{\widehat{j}}\right] \omega^{I_{\widehat{j}}} \, \nu^j \, \frac{\partial \omega^{I_{\widehat{i}}}}{\partial x_i} = \langle \nu \wedge \omega ; d\omega \rangle - \sum_{I \in \mathcal{T}^{k+1}} \sum_{i \in I} \omega^{I_{\widehat{i}}} \, \nu^i \, \frac{\partial \omega^{I_{\widehat{i}}}}{\partial x_i} . \tag{28}$$

Substituting (28) in (27), we obtain (23). Analogous calculations yield (24), starting from integration by parts of (26). ■

We now prove a theorem for piecewise C^2 Lipschitz domains, which can be viewed as a generalization of Theorem 3.1.1.2 in [22] valid for $k = 1$.

Theorem 28 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open Lipschitz set with piecewise C^2 boundary, i.e. $\partial\Omega = \cup_{s=1}^{N} \overline{\Gamma_s}$, where the Γ_s are C^2 and relatively open subset of $\partial\Omega$ and $\partial\Omega \setminus \cup_{s=1}^{N} \Gamma_s$ has zero surface measure. Let $E_1, \cdots, E_{n-1}$ be an orthonormal frame field of principal directions of $\cup_{s=1}^{N} \Gamma_s$ with associated principal curvatures $\gamma_1, \cdots, \gamma_{n-1}$. Then*

$$\|d\omega\|^2 + \|\delta\omega\|^2 - \|\nabla\omega\|^2 = \sum_{s=1}^{N} \int_{\Gamma_s} \left(\sum_{l=1}^{n-1} \gamma_l \left| E_l \wedge \omega \right|^2 \right) \quad \textit{for every } \omega \in C_T^1\left(\overline{\Omega}; \Lambda^k\right) \tag{29}$$

$$\|d\omega\|^2 + \|\delta\omega\|^2 - \|\nabla\omega\|^2 = \sum_{s=1}^{N} \int_{\Gamma_s} \left(\sum_{l=1}^{n-1} \gamma_l \left| E_l \lrcorner\, \omega \right|^2 \right) \quad \textit{for every } \omega \in C_N^1\left(\overline{\Omega}; \Lambda^k\right) . \tag{30}$$

Remark 29 *(i)* By standard regularization, the theorem is valid for $\omega \in W_T^{1,2}\left(\Omega; \Lambda^k\right)$ (respectively $\omega \in W_N^{1,2}\left(\Omega; \Lambda^k\right)$) if $\partial\Omega$ is (fully) C^2.

(ii) Note that the two identities above are the same as those appearing at the end of Theorem 33.

Proof *Step 1.* We first show (29). Since $\partial\Omega \setminus \cup_{s=1}^{N} \Gamma_s$ has zero surface measure, we obtain from (23)

$$\begin{aligned}&\|d\omega\|^2 + \|\delta\omega\|^2 - \|\nabla\omega\|^2 \\ &= \int_{\partial\Omega} \langle \nu \wedge \omega ; d\omega \rangle - \sum_{s=1}^{N} \int_{\Gamma_s} \sum_{I \in \mathcal{T}^{k+1}} \sum_{i,j \in I} \operatorname{sgn}\left[i, I_{\widehat{i}}\right] \operatorname{sgn}\left[j, I_{\widehat{j}}\right] \omega^{I_{\widehat{j}}} \, \nu^i \, \tfrac{\partial \omega^{I_{\widehat{i}}}}{\partial x_j} .\end{aligned} \tag{31}$$

We argue on each of the Γ_s. Since $E_1, \cdots, E_{n-1}$ denote a frame of tangent vectors at each point of Γ_s, we can write $e_j = \sum_{l=1}^{n-1} E_l^j \, E_l + \nu^j \, \nu$. Thus, for any $j = 1, \cdots, n$,

$$\frac{\partial \omega^{I_{\widehat{i}}}}{\partial x_j} = \sum_{l=1}^{n-1} E_l^j \, \frac{\partial \omega^{I_{\widehat{i}}}}{\partial E_l} + \nu^j \, \frac{\partial \omega^{I_{\widehat{i}}}}{\partial \nu} . \tag{32}$$

Step 1.1. We set

$$A=\sum_{I\in\mathcal{T}^{k+1}}\sum_{i,j\in I}\operatorname{sgn}\left[i,I_{\widehat{i}}\right]\operatorname{sgn}\left[j,I_{\widehat{j}}\right]\omega^{I_{\widehat{j}}}\,\nu^{i}\,\frac{\partial\omega^{I_{\widehat{i}}}}{\partial x_{j}}$$

and note, in view of (32), that $A=B+C$ where

$$B=\sum_{I\in\mathcal{T}^{k+1}}\left(\sum_{i\in I}\operatorname{sgn}\left[i,I_{\widehat{i}}\right]\nu^{i}\,\frac{\partial\omega^{I_{\widehat{i}}}}{\partial\nu}\right)\left(\sum_{j\in I}\operatorname{sgn}\left[j,I_{\widehat{j}}\right]\nu^{j}\,\omega^{I_{\widehat{j}}}\right)$$

$$C=\sum_{I\in\mathcal{T}^{k+1}}\sum_{i,j\in I}\operatorname{sgn}\left[i,I_{\widehat{i}}\right]\operatorname{sgn}\left[j,I_{\widehat{j}}\right]\omega^{I_{\widehat{j}}}\,\nu^{i}\left(\sum_{l=1}^{n-1}E_{l}^{j}\,\frac{\partial\omega^{I_{\widehat{i}}}}{\partial E_{l}}\right).$$

(i) We first observe that

$$B=\sum_{I\in\mathcal{T}^{k+1}}\left(\nu\wedge\frac{\partial\omega}{\partial\nu}\right)^{I}(\nu\wedge\omega)^{I}=\left\langle\nu\wedge\omega;\nu\wedge\frac{\partial\omega}{\partial\nu}\right\rangle.$$

(ii) We next prove that

$$C=\sum_{l=1}^{n-1}\left\langle E_{l}\wedge\omega;\frac{\partial}{\partial E_{l}}(\nu\wedge\omega)\right\rangle-\sum_{l=1}^{n-1}\gamma_{l}\left|E_{l}\wedge\omega\right|^{2}.$$

Indeed note that, for any $I\in\mathcal{T}^{k+1}$ and any $l=1,\cdots,n-1$,

$$\frac{\partial}{\partial E_{l}}(\nu\wedge\omega)^{I}=\frac{\partial}{\partial E_{l}}\left(\sum_{i\in I}\operatorname{sgn}\left[i,I_{\widehat{i}}\right]\nu^{i}\,\omega^{I_{\widehat{i}}}\right)=\sum_{i\in I}\operatorname{sgn}\left[i,I_{\widehat{i}}\right]\nu^{i}\,\frac{\partial\omega^{I_{\widehat{i}}}}{\partial E_{l}}+\sum_{i\in I}\operatorname{sgn}\left[i,I_{\widehat{i}}\right]\omega^{I_{\widehat{i}}}\,\frac{\partial\nu^{i}}{\partial E_{l}}.$$

For any $j\in I$, multiplying by $\operatorname{sgn}\left[j,I_{\widehat{j}}\right]\omega^{I_{\widehat{j}}}E_{l}^{j}$ and summing over $l=1,\cdots,n-1$ and $j\in I$, we deduce (recalling that $\gamma_{l}\,E_{l}^{i}=\partial\nu^{i}/\partial E_{l}$) that

$$\begin{aligned}
C^{I}&=\sum_{i,j\in I}\operatorname{sgn}\left[i,I_{\widehat{i}}\right]\operatorname{sgn}\left[j,I_{\widehat{j}}\right]\omega^{I_{\widehat{j}}}\,\nu^{i}\left(\sum_{l=1}^{n-1}E_{l}^{j}\,\frac{\partial\omega^{I_{\widehat{i}}}}{\partial E_{l}}\right)\\
&=\sum_{j\in I}\sum_{l=1}^{n-1}\operatorname{sgn}\left[j,I_{\widehat{j}}\right]\omega^{I_{\widehat{j}}}E_{l}^{j}\,\frac{\partial}{\partial E_{l}}(\nu\wedge\omega)^{I}-\sum_{i,j\in I}\operatorname{sgn}\left[i,I_{\widehat{i}}\right]\operatorname{sgn}\left[j,I_{\widehat{j}}\right]\omega^{I_{\widehat{i}}}\omega^{I_{\widehat{j}}}\left(\sum_{l=1}^{n-1}E_{l}^{j}\,\frac{\partial\nu^{i}}{\partial E_{l}}\right)\\
&=\sum_{l=1}^{n-1}(E_{l}\wedge\omega)^{I}\,\frac{\partial}{\partial E_{l}}(\nu\wedge\omega)^{I}-\sum_{i,j\in I}\operatorname{sgn}\left[i,I_{\widehat{i}}\right]\operatorname{sgn}\left[j,I_{\widehat{j}}\right]\omega^{I_{\widehat{i}}}\omega^{I_{\widehat{j}}}\left(\sum_{l=1}^{n-1}\gamma_{l}\,E_{l}^{j}\,E_{l}^{i}\right)
\end{aligned}$$

and thus

$$\begin{aligned}
C^{I}&=\sum_{l=1}^{n-1}(E_{l}\wedge\omega)^{I}\,\frac{\partial}{\partial E_{l}}(\nu\wedge\omega)^{I}-\sum_{l=1}^{n-1}\gamma_{l}\left(\sum_{i\in I}\operatorname{sgn}\left[i,I_{\widehat{i}}\right]E_{l}^{i}\,\omega^{I_{\widehat{i}}}\right)\left(\sum_{j\in I}\operatorname{sgn}\left[j,I_{\widehat{j}}\right]E_{l}^{j}\,\omega^{I_{\widehat{j}}}\right)\\
&=\sum_{l=1}^{n-1}(E_{l}\wedge\omega)^{I}\,\frac{\partial}{\partial E_{l}}(\nu\wedge\omega)^{I}-\sum_{l=1}^{n-1}\gamma_{l}\,(E_{l}\wedge\omega)^{I}\,(E_{l}\wedge\omega)^{I}
\end{aligned}$$

which is our claim, since $C=\sum_{I\in\mathcal{T}^{k+1}}C^I$.

Combining (i) and (ii) we have obtained that

$$A=\left\langle \nu\wedge\omega;\nu\wedge\frac{\partial\omega}{\partial\nu}\right\rangle+\sum_{l=1}^{n-1}\left\langle E_l\wedge\omega;\frac{\partial}{\partial E_l}\left(\nu\wedge\omega\right)\right\rangle-\sum_{l=1}^{n-1}\gamma_l\left|E_l\wedge\omega\right|^2.$$

Step 1.2. Combining (31) and Step 1.1, we just proved that

$$\begin{aligned}&\|d\omega\|^2+\|\delta\omega\|^2-\|\nabla\omega\|^2\\&=\sum_{s=1}^{N}\int_{\Gamma_s}\left[\langle\nu\wedge\omega;d\omega\rangle-\left\langle\nu\wedge\omega;\nu\wedge\frac{\partial\omega}{\partial\nu}\right\rangle-\sum_{l=1}^{n-1}\left\langle E_l\wedge\omega;\frac{\partial}{\partial E_l}\left(\nu\wedge\omega\right)\right\rangle+\sum_{l=1}^{n-1}\gamma_l\left|E_l\wedge\omega\right|^2\right].\end{aligned}$$

Thus, if $\nu\wedge\omega=0$ on Γ_s for each $s=1,\cdots,N$, we obtain (29).

Step 2. Analogous calculations, starting from (24), establishes the identity

$$\begin{aligned}&\|d\omega\|^2+\|\delta\omega\|^2-\|\nabla\omega\|^2\\&=\sum_{s=1}^{N}\int_{\Gamma_s}\left[\langle\nu\lrcorner\omega;\delta\omega\rangle-\left\langle\nu\lrcorner\omega;\nu\lrcorner\frac{\partial\omega}{\partial\nu}\right\rangle-\sum_{l=1}^{n-1}\left\langle E_l\lrcorner\omega;\frac{\partial}{\partial E_l}\left(\nu\lrcorner\omega\right)\right\rangle+\sum_{l=1}^{n-1}\gamma_l\left|E_l\lrcorner\omega\right|^2\right].\end{aligned}$$

Using that $\nu\lrcorner\omega=0$ on Γ_s for each $s=1,\cdots,N$, this yields (30). This finishes the proof. ■

We finally are ready to prove the theorem.

Proof (Theorem 20). By Theorem 28, the result is immediate since for a generalized polytope, the principal curvatures on every face are 0. ■

Theorem 28 also immediately implies the following.

Theorem 30 *Let $\Omega\subset\mathbb{R}^n$ be a bounded open Lipschitz set with piecewise C^2 boundary, i.e. $\partial\Omega=\cup_{s=1}^{N}\overline{\Gamma_s}$, where the Γ_s are C^2 and relatively open subset of $\partial\Omega$ and $\partial\Omega\setminus\cup_{s=1}^{N}\Gamma_s$ has zero surface measure. If the principal curvatures are all nonnegative at every point on $\cup_{s=1}^{N}\Gamma_s$, then*

$$\|\nabla\omega\|^2\leq\|d\omega\|^2+\|\delta\omega\|^2,\quad\forall\,\omega\in C_T^1\left(\overline{\Omega};\Lambda^k\right)\cup C_N^1\left(\overline{\Omega};\Lambda^k\right).$$

Remark 31 Note that unlike the case of smooth domains, here the hypothesis of all principal curvatures being non-negative does not imply that the domain is convex. For example, the domain given in polar coordinates by

$$\Omega=\{(r,\theta):\theta_0<\theta<2\pi-\theta_0,\ r\in[0,1)\}\subset\mathbb{R}^2,$$

for some $0<\theta_0<\pi/2$, is a piecewise C^2 Lipschitz domain which satisfies the property, but is neither convex, nor can be approximated from the inside by smooth $1-$convex domains since smooth $1-$convex domains are necessarily convex. Hence, the result for such domains is not covered by the result in [27].

For $k=1$, Theorem 28 also immediately implies, as a corollary, the following variant of Korn inequality with the precise constant, which was already observed in Bauer-Pauly [3].

Corollary 32 *Let $\Omega\subset\mathbb{R}^n$ be a bounded open Lipschitz set with piecewise C^2 boundary, i.e. $\partial\Omega=\cup_{s=1}^{N}\overline{\Gamma_s}$, where the Γ_s are C^2 and relatively open subset of $\partial\Omega$ and $\partial\Omega\setminus\cup_{s=1}^{N}\Gamma_s$ has zero* *surface measure. If the principal curvatures are all nonpositive at every point on* $\cup_{s=1}^{N}\Gamma_s$ *, then*

$$\|\nabla u\|^2 \leq 2\,\|\nabla^{sym}u\|^2\,, \quad \forall\, u \in C_T^1\left(\overline{\Omega};\Lambda^1\right) \cup C_N^1\left(\overline{\Omega};\Lambda^1\right),$$

where the symmetric gradient $\nabla^{sym}u$ *is defined by*

$$\left(\nabla^{sym}u\right)_{ij} = \frac{1}{2}\left(\frac{\partial u^i}{\partial x_j} + \frac{\partial u^j}{\partial x_i}\right), \quad \textit{for } i,j = 1,\cdots,n.$$

Proof Theorem 28 in this case implies,

$$\|\nabla u\|^2 \geq \|\operatorname{curl} u\|^2 + \|\operatorname{div} u\|^2\,, \quad \forall\, u \in C_T^1\left(\overline{\Omega};\Lambda^1\right) \cup C_N^1\left(\overline{\Omega};\Lambda^1\right).$$

Integrating the pointwise identity

$$|\nabla u|^2 = |\nabla^{sym}u|^2 + \frac{1}{2}\,|\operatorname{curl} u|^2$$

and combining with the inequality above gives,

$$\|\nabla^{sym}u\|^2 - \frac{1}{2}\,\|\nabla u\|^2 = \frac{1}{2}\left(\|\nabla u\|^2 - \|\operatorname{curl} u\|^2\right) \geq \frac{1}{2}\,\|\operatorname{div} u\|^2 \geq 0.$$

This completes the proof. ■

7 Appendix: an integral identity for differential forms

We recall that $\Omega \subset \mathbb{R}^n$ is a bounded open smooth set with exterior unit normal ν. When we say that $E_1,\cdots,E_{n-1}$ is an *orthonormal frame field of principal directions* of $\partial\Omega$ with associated principal curvatures $\gamma_1,\cdots,\gamma_{n-1}$, we mean that $\{\nu,E_1,\cdots,E_{n-1}\}$ form an orthonormal frame field of $\mathbb{R}^n$ and, for every $1 \leq i \leq n-1$,

$$\sum_{j=1}^{n}\left(E_i^j \nu_{x_j}^l\right) = \gamma_i\, E_i^l \quad \Rightarrow \quad \gamma_i = \sum_{j,l=1}^{n}\left(E_i^l E_i^j \nu_{x_j}^l\right).$$

The following proposition was used only implicitly and is another version of the identity obtained in [13] (see also Theorem 5.7 in [14]). We first state and prove it in the Euclidean setting and then for general Riemannian manifolds, using then the notation of differential geometry and referring to [12].

Theorem 33 *Let* $0 \leq k \leq n$. *Let* $E_1,\cdots,E_{n-1}$ *be an orthonormal frame field of principal directions of* $\partial\Omega$ *with associated principal curvatures* $\gamma_1,\cdots,\gamma_{n-1}$ *. Then every* α, $\beta \in C^1\left(\overline{\Omega};\Lambda^k\right)$ *satisfy the equation*

$$\begin{aligned}
&\int_\Omega \left(\langle d\alpha; d\beta\rangle + \langle \delta\alpha; \delta\beta\rangle - \langle \nabla\alpha; \nabla\beta\rangle\right)\\
&= -\int_{\partial\Omega} \left(\langle \nu \wedge d\left(\nu \lrcorner \alpha\right); \nu \wedge \beta\rangle + \langle \nu \lrcorner \delta\left(\nu \wedge \alpha\right); \nu \lrcorner \beta\rangle\right)\\
&+ \int_{\partial\Omega} \sum_{1\leq i_1<\cdots<i_k\leq n-1} \langle \alpha; E_{i_1\cdots i_k}\rangle \langle \beta; E_{i_1\cdots i_k}\rangle \sum_{j\in\{i_1,\cdots,i_k\}} \gamma_j\\
&+ \int_{\partial\Omega} \sum_{1\leq i_1<\cdots<i_{k-1}\leq n-1} \left\langle \alpha; \nu \wedge E_{i_1\cdots i_{k-1}}\right\rangle \left\langle \beta; \nu \wedge E_{i_1\cdots i_{k-1}}\right\rangle \sum_{j\notin\{i_1,\cdots,i_{k-1}\}} \gamma_j\,,
\end{aligned}$$

where $E_{i_1\cdots i_k} = E_{i_1}\wedge\cdots\wedge E_{i_k}$. *In particular the following two identities hold. For every* $\alpha\in W_T^{1,2}\left(\Omega;\Lambda^k\right)$

$$\int_\Omega\left(|d\alpha|^2+|\delta\alpha|^2-|\nabla\alpha|^2\right)=\int_{\partial\Omega}\sum_{1\leq i_1<\cdots<i_{k-1}\leq n-1}\left|\left\langle\alpha;\nu\wedge E_{i_1\cdots i_{k-1}}\right\rangle\right|^2\sum_{j\notin\{i_1,\cdots,i_{k-1}\}}\gamma_j$$

and for every $\alpha\in W_N^{1,2}\left(\Omega;\Lambda^k\right)$

$$\int_\Omega\left(|d\alpha|^2+|\delta\alpha|^2-|\nabla\alpha|^2\right)=\int_{\partial\Omega}\sum_{1\leq i_1<\cdots<i_k\leq n-1}\left|\left\langle\alpha;E_{i_1\cdots i_k}\right\rangle\right|^2\sum_{j\in\{i_1,\cdots,i_k\}}\gamma_j\,.$$

Remark 34 If $k=0$ or $k=n$, then the the right hand side has to be understood as 0 by definition. If $k=1$, the theorem reads as

$$\begin{aligned}&\int_\Omega\left(\langle d\alpha;d\beta\rangle+\langle\delta\alpha;\delta\beta\rangle-\langle\nabla\alpha;\nabla\beta\rangle\right)\\&=-\int_{\partial\Omega}\left(\langle\nu\wedge d\left(\nu\lrcorner\alpha\right);\nu\wedge\beta\rangle+\langle\nu\lrcorner\delta\left(\nu\wedge\alpha\right);\nu\lrcorner\beta\rangle\right)\\&+\int_{\partial\Omega}\sum_{i=1}^{n-1}\gamma_i\left\langle\alpha;E_i\right\rangle\left\langle\beta;E_i\right\rangle+\int_{\partial\Omega}\left\langle\alpha;\nu\right\rangle\left\langle\beta;\nu\right\rangle\sum_{j=1}^{n-1}\gamma_j\end{aligned}$$

and, in particular,

$$\int_\Omega\left(|d\alpha|^2+|\delta\alpha|^2-|\nabla\alpha|^2\right)=\begin{cases}\displaystyle\int_{\partial\Omega}\left(\gamma_1+\cdots+\gamma_{n-1}\right)|\langle\alpha;\nu\rangle|^2 & \text{if }\alpha\in W_T^{1,2}\left(\Omega;\Lambda^1\right)\\ \displaystyle\sum_{i=1}^{n-1}\int_{\partial\Omega}\gamma_i\,|\langle\alpha;E_i\rangle|^2 & \text{if }\alpha\in W_N^{1,2}\left(\Omega;\Lambda^1\right).\end{cases}$$

Proof The theorem follows from Theorem 5.7 in [14] and Lemma 14. The last two identities also follow from (29), respectively (30), of Theorem 28 and the remark following it. ■

We now discuss the more general version of the identity. We assume that Ω is an $n-$dimensional compact orientable smooth Riemannian manifold with boundary $\partial\Omega$. We also adopt the following abbreviations

$$\begin{gathered}\mathcal{T}_{n-1}^k=\left\{I=(i_1,\cdots,i_k)\in\mathbb{N}^k:1\leq i_1<\cdots<i_k\leq n-1\right\}\\ E_I=\left(E_{i_1},\cdots,E_{i_k}\right)\quad\text{for }I\in\mathcal{T}_{n-1}^k\\ I^c\quad\text{is the complement of }I\text{ in }\{1,\cdots,n-1\}\,.\end{gathered}$$

In the next theorem the quantity F_k is the linear $0-$th order (not differential) operator given by the Bochner-Weitzenböck formula, see [12] Section 2.2 for more details. On a flat manifold, for instance $\mathbb{R}^n$, the operator F_k is equal to 0 (which explains its absence in Theorem 33).

Theorem 35 *Let* $0\leq k\leq n$. *Then every* $\alpha,\,\beta\in C^1\left(\overline{\Omega};\Lambda^k\right)$ *satisfy the equation*

$$\begin{aligned}&\int_\Omega\left(\langle d\alpha;d\beta\rangle+\langle\delta\alpha;\delta\beta\rangle-\langle\nabla\alpha;\nabla\beta\rangle\right)-\int_\Omega\langle F_k\alpha;\beta\rangle\\&=-\int_{\partial\Omega}\left(\langle\nu\wedge d\left(\nu\lrcorner\alpha\right);\nu\wedge\beta\rangle+\langle\nu\lrcorner\delta\left(\nu\wedge\alpha\right);\nu\lrcorner\beta\rangle\right)\\&+\int_{\partial\Omega}\sum_{I\in\mathcal{T}_{n-1}^k}\alpha\left(E_I\right)\beta\left(E_I\right)\sum_{j\in I}\gamma_j+\int_{\partial\Omega}\sum_{I\in\mathcal{T}_{n-1}^{k-1}}\alpha\left(\nu,E_I\right)\beta\left(\nu,E_I\right)\sum_{j\in I^c}\gamma_j\,.\end{aligned}$$

Remark 36 In the case $\alpha = \beta$ a similar form of this identity is known as Reilly formula, see Theorem 3 in [32] and the references there. In that case the identity simplifies: using partial integration twice one obtains, see Remark 3.8 (iv) in [12],

$$\int_{\partial\Omega} \left(\langle \nu \wedge d\left(\nu \lrcorner \alpha\right); \nu \wedge \alpha \rangle + \langle \nu \lrcorner \delta\left(\nu \wedge \alpha\right); \nu \lrcorner \alpha \rangle \right) = 2 \int_{\partial\Omega} \langle \nu \lrcorner \delta\left(\nu \wedge \alpha\right); \nu \lrcorner \alpha \rangle .$$

Proof The theorem follows from [12] Theorem 3.7 and Remark 3.8 (ii) and from Lemma 37 below. ■

The following Lemma is the analogue of Lemma 14. β_N (β_T) denotes the normal (respectively tangential) component of β (see [12]).

Lemma 37 *Let S_k be defined as in Definition 3.3 in [12]. Then the following identities hold*

$$\textit{(i)} \qquad \langle S_k \alpha, \beta_N \rangle = \sum_{I \in \mathcal{T}_{n-1}^{k-1}} \left[\alpha\left(\nu, E_I\right) \beta\left(\nu, E_I\right) \sum_{j \in I^c} \gamma_j \right]$$

$$\textit{(ii)} \qquad \langle S_{n-k}(*\alpha), *(\beta_T) \rangle = \sum_{I \in \mathcal{T}_{n-1}^{k}} \left[\alpha\left(E_I\right) \beta\left(E_I\right) \sum_{j \in I} \gamma_j \right].$$

Proof *Step 1.* We first prove (i). Since $E_1, \cdots, E_{n-1}$ are principal directions of $\partial\Omega$ they satisfy that

$$\nabla_{E_i} \nu = \gamma_i \, E_i \quad \text{and} \quad \langle E_i; E_j \rangle = \delta_{ij} \, .$$

We now choose $(\nu, E_1, \cdots, E_{n-1})$ as an orthonormal basis of $\mathbb{R}^n$ to evaluate the scalar product $\langle S_k \alpha; \beta_N \rangle$. Tuples $(E_{i_1}, \cdots, E_{i_k})$ have no contribution in $\langle S_k \alpha; \beta_N \rangle$, because β_N is normal (actually $S_k \alpha$ too). We therefore obtain

$$\langle S_k \alpha; \beta_N \rangle = \sum_{1 \le i_1 < \cdots < i_{k-1} \le n-1} S_k \alpha \left(\nu, E_{i_1}, \cdots, E_{i_{k-1}}\right) \beta\left(\nu, E_{i_1}, \cdots, E_{i_{k-1}}\right), \tag{33}$$

where we have used that $\beta_N\left(\nu, E_{i_1}, \cdots, E_{i_{k-1}}\right) = \beta\left(\nu, E_{i_1}, \cdots, E_{i_{k-1}}\right)$ by definition of the normal component. By definition of S_k , we find

$$\begin{aligned} S_k \alpha \left(\nu, E_{i_1}, \cdots, E_{i_{k-1}}\right) &= -\sum_{j=1}^{n-1} \sum_{l=1}^{k-1} \alpha\left(E_j, E_{i_1}, \cdots, E_{i_{l-1}}, \mathrm{II}\left(E_j, E_{i_l}\right), E_{i_{l+1}}, \cdots E_{i_{k-1}}\right) \\ &\quad - \sum_{j=1}^{n-1} \alpha\left(\mathrm{II}\left(E_j, E_j\right), E_{i_1}, \cdots, E_{i_{k-1}}\right) \\ &= A_{(i_1, \cdots, i_{k-1})} + B_{(i_1, \cdots, i_{k-1})}, \end{aligned}$$

where II is the second fundamental form. We have also used that $(E_r)_T = E_r$ for any $r = 1, \cdots, n-1$, since they are tangent vectors. We therefore get for any r, s

$$\mathrm{II}\left(E_s, E_r\right) = \langle \nabla_{E_s} E_r; \nu \rangle \, \nu = - \langle E_r; \nabla_{E_s} \nu \rangle \, \nu = -\gamma_r \, \delta_{rs} \, \nu.$$

This shows that

$$B_{(i_1, \cdots, i_{k-1})} = \alpha\left(\nu, E_{i_1}, \cdots, E_{i_{k-1}}\right) \sum_{j=1}^{n-1} \gamma_j \, .$$

In the same way we get

$$\begin{aligned}A_{(i_1,\cdots,i_{k-1})} &= -\sum_{l=1}^{k-1}\alpha\left(E_{i_l},E_{i_1},\cdots,E_{i_{l-1}},\mathrm{II}\left(E_{i_l},E_{i_l}\right),E_{i_{l+1}},\cdots E_{i_{k-1}}\right)\\ &= \sum_{l=1}^{k-1}\gamma_{i_l}\alpha\left(E_{i_l},E_{i_1},\cdots,E_{i_{l-1}},\nu,E_{i_{l+1}},\cdots E_{i_{k-1}}\right)\\ &= -\alpha\left(\nu,E_{i_1},\cdots,E_{i_{k-1}}\right)\sum_{l=1}^{k-1}\gamma_{i_l}\,.\end{aligned}$$

So we obtain that

$$\begin{aligned}S_k\alpha\left(\nu,E_{i_1},\cdots,E_{i_{k-1}}\right) &= \alpha\left(\nu,E_{i_1},\cdots,E_{i_{k-1}}\right)\left(\sum_{j=1}^{n-1}\gamma_j-\sum_{l=1}^{k-1}\gamma_{i_l}\right)\\ &= \alpha\left(\nu,E_{i_1},\cdots,E_{i_{k-1}}\right)\sum_{j\in I^c}\gamma_j\,.\end{aligned}$$

From this last equation and (33) the first identity (i) follows.

Step 2. We now deduce (ii) from (i) in the following way

$$\langle S_{n-k}\left(*\alpha\right);*\left(\beta_T\right)\rangle = \sum_{I\in\mathcal{T}_{n-1}^{n-k-1}}\left[\left(*\alpha\right)\left(\nu,E_I\right)\left(*\beta_T\right)\left(\nu,E_I\right)\sum_{j\in I^c}\gamma_j\right].$$

Note that the Hodge $*$ operator computes on $k-$forms ω as

$$\left(*\omega\right)\left(X_{k+1},\cdots,X_n\right)=\omega\left(X_1,\cdots,X_k\right)$$

whenever $\left(X_1,\cdots,X_n\right)$ is an orthonormal basis of $\mathbb{R}^n$. We apply this to $X=\left(\nu,E_I,E_{I^c}\right)$. Thus we obtain that for each $I\in\mathcal{T}^{n-k-1}$

$$\left(*\alpha\right)\left(\nu,E_I\right)\left(*\beta_T\right)\left(\nu,E_I\right)=\alpha\left(E_{I^c}\right)\beta\left(E_{I^c}\right).$$

This leads to, renaming the summation index $I\to I^c$,

$$\begin{aligned}\langle S_{n-k}\left(*\alpha\right);*\left(\beta_T\right)\rangle &= \sum_{I\in\mathcal{T}_{n-1}^{(n-1)-k}}\left[\alpha\left(E_{I^c}\right)\beta\left(E_{I^c}\right)\sum_{j\in I^c}\gamma_j\right]\\ &= \sum_{I\in\mathcal{T}_{n-1}^{k}}\left[\alpha\left(E_I\right)\beta\left(E_I\right)\sum_{j\in I}\gamma_j\right],\end{aligned}$$

which proves (ii). ■

Acknowledgement 38 *We would like to thank M. Troyanov for providing us with the references [1] and [6]. We also want to thank two anonymous referees for their comments.*

References

[1] Alexander S., Locally convex hypersurfaces of negatively curved spaces, *Proc. Amer. Math. Soc.* **64** (1977), 321–325.

[2] Amrouche C., Bernardi C., Dauge M. and Girault V., Vector potentials in three-dimensional non-smooth domains, *Math. Methods Appl. Sci.,* **21** (1998), 823–864.

[3] Bauer S. and Pauly D., On Korn's first inequality for tangential or normal boundary conditions with explicit constants, *Math. Methods Appl. Sci.* **39** (2016), 5695–5704.

[4] Ben Belgacem F., Bernardi C., Costabel M. and Dauge M., Un résultat de densité pour les équations de Maxwell, *C. R. Acad. Sci. Paris Sér. I Math,* **324** (1997), 731–736.

[5] Bolik J., H Weyl's boundary value problems for differential forms, *Differential and Integral Equations,* **14** (2001), 937–952.

[6] Chern S.S. and Lashof R.K., On the total curvature of immersed manifolds II, *Michigan Math. J.* **5** (1958), 5–12.

[7] Ciarlet P., Hazard C. and Lohrengel S., Les équations de Maxwell dans un polyèdre: un résultat de densité, *C. R. Acad. Sci. Paris Sér. I Math.,* **326** (1998), 1305–1310.

[8] Costabel M., A remark on the regularity of solutions of Maxwell's equations on Lipschitz domains, *Math. Methods Appl. Sci.,* **12** (1990), 365–368.

[9] Costabel M., A coercive bilinear form for Maxwell's equations, *J. Math. Anal. Appl.,* **157** (1991), 527–541.

[10] Costabel M. and Dauge M., Un résultat de densité pour les équations de Maxwell régularisées dans un domaine lipschitzien, *C. R. Acad. Sci. Paris Sér. I Math.,* **327** (1998), 849–854.

[11] Costabel M. and Dauge M., Maxwell and Lamé eigenvalues on polyhedra, *Math. Methods Appl. Sci.* **22** (1999), 243–258.

[12] Csató G., On an integral formula for differential forms and its applications on manifolds with boundary, *Analysis (Berlin)* **33** (2013), 349–366.

[13] Csató G. and Dacorogna B., An identity involving exterior derivatives and applications to Gaffney inequality, *Discrete and Continuous Dynamical Systems, Series S* **5** (2012), 531–544.

[14] Csató G., Dacorogna B. and Kneuss O., *The pullback equation for differential forms*, Birkhäuser, 2012.

[15] Csató G., Kneuss O. and Rajendran D., On the boundary conditions in estimating $\nabla\omega$ by $\operatorname{div}\omega$ and $\operatorname{curl}\omega$; to appear.

[16] Dautray R. and Lions J.L., *Analyse mathématique et calcul numérique*, Masson, Paris, 1988.

[17] Desvillettes L. and Villani C., On a variant of Korn's inequality arising in statistical mechanics. A tribute to J. L. Lions; *ESAIM Control Optim. Calc. Var.* **8** (2002), 603–619.

[18] Friedrichs K. O., Differential forms on Riemannian manifolds, *Comm. Pure Appl. Math.,* **8** (1955), 551–590.

[19] Gaffney M.P., The harmonic operator for exterior differential forms, *Proc. Nat. Acad. of Sci. U. S. A.,* **37** (1951), 48–50.

[20] Gaffney M.P., Hilbert space methods in the theory of harmonic integrals, *Trans. Amer. Math. Soc,* **78** (1955), 426–444.

[21] Girault V. and Raviart P.A., *Finite element approximation of the Navier-Stokes equations,* Lecture Notes in Math. **749**, Springer-Verlag, Berlin, 1979.

[22] Grisvard P., *Elliptic problems in nonsmooth domains*, Classics in Applied Mathematics, **69**, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2011.

[23] Iwaniec T. and Martin G., *Geometric function theory and non-linear analysis,* Oxford University Press, Oxford, 2001.

[24] Iwaniec T., Scott C. and Stroffolini B., Nonlinear Hodge theory on manifolds with boundary, *Annali Mat. Pura Appl.,* **177** (1999), 37–115.

[25] Lewis R.T., Li J. and Li Y., A geometric characterization of a sharp Hardy inequality. *J. Funct. Anal.* **262** (2012), 3159–3185.

[26] Marcus M., Mizel V.J. and Pinchover Y., On the best constant for Hardy's inequality in $\mathbb{R}^n$, *Trans. Amer. Math. Soc.* **350** (1998), 3237–3255.

[27] Mitrea M., Dirichlet integrals and Gaffney-Friedrichs inequalities in convex domains, *Forum Math.* **13** (2001), 531–567.

[28] Mitrea D. and Mitrea M., Finite energy solutions of Maxwell's equations and constructive Hodge decompositions on nonsmooth Riemannian manifolds, *J. Funct. Anal.,* **190** (2002), 339–417.

[29] Morrey C.B., A variational method in the theory of harmonic integrals II, *Amer. J. Math.,* **78** (1956), 137–170.

[30] Morrey C.B., *Multiple integrals in the calculus of variations,* Springer-Verlag, Berlin, 1966.

[31] Morrey C.B. and Eells J., A variational method in the theory of harmonic integrals, *Ann. of Math.,* **63** (1956), 91–128.

[32] Raulot S. and Savo A., A Reilly formula and eigenvalue estimates for differential forms, *J. Geom. Anal.* **21** (2011), 620–640.

[33] Schwarz G., *Hodge decomposition - A method for solving boundary value problems*, Lecture Notes in Math. **1607**, Springer-Verlag, Berlin, 1995.

[34] Sha Ji-Ping, $p-$convex Riemannian manifolds, *Invent. Math.* **83** (1986), 437–447.

[35] Taylor M.E., *Partial differential equations,* Vol. 1, Springer-Verlag, New York, 1996.

[36] Von Wahl W., Estimating ∇u by $\operatorname{div} u$ and $\operatorname{curl} u$, *Math. Methods Appl. Sci.,* **15** (1992), 123–143.