

The pullback equation

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1 Introduction

The aim of this course is the study of the *pullback equation*

$$\varphi^*(g) = f. \tag{1}$$

More precisely we want to find a map $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n$, preferably we want this map to be a diffeomorphism, that satisfies the above equation, where f, g are differential k -forms, $0 \leq k \leq n$. Most of the time we will require these two forms to be closed. Before going further let us examine the exact meaning of (1). We write

$$g(x) = \sum_{1 \leq i_1 < \dots < i_k \leq n} g_{i_1 \dots i_k}(x) dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

and similarly for f . The meaning of (1) is that

$$\sum_{1 \leq i_1 < \dots < i_k \leq n} g_{i_1 \dots i_k}(\varphi) d\varphi^{i_1} \wedge \dots \wedge d\varphi^{i_k} = \sum_{1 \leq i_1 < \dots < i_k \leq n} f_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

where

$$d\varphi^i = \sum_{j=1}^n \frac{\partial \varphi^i}{\partial x_j} dx^j.$$

This turns out to be a *non-linear* (if $2 \leq k \leq n$) homogeneous of degree k (in the derivatives) first order system of $\binom{n}{k}$ partial differential equations. Let us see the form that the equation takes when $k = 0, 1, 2, n$.

Case: $k = 0$. The equation (1) reads as

$$g(\varphi(x)) = f(x)$$

while

$$dg = 0 \quad \Leftrightarrow \quad \text{grad } g = 0.$$

We will be, only marginally, interested in this elementary case, which is trivial for closed forms. In any case (1) is *not*, when $k = 0$, a differential equation.

Case: $k = 1$. The form g , and analogously for f , can be written as

$$g(x) = \sum_{i=1}^n g_i(x) dx^i.$$

The equation (1) becomes then

$$\sum_{i=1}^n g_i(\varphi(x)) d\varphi^i = \sum_{i=1}^n f_i(x) dx^i$$

while

$$dg = 0 \quad \Leftrightarrow \quad \text{curl } g = 0 \quad \Leftrightarrow \quad \frac{\partial g_i}{\partial x_j} - \frac{\partial g_j}{\partial x_i} = 0, \quad 1 \leq i < j \leq n.$$

Writing

$$d\varphi^i = \sum_{j=1}^n \frac{\partial \varphi^i}{\partial x_j} dx^j$$

and substituting into the equation, we find that (1) is equivalent to

$$\sum_{j=1}^n g_j(\varphi(x)) \frac{\partial \varphi^j}{\partial x_i}(x) = f_i(x), \quad 1 \leq i \leq n.$$

This is a system of $\binom{n}{1} = n$ first order *linear* (in the first derivatives) partial differential equations.

Case: $k = 2$. The form g , and analogously for f , can be written as

$$g = \sum_{1 \leq i < j \leq n} g_{ij}(x) dx^i \wedge dx^j$$

while

$$dg = 0 \quad \Leftrightarrow \quad \frac{\partial g_{ij}}{\partial x_k} - \frac{\partial g_{ik}}{\partial x_j} + \frac{\partial g_{jk}}{\partial x_i} = 0, \quad 1 \leq i < j < k \leq n.$$

The equation $\varphi^*(g) = f$ becomes

$$\sum_{1 \leq p < q \leq n} g_{pq}(\varphi(x)) d\varphi^p \wedge d\varphi^q = \sum_{1 \leq i < j \leq n} f_{ij}(x) dx^i \wedge dx^j.$$

We get, as before, that (1) is equivalent, for every $1 \leq i < j \leq n$, to

$$\sum_{1 \leq p < q \leq n} g_{pq}(\varphi(x)) \left(\frac{\partial \varphi^p}{\partial x_i} \frac{\partial \varphi^q}{\partial x_j} - \frac{\partial \varphi^p}{\partial x_j} \frac{\partial \varphi^q}{\partial x_i} \right) = f_{ij}(x)$$

which is a *non-linear* homogeneous of degree 2 (in the derivatives) system of $\binom{n}{2} = \frac{n(n-1)}{2}$ first order partial differential equations.

Case: $k = n$. In this case we always have $df = dg = 0$. By abuse of notations, if we identify volume forms and functions, we get that the equation $\varphi^*(g) = f$ becomes

$$g(\varphi(x)) \det \nabla \varphi(x) = f(x).$$

It is then a non-linear homogeneous of degree n (in the derivatives) first order partial differential equation.

The main questions that we will discuss are the following.

1) Algebraic case. When the forms are constants, it is natural to seek for solutions φ of the form $\varphi(x) = Ax$ where A is an invertible $n \times n$ matrix. Therefore the problem turns out to be of linear algebraic nature. For example when $k = 2$ we can associate, in a unique way, to any 2-form

$$g = \sum_{1 \leq i < j \leq n} g_{ij} dx^i \wedge dx^j$$

a *skew symmetric matrix* $G \in \mathbb{R}^{n \times n}$ (i.e. $G^t = -G$)

$$G = \begin{pmatrix} 0 & g_{12} & g_{13} & \cdots & g_{1n} \\ -g_{12} & 0 & g_{23} & \cdots & g_{2n} \\ -g_{13} & -g_{23} & 0 & \cdots & g_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -g_{1n} & -g_{2n} & -g_{3n} & \cdots & 0 \end{pmatrix}.$$

We therefore have

$$\varphi^*(g) = f \Leftrightarrow AGA^t = F.$$

Since any skew symmetric matrix has even rank, we have that if

$$\text{rank } G = \text{rank } F = 2m \leq n$$

then it is always possible to find an invertible matrix A . The canonical form is

then

$$J_m = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

2) Local existence. This is the easiest question. We will handle fairly completely the case of closed 2-forms, which is the case of Darboux theorem. The cases of 1 and $(n - 1)$ -forms as well as the case of n -forms will also be dealt with. It will turn out that the case $3 \leq k \leq n - 2$ is much more difficult and we will be able to handle only closed k -forms with special structure.

3) Existence of canonical forms. It will turn out that (for closed forms):

- when $k = 1$, then the canonical form is $g = dx^1$
- when $k = 2$, then the canonical form is, depending of the rank of the form,

$$g = \sum_{i=1}^m dx^{2i-1} \wedge dx^{2i}$$

- when $k = n - 1$, then the canonical form is

$$g = dx^1 \wedge \cdots \wedge dx^{n-1}$$

- when $k = n$, then the canonical form is

$$g = dx^1 \wedge \cdots \wedge dx^n.$$

4) Global existence. This is a much more difficult problem. We will obtain results in the case of volume forms and of closed 2-forms.

5) Regularity. A special emphasis will be given on getting sharp regularity results. For this reason we will have to work with Hölder spaces $C^{r,\alpha}$, $0 < \alpha < 1$, and not with spaces C^r . We will not deal with Sobolev spaces. In the present context the reason is that Hölder spaces form an algebra contrary to Sobolev spaces (with low exponents).

6) Selection principle. In all cases discussed here, once the existence part has been settled, it turns out that there are, in general, infinitely many solutions. Therefore the problem of selecting a solution with further properties becomes an important one. We will discuss very briefly this difficult problem in the cases $k = 2$ and $k = n$.

7) Invariants. Finally the question of the invariants will be discussed. We will see that the rank and the closedness are two invariants. They are the only

one (at least for the local problem) when $k = 1, 2, n - 1, n$, but there are others when $3 \leq k \leq n - 2$.

The course is based on the recent book [17], to which we refer for all missing proofs.

2 Algebraic preliminaries

We now gather some algebraic results about exterior forms that are used throughout the course. Let $1 \leq k \leq n$ be an integer (if $k > n$, we set $f = 0$). An exterior k -form will be denoted by

$$f = \sum_{1 \leq i_1 < \dots < i_k \leq n} f_{i_1 \dots i_k} e^{i_1} \wedge \dots \wedge e^{i_k}.$$

The set of exterior k -forms over \mathbb{R}^n is a vector space and is denoted $\Lambda^k(\mathbb{R}^n)$ and its dimension is

$$\dim(\Lambda^k(\mathbb{R}^n)) = \binom{n}{k}.$$

If $k = 0$, we set

$$\Lambda^0(\mathbb{R}^n) = \mathbb{R}.$$

By abuse of notations, we will, when convenient and in order not to burden the notations, identify k -forms with vectors in $\mathbb{R}^{\binom{n}{k}}$.

(i) The *exterior product* of $f \in \Lambda^k(\mathbb{R}^n)$ with $g \in \Lambda^l(\mathbb{R}^n)$, denoted by $f \wedge g$, is defined as usual and it belongs to $\Lambda^{k+l}(\mathbb{R}^n)$. For example if $k = l = 1$,

$$f = \sum_{1 \leq i \leq n} f_i e^i \quad \text{and} \quad g = \sum_{1 \leq j \leq n} g_j e^j$$

then

$$f \wedge g = \sum_{1 \leq i, j \leq n} f_i g_j e^i \wedge e^j = \sum_{1 \leq i < j \leq n} (f_i g_j - f_j g_i) e^i \wedge e^j.$$

If $k = 2$ and $l = 1$,

$$f = \sum_{1 \leq i < j \leq n} f_{ij} e^i \wedge e^j \quad \text{and} \quad g = \sum_{1 \leq l \leq n} g_l e^l$$

then

$$f \wedge g = \sum_{1 \leq i < j < l \leq n} (f_{ij} g_l - f_{il} g_j + f_{jl} g_i) e^i \wedge e^j \wedge e^l.$$

(ii) The *scalar product* between two k -forms f and g is denoted by

$$\langle g; f \rangle = \sum_{1 \leq i_1 < \dots < i_k \leq n} g_{i_1 \dots i_k} f_{i_1 \dots i_k}.$$

(iii) The *Hodge star operator* associates to $f \in \Lambda^k(\mathbb{R}^n)$ a form $(*f) \in \Lambda^{n-k}(\mathbb{R}^n)$ defined by

$$f \wedge g = \langle *f; g \rangle e^1 \wedge \dots \wedge e^n$$

for every $g \in \Lambda^{n-k}(\mathbb{R}^n)$. For example if $k = 1$, $n = 3$ and

$$f = \sum_{1 \leq i \leq 3} f_i e^i = f_1 e^1 + f_2 e^2 + f_3 e^3$$

then

$$*f = f_3 e^1 \wedge e^2 - f_2 e^1 \wedge e^3 + f_1 e^2 \wedge e^3.$$

(iv) We define, for $0 \leq l \leq k \leq n$, the *interior product* of $f \in \Lambda^k(\mathbb{R}^n)$ with $g \in \Lambda^l(\mathbb{R}^n)$ by

$$g \lrcorner f = (-1)^{n(k-l)} * (g \wedge (*f)) \in \Lambda^{k-l}(\mathbb{R}^n).$$

For example if $k = l$, then

$$g \lrcorner f = \langle g; f \rangle$$

or if $k = 1$ and $l = 2$, then

$$g \lrcorner f = \sum_{j=1}^n \left[\sum_{i=1}^n f_{ij} g_i \right] e^j \in \Lambda^1(\mathbb{R}^n).$$

These definitions are linked through the following elementary facts. For every $f \in \Lambda^k(\mathbb{R}^n)$, $g \in \Lambda^{k+1}(\mathbb{R}^n)$ and $h \in \Lambda^1(\mathbb{R}^n)$

$$|h|^2 f = h \lrcorner (h \wedge f) + h \wedge (h \lrcorner f)$$

$$\langle h \wedge f; g \rangle = \langle f; h \lrcorner g \rangle.$$

(v) Let $A \in \mathbb{R}^{n \times n}$ be a matrix and $f \in \Lambda^k(\mathbb{R}^n)$ be given by

$$f = \sum_{1 \leq i_1 < \dots < i_k \leq n} f_{i_1 \dots i_k} e^{i_1} \wedge \dots \wedge e^{i_k}.$$

We define the *pullback* of f by A , denoted $A^*(f)$, by

$$A^*(f) = \sum_{1 \leq i_1 < \dots < i_k \leq n} f_{i_1 \dots i_k} A^{i_1} \wedge \dots \wedge A^{i_k} \in \Lambda^k(\mathbb{R}^n)$$

where A^j is the j -th row of A and is identified with

$$A^j = \sum_{k=1}^n A_k^j e^k \in \Lambda^1(\mathbb{R}^n).$$

If $k = 0$, we then let

$$A^*(f) = f.$$

The present definition is consistent with the one given in the introduction; just set $\varphi(x) = Ax$ in (1).

(vi) We next define the notion of *rank* of $f \in \Lambda^k(\mathbb{R}^n)$. We first associate to the linear map

$$g \in \Lambda^1(\mathbb{R}^n) \rightarrow g \lrcorner f \in \Lambda^{k-1}(\mathbb{R}^n)$$

a matrix $\bar{f} \in \mathbb{R}^{\binom{n}{k-1} \times n}$ such that, by abuse of notations,

$$g \lrcorner f = \bar{f} g \quad \text{for every } g \in \Lambda^1(\mathbb{R}^n).$$

In this case, we have

$$\begin{aligned} g \lrcorner f &= \sum_{1 \leq j_1 < \dots < j_{k-1} \leq n} \left(\sum_{\gamma=1}^k (-1)^{\gamma-1} \sum_{j_{\gamma-1} < i < j_{\gamma}} f_{j_1 \dots j_{\gamma-1} i j_{\gamma} \dots j_{k-1}} g_i \right) e^{j_1} \wedge \dots \wedge e^{j_{k-1}}. \end{aligned}$$

More explicitly, using the lexicographical order for the columns (index below) and the rows (index above) of the matrix \bar{f} , we have

$$(\bar{f})_i^{j_1 \dots j_{k-1}} = f_{i j_1 \dots j_{k-1}}$$

for $1 \leq i \leq n$ and $1 \leq j_1 < \dots < j_{k-1} \leq n$. The rank of the k -form f is then the rank of the $\binom{n}{k-1} \times n$ matrix \bar{f} . We then write

$$\text{rank}[f] = \text{rank}(\bar{f}).$$

Example 1 For example if $k = 2$, then

$$(\bar{f})_i^j = f_{ij}$$

i.e. when $n = 4$

$$\bar{f} = \begin{pmatrix} 0 & f_{12} & f_{13} & f_{14} \\ f_{21} = -f_{12} & 0 & f_{23} & f_{24} \\ f_{31} = -f_{13} & f_{32} = -f_{23} & 0 & f_{34} \\ f_{41} = -f_{14} & f_{42} = -f_{24} & f_{43} = -f_{34} & 0 \end{pmatrix}.$$

Since $f_{ij} = -f_{ji}$, we have that $\bar{f} \in \mathbb{R}^{n \times n}$ is skew symmetric and therefore can never be invertible if n is odd. The canonical form when $n = 4$ is the standard symplectic matrix

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Example 2 When $k = n$, then, identifying the form with its component, we have that $\bar{f} \in \mathbb{R}^{n \times n}$ and, up to a sign,

$$\bar{f} = \begin{pmatrix} f & 0 & \cdots & 0 \\ 0 & -f & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (-1)^{n-1} f \end{pmatrix}.$$

Note that only when $k = 2$ or $k = n$ the matrix \bar{f} is a square matrix. Our best results are obtained precisely in these cases and when the matrix \bar{f} is invertible.

We then have the following elementary result.

Proposition 3 Let $f \in \Lambda^k(\mathbb{R}^n)$, $f \neq 0$.

- (i) If $k = 1$, then the rank of f is always 1.
- (ii) If $k = 2$, then the rank of f is even. The forms

$$\omega_m = \sum_{i=1}^m e^{2i-1} \wedge e^{2i}$$

are such that $\text{rank}[\omega_m] = 2m$. Moreover $\text{rank}[f] = 2m$ if and only if

$$f^m \neq 0 \quad \text{and} \quad f^{m+1} = 0$$

where $f^m = \underbrace{f \wedge \cdots \wedge f}_{m\text{-times}}$.

- (iii) If $3 \leq k \leq n$, then

$$\text{rank}[f] \in \{k, k+2, \dots, n\}$$

and any of the values in $\{k, k+2, \dots, n\}$ can be achieved by the rank of a k -form. In particular if $k = n-1$, then $\text{rank}[f] = n-1$, while if $k = n$, then $\text{rank}[f] = n$.

- (iv) If $\text{rank}[f] = k$, then there exist $f_1, \dots, f_k \in \Lambda^1(\mathbb{R}^n)$ such that

$$f = f_1 \wedge \cdots \wedge f_k.$$

Remark 4 The rank is an invariant for the pullback equation. More precisely if there exists $A \in \text{GL}(n)$, i.e. A is an invertible $n \times n$ matrix, such that

$$A^*(g) = f$$

(or equivalently if $\varphi(x) = Ax$, then the above equation is equivalent to $\varphi^*(g) = f$) then (cf. Theorem 64)

$$\text{rank}[g] = \text{rank}[f].$$

Conversely, when $k = 1, 2, n-1, n$, if $\text{rank}[g] = \text{rank}[f]$, then there exists $A \in \text{GL}(n)$ such that

$$A^*(g) = f.$$

However the converse is not anymore true, in general, if $3 \leq k \leq n-2$ (cf. Examples 60 and 61).

3 Harmonic fields and Poincaré lemma

3.1 Preliminaries

Definition 5 Let $\Omega \subset \mathbb{R}^n$ be open and $f \in C^1(\Omega; \Lambda^k)$, namely

$$f = \sum_{1 \leq i_1 < \dots < i_k \leq n} f_{i_1 \dots i_k}(x) dx^{i_1} \wedge \dots \wedge dx^{i_k}.$$

(i) The exterior derivative of f denoted df belongs to $C^0(\Omega; \Lambda^{k+1})$ and is defined by

$$df = \sum_{1 \leq i_1 < \dots < i_k \leq n} \sum_{m=1}^n \frac{\partial f_{i_1 \dots i_k}}{\partial x_m} dx^m \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}.$$

If $k = n$, then $df = 0$.

(ii) The interior derivative or codifferential of f denoted δf belongs to $C^0(\Omega; \Lambda^{k-1})$ and is defined by

$$\delta f = (-1)^{n(k-1)} * (d(*f)).$$

By abuse of notations one can write

$$df = \nabla \wedge f \quad \text{and} \quad \delta f = \nabla \lrcorner f.$$

Note that, for example, when $k = 2$

$$df = \sum_{1 \leq i < j < l \leq n} \left(\frac{\partial f_{ij}}{\partial x_l} - \frac{\partial f_{il}}{\partial x_j} + \frac{\partial f_{jl}}{\partial x_i} \right) dx^i \wedge dx^j \wedge dx^l.$$

Remark 6 (i) If $k = 0$, then the operator d can be identified with the gradient operator, while $\delta f = 0$ for any f .

(ii) If $k = 1$, then the operator d can be identified with the curl operator, while the operator δ is the divergence operator.

We next gather some well known properties of the operators d and δ .

Theorem 7 Let $f \in C^2(\Omega; \Lambda^k)$ and $g \in C^2(\Omega; \Lambda^l)$. Then

$$\begin{aligned} d(f \wedge g) &= df \wedge g + (-1)^k f \wedge dg \\ \delta(f \lrcorner g) &= (-1)^{k+l} df \lrcorner g - f \lrcorner \delta g \\ ddf &= 0, \quad \delta\delta f = 0 \quad \text{and} \quad d\delta f + \delta df = \Delta f. \end{aligned}$$

We also need the following definition. In the sequel we will denote the exterior unit normal of $\partial\Omega$ by ν .

Definition 8 The tangential component of a k -form f on $\partial\Omega$ is the $(k+1)$ -form

$$\nu \wedge f \in \Lambda^{k+1}.$$

The normal component of a k -form f on $\partial\Omega$ is the $(k-1)$ -form

$$\nu \lrcorner f \in \Lambda^{k-1}.$$

Example 9 If f is a 1-form and

$$\Omega = \{x \in \mathbb{R}^n : x_n > 0\}$$

then $\nu = -e_n$ and

$$\nu \wedge f = 0 \quad \Leftrightarrow \quad f_1 = \cdots = f_{n-1} = 0$$

$$\nu \lrcorner f = 0 \quad \Leftrightarrow \quad f_n = 0$$

We easily deduce the following properties.

Proposition 10 Let $0 \leq k \leq n$ and $f \in C^1(\bar{\Omega}; \Lambda^k)$, then

$$\nu \wedge f = 0 \text{ on } \partial\Omega \quad \Rightarrow \quad \nu \wedge df = 0 \text{ on } \partial\Omega$$

$$\nu \lrcorner f = 0 \text{ on } \partial\Omega \quad \Rightarrow \quad \nu \lrcorner \delta f = 0 \text{ on } \partial\Omega.$$

We will constantly use the integration by parts formula.

Theorem 11 (integration by parts formula) Let $1 \leq k \leq n$, $f \in C^1(\bar{\Omega}; \Lambda^{k-1})$ and $g \in C^1(\bar{\Omega}; \Lambda^k)$. Then

$$\int_{\Omega} \langle df; g \rangle + \int_{\Omega} \langle f; \delta g \rangle = \int_{\partial\Omega} \langle \nu \wedge f; g \rangle = \int_{\partial\Omega} \langle f; \nu \lrcorner g \rangle.$$

We will adopt the following notations.

Notation 12 Let $\Omega \subset \mathbb{R}^n$ be open, $r \geq 0$ be an integer and $0 \leq \alpha \leq 1 \leq p \leq \infty$. Spaces with vanishing tangential or normal component will be denoted in the following way

$$C_T^{r,\alpha}(\bar{\Omega}; \Lambda^k) = \{f \in C^{r,\alpha}(\bar{\Omega}; \Lambda^k) : \nu \wedge f = 0 \text{ on } \partial\Omega\}$$

$$C_N^{r,\alpha}(\bar{\Omega}; \Lambda^k) = \{f \in C^{r,\alpha}(\bar{\Omega}; \Lambda^k) : \nu \lrcorner f = 0 \text{ on } \partial\Omega\}$$

$$W_T^{r+1,p}(\Omega; \Lambda^k) = \{f \in W^{r+1,p}(\Omega; \Lambda^k) : \nu \wedge f = 0 \text{ on } \partial\Omega\}$$

$$W_N^{r+1,p}(\Omega; \Lambda^k) = \{f \in W^{r+1,p}(\Omega; \Lambda^k) : \nu \lrcorner f = 0 \text{ on } \partial\Omega\}.$$

The different sets of harmonic fields will be denoted by

$$\mathcal{H}(\Omega; \Lambda^k) = \{f \in W^{1,2}(\Omega; \Lambda^k) : df = 0 \text{ and } \delta f = 0 \text{ in } \Omega\}$$

$$\mathcal{H}_T(\Omega; \Lambda^k) = \{f \in \mathcal{H}(\Omega; \Lambda^k) : \nu \wedge f = 0 \text{ on } \partial\Omega\}$$

$$\mathcal{H}_N(\Omega; \Lambda^k) = \{f \in \mathcal{H}(\Omega; \Lambda^k) : \nu \lrcorner f = 0 \text{ on } \partial\Omega\}.$$

We now list some properties of the harmonic fields.

Theorem 13 *Let $\Omega \subset \mathbb{R}^n$ be an open set. Then*

$$\mathcal{H}(\Omega; \Lambda^k) \subset C^\infty(\Omega; \Lambda^k).$$

Moreover if Ω is bounded and smooth, then the next statements are valid.

(i) *The following inclusion holds*

$$\mathcal{H}_T(\Omega; \Lambda^k) \cup \mathcal{H}_N(\Omega; \Lambda^k) \subset C^\infty(\overline{\Omega}; \Lambda^k).$$

Furthermore if $r \geq 0$ is an integer and $0 \leq \alpha \leq 1$, then there exists $C = C(r, \Omega)$ such that, for every $f \in \mathcal{H}_T(\Omega; \Lambda^k) \cup \mathcal{H}_N(\Omega; \Lambda^k)$,

$$\|f\|_{W^{r,2}} \leq C\|f\|_{L^2} \quad \text{and} \quad \|f\|_{C^{r,\alpha}} \leq C\|f\|_{C^0}.$$

(ii) *The spaces $\mathcal{H}_T(\Omega; \Lambda^k)$ and $\mathcal{H}_N(\Omega; \Lambda^k)$ are finite dimensional and closed in $L^2(\Omega; \Lambda^k)$.*

(iii) *Furthermore if Ω is contractible, then*

$$\begin{aligned} \mathcal{H}_T(\Omega; \Lambda^k) &= \{0\} \quad \text{if } 0 \leq k \leq n-1 \\ \mathcal{H}_N(\Omega; \Lambda^k) &= \{0\} \quad \text{if } 1 \leq k \leq n. \end{aligned}$$

(iv) *If $k = 0$ or $k = n$ and $h \in \mathcal{H}(\Omega; \Lambda^k)$, then h is constant on each connected component of Ω . In particular $\mathcal{H}_T(\Omega; \Lambda^0) = \{0\}$ and $\mathcal{H}_N(\Omega; \Lambda^n) = \{0\}$.*

Remark 14 *If $k = 1$ and assuming that Ω is smooth, then the sets \mathcal{H}_T and \mathcal{H}_N can be rewritten, as usual by abuse of notations, as*

$$\begin{aligned} \mathcal{H}_T(\Omega; \Lambda^1) &= \left\{ f \in C^\infty(\overline{\Omega}; \mathbb{R}^n) : \begin{cases} \text{curl } f = 0 \text{ and } \text{div } f = 0 \\ f_i \nu_j - f_j \nu_i = 0, \forall 1 \leq i < j \leq n \end{cases} \right\} \\ \mathcal{H}_N(\Omega; \Lambda^1) &= \left\{ f \in C^\infty(\overline{\Omega}; \mathbb{R}^n) : \begin{cases} \text{curl } f = 0 \text{ and } \text{div } f = 0 \\ \sum_{i=1}^n f_i \nu_i = 0 \end{cases} \right\}. \end{aligned}$$

Moreover if Ω is simply connected, then

$$\mathcal{H}_T(\Omega; \Lambda^1) = \mathcal{H}_N(\Omega; \Lambda^1) = \{0\}.$$

In particular if $n = 2$ and $\Omega = B_1 \setminus \{(0, 0)\}$, then

$$f = \frac{-x_2}{x_1^2 + x_2^2} dx^1 + \frac{x_1}{x_1^2 + x_2^2} dx^2 \in \mathcal{H}_N(\Omega; \Lambda^1).$$

Proof We only prove the inclusion

$$\mathcal{H}(\Omega; \Lambda^k) \subset C^\infty(\Omega; \Lambda^k)$$

which follows from Weyl Lemma (cf. for example [23]). Indeed let $\phi \in C_0^\infty(\Omega; \Lambda^k)$

$$\int_{\Omega} \langle \omega, \Delta \phi \rangle = \int_{\Omega} \langle \omega, d\delta\phi + \delta d\phi \rangle = \int_{\Omega} \langle d\omega, d\phi \rangle + \int_{\Omega} \langle \delta\omega, \delta\phi \rangle = 0$$

Choose $\phi = \varphi dx^I$, $\varphi \in C_0^\infty(\Omega)$ and thus $\omega_I \in C^\infty(\Omega)$. ■

3.2 The Hodge-Morrey decomposition

We now turn to the celebrated Hodge-Morrey decomposition and an equivalent formulation (see [17] for details).

Theorem 15 (Hodge-Morrey decomposition) *Let $\Omega \subset \mathbb{R}^n$ be a bounded open smooth set and ν be the exterior unit normal. Let $0 \leq k \leq n$ and $f \in L^2(\Omega; \Lambda^k)$. Then there exist*

$$\begin{aligned} \alpha &\in W_T^{1,2}(\Omega; \Lambda^{k-1}), \quad \beta \in W_T^{1,2}(\Omega; \Lambda^{k+1}), \\ h &\in \mathcal{H}_T(\Omega; \Lambda^k) \quad \text{and} \quad \omega \in W_T^{2,2}(\Omega; \Lambda^k) \end{aligned}$$

such that, in Ω ,

$$f = d\alpha + \delta\beta + h, \quad \alpha = \delta\omega \quad \text{and} \quad \beta = d\omega.$$

Moreover the decomposition is an orthogonal one, i.e.

$$\int_{\Omega} \langle d\alpha; \delta\beta \rangle = \int_{\Omega} \langle d\alpha; h \rangle = \int_{\Omega} \langle \delta\beta; h \rangle = 0.$$

Remark 16 (i) *We have quoted only one of the three decompositions. Another one, completely similar, is by replacing T by N and the other one mixing both T and N .*

(ii) *If $k \leq n - 1$ and if the domain Ω is contractible, then $h = 0$.*

(iii) *If $k = 0$, then the theorem reads as*

$$f = \delta\beta = \delta d\omega = \Delta\omega \quad \text{in } \Omega \quad \text{with} \quad \omega = 0 \quad \text{on } \partial\Omega.$$

(iv) *When $k = 1$ and $n = 3$, the decomposition reads as follows. For any $f \in L^2(\Omega; \mathbb{R}^3)$, there exist*

$$\omega \in W^{2,2}(\Omega; \mathbb{R}^3) \quad \text{with } \omega_i \nu_j - \omega_j \nu_i = 0 \quad \text{on } \partial\Omega, \quad \forall 1 \leq i < j \leq 3$$

$$\alpha \in W_0^{1,2}(\Omega) \quad \text{and} \quad \alpha = \operatorname{div} \omega$$

$$\beta \in W^{1,2}(\Omega; \mathbb{R}^3) \quad \text{with } \beta = -\operatorname{curl} \omega \quad \text{and} \quad \langle \nu; \beta \rangle = 0 \quad \text{on } \partial\Omega$$

$$h \in \left\{ h \in C^\infty(\bar{\Omega}; \mathbb{R}^3) : \begin{cases} \operatorname{curl} h = 0 \quad \text{and} \quad \operatorname{div} h = 0 \\ h_i \nu_j - h_j \nu_i = 0, \quad \forall 1 \leq i < j \leq 3 \end{cases} \right\}$$

such that

$$f = \operatorname{grad} \alpha + \operatorname{curl} \beta + h \quad \text{in } \Omega.$$

Furthermore if Ω is simply connected, then $h = 0$.

(v) *If f is more regular than in L^2 , then α, β and ω are in the corresponding class of regularity. More precisely if, for example, $r \geq 0$ is an integer, $0 < q < 1$ and $f \in C^{r,q}(\bar{\Omega}; \Lambda^k)$, then*

$$\alpha \in C^{r+1,q}(\bar{\Omega}; \Lambda^{k-1}), \quad \beta \in C^{r+1,q}(\bar{\Omega}; \Lambda^{k+1}) \quad \text{and} \quad \omega \in C^{r+2,q}(\bar{\Omega}; \Lambda^k).$$

(vi) The proof of Morrey uses the direct methods of the calculus of variations. One minimizes

$$D_f(\omega) = \int_{\Omega} \left(\frac{1}{2} |d\omega|^2 + \frac{1}{2} |\delta\omega|^2 + \langle f; \omega \rangle \right)$$

in an appropriate space, Gaffney inequality giving the coercivity of the integral.

It turns out that the Hodge-Morrey decomposition is in fact equivalent to solving the first order system

$$\begin{cases} d\omega = f & \text{and} & \delta\omega = g & \text{in } \Omega \\ \nu \wedge \omega = \nu \wedge \omega_0 & & & \text{on } \partial\Omega \end{cases}$$

or the similar one

$$\begin{cases} d\omega = f & \text{and} & \delta\omega = g & \text{in } \Omega \\ \nu \lrcorner \omega = \nu \lrcorner \omega_0 & & & \text{on } \partial\Omega. \end{cases}$$

We here state a simplified version for the first system.

Theorem 17 *Let $r \geq 0$ and $1 \leq k \leq n - 2$ be integers, $0 < q < 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded contractible open smooth set and with exterior unit normal ν . The following statements are then equivalent.*

(i) *Let $g \in C^{r,q}(\bar{\Omega}; \Lambda^{k-1})$ and $f \in C^{r,q}(\bar{\Omega}; \Lambda^{k+1})$ be such that*

$$\delta g = 0 \text{ in } \Omega, \quad df = 0 \text{ in } \Omega \quad \text{and} \quad \nu \wedge f = 0 \text{ on } \partial\Omega.$$

(ii) *There exists $\omega \in C^{r+1,q}(\bar{\Omega}; \Lambda^k)$, such that*

$$\begin{cases} d\omega = f & \text{and} & \delta\omega = g & \text{in } \Omega \\ \nu \wedge \omega = 0 & & & \text{on } \partial\Omega. \end{cases}$$

Remark 18 (i) *When $k = n - 1$, the result is valid provided*

$$\int_{\Omega} f = 0.$$

Note that in this case the conditions $df = 0$ and $\nu \wedge f = 0$ are automatically fulfilled.

(ii) *Completely analogous results can be given for Sobolev spaces.*

(iii) *If the domain Ω is not contractible or if $\omega_0 \neq 0$, then additional necessary conditions have to be added, namely $\nu \wedge \omega_0 \in C^{r+1,q}(\partial\Omega; \Lambda^{k+1})$,*

$$\nu \wedge d\omega_0 = \nu \wedge f \text{ on } \partial\Omega$$

and, for every $\chi \in \mathcal{H}_T(\Omega; \Lambda^{k+1})$ and $\psi \in \mathcal{H}_T(\Omega; \Lambda^{k-1})$,

$$\int_{\Omega} \langle f; \chi \rangle - \int_{\partial\Omega} \langle \nu \wedge \omega_0; \chi \rangle = 0 \quad \text{and} \quad \int_{\Omega} \langle g; \psi \rangle = 0.$$

(iv) When $k = 1$ and $n = 3$, the theorem reads as follows. Let $\Omega \subset \mathbb{R}^3$ be a bounded contractible smooth open set, $g \in C^{r,q}(\overline{\Omega})$ and $f \in C^{r,q}(\overline{\Omega}; \mathbb{R}^3)$ be such that

$$\operatorname{div} f = 0 \text{ in } \Omega \quad \text{and} \quad \langle f; \nu \rangle = 0 \text{ on } \partial\Omega.$$

Then there exists $\omega \in C^{r+1,q}(\overline{\Omega}; \mathbb{R}^3)$, such that

$$\begin{cases} \operatorname{curl} \omega = f & \text{and} & \operatorname{div} \omega = g & \text{in } \Omega \\ \omega_i \nu_j - \omega_j \nu_i = 0 & \forall 1 \leq i < j \leq 3 & & \text{on } \partial\Omega. \end{cases}$$

3.3 Poincaré lemma

We now have a global version of Poincaré lemma with optimal regularity.

Theorem 19 Let $r \geq 0$ and $0 \leq k \leq n - 1$ be integers, $0 < \alpha < 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded open smooth set. The following statements are equivalent.

(i) Let $f \in C^{r,\alpha}(\overline{\Omega}; \Lambda^{k+1})$, be such that

$$df = 0 \text{ in } \Omega \quad \text{and} \quad \int_{\Omega} \langle f; \psi \rangle = 0 \text{ for every } \psi \in \mathcal{H}_N(\Omega; \Lambda^{k+1}).$$

(ii) There exists $\omega \in C^{r+1,\alpha}(\overline{\Omega}; \Lambda^k)$, such that

$$d\omega = f \quad \text{in } \Omega.$$

Moreover there exists a constant $C = C(r, \alpha, \Omega)$ such that

$$\|\omega\|_{C^{r+1,\alpha}} \leq C \|f\|_{C^{r,\alpha}}.$$

Remark 20 (i) When $k = n - 1$ in Theorem 19, there is no restriction on the solvability of $d\omega = f$ (since $df = 0$ automatically and $\mathcal{H}_N(\Omega; \Lambda^n) = \{0\}$).

(ii) If $r = 0$, then the conditions $df = 0$ have to be understood in the sense of distributions.

(iii) The above results remain valid if Ω is $C^{r+3,\alpha}$.

(iv) The same result holds true for Sobolev spaces.

(v) The construction is linear and universal.

Proof (ii) \Rightarrow (i). Suppose first that there exists $\omega \in C^{r+1,\alpha}(\overline{\Omega}; \Lambda^k)$ such that $f = d\omega$. Clearly $df = 0$ and the other assertion follows by partial integration, since, for every $\psi \in \mathcal{H}_N$,

$$\int_{\Omega} \langle f; \psi \rangle = \int_{\Omega} \langle d\omega; \psi \rangle = - \int_{\Omega} \langle \omega; \delta\psi \rangle + \int_{\partial\Omega} \langle \omega; \nu \lrcorner \psi \rangle = 0.$$

(i) \Rightarrow (ii). Suppose now that

$$df = 0 \text{ in } \Omega \quad \text{and} \quad \int_{\Omega} \langle f; \psi \rangle = 0 \text{ for every } \psi \in \mathcal{H}_N(\Omega; \Lambda^{k+1}).$$

We then appeal to the dual version of Theorem 17 to solve the problem

$$\begin{cases} d\omega = f & \text{and } \delta\omega = 0 & \text{in } \Omega \\ \nu \lrcorner \omega = 0 & & \text{on } \partial\Omega. \end{cases}$$

This concludes the proof. ■

A much more elementary proof can be obtained in a star shaped domain if we are ready to give up the gain of regularity. The formula is standard and the proof is done by straightforward differentiation (see [25]).

Theorem 21 *Let $0 \leq k \leq n-1$, $\Omega \subset \mathbb{R}^n$ be a star shaped (with respect to the origin) open set and $f \in C^1(\Omega; \Lambda^{k+1})$ be such that $df = 0$. Then $F \in C^1(\Omega; \Lambda^k)$ defined by*

$$F(x) = \int_0^1 [x \lrcorner f(tx)] t^k dt$$

verifies $dF = f$.

Remark 22 (i) *The case $k = 0$ is the most classical and reads as*

$$F(x) = \sum_{j=1}^n \int_0^1 [x_j f_j(tx)] dt \quad \Rightarrow \quad F_{x_i} = f_i \quad (\text{i.e. } \text{grad } F = f).$$

When $k = 1$ the formula becomes

$$F^j(x) = \sum_{i=1}^n \int_0^1 [x_i f_{ij}(tx)] t dt \quad \Rightarrow \quad F_{x_i}^j - F_{x_j}^i = f_{ij} \quad (\text{i.e. } \text{curl } F = f).$$

(ii) *Using the Hodge $*$ operator, we have the dual version, namely if $1 \leq k \leq n$ and $\varphi \in C^1(\Omega; \Lambda^{k-1})$ is such that $\delta\varphi = 0$, then $\Phi \in C^1(\Omega; \Lambda^k)$ defined by*

$$\Phi(x) = \int_0^1 [x \wedge \varphi(tx)] t^{n-k} dt$$

verifies $\delta\Phi = \varphi$. In particular when $k = 1$ the formula reads as

$$\Phi(x) = x \int_0^1 [\varphi(tx)] t^{n-1} dt \quad \Rightarrow \quad \text{div } \Phi = \varphi.$$

3.4 Poincaré lemma on the boundary

We start with a slight improvement of a lemma proved in Dacorogna-Moser [34].

Lemma 23 *Let $r \geq 0$ be an integer, $0 \leq \alpha \leq 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded open $C^{r+1,\alpha}$ set with exterior unit normal ν . Let $c \in C^{r,\alpha}(\partial\Omega)$. Then there exists*

$$b \in C^{r+1,\alpha}(\overline{\Omega})$$

satisfying all over $\partial\Omega$

$$\text{grad } b = c\nu \quad \text{and} \quad b = 0.$$

Furthermore there exists a constant $C = C(r, \Omega) > 0$ such that

$$\|b\|_{C^{r+1,\alpha}(\bar{\Omega})} \leq C \|c\|_{C^{r,\alpha}(\partial\Omega)}.$$

Proof If one is not interested in the sharp regularity result a solution of the problem is given by

$$b(x) = -c(x) \zeta(d(x, \partial\Omega))$$

where c has been extended to $\bar{\Omega}$ and $d(x, \partial\Omega)$ stands for the distance from x to the boundary (recalling that the distance function is as regular as the set Ω near the boundary, see for example [36]) and ζ is a smooth function so that $\zeta(0) = 0$, $\zeta'(0) = 1$ and $\zeta \equiv 0$ outside a small neighborhood of 0.

We give here a proof that uses elliptic regularity and hence only works whenever $0 < \alpha < 1$ (and also works in L^p for $1 < p < \infty$) in this case the constant obtained depends also on α . Another proof exists which is valid also when $\alpha = 0, 1$ (cf. [17]).

The desired solution b is obtained by solving

$$\begin{cases} \Delta^2 b = 0 & \text{in } \Omega \\ b = 0 \text{ and } \frac{\partial b}{\partial \nu} = c & \text{on } \partial\Omega. \end{cases}$$

The solution b is in $C^\infty(\Omega) \cap C^{r+1,\alpha}(\bar{\Omega})$ and satisfies the estimate

$$\|b\|_{C^{r+1,\alpha}(\bar{\Omega})} \leq C \|c\|_{C^{r,\alpha}(\partial\Omega)}.$$

Clearly b solves on $\partial\Omega$

$$\text{grad } b = c\nu \quad \text{and} \quad b = 0.$$

This concludes the proof. ■

We now need a generalization of the above lemma to differential forms, as achieved in [21].

Lemma 24 *Let $r \geq 0$ and $1 \leq k \leq n - 1$ be integers, $0 \leq \alpha \leq 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded open $C^{r+1,\alpha}$ set with exterior unit normal ν .*

(i) *If $c \in C^{r,\alpha}(\partial\Omega; \Lambda^k)$ is such that*

$$\nu \wedge c = 0 \quad \text{on } \partial\Omega,$$

then there exists $b \in C^{r+1,\alpha}(\bar{\Omega}; \Lambda^{k-1})$ satisfying all over $\partial\Omega$

$$db = c, \quad \delta b = 0 \quad \text{and} \quad b = 0.$$

Moreover there exists a constant $C = C(r, \Omega) > 0$ such that

$$\|b\|_{C^{r+1,\alpha}(\overline{\Omega})} \leq C \|c\|_{C^{r,\alpha}(\partial\Omega)}.$$

(ii) If $c \in C^{r,\alpha}(\partial\Omega; \Lambda^k)$ is such that

$$\nu \lrcorner c = 0 \quad \text{on } \partial\Omega,$$

then there exists $b \in C^{r+1,\alpha}(\overline{\Omega}; \Lambda^{k+1})$ satisfying all over $\partial\Omega$

$$\delta b = c, \quad db = 0 \quad \text{and} \quad b = 0.$$

Furthermore there exists a constant $C = C(r, \Omega) > 0$ such that

$$\|b\|_{C^{r+1,\alpha}(\overline{\Omega})} \leq C \|c\|_{C^{r,\alpha}(\partial\Omega)}.$$

Remark 25 (i) If $k = 0$ in Statement (ii) (and analogously if $k = n$ in Statement (i)) and $0 < \alpha < 1$, then it is easy to find b such that (and without any restriction on c)

$$\delta b = c \quad \text{and} \quad db = 0 \quad \text{in } \overline{\Omega}$$

where c has been extended to $\overline{\Omega}$ with the appropriate regularity. Indeed choose $b = \text{grad } B$ where B solves

$$\begin{cases} \Delta B = c & \text{in } \Omega \\ B = 0 & \text{on } \partial\Omega. \end{cases}$$

(ii) The above result remains valid, with the same proof, in the Sobolev setting. More precisely Statement (i) (and similarly for Statement (ii)) reads as follows. Let $r \geq 1$ be an integer, $1 < p < \infty$ and $\Omega \subset \mathbb{R}^n$ be a bounded open C^{r+1} set with exterior unit normal ν . Let $c \in W^{r,p}(\Omega; \Lambda^k)$, then there exists

$$b \in W^{r+1,p}(\Omega; \Lambda^{k-1})$$

satisfying all over $\partial\Omega$

$$db = c, \quad \delta b = 0 \quad \text{and} \quad b = 0.$$

Moreover there exists a constant $C = C(r, p, \Omega) > 0$ such that

$$\|b\|_{W^{r+1,p}(\Omega)} \leq C \|c\|_{W^{r-1/p,p}(\partial\Omega)}.$$

Proof Step 1. We start with the case (i). First solve with Lemma 23 the problem, on $\partial\Omega$,

$$\text{grad } b_{i_1 \dots i_{k-1}} = (\nu \lrcorner c)_{i_1 \dots i_{k-1}} \nu \quad \text{and} \quad b_{i_1 \dots i_{k-1}} = 0$$

for every multiindex $1 \leq i_1 < \dots < i_{k-1} \leq n$ and set

$$b = \sum_{1 \leq i_1 < \dots < i_{k-1} \leq n} b_{i_1 \dots i_{k-1}} dx^{i_1} \wedge \dots \wedge dx^{i_{k-1}}.$$

The classical formulas immediately imply that, on $\partial\Omega$,

$$db = \nu \wedge (\nu \lrcorner c) \quad \text{and} \quad \delta b = \nu \lrcorner (\nu \lrcorner c) = 0.$$

We combine the first equation with the hypothesis $\nu \wedge c = 0$ and use the fact that

$$c = \nu \lrcorner (\nu \wedge c) + \nu \wedge (\nu \lrcorner c) = \nu \wedge (\nu \lrcorner c)$$

to get

$$db = \nu \wedge (\nu \lrcorner c) = \nu \wedge (\nu \lrcorner c) + \nu \lrcorner (\nu \wedge c) = c \quad \text{on } \partial\Omega$$

We have therefore proved the assertion.

Step 2. For (ii) we first solve, on $\partial\Omega$,

$$\text{grad } b_{i_1 \dots i_{k+1}} = (\nu \wedge c)_{i_1 \dots i_{k+1}} \nu \quad \text{and} \quad b_{i_1 \dots i_{k+1}} = 0$$

and then proceed exactly as in Step 1. This concludes the proof of the lemma. \blacksquare

3.5 Poincaré lemma with Dirichlet boundary data

We now consider the boundary value problems

$$\begin{cases} d\omega = f & \text{in } \Omega \\ \omega = \omega_0 & \text{on } \partial\Omega \end{cases} \quad \text{and} \quad \begin{cases} \delta\omega = g & \text{in } \Omega \\ \omega = \omega_0 & \text{on } \partial\Omega. \end{cases}$$

In contrast to the problems studied in the previous sections $\delta\omega$ (respectively $d\omega$) is not prescribed, but however both the tangential and normal components of ω are given on the boundary. It turns out that the problems can be solved under exactly the same hypotheses on f , g and ω_0 as above. We follow exactly the construction in Dacorogna [21] for Hölder spaces; a very similar method is used in Schwarz [51] for Sobolev spaces.

Theorem 26 *Let $r \geq 0$ and $0 \leq k \leq n - 1$ be integers, $0 < \alpha < 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded connected open smooth set with exterior unit normal ν . Let $f : \overline{\Omega} \rightarrow \Lambda^{k+1}$. Then the following statements are equivalent.*

(i) *Let $f \in C^{r,\alpha}(\overline{\Omega}; \Lambda^{k+1})$ satisfy*

$$df = 0 \text{ in } \Omega, \quad \nu \wedge f = 0 \text{ on } \partial\Omega \tag{A1}$$

and, for every $\chi \in \mathcal{H}_T(\Omega; \Lambda^{k+1})$,

$$\int_{\Omega} \langle f; \chi \rangle = 0. \tag{A2}$$

(ii) *There exists $\omega \in C^{r+1,\alpha}(\overline{\Omega}; \Lambda^k)$ such that*

$$\begin{cases} d\omega = f & \text{in } \Omega \\ \omega = 0 & \text{on } \partial\Omega \end{cases}$$

and there exists a constant $C = C(r, \alpha, \Omega)$ such that

$$\|\omega\|_{C^{r+1,\alpha}(\overline{\Omega})} \leq C\|f\|_{C^{r,\alpha}(\overline{\Omega})}.$$

Remark 27 (i) Instead of imposing the boundary data to be 0 we can have any boundary data ω_0 satisfying, in addition to $df = 0$,

$$\nu \wedge d\omega_0 = \nu \wedge f \text{ on } \partial\Omega$$

and, for every $\chi \in \mathcal{H}_T(\Omega; \Lambda^{k+1})$,

$$\int_{\Omega} \langle f; \chi \rangle = \int_{\partial\Omega} \langle \nu \wedge \omega_0; \chi \rangle.$$

(ii) In the case $k = n - 1$, the conditions (A1) are trivially satisfied, while (A2) reads as

$$\int_{\Omega} f = 0.$$

(iii) When $k = 0$, then the result is still valid for $\alpha = 0, 1$.

(iv) If $r = 0$, then the condition $df = 0$ is understood in the sense of distributions.

(v) The above results remain valid if the set Ω is $C^{r+3,\alpha}$.

(vi) If Ω is contractible, then $\mathcal{H}_T(\Omega; \Lambda^k) = \{0\}$.

(vii) The construction is linear and universal.

(viii) Analogous results hold in Sobolev spaces.

(ix) If f has compact support, one can find (see [52], following the earlier work [6]) a solution ω with compact support and optimal regularity.

Proof The implication (ii) \Rightarrow (i) is straightforward using partial integration, cf. Theorem 11. To show the other implication, we first use Theorem 17 to find a solution $u \in C^{r+1,\alpha}(\overline{\Omega}; \Lambda^k)$ of the problem

$$\begin{cases} du = f & \text{and } \delta u = 0 & \text{in } \Omega \\ \nu \wedge u = 0 & & \text{on } \partial\Omega. \end{cases}$$

Since $\nu \wedge u = 0$, we can apply Lemma 24 Part (i) to find $\beta \in C^{r+2,\alpha}(\overline{\Omega}; \Lambda^{k-1})$ such that

$$d\beta = -u \text{ on } \partial\Omega.$$

We finally set $\omega = u + d\beta$ to obtain the result. ■

We have now the dual version.

Theorem 28 Let $r \geq 0$ and $1 \leq k \leq n$ be integers, $0 < \alpha < 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded connected open smooth set with exterior unit normal ν . Let $g : \overline{\Omega} \rightarrow \Lambda^{k-1}$. Then the following claims are equivalent.

(i) Let $g \in C^{r,\alpha}(\bar{\Omega}; \Lambda^{k-1})$ satisfy

$$\delta g = 0 \text{ in } \Omega, \quad \nu \lrcorner g = 0 \text{ on } \partial\Omega \quad (\text{C1})$$

and, for every $\chi \in \mathcal{H}_N(\Omega; \Lambda^{k-1})$,

$$\int_{\Omega} \langle g; \chi \rangle = 0. \quad (\text{C2})$$

(ii) There exists $\omega \in C^{r+1,\alpha}(\bar{\Omega}; \Lambda^k)$ such that

$$\begin{cases} \delta\omega = g & \text{in } \Omega \\ \omega = 0 & \text{on } \partial\Omega \end{cases}$$

and there exists a constant $C = C(r, \alpha, \Omega)$ such that

$$\|\omega\|_{C^{r+1,\alpha}(\bar{\Omega})} \leq C \|g\|_{C^{r,\alpha}(\bar{\Omega})}.$$

Remark 29 (i) Instead of imposing the boundary data to be 0 we can have any boundary data ω_0 satisfying, in addition to $\delta g = 0$,

$$\nu \lrcorner \delta\omega_0 = \nu \lrcorner g \text{ on } \partial\Omega$$

and, for every $\chi \in \mathcal{H}_N(\Omega; \Lambda^{k-1})$,

$$\int_{\Omega} \langle g; \chi \rangle = \int_{\partial\Omega} \langle \nu \lrcorner \omega_0; \chi \rangle.$$

(ii) In the case $k = 1$ (cf. Theorem 30 below), then $\delta\omega = \text{div } \omega$ and the conditions (C1) are trivially satisfied, while (C2) reads as

$$\int_{\Omega} g = 0.$$

(iii) When $k = n$, then the result is still valid for $\alpha = 0, 1$ with a simple argument.

(iv) If $r = 0$, then the condition $\delta g = 0$ is understood in the sense of distributions.

(v) The above results remains valid if Ω is $C^{r+3,\alpha}$.

(vi) If Ω is contractible, then $\mathcal{H}_N(\Omega; \Lambda^k) = \{0\}$.

(vii) The construction is linear and universal.

(viii) Analogous results hold in Sobolev spaces.

As a corollary, but it can be proved in a more direct way, we have the following.

Theorem 30 Let $r \geq 0$ be an integer and $0 < \alpha < 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open $C^{r+2,\alpha}$ set. The following conditions are then equivalent.

(i) $f \in C^{r,\alpha}(\overline{\Omega})$ satisfies

$$\int_{\Omega} f = 0.$$

(ii) There exists $u \in C^{r+1,\alpha}(\overline{\Omega}; \mathbb{R}^n)$ verifying

$$\begin{cases} \operatorname{div} u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (2)$$

Furthermore the correspondence $f \rightarrow u$ can be chosen linear and there exists $C = C(r, \alpha, \Omega) > 0$ such that

$$\|u\|_{C^{r+1,\alpha}} \leq C \|f\|_{C^{r,\alpha}}.$$

Remark 31 (i) The result is false when $r = \alpha = 0$, as established in [10], [26], [43] and [48].

(ii) Similar results (see [16]) can be obtained for the inhomogeneous problem

$$\begin{cases} \operatorname{div} u + \langle a; u \rangle = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where a is a vector field and $\langle \cdot; \cdot \rangle$ stands for the scalar product in \mathbb{R}^n .

4 The flow method

We start by recalling a well known result of differential geometry. In the sequel we will write

$$u = u(t, x) = u_t(x) \quad \text{and} \quad \varphi = \varphi(t, x) = \varphi_t(x).$$

Theorem 32 Let $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ be open sets, $T > 0$ and $1 \leq k \leq n$ be an integer. Let

$$u \in C^1([0, T] \times \Omega_2; \mathbb{R}^n) \quad \text{and} \quad \varphi \in C^1([0, T] \times \Omega_1; \Omega_2)$$

be such that, in Ω_1 ,

$$\frac{d}{dt} \varphi_t = u_t \circ \varphi_t = u_t(t, \varphi_t(t, x)), \quad \text{for every } 0 \leq t \leq T. \quad (3)$$

Then, for every $f \in C^1([0, T] \times \Omega_2; \Lambda^k)$, the following equality holds in Ω_1 and for $0 \leq t \leq T$

$$\frac{d}{dt} [\varphi_t^*(f_t)] = \varphi_t^* \left(\frac{d}{dt} f_t + d(u_t \lrcorner f_t) + u_t \lrcorner (df_t) \right) \quad (4)$$

where u_t has been identified with a 1-form.

Remark 33 (i) The right-hand side of (4) is related to the so called Lie derivative which we define now. Let $u \in C^1(U; \mathbb{R}^n)$ and $f \in C^1(U; \Lambda^k)$. The Lie derivative $\mathcal{L}_u(f)$ is defined by

$$\mathcal{L}_u(f) = \left. \frac{d}{dt} \right|_{t=0} \varphi_t^*(f)$$

where $\varphi = \varphi(t, x) = \varphi_t(x)$ is the flow associated to the vector field u , that is

$$\begin{cases} \frac{d}{dt} \varphi_t = u \circ \varphi_t \\ \varphi_0 = \text{id} \end{cases}$$

for t small enough. Cartan formula gives that

$$\mathcal{L}_u(f) = d(u \lrcorner f) + u \lrcorner df.$$

(ii) When $k = n$ the formula reads as

$$\frac{d}{dt} [f_t(\varphi_t) \det \nabla \varphi_t] = \frac{d}{dt} [\det \nabla \varphi_t] f_t(\varphi_t) + \det \nabla \varphi_t \left[\left(\frac{d}{dt} f_t \right) (\varphi_t) + \langle \nabla f_t(\varphi_t); \frac{d}{dt} \varphi_t \rangle \right].$$

Proof We prove this result only for $k = n$. Note that when $k = n$, then necessarily $df_t = 0$ and (identifying as usual the $(n-1)$ -form $u_t \lrcorner f_t$ with a vector field whose components have the appropriate sign)

$$d(u_t \lrcorner f_t) = \text{div}(f_t u_t)$$

Step 1. First recall the well known Abel formula. Any solution of

$$F'(t) = A(t) F(t)$$

satisfies

$$\det F(t) = [\det F(0)] \exp \left[\int_0^t \text{trace } A(s) ds \right].$$

It follows that in the nonlinear case (cf., for example, Theorem 7.2 in Chapter 1 of Coddington-Levinson [14]) the solution of (3) satisfies

$$\det \nabla \varphi_t(x) = [\det \nabla \varphi_0(x)] \exp \left[\int_0^t (\text{div } u_s)(\varphi_s(x)) ds \right].$$

Since the right hand side of the above identity is C^1 in t , we get

$$\frac{d}{dt} [\det \nabla \varphi_t(x)] = \det \nabla \varphi_t(x) \cdot (\text{div } u_t)(\varphi_t(x)). \quad (5)$$

Step 2. Using (3), we obtain

$$\begin{aligned} \frac{d}{dt} [\varphi_t^*(f_t)] &= \frac{d}{dt} [\det \nabla \varphi_t \cdot f_t(\varphi_t)] \\ &= \frac{d}{dt} [\det \nabla \varphi_t] f_t(\varphi_t) + \det \nabla \varphi_t \left[\left(\frac{d}{dt} f_t \right) (\varphi_t) + \langle \nabla f_t(\varphi_t); \frac{d}{dt} \varphi_t \rangle \right] \end{aligned}$$

and thus, appealing to (5), we find

$$\begin{aligned} \frac{d}{dt}[\varphi_t^*(f_t)] &= \det \nabla \varphi_t \left[(\operatorname{div} u_t)(\varphi_t) \cdot f_t(\varphi_t) + \left(\frac{d}{dt} f_t\right)(\varphi_t) + \langle \nabla f_t(\varphi_t); u_t(\varphi_t) \rangle \right] \\ &= \det \nabla \varphi_t \left[\left(\frac{d}{dt} f_t\right)(\varphi_t) + \operatorname{div}(f_t u_t)(\varphi_t) \right] = \varphi_t^* \left(\frac{d}{dt} f_t + \operatorname{div}(f_t u_t) \right) \end{aligned}$$

which concludes the proof. ■

As a consequence we have the following result essentially established by Moser [47].

Theorem 34 *Let $r \geq 1$ and $1 \leq k \leq n$ be integers, $0 \leq \alpha \leq 1$, $T > 0$ and $\Omega \subset \mathbb{R}^n$ be a bounded open Lipschitz set. Let*

$$u \in C^{r,\alpha}([0, T] \times \bar{\Omega}; \mathbb{R}^n) \quad \text{and} \quad f \in C^{r,\alpha}([0, T] \times \bar{\Omega}; \Lambda^k)$$

be such that, for every $t \in [0, T]$,

$$u_t = 0 \quad \text{on } \partial\Omega, \quad df_t = 0 \quad \text{in } \Omega$$

$$d(u_t \lrcorner f_t) = -\frac{d}{dt} f_t \quad \text{in } \Omega.$$

Then, for every $t \in [0, T]$, the solution φ_t of

$$\begin{cases} \frac{d}{dt} \varphi_t = u_t \circ \varphi_t & 0 \leq t \leq T \\ \varphi_0 = \operatorname{id} \end{cases} \quad (6)$$

belongs to $\operatorname{Diff}^{r,\alpha}(\bar{\Omega}; \bar{\Omega})$, satisfies $\varphi_t = \operatorname{id}$ on $\partial\Omega$ and

$$\varphi_t^*(f_t) = f_0 \quad \text{in } \Omega.$$

Proof Standard results show that, for every $0 \leq t \leq T$, the solution φ_t of (6) belongs to $\operatorname{Diff}^{r,\alpha}(\bar{\Omega}; \bar{\Omega})$ and verifies $\varphi_t = \operatorname{id}$ on $\partial\Omega$. Moreover, defining $\varphi : [0, T] \times \bar{\Omega} \rightarrow \bar{\Omega}$ by $\varphi(t, x) = \varphi_t(x)$, we have

$$\varphi \in C^{r,\alpha}([0, T] \times \bar{\Omega}; \bar{\Omega}).$$

Using Theorem 32 and the hypotheses on u_t and f_t , we find that, in Ω ,

$$\frac{d}{dt}[\varphi_t^*(f_t)] = \varphi_t^* \left(\frac{d}{dt} f_t + d(u_t \lrcorner f_t) + u_t \lrcorner (df_t) \right) = 0,$$

which implies the result since $\varphi_0 = \operatorname{id}$. ■

5 The case of volume forms

5.1 Statement of the problem

We start with the case $k = n$. We first observe that the local problem is here elementary. Indeed if we want to solve (if $g \equiv 1$, the general case being treated similarly)

$$\varphi^*(1) = f \quad \Leftrightarrow \quad \det \nabla \varphi = f$$

we just choose $\varphi(x) = (\varphi^1(x), x_2, \dots, x_n)$ where

$$\varphi^1(x_1, x_2, \dots, x_n) = \int^{x_1} f(t, x_2, \dots, x_n) dt.$$

Note however that this does not give the optimal regularity for φ . A less trivial problem, discussed in Theorem 39, concerns the case where several equations are considered, namely for $1 \leq i \leq n$

$$\varphi^*(g_i) = f_i \quad \Leftrightarrow \quad g_i(\varphi) \det \nabla \varphi = f_i.$$

In the case $k = n$ we will therefore (apart from Theorem 39) discuss only the global case which takes the following form. Given Ω a bounded open set in \mathbb{R}^n and $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$, we want to find $\varphi : \overline{\Omega} \rightarrow \mathbb{R}^n$ verifying

$$\begin{cases} g(\varphi(x)) \det \nabla \varphi(x) = f(x) & x \in \Omega \\ \varphi(x) = x & x \in \partial\Omega. \end{cases} \quad (7)$$

Writing the functions f and g as volume forms through the straightforward identification

$$g = g(x) dx^1 \wedge \dots \wedge dx^n \quad \text{and} \quad f = f(x) dx^1 \wedge \dots \wedge dx^n,$$

the problem (7) can be written as

$$\begin{cases} \varphi^*(g) = f & \text{in } \Omega \\ \varphi = \text{id} & \text{on } \partial\Omega \end{cases}$$

where $\varphi^*(g)$ is the pullback of g by φ .

The following preliminary remarks are in order.

(i) The case $n = 1$ is completely elementary and is discussed in the section below.

(ii) When $n \geq 2$, the equation in (7) is a non-linear first order *partial differential equation* homogeneous of degree n in the derivatives. It is *underdetermined*, in the sense that we have n unknowns (the components of φ) and only one equation. Related to this observation, we have that if there exists a solution to our problem then there are infinitely many ones. Indeed, for example, if $n = 2$, Ω

is the unit ball and $f = g = 1$, the maps φ_m (written in polar and in Cartesian coordinates) defined by

$$\begin{aligned}\varphi_m(x) = \varphi_m(x_1, x_2) &= \begin{pmatrix} r \cos(\theta + 2m\pi r^2) \\ r \sin(\theta + 2m\pi r^2) \end{pmatrix} \\ &= \begin{pmatrix} x_1 \cos(2m\pi(x_1^2 + x_2^2)) - x_2 \sin(2m\pi(x_1^2 + x_2^2)) \\ x_2 \cos(2m\pi(x_1^2 + x_2^2)) + x_1 \sin(2m\pi(x_1^2 + x_2^2)) \end{pmatrix}\end{aligned}$$

satisfy (7) for every $m \in \mathbb{Z}$.

(iii) An integration by parts, or, what amounts to the same thing, an elementary topological degree argument immediately gives the *necessary condition* (independently of the fact that φ is a diffeomorphism or not and of the fact that $\varphi(\Omega)$ contains strictly or not the domain Ω)

$$\int_{\Omega} f = \int_{\Omega} g. \quad (8)$$

In most of our analysis, it will turn out that this condition is also sufficient.

(iv) We will always assume that $g > 0$. If g is not strictly positive, then other hypotheses than (8) are necessary; for example f cannot be strictly positive. Indeed if for example $f \equiv 1$ and g is allowed to vanish even at a single point, then no C^1 solution of our problem exists. However in some very special cases, one can deal with functions f and g that *both* change sign.

(v) We will however allow f to change sign, but the analysis is very different if $f > 0$ or if f vanishes, even at a single point, let alone if it becomes negative. The first problem will be discussed below, while the second is discussed in [17] and [19]. One of the main differences is that in the first case any solution of (7) is necessarily a diffeomorphism, while this is never true in the second case.

(vi) It is easy to see that any solution of (7) satisfies

$$\varphi(\Omega) \supset \Omega \quad \text{and} \quad \varphi(\overline{\Omega}) \supset \overline{\Omega}. \quad (9)$$

If $f > 0$, we have, since φ is a diffeomorphism, that

$$\varphi(\Omega) = \Omega \quad \text{and} \quad \varphi(\overline{\Omega}) = \overline{\Omega}.$$

If this is not the case, then, in general the inclusions can be strict. This matter is discussed in details in [17].

(vii) Problem (7) admits a *weak formulation*. Indeed if φ is a diffeomorphism, we can write the equation $g(\varphi) \det \nabla \varphi = f$ as

$$\int_{\varphi(E)} g = \int_E f \quad \text{for every open set } E \subset \Omega$$

or equivalently

$$\int_{\Omega} g \zeta(\varphi^{-1}) = \int_{\Omega} f \zeta \quad \text{for every } \zeta \in C_0^\infty(\Omega).$$

We observe that both new writings make sense if φ is only a homeomorphism.

(viii) The problem can be seen as a question of *mass transportation*. Indeed we want to transport the mass distribution g to the mass distribution f , without moving the points of the boundary of Ω . In this context the equation is usually written as

$$\int_E g = \int_{\varphi^{-1}(E)} f \quad \text{for every open set } E \subset \Omega.$$

The problem of *optimal* mass transportation has received considerable attention. We should point out that our analysis is not in this framework (except in an indirect way in Section 5.6.4). The two main strong points of our analysis are that we are able to find smooth solutions, sometimes with the optimal regularity, and to deal with fixed boundary data.

5.2 The one dimensional case

As already said the case $n = 1$ is completely elementary, but it exhibits some striking differences with the case $n \geq 2$. However it may shed some light on some issues that we will discuss in the higher dimensional case. Let $\Omega = (a, b)$,

$$F(x) = \int_a^x f(t) dt \quad \text{and} \quad G(x) = \int_a^x g(t) dt.$$

Then Problem (7) becomes

$$\begin{cases} G(\varphi(x)) = F(x) & \text{if } x \in (a, b) \\ \varphi(a) = a & \text{and } \varphi(b) = b. \end{cases}$$

If G is invertible, and this happens if, for example, $g > 0$ and if

$$F([a, b]) \subset G(\mathbb{R}), \tag{10}$$

and this happens if, for example, $g \geq g_0 > 0$, then the problem has the solution

$$\varphi(x) = G^{-1}(F(x)).$$

The necessary condition (8)

$$\int_a^b f = \int_a^b g$$

ensures that

$$\varphi(a) = a \quad \text{and} \quad \varphi(b) = b.$$

This very elementary analysis leads to the following conclusions.

- 1) Contrary to the case $n \geq 2$, the necessary condition (8) is not sufficient. We need the extra condition (10).
- 2) The problem has a *unique* solution, contrary to the case $n \geq 2$.
- 3) If f and g are in the space C^r , then the solution φ is in C^{r+1} .

- 4) If $f > 0$, then φ is a diffeomorphism from $[a, b]$ onto itself.
 5) If f is allowed to change sign, then, in general,

$$[a, b] \subsetneq \varphi([a, b]).$$

For example, this always happens if $f(a) < 0$ or $f(b) < 0$.

5.3 The case $f \cdot g > 0$

We now discuss the problem (7) when $f \cdot g > 0$. It will be seen that (8) is sufficient to solve (7) and that any solution is in fact a diffeomorphism from $\bar{\Omega}$ to $\bar{\Omega}$. This last observation implies, in particular, a symmetry in f and g and allows us to restrict ourselves, without loss of generality, to the case $g \equiv 1$. Our main result will be the following.

Theorem 35 (Dacorogna-Moser theorem) *Let $r \geq 0$ be an integer and $0 < \alpha < 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open $C^{r+2, \alpha}$ set. Then the two following statements are equivalent.*

- (i) *The function $f \in C^{r, \alpha}(\bar{\Omega})$, $f > 0$ in $\bar{\Omega}$ and satisfies*

$$\int_{\Omega} f = \text{meas } \Omega.$$

- (ii) *There exists $\varphi \in \text{Diff}^{r+1, \alpha}(\bar{\Omega}; \bar{\Omega})$ satisfying*

$$\begin{cases} \det \nabla \varphi(x) = f(x) & x \in \Omega \\ \varphi(x) = x & x \in \partial\Omega. \end{cases}$$

Furthermore if $c > 0$ is such that

$$\left\| \frac{1}{f} \right\|_{C^0}, \|f\|_{C^{0, \alpha}} \leq c,$$

then there exists a constant $C = C(c, r, \alpha, \Omega) > 0$ such that

$$\|\varphi - \text{id}\|_{C^{r+1, \alpha}} \leq C \|f - 1\|_{C^{r, \alpha}}.$$

The study of this problem originated in the seminal work of Moser [47]. This result has generated a considerable amount of work, notably by Banyaga [3], Dacorogna [20], Reimann [49], Tartar [53], Zehnder [58]. The above optimal theorem was obtained by Dacorogna-Moser [34], the estimate is however new. Posterior contributions can be found in Carlier-Dacorogna [13], Rivière-Ye [50] and Ye [57]. Burago-Kleiner [11] and Mc Mullen [43], independently, proved that the result is false if $r = \alpha = 0$, suggesting that the gain of regularity is to be expected only when $0 < \alpha < 1$.

Corollary 36 *Let $r \geq 0$ be an integer and $0 < \alpha < 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open $C^{r+2,\alpha}$ set. Let $f, g \in C^{r,\alpha}(\overline{\Omega})$ be such that $f \cdot g > 0$ in $\overline{\Omega}$ and*

$$\int_{\Omega} f = \int_{\Omega} g. \quad (11)$$

Then there exists $\varphi \in \text{Diff}^{r+1,\alpha}(\overline{\Omega}; \overline{\Omega})$ satisfying

$$\begin{cases} g(\varphi(x)) \det \nabla \varphi(x) = f(x) & x \in \Omega \\ \varphi(x) = x & x \in \partial\Omega. \end{cases} \quad (12)$$

Remark 37 (i) *Recall that $\text{Diff}^{r,\alpha}(\overline{\Omega}; \overline{\Omega})$ denotes the set of diffeomorphisms φ so that $\varphi(\overline{\Omega}) = \overline{\Omega}$, $\varphi \in C^{r,\alpha}(\overline{\Omega}; \mathbb{R}^n)$ and $\varphi^{-1} \in C^{r,\alpha}(\overline{\Omega}; \mathbb{R}^n)$.*

(ii) *If the domain is not connected, then the condition (11) has to hold on each connected component.*

(iii) *The sufficient conditions are also necessary. More precisely if φ satisfies (12), then necessarily, for non-vanishing f and g , we have $f \cdot g > 0$ in $\overline{\Omega}$ and (11) holds. Moreover the function*

$$\frac{f}{g \circ \varphi} \in C^{r,\alpha}(\overline{\Omega}),$$

hence, if one of the functions f or g is in $C^{r,\alpha}$, then so is the other one.

Proof (Corollary 36). First find, by Theorem 35,

$$\psi_1, \psi_2 \in \text{Diff}^{r+1,\alpha}(\overline{\Omega}; \overline{\Omega})$$

satisfying

$$\begin{cases} \det \nabla \psi_2(x) = \frac{f(x) \text{meas } \Omega}{\int_{\Omega} f(x) dx} & x \in \Omega \\ \det \nabla \psi_1(x) = \frac{g(x) \text{meas } \Omega}{\int_{\Omega} g(x) dx} & x \in \Omega \\ \psi_1(x) = \psi_2(x) = x & x \in \partial\Omega. \end{cases}$$

It is then easy to see that $\varphi = \psi_1^{-1} \circ \psi_2$ satisfies (12). ■

There are other more constructive methods to solve the equation, cf. for example Dacorogna-Moser [34]. These methods do not use the regularity of elliptic differential operators; in this sense they are more elementary. The drawback is that they do not provide any gain of regularity, which is the strong point of the above theorem. However the advantage is that they are much more flexible. For example, if we assume in (7) that

$$\text{supp}(f - g) \subset \Omega$$

then we are able to find φ such that

$$\text{supp}(\varphi - \text{id}) \subset \Omega.$$

In this last case see also [37].

5.4 The case with no sign hypothesis on f

We start by observing that if f vanishes even at a single point, then the solution φ cannot be anymore a diffeomorphism, though it can be a homeomorphism. In any case if f is negative somewhere, it can never be a homeomorphism. Furthermore if f is negative in some parts of the boundary, then any solution φ must go out of the domain, more precisely

$$\bar{\Omega} \subsetneq \varphi(\bar{\Omega}).$$

A special case of our theorem (cf. Cupini-Dacorogna-Kneuss [19]) is the following.

Theorem 38 *Let $n \geq 2$ and $r \geq 1$ be integers. Let $B_1 \subset \mathbb{R}^n$ be the open unit ball. Let $f \in C^r(\bar{B}_1)$ be such that*

$$\int_{B_1} f = \text{meas } B_1.$$

Then there exists $\varphi \in C^r(\bar{B}_1; \mathbb{R}^n)$ satisfying

$$\begin{cases} \det \nabla \varphi(x) = f(x) & x \in B_1 \\ \varphi(x) = x & x \in \partial B_1. \end{cases}$$

Furthermore the following conclusions also hold.

(i) *If either $f > 0$ on ∂B_1 or $f \geq 0$ in \bar{B}_1 , then φ can be chosen so that*

$$\varphi(\bar{B}_1) = \bar{B}_1.$$

(ii) *If $f \geq 0$ in \bar{B}_1 and $f^{-1}(0) \cap B_1$ is countable, then φ can be chosen as a homeomorphism from \bar{B}_1 onto \bar{B}_1 .*

5.5 Multiple Jacobian equations

When dealing with several equations we have the following local result established in [31].

Theorem 39 *Let $n, r \geq 2$ be integers, $x_0 \in \mathbb{R}^n$ and $g_i, f_i \in C^r(\mathbb{R}^n)$, $1 \leq i \leq n$, be such that $g_i(x_0), f_i(x_0) \neq 0$ for every $1 \leq i \leq n$,*

$$\text{rank} \left[\begin{pmatrix} \nabla(g_2/g_1) \\ \vdots \\ \nabla(g_n/g_1) \end{pmatrix} (x_0) \right] = \text{rank} \left[\begin{pmatrix} \nabla(f_2/f_1) \\ \vdots \\ \nabla(f_n/f_1) \end{pmatrix} (x_0) \right] = n - 1 \quad (13)$$

and

$$\frac{g_i}{g_1}(x_0) = \frac{f_i}{f_1}(x_0) \quad \text{for every } 2 \leq i \leq n. \quad (14)$$

Part 1 (Existence and regularity). *There exist a neighborhood U of x_0 and $\varphi \in \text{Diff}^{r-1}(U; \varphi(U))$ such that $\varphi(x_0) = x_0$ and*

$$g_i(\varphi) \det \nabla \varphi = f_i \quad \text{in } U \text{ for every } 1 \leq i \leq n. \quad (15)$$

The regularity is, in general, optimal.

Part 2 (Uniqueness). *Let $h \in C^{r-1}(\mathbb{R}^n)$ be such that $h(x_0) = 0$ and*

$$\det \left[\begin{pmatrix} \nabla h \\ \nabla(f_2/f_1) \\ \vdots \\ \nabla(f_n/f_1) \end{pmatrix} (x_0) \right] \neq 0.$$

Let U be a neighborhood of x_0 , $\varphi \in \text{Diff}^{r-1}(U; \varphi(U))$ and $\psi \in \text{Diff}^{r-1}(U; \psi(U))$ be two solutions of (15) verifying

$$\varphi = \psi \quad \text{on} \quad \{x \in U : h(x) = 0\}.$$

Then, up to further restricting U ,

$$\varphi \equiv \psi \quad \text{on } U.$$

Remark 40 (i) *The fact that the regularity that we obtain is optimal (even in Hölder spaces), is, at first glance, surprising.*

(ii) *The hypothesis (14) is obviously necessary to have $\varphi(x_0) = x_0$. The hypothesis (13) (although not necessary in general) is very reasonable: for example if $n = 2$ and $g_1 = g_2$ near x_0 (and thus $\nabla(g_2/g_1) = 0$), then obviously f_2 has to be equal to f_1 near x_0 to be able to solve (15) and vice versa.*

(iii) *The solution in Part 1 of the previous theorem is easily seen not to be unique. However (cf. Part 2) it becomes unique as soon as the value of the solution is prescribed not only at the point x_0 but on a $(n - 1)$ surface (near x_0) compatible with the data.*

(iv) *It is to be noted that the equivalent problem for 1 and 2 forms has already been considered in Chapter 15 of [17] (for a particular case see Theorem 63).*

Proof We discuss here only the existence part, for the other statements we refer to [31]. With no loss of generality we can assume throughout the proof that $x_0 = 0$. Obviously (15) is equivalent to $\varphi(0) = 0$,

$$g_1(\varphi) \det \nabla \varphi = f_1 \quad \text{and} \quad \frac{g_i}{g_1}(\varphi) = \frac{f_i}{f_1} \quad 2 \leq i \leq n.$$

We claim that $\varphi = G^{-1} \circ F$ has all the desired properties where

$$G = \left(\langle a; x \rangle, \frac{g_2}{g_1}, \dots, \frac{g_n}{g_1} \right) \quad \text{and} \quad F = \left(u, \frac{f_2}{f_1}, \dots, \frac{f_n}{f_1} \right)$$

where $a \in \mathbb{R}^n$ and $u : \mathbb{R}^n \rightarrow \mathbb{R}$ are determined as follows. Using (13) we can select $a \in \mathbb{R}^n$ such that $\det \nabla G(0) \neq 0$ which immediately implies that $G \in \text{Diff}^r(B_\epsilon; G(B_\epsilon))$ for $\epsilon > 0$ small enough (B_ϵ being the ball centered at 0 and of radius ϵ). Then, for any $u \in C^{r-1}$ with $u(0) = 0$, we have that $\varphi = G^{-1} \circ F$ satisfies, using (14), $\varphi(0) = 0$ and

$$\frac{g_i}{g_1}(\varphi) = \frac{f_i}{f_1} \quad \text{near } 0, \text{ for every } 2 \leq i \leq n.$$

In view of the previous considerations, it only remains to find $u \in C^{r-1}$ such that $u(0) = 0$ and

$$g_1(\varphi) \det \nabla \varphi = f_1 \quad \text{near } 0 \tag{16}$$

(note that the above equation implies, in particular, that φ is a local diffeomorphism) or equivalently

$$g_1(G^{-1} \circ F) \cdot (\det \nabla(G^{-1}))(F) \cdot \det \nabla F = f_1.$$

Let us investigate the terms in the left hand side of the last equation. Note that

$$g_1(G^{-1} \circ F)(x) \quad \text{and} \quad (\det \nabla(G^{-1}))(F(x))$$

have, respectively, the form

$$\alpha(x, u(x)) \quad \text{and} \quad \beta(x, u(x))$$

where $\alpha \in C^r$ with $\alpha(0, 0) \neq 0$ and $\beta \in C^{r-1}$ with $\beta(0, 0) \neq 0$. Finally, using (13), we obtain that

$$\det \nabla F = \langle \nabla u; H \rangle$$

for some $H \in C^{r-1}(\mathbb{R}^n; \mathbb{R}^n)$ with $H(0) \neq 0$. We hence deduce that (16) can be written, near 0, as

$$\langle \nabla u(x); H(x) \rangle = \gamma(x, u(x)) = \frac{f_1(x)}{\alpha(x, u(x))\beta(x, u(x))}$$

where $\gamma \in C^{r-1}$. Using the method of characteristics, we can find, near 0, $u \in C^{r-1}$ verifying the last equation as well as $u(0) = 0$. This concludes the proof of the existence part. ■

5.6 Proof of the main theorem

Before proving the theorem (without the estimates) we prove two intermediate results.

5.6.1 The flow method

We first present the flow method introduced by Moser [47] (cf. also Section 4), who did not however consider the boundary condition. Note that the theorem does not provide the optimal regularity.

Theorem 41 *Let $r \geq 1$ be a integer, $0 \leq \alpha \leq 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded connected open $C^{r+2,\alpha}$ set. Let also $f \in C^{r,\alpha}(\overline{\Omega})$ be such that $f > 0$ in $\overline{\Omega}$ and*

$$\int_{\Omega} f = \text{meas } \Omega.$$

Then there exists $\varphi \in \text{Diff}^{r,\alpha}(\overline{\Omega}; \overline{\Omega})$ satisfying

$$\begin{cases} \det \nabla \varphi(x) = f(x) & x \in \Omega \\ \varphi(x) = x & x \in \partial\Omega. \end{cases} \quad (17)$$

Proof Define, for $0 \leq t \leq 1$, $x \in \overline{\Omega}$,

$$f_t(x) = (1-t)f(x) + t$$

and

$$u_t(x) = \frac{u(x)}{f_t(x)} \quad (18)$$

where $u \in C^{r,\alpha}(\overline{\Omega}; \mathbb{R}^n)$ (if $0 < \alpha < 1$, then $u \in C^{r+1,\alpha}(\overline{\Omega}; \mathbb{R}^n)$) satisfies

$$\begin{cases} \text{div } u = f - 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (19)$$

Such a u exists by Theorem 28 (cf. in particular the remark following the theorem) or Theorem 30. Note however that u_t (see (18)) is only in $C^{r,\alpha}$ (even if $0 < \alpha < 1$), since f is only in $C^{r,\alpha}$. Since (18) and (19) hold, we have

$$\begin{cases} \text{div}(u_t f_t) = -\frac{d}{dt} f_t = f - 1 & \text{in } \Omega \\ u_t = 0 & \text{on } \partial\Omega. \end{cases} \quad (20)$$

We can then apply Theorem 34 and have, defining $\phi_t : \overline{\Omega} \rightarrow \mathbb{R}^n$ for every $t \in [0, 1]$ as the solution of

$$\begin{cases} \frac{d}{dt} \phi_t = u_t \circ \phi_t & 0 \leq t \leq 1 \\ \phi_0 = \text{id}, \end{cases}$$

that

$$\varphi = \phi_1$$

has all the desired properties. ■

5.6.2 The fixed point method

We now prove Theorem 36 when $g \equiv 1$ and under a smallness assumption on the $C^{0,\gamma}$ norm of $f - 1$. The following result is in Dacorogna-Moser [34] and follows earlier considerations by Zehnder [58].

Theorem 42 Let $r \geq 0$ be an integer and $0 < \gamma \leq \alpha < 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open $C^{r+2,\alpha}$ set. Let $f \in C^{r,\alpha}(\overline{\Omega})$, $f > 0$ in $\overline{\Omega}$ and

$$\int_{\Omega} f = \text{meas } \Omega.$$

Then there exists $\epsilon = \epsilon(r, \alpha, \gamma, \Omega) > 0$ such that if $\|f - 1\|_{C^{0,\gamma}} \leq \epsilon$, then there exists $\varphi \in \text{Diff}^{r+1,\alpha}(\overline{\Omega}; \overline{\Omega})$ satisfying

$$\begin{cases} \det \nabla \varphi(x) = f(x) & x \in \Omega \\ \varphi(x) = x & x \in \partial\Omega. \end{cases} \quad (21)$$

Moreover there exists a constant $c = c(r, \alpha, \gamma, \Omega) > 0$ such that if $\|f - 1\|_{C^{0,\gamma}} \leq \epsilon$, then φ satisfies

$$\|\varphi - \text{id}\|_{C^{r+1,\alpha}} \leq c \|f - 1\|_{C^{r,\alpha}} \quad \text{and} \quad \|\varphi - \text{id}\|_{C^{1,\gamma}} \leq c \|f - 1\|_{C^{0,\gamma}}.$$

Proof For the convenience of the reader we will not use the abstract fixed point theorem (cf. Theorem 80) but we will redo the proof. We divide the proof into two steps.

Step1. We start by introducing some notations.

(i) Let

$$\begin{aligned} X &= \{a \in C^{r+1,\alpha}(\overline{\Omega}; \mathbb{R}^n) : a = 0 \text{ on } \partial\Omega\} \\ Y &= \{b \in C^{r,\alpha}(\overline{\Omega}) : \int_{\Omega} b = 0\}. \end{aligned}$$

Define $L : X \rightarrow Y$ by $La = \text{div } a$. Note that L is well defined by the divergence theorem. As seen in Theorem 28, there exist a bounded linear operator $L^{-1} : Y \rightarrow X$ and a constant $K_1 > 0$, such that

$$LL^{-1} = \text{id}, \quad \text{in } Y$$

$$\|L^{-1}b\|_{C^{1,\gamma}} \leq K_1 \|b\|_{C^{0,\gamma}} \quad (22)$$

$$\|L^{-1}b\|_{C^{r+1,\alpha}} \leq K_1 \|b\|_{C^{r,\alpha}}. \quad (23)$$

(ii) Let for ξ , any $n \times n$ matrix,

$$Q(\xi) = \det(I + \xi) - 1 - \text{trace}(\xi) \quad (24)$$

where I stands for the identity matrix. Note that Q is a sum of monomials of degree t , $2 \leq t \leq n$. Hence there exists a constant $k > 0$ such that, for every $\xi, \eta \in \mathbb{R}^{n \times n}$,

$$|Q(\xi) - Q(\eta)| \leq k \left(|\xi| + |\eta| + |\xi|^{n-1} + |\eta|^{n-1} \right) |\xi - \eta|.$$

With the same method, we can find (cf. Theorem 74) a constant $K_2 > 0$ such that if $v, w \in C^{r+1,\alpha}$ with $\|v\|_{C^{1,\gamma}}, \|w\|_{C^{1,\gamma}} \leq 1$, then

$$\begin{aligned} \|Q(\nabla v) - Q(\nabla w)\|_{C^{0,\gamma}} &\leq K_2 (\|v\|_{C^{1,\gamma}} + \|w\|_{C^{1,\gamma}}) \|v - w\|_{C^{1,\gamma}} \\ \|Q(\nabla v)\|_{C^{r,\alpha}} &\leq K_2 \|v\|_{C^1} \|v\|_{C^{r+1,\alpha}}. \end{aligned} \quad (25)$$

Step 2. In order to solve (21) we set $v(x) = \varphi(x) - x$ and we rewrite it as

$$\begin{cases} \operatorname{div} v = f - 1 - Q(\nabla v) & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases} \quad (26)$$

If we set

$$N(v) = f - 1 - Q(\nabla v)$$

then (26) is satisfied for any $v \in X$ with

$$v = L^{-1}N(v). \quad (27)$$

Note first that the equation is well defined (i.e. $N : X \rightarrow Y$), since if $v = 0$ on $\partial\Omega$ then $\int_{\Omega} N(v(x)) dx = 0$. Indeed from (24) we have that

$$\begin{aligned} \int_{\Omega} N(v(x)) dx &= \int_{\Omega} [f(x) - 1 - Q(\nabla v(x))] dx \\ &= \int_{\Omega} [f(x) + \operatorname{div} v(x) - \det(I + \nabla v(x))] dx; \end{aligned}$$

since $v = 0$ on $\partial\Omega$ and $\int_{\Omega} f = \operatorname{meas} \Omega$, it follows immediately that the right hand side of the above identity is 0.

We now solve (27) by the contraction principle. We first let

$$B = \left\{ u \in C^{r+1,\alpha}(\bar{\Omega}; \mathbb{R}^n) : \begin{array}{l} u = 0 \text{ on } \partial\Omega \\ \|u\|_{C^{1,\gamma}} \leq 2K_1 \|f - 1\|_{C^{0,\gamma}} \\ \|u\|_{C^{r+1,\alpha}} \leq 2K_1 \|f - 1\|_{C^{r,\alpha}} \end{array} \right\}.$$

We endow B with the $C^{1,\gamma}$ norm. We observe that B is complete and we will show that by choosing $\|f - 1\|_{C^{0,\gamma}}$ small enough, then $L^{-1}N : B \rightarrow B$ is a contraction mapping. The contraction principle will then immediately lead to a solution $v \in B$ and hence in $C^{r+1,\alpha}$ of (27). Indeed let

$$\|f - 1\|_{C^{0,\gamma}} \leq \min \left\{ \frac{1}{8K_1^2 K_2}, \frac{1}{2K_1} \right\}. \quad (28)$$

If $v, w \in B$ (note that by construction $2K_1 \|f - 1\|_{C^{0,\gamma}} \leq 1$), we will show that

$$\|L^{-1}N(v) - L^{-1}N(w)\|_{C^{1,\gamma}} \leq \frac{1}{2} \|v - w\|_{C^{1,\gamma}} \quad (29)$$

$$\|L^{-1}N(v)\|_{C^{1,\gamma}} \leq 2K_1 \|f - 1\|_{C^{0,\gamma}}, \quad \|L^{-1}N(v)\|_{C^{r+1,\alpha}} \leq 2K_1 \|f - 1\|_{C^{r,\alpha}}. \quad (30)$$

The inequality (29) follows from (22), (25) and (28) through

$$\begin{aligned} \|L^{-1}N(v) - L^{-1}N(w)\|_{C^{1,\gamma}} &\leq K_1 \|N(v) - N(w)\|_{C^{0,\gamma}} \\ &= K_1 \|Q(\nabla v) - Q(\nabla w)\|_{C^{0,\gamma}} \\ &\leq K_1 K_2 (\|v\|_{C^{1,\gamma}} + \|w\|_{C^{1,\gamma}}) \|v - w\|_{C^{1,\gamma}} \\ &\leq 4K_1^2 K_2 \|f - 1\|_{C^{0,\gamma}} \|v - w\|_{C^{1,\gamma}} \\ &\leq \frac{1}{2} \|v - w\|_{C^{1,\gamma}}. \end{aligned}$$

To obtain the first inequality in (30) we observe that

$$\|L^{-1}N(0)\|_{C^{1,\gamma}} \leq K_1 \|N(0)\|_{C^{0,\gamma}} = K_1 \|f-1\|_{C^{0,\gamma}}$$

and hence combining (29) with the above inequality we have immediately the first inequality in (30). To obtain the second one we just have to observe that

$$\|L^{-1}N(v)\|_{C^{r+1,\alpha}} \leq K_1 \|N(v)\|_{C^{r,\alpha}} \leq K_1 \|f-1\|_{C^{r,\alpha}} + K_1 \|Q(\nabla v)\|_{C^{r,\alpha}} \quad (31)$$

and use the second inequality in (25) to get, recalling that $v \in B$,

$$\begin{aligned} \|Q(\nabla v)\|_{C^{r,\alpha}} &\leq K_2 \|v\|_{C^1} \|v\|_{C^{r+1,\alpha}} \leq K_2 \|v\|_{C^{1,\gamma}} \|v\|_{C^{r+1,\alpha}} \\ &\leq 2K_1 K_2 \|f-1\|_{C^{0,\gamma}} \|v\|_{C^{r+1,\alpha}} . \end{aligned}$$

The above inequality combined with (28) gives

$$\|Q(\nabla v)\|_{C^{r,\alpha}} \leq \frac{1}{4K_1} \|v\|_{C^{r+1,\alpha}} .$$

Combining this last inequality, (31) and the fact that $v \in B$ we deduce that

$$\|L^{-1}N(v)\|_{C^{r+1,\alpha}} \leq 2K_1 \|f-1\|_{C^{r,\alpha}} .$$

Thus the contraction principle gives immediately the existence of a $C^{r+1,\alpha}$ solution.

It now remains to show that $\varphi(x) = v(x) + x$ is a diffeomorphism. This is a consequence of the fact that $\det \nabla \varphi = f > 0$ and $\varphi(x) = x$ on $\partial\Omega$. The estimate in the statement of the theorem follows by construction, since $v \in B$. ■

5.6.3 Proof of the main theorem

We prove Theorem 35, following the original proof of Dacorogna-Moser [34].

Proof We divide the proof into three steps. The first step is to prove that (ii) \Rightarrow (i) and the two others to prove the reverse implication.

Step 1. Assume that $\varphi \in \text{Diff}^{r+1,\alpha}(\overline{\Omega}; \overline{\Omega})$ satisfies

$$\begin{cases} \det \nabla \varphi(x) = f(x) & x \in \Omega \\ \varphi(x) = x & x \in \partial\Omega. \end{cases}$$

Then clearly $f \in C^{r,\alpha}(\overline{\Omega})$. We easily have that $f > 0$ in $\overline{\Omega}$ and

$$\int_{\Omega} f = \text{meas } \Omega.$$

Step 2 (approximation). We first approximate $f \in C^{r,\alpha}$ by a function $h \in C^{\infty}(\overline{\Omega})$ with $h > 0$ in $\overline{\Omega}$ so that

$$\int_{\Omega} \frac{f}{h} = \text{meas } \Omega$$

$$\left\| \frac{f}{h} - 1 \right\|_{C^{0,\alpha/2}} \leq \epsilon \quad (32)$$

where ϵ is as in the statement of Theorem 42.

Step 3 (conclusion). Using (32) and Theorem 42 we can find $\varphi_1 \in \text{Diff}^{r+1,\alpha}(\overline{\Omega}; \overline{\Omega})$ a solution of

$$\begin{cases} \det \nabla \varphi_1(x) = \frac{f(x)}{h(x)} & x \in \Omega \\ \varphi_1(x) = x & x \in \partial\Omega. \end{cases}$$

We further let $\varphi_2 \in \text{Diff}^{r+1,\alpha}(\overline{\Omega}; \overline{\Omega})$ to be a solution of

$$\begin{cases} \det \nabla \varphi_2(y) = h(\varphi_1^{-1}(y)) & y \in \Omega \\ \varphi_2(y) = y & y \in \partial\Omega. \end{cases}$$

Such a solution exists by Theorem 41, since $h \circ \varphi_1^{-1} \in C^{r+1,\alpha}(\overline{\Omega})$ and

$$\int_{\Omega} h(\varphi_1^{-1}(y)) dy = \int_{\Omega} h(x) \det \nabla \varphi_1(x) dx = \int_{\Omega} f(x) dx = \text{meas } \Omega.$$

Finally observe that the function $\varphi = \varphi_2 \circ \varphi_1$ has all the claimed properties. ■

5.6.4 An alternative proof

We here outline the approach of Carlier-Dacorogna [13] using known results on Monge-Ampère equation. The proof that we briefly discuss below is the nonlinear analogue of the solution (for zero mean f) of

$$\text{div } \varphi = f \text{ in } \Omega \quad \text{and} \quad \varphi = 0 \text{ on } \partial\Omega.$$

Indeed, the "optimal" way to solve this linear problem (cf. Theorem 30) is to look for solutions of the form $\varphi = \nabla\Phi + s$.

(i) We first solve the Neumann problem

$$\Delta\Phi = f \text{ in } \Omega \quad \text{and} \quad \frac{\partial\Phi}{\partial\nu} = 0 \text{ on } \partial\Omega.$$

(ii) We next seek s so that $\text{div } s = 0$ (cf. Lemma 24) and

$$s = -\nabla\Phi \quad \text{on } \partial\Omega.$$

We proceed similarly for the nonlinear problem. We look for solutions of the form $\varphi = s \circ \nabla\Phi$.

(i) We first solve, using Caffarelli result [12],

$$\det \nabla^2 \Phi = f \text{ in } \Omega, \quad \Phi \text{ convex} \quad \text{and} \quad \nabla\Phi(\Omega) = \Omega.$$

(ii) We then find s such that

$$\det \nabla s = 1 \text{ in } \Omega \quad \text{and} \quad s = \nabla\Phi^{-1} = \nabla\Phi^* \text{ on } \partial\Omega. \quad (33)$$

This last construction goes as follows.

- We first solve, using again [12],

$$\det \nabla^2 \Psi_t = (1-t) + t \det \nabla \Phi^* \text{ in } \Omega, \quad \Psi_t \text{ convex} \quad \text{and} \quad \nabla \Psi_t(\Omega) = \Omega.$$

Note that $\nabla \Psi_0 = I$ while $\nabla \Psi_1 = \nabla \Phi^*$.

- We next find w_t through

$$w_t(\nabla \Psi_t) = \frac{d}{dt} \nabla \Psi_t.$$

It is easy to see that $\langle w_t; \nu \rangle = 0$ on $\partial\Omega$.

- We then solve (cf. Theorem 30)

$$\operatorname{div} v_t = -\operatorname{div} w_t \text{ in } \Omega \quad \text{and} \quad v_t = 0 \text{ on } \partial\Omega.$$

This is possible since $\langle w_t; \nu \rangle = 0$ on $\partial\Omega$.

- Letting $u_t = v_t + w_t$, we define s_t by the flow method (cf. Theorem 41)

$$\frac{d}{dt} s_t = u_t(s_t) \quad \text{and} \quad s_0 = \operatorname{id}.$$

By construction $\det \nabla s_t \equiv 1$ and, by uniqueness, $s_t = \nabla \Psi_t$ on $\partial\Omega$. The map $s = s_1$ therefore solves (33).

6 The case $k = 2$

Our best results besides the ones for volume forms are in the case $k = 2$.

6.1 The case of constant forms

Recall what we have seen in the introduction. To any 2-form g

$$g = \sum_{1 \leq i < j \leq n} g_{ij} dx^i \wedge dx^j$$

we can associate, in a unique way, a *skew symmetric matrix* $G = (g_{ij}) \in \mathbb{R}^{n \times n}$ (i.e. $G^t = -G$). We therefore have, if we choose $\varphi(x) = Ax$,

$$\varphi^*(g) = f \quad \Leftrightarrow \quad AGA^t = F.$$

The following theorem is standard in linear algebra (cf. Theorem 65 for a sharper version).

Theorem 43 *Let $F, G \in \mathbb{R}^{n \times n}$ be two skew symmetric matrices with*

$$\operatorname{rank} G = \operatorname{rank} F = 2m \leq n.$$

Then there exists an invertible matrix $A \in \mathbb{R}^{n \times n}$ such that $AGA^t = F$.

The theorem says, in particular, that any skew symmetric matrix of rank $2m$ is equivalent to the canonical matrix J_m given by

$$J_m = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

6.2 Darboux theorem with optimal regularity

The following result is the classical Darboux theorem for closed 2-forms but with optimal regularity. This is a delicate point and it has been obtained by Bandyopadhyay-Dacorogna [1]. The other existing results provide solutions that are, at best, only in $C^{r,\alpha}$, while in the theorem below we find a solution which belongs to $C^{r+1,\alpha}$.

Theorem 44 (Darboux theorem with optimal regularity) *Let $r \geq 0$ and $n = 2m \geq 4$ be integers. Let $0 < \alpha < 1$ and $x_0 \in \mathbb{R}^n$. Let ω_m be the standard symplectic form*

$$\omega_m = \sum_{i=1}^m dx^{2i-1} \wedge dx^{2i}.$$

Let ω be a 2-form. The two following statements are then equivalent.

(i) *The 2-form ω is closed, is in $C^{r,\alpha}$ in a neighborhood of x_0 and verifies*

$$\text{rank}[\omega(x_0)] = n.$$

(ii) *There exist a neighborhood U of x_0 and $\varphi \in \text{Diff}^{r+1,\alpha}(U; \varphi(U))$ such that*

$$\varphi^*(\omega_m) = \omega \text{ in } U \quad \text{and} \quad \varphi(x_0) = x_0.$$

Remark 45 (i) *When $r = 0$, the hypothesis $d\omega = 0$ is to be understood in the sense of distributions.*

(ii) *The theorem is still valid when $n = 2$, but it is then the result of Dacorogna-Moser [34] (cf. Theorem 36).*

(iii) *One possible proof of the theorem could be to use Theorem 56 with $n = 2m$, i.e. the result for 1-forms. We however will go the other way around and prove Theorem 56 using Theorem 44.*

Proof The necessary part is obvious and we discuss only the sufficient part. We divide the proof into four steps.

Step 1. Without loss of generality we can always assume (cf. Theorem 43) that

$$x_0 = 0 \quad \text{and} \quad \omega(0) = \omega_m.$$

Step 2. Our theorem will follow from Theorem 80. So we need to define the spaces and the operators and then check all the hypotheses.

1) We choose V a sufficiently small ball centered at 0 and we define the sets

$$X_1 = C^{1,\alpha}(\bar{V}; \mathbb{R}^n) \quad \text{and} \quad Y_1 = C^{0,\alpha}(\bar{V}; \Lambda^2)$$

$$X_2 = C^{r+1,\alpha}(\bar{V}; \mathbb{R}^n) \quad \text{and} \quad Y_2 = \{b \in C^{r,\alpha}(\bar{V}; \Lambda^2) : db = 0 \text{ in } V\}.$$

It is easy to see that (H_{XY}) of Theorem 80 is fulfilled.

2) Define $L : X_2 \rightarrow Y_2$ by

$$La = d[a \lrcorner \omega_m] = b.$$

We will show that there exists $L^{-1} : Y_2 \rightarrow X_2$ a linear right inverse of L and a constant $C_1 = C_1(r, \alpha, V)$ such that

$$\|L^{-1}b\|_{X_i} \leq C_1 \|b\|_{Y_i} \quad \text{for every } b \in Y_2 \text{ and } i = 1, 2.$$

Once shown this, (H_L) of Theorem 80 will be satisfied. First, using Theorem 19, find $w \in C^{r+1,\alpha}(\bar{V}; \Lambda^1)$ and $C_1 = C_1(r, \alpha, V) > 0$ such that

$$dw = b \quad \text{in } V$$

$$\|w\|_{C^{r+1,\alpha}} \leq C_1 \|b\|_{C^{r,\alpha}} \quad \text{and} \quad \|w\|_{C^{1,\alpha}} \leq C_1 \|b\|_{C^{0,\alpha}}.$$

Moreover the correspondence $b \rightarrow w$ can be chosen to be linear. Next, define $a \in C^{r+1,\alpha}(\bar{V}; \mathbb{R}^n)$ by

$$a_{2i-1} = w_{2i} \quad \text{and} \quad a_{2i} = -w_{2i-1}, \quad 1 \leq i \leq m$$

and note that

$$a \lrcorner \omega_m = w.$$

Finally, defining $L^{-1} : Y_2 \rightarrow X_2$ by $L^{-1}(b) = a$, we easily check that L^{-1} is linear,

$$LL^{-1} = \text{id} \quad \text{on } Y_2$$

and

$$\|L^{-1}b\|_{X_i} \leq C_1 \|b\|_{Y_i} \quad \text{for every } b \in Y_2 \text{ and } i = 1, 2.$$

So that (H_L) of Theorem 80 is satisfied.

3) We then let Q be defined by

$$Q(u) = \omega_m - (\text{id} + u)^* \omega_m + d[u \lrcorner \omega_m].$$

Since

$$d[u \lrcorner \omega_m] = \sum_{i=1}^m [du^{2i-1} \wedge dx^{2i} + dx^{2i-1} \wedge du^{2i}]$$

$$\omega_m - (\text{id} + u)^* \omega_m = \sum_{i=1}^m [dx^{2i-1} \wedge dx^{2i} - (dx^{2i-1} + du^{2i-1}) \wedge (dx^{2i} + du^{2i})]$$

we get

$$Q(u) = - \sum_{i=1}^m du^{2i-1} \wedge du^{2i}.$$

4) Note that $Q(0) = 0$ and $dQ(u) = 0$ in V . Appealing to Theorem 74, there exists a constant $C_2 = C_2(r, V)$ such that, for every $u, v \in C^{r+1, \alpha}(\bar{\Omega}; \mathbb{R}^n)$, the following estimates hold

$$\begin{aligned} \|Q(u) - Q(v)\|_{C^{0, \alpha}} &\leq \sum_{i=1}^m \|du^{2i-1} \wedge du^{2i} - dv^{2i-1} \wedge dv^{2i}\|_{C^{0, \alpha}} \\ &\leq \sum_{i=1}^m \|du^{2i-1} \wedge (du^{2i} - dv^{2i})\|_{C^{0, \alpha}} \\ &\quad + \sum_{i=1}^m \|(dv^{2i-1} - du^{2i-1}) \wedge dv^{2i}\|_{C^{0, \alpha}} \\ &\leq C_2(\|u\|_{C^{1, \alpha}} + \|v\|_{C^{1, \alpha}})\|u - v\|_{C^{1, \alpha}} \end{aligned}$$

and

$$\begin{aligned} \|Q(u)\|_{C^{r, \alpha}} &\leq \sum_{i=1}^m \|du^{2i-1} \wedge du^{2i}\|_{C^{r, \alpha}} \\ &\leq C \sum_{i=1}^m [\|du^{2i-1}\|_{C^{r, \alpha}} \|du^{2i}\|_{C^0} + \|du^{2i}\|_{C^{r, \alpha}} \|du^{2i-1}\|_{C^0}] \\ &\leq C_2 \|u\|_{C^{1, \alpha}} \|u\|_{C^{r+1, \alpha}}. \end{aligned}$$

We therefore see that property (H_Q) is valid for every ρ and we choose $\rho = 1/(2n)$, where $c = C_2$.

5) Setting $\varphi = \text{id} + u$, we can rewrite the equation $\varphi^*(\omega_m) = \omega$ as

$$\begin{aligned} Lu = d[u \lrcorner \omega_m] &= \omega - (\text{id} + u)^* \omega_m + d[u \lrcorner \omega_m] \\ &= \omega - \omega_m + [\omega_m - (\text{id} + u)^* \omega_m + d[u \lrcorner \omega_m]] \\ &= \omega - \omega_m + Q(u). \end{aligned}$$

Step 3. We may now apply Theorem 80 and get that there exists $\psi \in \text{Diff}^{r+1, \alpha}(\bar{V}; \psi(\bar{V}))$ such that

$$\psi^*(\omega_m) = \omega \text{ in } V \quad \text{and} \quad \|\nabla \psi - I\|_{C^0} \leq 1/(2n),$$

provided

$$\|\omega - \omega_m\|_{C^{0, \alpha}} \leq \frac{1}{2C_1 \max\{4C_1 C_2, 2n\}}. \quad (34)$$

Setting $\varphi(x) = \psi(x) - \psi(0)$, we have indeed proved that there exists $\varphi \in \text{Diff}^{r+1,\alpha}(\bar{V}; \varphi(\bar{V}))$ satisfying

$$\varphi^*(\omega_m) = \omega \text{ in } V, \quad \|\nabla\varphi - I\|_{C^0} \leq 1/(2n) \quad \text{and} \quad \varphi(0) = 0.$$

Step 4. We may now conclude the proof of the theorem.

Step 4.1. Let $0 < \epsilon < 1$ and define

$$\omega^\epsilon(x) = \omega(\epsilon x).$$

Observe that $\omega^\epsilon \in C^{r,\alpha}(\bar{V}; \Lambda^2)$, $d\omega^\epsilon = 0$, $\omega^\epsilon(0) = \omega_m$ and

$$\|\omega^\epsilon - \omega_m\|_{C^{0,\alpha}(\bar{V})} \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0.$$

Choose ϵ sufficiently small so that

$$\|\omega^\epsilon - \omega_m\|_{C^{0,\alpha}(\bar{V})} \leq \frac{1}{2C_1 \max\{4C_1C_2, 2n\}}.$$

Apply Step 3 to find $\psi_\epsilon \in \text{Diff}^{r+1,\alpha}(\bar{V}; \psi_\epsilon(\bar{V}))$ satisfying

$$\psi_\epsilon^*(\omega_m) = \omega^\epsilon \text{ in } V, \quad \|\nabla\psi_\epsilon - I\|_{C^0} \leq 1/(2n) \quad \text{and} \quad \psi_\epsilon(0) = 0.$$

Step 4.2. Let

$$\chi_\epsilon(x) = \frac{x}{\epsilon}$$

and define

$$\varphi = \epsilon\psi_\epsilon \circ \chi_\epsilon.$$

Define $U = \epsilon V$. It is easily seen that $\varphi \in C^{r+1,\alpha}(\bar{U}; \varphi(U))$

$$\varphi^*(\omega_m) = \omega \text{ in } U \quad \text{and} \quad \varphi(0) = 0.$$

Note in particular that

$$\|\nabla\varphi - I\|_{C^0(\bar{U})} = \|\nabla\psi_\epsilon - I\|_{C^0(\bar{V})} \leq 1/(2n)$$

and therefore $\det \nabla\varphi > 0$ in \bar{U} . Hence, restricting U , if necessary, we can assume that $\varphi \in \text{Diff}^{r+1,\alpha}(U; \varphi(U))$. This concludes the proof of the theorem. ■

6.3 Darboux theorem for lower rank forms

We next discuss the case of forms of lower rank. This is also well known in the literature. However our theorem (proved in [2] by Bandyopadhyay-Dacorogna-Kneuss) provides, as the previous theorem, one class higher degree of regularity than the other results. Indeed in all other theorems it is proved that if $\omega \in C^{r,\alpha}$, then, at best, $\varphi \in C^{r-1,\alpha}$. It may appear that the theorem below is still not optimal, since it only shows that $\varphi \in C^{r,\alpha}$ when $\omega \in C^{r,\alpha}$. But since there are some missing variables, it is probably the best possible regularity (in [18], we can allow in the theorem below $\alpha = 0, 1$).

Theorem 46 Let $n \geq 3$, $r, m \geq 1$ be integers and $0 < \alpha < 1$. Let $x_0 \in \mathbb{R}^n$ and ω_m be the standard symplectic form with $\text{rank} [\omega_m] = 2m < n$, namely

$$\omega_m = \sum_{i=1}^m dx^{2i-1} \wedge dx^{2i}.$$

Let ω be a $C^{r,\alpha}$ closed 2-form such that

$$\text{rank} [\omega] = 2m \quad \text{in a neighborhood of } x_0.$$

Then there exist a neighborhood U of x_0 and $\varphi \in \text{Diff}^{r,\alpha}(U; \varphi(U))$ such that

$$\varphi^*(\omega_m) = \omega \quad \text{in } U \quad \text{and} \quad \varphi(x_0) = x_0.$$

Proof Step 1. Without loss of generality, we can assume $x_0 = 0$. We first find, appealing to Theorem 47 below, a neighborhood $V \subset \mathbb{R}^n$ of 0 and $\psi \in \text{Diff}^{r,\alpha}(V; \psi(V))$ with $\psi(0) = 0$ and

$$\psi^*(\omega)(x_1, \dots, x_n) = \tilde{\omega}(x_1, \dots, x_{2m}) = \sum_{1 \leq i < j \leq 2m} \tilde{\omega}_{ij}(x_1, \dots, x_{2m}) dx^i \wedge dx^j.$$

Therefore $\psi^*(\omega) = \tilde{\omega} \in C^{r-1,\alpha}$ in a neighborhood of 0 in \mathbb{R}^{2m} and $\text{rank} [\tilde{\omega}] = 2m$ in a neighborhood of 0.

Step 2. We then apply Theorem 44 to $\tilde{\omega}$ and find a neighborhood $W \subset \mathbb{R}^{2m}$ of 0 and $\chi \in \text{Diff}^{r,\alpha}(W; \chi(W))$, with $\chi(0) = 0$, such that

$$\chi^*(\omega_m) = \tilde{\omega} \quad \text{in } W.$$

We set

$$\tilde{\chi}(x) = \tilde{\chi}(x_1, \dots, x_{2m}, x_{2m+1}, \dots, x_n) = (\chi(x_1, \dots, x_{2m}), x_{2m+1}, \dots, x_n)$$

We then choose V smaller, if necessary, so that

$$V \subset W \times \mathbb{R}^{n-2m}.$$

We finally have that $U = \psi(V)$ and $\varphi = \tilde{\chi} \circ \psi^{-1}$ have all the desired properties.

■

In the above theorem we used a very useful result (cf. Theorem 4.5 in [17]).

Theorem 47 (Reduction of dimension) Let $r \geq 1$, $1 \leq k \leq l \leq n - 1$ be integers and $x_0 \in \mathbb{R}^n$. Let g be a C^r k -form verifying

$$dg = 0 \quad \text{and} \quad \text{rank} [g] = l \quad \text{in a neighborhood of } x_0.$$

Then, there exist a neighborhood U of x_0 and $\varphi \in \text{Diff}^r(U; \varphi(U))$ with $\varphi(x_0) = x_0$ and such that, for every $x = (x_1, \dots, x_n) \in U$,

$$\begin{aligned} \varphi^*(g)(x_1, \dots, x_n) &= f(x_1, \dots, x_l) \\ &= \sum_{1 \leq i_1 < \dots < i_k \leq l} f_{i_1 \dots i_k}(x_1, \dots, x_l) dx^{i_1} \wedge \dots \wedge dx^{i_k}. \end{aligned}$$

Thus $f = \varphi^*(g)$ can be seen as a k -form with maximal rank (i.e. $\text{rank} [f] = l$) on \mathbb{R}^l .

Remark 48 (i) *The result is still valid in Hölder spaces.*

(ii) *Note that $\varphi^*(g)$ is only in C^{r-1} although $g \in C^r$.*

6.4 A global result

6.4.1 The main result

We now state our main theorem. It has been obtained under slightly more restrictive hypotheses by Bandyopadhyay-Dacorogna [1]; as stated it is due to Dacorogna-Kneuss [30]. We will only outline the main steps of the proof (for complete details see [17]).

Theorem 49 *Let $n > 2$ be even and $\Omega \subset \mathbb{R}^n$ be a bounded open smooth set with exterior unit normal ν . Let $0 < \alpha < 1$ and $r \geq 1$ be an integer. Let $f, g \in C^{r,\alpha}(\bar{\Omega}; \Lambda^2)$ satisfying $df = dg = 0$ in Ω ,*

$$\begin{aligned} \nu \wedge f, \nu \wedge g &\in C^{r+1,\alpha}(\partial\Omega; \Lambda^3) \quad \text{and} \quad \nu \wedge f = \nu \wedge g \quad \text{on } \partial\Omega \\ \int_{\Omega} \langle f; \psi \rangle dx &= \int_{\Omega} \langle g; \psi \rangle dx, \quad \text{for every } \psi \in \mathcal{H}_T(\Omega; \Lambda^2) \end{aligned} \quad (35)$$

and, for every $t \in [0, 1]$,

$$\text{rank}[tg + (1-t)f] = n, \quad \text{in } \bar{\Omega}.$$

Then there exists $\varphi \in \text{Diff}^{r+1,\alpha}(\bar{\Omega}; \bar{\Omega})$ such that

$$\varphi^*(g) = f \quad \text{in } \Omega \quad \text{and} \quad \varphi = \text{id} \quad \text{on } \partial\Omega.$$

Remark 50 (i) *We can consider, in a similar way, a general homotopy f_t with $f_0 = f, f_1 = g$,*

$$\begin{aligned} df_t = 0, \quad \nu \wedge f_t &= \nu \wedge f_0 \quad \text{on } \partial\Omega \quad \text{and} \quad \text{rank}[f_t] = n \quad \text{in } \bar{\Omega} \\ \int_{\Omega} \langle f_t; \psi \rangle dx &= \int_{\Omega} \langle f_0; \psi \rangle dx, \quad \text{for every } \psi \in \mathcal{H}_T(\Omega; \Lambda^2). \end{aligned}$$

Note that the non-degeneracy condition $\text{rank}[f_t] = n$ implies (identifying, as usual, volume forms with functions)

$$f^{n/2} \cdot g^{n/2} > 0 \quad \text{in } \bar{\Omega}.$$

(ii) *The non-degeneracy condition*

$$\text{rank}[tg + (1-t)f] = n, \quad \text{for every } t \in [0, 1]$$

is equivalent to the condition that the matrix $(\bar{g})(\bar{f})^{-1}$ has no negative eigenvalues.

(iii) *If Ω is contractible, then $\mathcal{H}_T(\Omega; \Lambda^2) = \{0\}$ and therefore (35) is automatically satisfied.*

(iv) Note that the extra regularity on f and g holds only on the boundary and only for their tangential parts. More precisely recall that, for $x \in \partial\Omega$, we denote by $\nu = \nu(x)$ the exterior unit normal to Ω . By

$$\nu \wedge f \in C^{r+1,\alpha}(\partial\Omega; \Lambda^3)$$

we mean that the tangential part of f is in $C^{r+1,\alpha}$, namely the 3-form F defined by

$$F(x) = \nu(x) \wedge f(x)$$

is such that

$$F \in C^{r+1,\alpha}(\partial\Omega; \Lambda^3).$$

(v) If the support of $(f - g)$ is compact in Ω , then one can find a diffeomorphism φ such that the support of $(\varphi - \text{id})$ is also compact in Ω , see [37].

The proof of Theorem 49 follows the same pattern as that of the case $k = n$ (cf. Theorem 35). In the first step (cf. Theorem 51) we establish the result, through the flow method, but without the optimal regularity. We next (cf. Theorem 52) prove the result under a smallness assumption. We finally combine the two intermediate theorems to get the claim. The technical details are however more delicate than those for the case $k = n$ and we prove only the first step and refer for details to [17].

6.4.2 The flow method

We now state and prove a weaker version, from the point of view of regularity, of Theorem 49. It has, however, the advantage of having a simple proof. It has been obtained by Bandyopadhyay-Dacorogna [1].

Theorem 51 *Let $n > 2$ be even and $\Omega \subset \mathbb{R}^n$ be a bounded smooth set with exterior unit normal ν . Let $r \geq 1$ be an integer, $0 < \alpha < 1$ and let $f, g \in C^{r,\alpha}(\overline{\Omega}; \Lambda^2)$ satisfy*

$$df = dg = 0 \text{ in } \Omega, \quad \nu \wedge f = \nu \wedge g \text{ on } \partial\Omega$$

$$\int_{\Omega} \langle f; \psi \rangle dx = \int_{\Omega} \langle g; \psi \rangle dx, \quad \text{for every } \psi \in \mathcal{H}_T(\Omega; \Lambda^2)$$

$$\text{rank}[tg + (1-t)f] = n, \quad \text{in } \overline{\Omega} \text{ and for every } t \in [0, 1].$$

Then there exists $\varphi \in \text{Diff}^{r,\alpha}(\overline{\Omega}; \overline{\Omega})$ such that

$$\varphi^*(g) = f \text{ in } \Omega \quad \text{and} \quad \varphi = \text{id} \text{ on } \partial\Omega.$$

Proof We first find $w \in C^{r+1,\alpha}(\overline{\Omega}; \Lambda^1)$ (cf. Theorem 26) such that

$$\begin{cases} dw = f - g & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $\text{rank}[tg + (1-t)f] = n$, we can find $u_t \in C^{r,\alpha}(\bar{\Omega}; \mathbb{R}^n)$ so that

$$u_t \lrcorner [tg + (1-t)f] = w \quad \Leftrightarrow \quad u_t = [t\bar{g} + (1-t)\bar{f}]^{-1} w.$$

We then apply Theorem 34 to u_t and $f_t = tg + (1-t)f$ to find φ satisfying

$$\varphi^*(g) = f \text{ in } \Omega \quad \text{and} \quad \varphi = \text{id} \text{ on } \partial\Omega.$$

The proof is therefore complete. ■

6.4.3 The fixed point method

As for the case of volume forms $k = n$, the second step in the proof is to obtain the theorem under a smallness condition on g and f . The case $k = 2$ is however more delicate and we refer to [17] for details of the proof of next theorem.

Theorem 52 *Let $n > 2$ be even and $\Omega \subset \mathbb{R}^n$ be a bounded open smooth set. Let $r \geq 1$ be an integer and $0 < \gamma \leq \alpha < 1$. Let $g \in C^{r+1,\alpha}(\bar{\Omega}; \Lambda^2)$ and $f \in C^{r,\alpha}(\bar{\Omega}; \Lambda^2)$ be such that*

$$\begin{aligned} df = dg = 0 \text{ in } \Omega, \quad \nu \wedge f = \nu \wedge g \text{ on } \partial\Omega \\ \int_{\Omega} \langle f; \psi \rangle dx = \int_{\Omega} \langle g; \psi \rangle dx \quad \text{for every } \psi \in \mathcal{H}_T(\Omega; \Lambda^2) \\ \text{rank}[g] = n \quad \text{in } \bar{\Omega}. \end{aligned}$$

Let $c > 0$ be such that

$$\|g\|_{C^0}, \left\| \frac{1}{[g]^{n/2}} \right\|_{C^0} \leq c$$

and define

$$\theta(g) = \frac{1}{\|g\|_{C^{1,\gamma}}^2} \min \left\{ \|g\|_{C^{1,\gamma}}, \frac{1}{\|g\|_{C^{2,\gamma}}}, \frac{1}{\|g\|_{C^{r+1,\alpha}}} \right\}.$$

There exists $C = C(c, r, \alpha, \gamma, \Omega) > 0$ such that if

$$\|f - g\|_{C^{0,\gamma}} \leq C\theta(g) \quad \text{and} \quad \|f - g\|_{C^{0,\gamma}} \leq C \frac{\|f - g\|_{C^{r,\alpha}}}{\|g\|_{C^{1,\gamma}} \|g\|_{C^{r+1,\alpha}}} \quad (36)$$

then there exists $\varphi \in \text{Diff}^{r+1,\alpha}(\bar{\Omega}; \bar{\Omega})$ verifying

$$\varphi^*(g) = f \text{ in } \Omega \quad \text{and} \quad \varphi = \text{id} \text{ on } \partial\Omega. \quad (37)$$

Furthermore there exists $\tilde{C} = \tilde{C}(c, r, \alpha, \gamma, \Omega) > 0$ such that

$$\|\varphi - \text{id}\|_{C^{r+1,\alpha}} \leq \tilde{C} \|g\|_{C^{r+1,\alpha}} \|f - g\|_{C^{r,\alpha}}.$$

Remark 53 *Note that since $g \in C^{r+1,\alpha}(\bar{\Omega}; \Lambda^2)$ and $\nu \wedge f = \nu \wedge g$ on $\partial\Omega$, then $\nu \wedge f \in C^{r+1,\alpha}(\partial\Omega; \Lambda^3)$.*

7 The other cases and necessary conditions

7.1 The cases $k = 0$ and $k = 1$

We start with the case $k = 0$ which is particularly elementary. The first theorem is of a local nature.

Theorem 54 *Let $r \geq 1$ be an integer, $x_0 \in \mathbb{R}^n$ and $f, g \in C^r$ in a neighborhood of x_0 and such that $f(x_0) = g(x_0)$,*

$$\nabla f(x_0) \neq 0 \quad \text{and} \quad \nabla g(x_0) \neq 0.$$

Then there exists a neighborhood U of x_0 and $\varphi \in \text{Diff}^r(U; \varphi(U))$ such that

$$\varphi^*(g) = f \text{ in } U \quad \text{and} \quad \varphi(x_0) = x_0.$$

We now have the following global result.

Theorem 55 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open Lipschitz set. Let $r \geq 1$ be an integer and $f, g \in C^r(\bar{\Omega})$ with $f = g$ on $\partial\Omega$ and*

$$\frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_i} > 0 \quad \text{in } \bar{\Omega}$$

for a certain $1 \leq i \leq n$. Then there exists a diffeomorphism $\varphi \in \text{Diff}^r(\bar{\Omega}; \bar{\Omega})$ satisfying

$$\varphi^*(g) = f \text{ in } \Omega \quad \text{and} \quad \varphi = \text{id} \text{ on } \partial\Omega.$$

Both results extend in a straightforward way to the case of closed 1-forms.

We now give a theorem for non-closed 1-forms. It can be considered as the 1-form version of Darboux theorem. It is easy to see that it leads to Darboux theorem for closed 2-forms.

Theorem 56 *Let $2 \leq 2m \leq n$ be integers, $x_0 \in \mathbb{R}^n$ and ω be a C^∞ 1-form such that $\omega(x_0) \neq 0$ and*

$$\text{rank}[d\omega] = 2m \quad \text{in a neighborhood of } x_0,$$

Then there exist an open set U and $\varphi \in \text{Diff}^\infty(U; \varphi(U))$ such that $\varphi(U)$ is a neighborhood of x_0 and

$$\varphi^*(\omega) = \begin{cases} \Omega_m(x) & \text{if } \omega \wedge (d\omega)^m = 0 \text{ in a neighborhood of } x_0 \\ \Omega_m(x) + dx^{2m+1} & \text{if } \omega \wedge (d\omega)^m \neq 0 \text{ in a neighborhood of } x_0 \end{cases}$$

where

$$\Omega_m(x) = \sum_{i=1}^m x_{2i-1} dx^{2i}.$$

Remark 57 (i) We recall that

$$(d\omega)^m = \underbrace{d\omega \wedge \cdots \wedge d\omega}_{m \text{ times}}.$$

(ii) Note that if $n = 2m$, then $\omega \wedge (d\omega)^m \equiv 0$.

(iii) Without further hypothesis it is, in general, impossible to guarantee that $\varphi(x_0) = x_0$.

(iv) Of particular interest is the case of contact forms where $n = 2m + 1$ and $\omega \wedge (d\omega)^m \neq 0$, see [32] for some improvements on the result.

7.2 The case $k = n - 1$

We have the following result (cf. Bandyopadhyay-Dacorogna-Kneuss [2], see also Barbarosie [4]).

Theorem 58 Let $x_0 \in \mathbb{R}^n$ and f be a $(n - 1)$ -form such that $f \in C^\infty$ in a neighborhood of x_0 and $f(x_0) \neq 0$. Then there exist an open set U and

$$\varphi \in \text{Diff}^\infty(U; \varphi(U))$$

such that $\varphi(U)$ is a neighborhood of x_0 and

$$f = \begin{cases} \nabla\varphi^1 \wedge \cdots \wedge \nabla\varphi^{n-1} & \text{if } df = 0 \text{ in a neighborhood of } x_0 \\ \varphi^n (\nabla\varphi^1 \wedge \cdots \wedge \nabla\varphi^{n-1}) & \text{if } df \neq 0 \text{ in a neighborhood of } x_0. \end{cases}$$

Remark 59 (i) The statement can be rewritten as follows

$$f = \begin{cases} \varphi^* (dx^1 \wedge \cdots \wedge dx^{n-1}) & \text{if } df = 0 \\ \varphi^* (x_n dx^1 \wedge \cdots \wedge dx^{n-1}) & \text{if } df \neq 0. \end{cases}$$

The present theorem, when $df = 0$, is a consequence of a theorem which is valid for k -forms of rank k .

(ii) With our usual abuse of notations, identifying a $(n - 1)$ -form with a vector field and observing that the d operator can then be essentially identified with the divergence operator, we can rewrite the theorem as follows. For any C^∞ vector field f such that $f(x_0) \neq 0$, there exist an open set U and

$$\varphi \in \text{Diff}^\infty(U; \varphi(U))$$

such that $\varphi(U)$ is a neighborhood of x_0 and

$$f = \begin{cases} * (\nabla\varphi^1 \wedge \cdots \wedge \nabla\varphi^{n-1}) & \text{if } \text{div } f = 0 \\ * (\varphi^n (\nabla\varphi^1 \wedge \cdots \wedge \nabla\varphi^{n-1})) & \text{if } \text{div } f \neq 0 \end{cases}$$

where $*$ denotes the Hodge $*$ operator.

(iii) When $df = 0$ we can also ensure that $\varphi(x_0) = x_0$; but not, in general, when $df \neq 0$.

7.3 The case $3 \leq k \leq n - 2$

We now turn to the case $3 \leq k \leq n - 2$ which is, as already said, much more difficult. This is so already at the algebraic level, since there are no known canonical forms. In particular the rank is not the only invariant (cf. Remark 4). And even when the algebraic setting is simple, the analytical situation is more complicated than in the cases $k = 0, 1, 2, n - 1, n$. We give three examples; the first two are purely algebraic and the third one is more analytic.

Example 60 (Example 2.36 in [17]). When $k = 3$, the forms

$$f = e^1 \wedge e^2 \wedge e^3 + e^4 \wedge e^5 \wedge e^6$$

$$g = e^1 \wedge e^2 \wedge e^3 + e^1 \wedge e^4 \wedge e^5 + e^2 \wedge e^4 \wedge e^6 + e^3 \wedge e^5 \wedge e^6$$

have both rank = 6, but there is no $A \in \text{GL}(6)$ so that

$$A^*(g) = f.$$

Example 61 (Example 2.35 in [17]). Similarly and more strikingly, when $k = 4$ and

$$f = e^1 \wedge e^2 \wedge e^3 \wedge e^4 + e^1 \wedge e^2 \wedge e^5 \wedge e^6 + e^3 \wedge e^4 \wedge e^5 \wedge e^6$$

there is no $A \in \text{GL}(6)$ such that

$$A^*(f) = -f$$

although

$$\text{rank}[f] = \text{rank}[-f] = 6.$$

Example 62 (Proposition 15.14 in [17]). Although every constant 3-form of rank = 5 is a linear pullback of

$$f = dx^1 \wedge dx^2 \wedge dx^5 + dx^3 \wedge dx^4 \wedge dx^5$$

we have the following result. There exists $g \in C^\infty(\mathbb{R}^5; \Lambda^3)$ with

$$dg = 0 \quad \text{and} \quad \text{rank}[g] = 5 \quad \text{in } \mathbb{R}^5$$

namely

$$g = -dx^1 \wedge dx^2 \wedge dx^5 + \left((x_3)^2 + 1\right) dx^1 \wedge dx^3 \wedge dx^4 + \left((x_3)^4 + 1\right) dx^2 \wedge dx^3 \wedge dx^4$$

which cannot be pulled back locally by a diffeomorphism to f .

The only cases that we are able to study are those that are combinations of 1 and 2-forms that we can handle separately. For 1-forms, we easily obtain local as well as global results. We now give a simple theorem (more general statements can be found in [17]) that deals with 3-forms obtained by product of a 1-form and a 2-form (in the same spirit, we can deal with some k -forms that are product of 1 and 2-forms).

Theorem 63 Let $n = 2m \geq 4$ be integers, $x_0 \in \mathbb{R}^n$ and f be a C^∞ symplectic (i.e. closed and with $\text{rank}[f] = n$) 2-form and a be a non-zero closed C^∞ 1-form. Then there exist a neighborhood U of x_0 and $\varphi \in \text{Diff}^\infty(U; \varphi(U))$ such that $\varphi(x_0) = x_0$ and

$$\varphi^*(\omega_m) = f \quad \text{and} \quad \varphi^*(dx^n) = a, \quad \text{in } U.$$

where

$$\omega_m = \sum_{i=1}^m dx^{2i-1} \wedge dx^{2i}.$$

In particular if

$$G = \left[\sum_{i=1}^{m-1} dx^{2i-1} \wedge dx^{2i} \right] \wedge dx^n = \omega_m \wedge dx^n$$

then

$$\varphi^*(G) = f \wedge a, \quad \text{in } U.$$

7.4 Necessary conditions

We point out the following necessary conditions.

Theorem 64 Let $\Omega \subset \mathbb{R}^n$ be a bounded open smooth set and $\varphi \in \text{Diff}^1(\overline{\Omega}; \varphi(\overline{\Omega}))$. Let $1 \leq k \leq n$, $f \in C^1(\overline{\Omega}; \Lambda^k)$ and $g \in C^1(\varphi(\overline{\Omega}); \Lambda^k)$ be such that

$$\varphi^*(g) = f \quad \text{in } \Omega.$$

(i) For every $x \in \Omega$, then

$$\text{rank}[g(\varphi(x))] = \text{rank}[f(x)] \quad \text{and} \quad \text{rank}[dg(\varphi(x))] = \text{rank}[df(x)].$$

In particular

$$dg = 0 \text{ in } \varphi(\Omega) \quad \Leftrightarrow \quad df = 0 \text{ in } \Omega.$$

(ii) If $\varphi(x) = x$ for $x \in \partial\Omega$, then

$$\nu \wedge f = \nu \wedge g \quad \text{on } \partial\Omega$$

where ν is the exterior unit normal to Ω .

8 Selection principle via ellipticity

We now very briefly discuss the question of selecting one specific solution of the pullback equation. The problem for $k = n$ has been intensively studied in the context of optimal mass transportation, but very little for general forms (see [27]). We here discuss (following [28]) the choice that can be made by considering appropriate elliptic systems in the cases $k = n$ and $k = 2$.

8.1 The case $k = n$

When $k = n$ a natural choice, which is the one obtained via optimal mass transportation, is $\varphi = \nabla\Phi$ transforming the pullback equation $\varphi^*(g) = f$ to the celebrated *Monge-Ampère equation*

$$g(\nabla\Phi(x)) \det \nabla^2\Phi(x) = f(x).$$

This equation, when coupled with appropriate boundary conditions, leads to uniqueness as well as regularity. In order to understand better the case $k = 2$, we rephrase the earlier considerations in the following way. For the sake of simplicity we assume that $g \equiv 1$, transforming the pullback equation to the single equation $\det \nabla\varphi = f$. We then couple this equation to the $[n(n-1)/2]$ equations $\text{curl} \varphi = 0$. It turns out that this system, when restricted to maps such that $\nabla\varphi + (\nabla\varphi)^t > 0$, is elliptic (see [28] for a definition of ellipticity of first order systems).

8.2 The case $k = 2$

It was essentially equivalent, in the case $k = n$, to choose $\varphi = \nabla\Phi$ or to add the equations $\text{curl} \varphi = 0$ in order to find an appropriate elliptic system. This is not any more true when $k = 2$.

8.2.1 The gradient case

It has been proved in [18] that for constant forms the choice $\varphi = \nabla\Phi$ (here requiring that $\varphi(x) = \nabla\Phi(x) = Ax$ amounts to say that A is symmetric) is appropriate.

Theorem 65 *Let n be even and $g, f \in \Lambda^2(\mathbb{R}^n)$ be such that $\text{rank}[g] = \text{rank}[f] = n$. Then there exists $A \in \text{GL}(n)$ such that*

$$A^*(g) = f \quad \text{and} \quad A^t = A.$$

However as soon as the forms are non-constant, the above theorem is, in general, false.

Proposition 66 *Let $f \in C^\infty(\mathbb{R}^4; \Lambda^2)$ be defined by*

$$f = (1 + x_3) dx^1 \wedge dx^2 + x_2 dx^1 \wedge dx^3 + 2 dx^3 \wedge dx^4.$$

Then there exists no $\Phi \in C^3(\mathbb{R}^4)$ such that near 0

$$(\nabla\Phi)^*(dx^1 \wedge dx^2 + dx^3 \wedge dx^4) = f$$

although there exists a local C^∞ diffeomorphism φ such that

$$\varphi^*(dx^1 \wedge dx^2 + dx^3 \wedge dx^4) = f.$$

8.2.2 The ellipticity criterion

A better choice is as follows (cf. [28]). We couple the pullback equation $\varphi^*(g) = f$, which is a first order system of $[n(n-1)/2]$ equations, to the single equation $\langle d\varphi; g \rangle = 0$, i.e.

$$\sum_{1 \leq i < j \leq n} (\varphi_{x_i}^j - \varphi_{x_j}^i) g^{ij} = 0.$$

It turns out that the new system, when restricted to maps such that $\nabla\varphi + (\nabla\varphi)^t > 0$, is elliptic. In fact one can prove the following theorem. We let $r \geq 0$ and $n = 2m$ be integers, $0 < \alpha < 1$ and ω_m be the standard symplectic form. We also let $\Omega \subset \mathbb{R}^n$ be a bounded contractible open smooth set with exterior unit normal ν .

Theorem 67 *Let $f \in C^{r,\alpha}(\overline{\Omega}; \Lambda^2)$ be closed. Then there exist ϵ, γ and c depending only on (r, α, Ω) such that if*

$$\|f - \omega_m\|_{C^{0,\alpha/2}} \leq \epsilon,$$

then there exists a unique $\varphi \in \text{Diff}^{r+1,\alpha}(\overline{\Omega}; \varphi(\overline{\Omega}))$ satisfying

$$\begin{cases} \varphi^*(\omega_m) = f & \text{and } d\varphi \lrcorner \omega_m = \langle d\varphi; \omega_m \rangle = 0 & \text{in } \Omega \\ \nu \lrcorner ((\varphi - \text{id}) \lrcorner \omega_m) = 0 & & \text{on } \partial\Omega \end{cases} \quad (38)$$

and such that

$$\begin{cases} \|\varphi - \text{id}\|_{C^{r+1,\alpha}} \leq c \|f - \omega_m\|_{C^{r,\alpha}} \\ |\langle [\nabla\varphi(x)] \xi; \xi \rangle| \geq \gamma |\xi|^2, \quad \forall \xi \in \mathbb{R}^n \text{ and } \forall x \in \overline{\Omega}. \end{cases} \quad (39)$$

Furthermore the system (38) is elliptic when restricted to maps satisfying the second inequality in (39).

The above theorem can be written as a second order system, which is the counterpart of Monge-Ampère equation when $n = 2$ and therefore f is a volume form.

Corollary 68 (Second order Darboux theorem) *Let $f \in C^{r,\alpha}(\overline{\Omega}; \Lambda^2)$ be closed. Then there exists $\epsilon = \epsilon(r, \alpha, \Omega)$ such that if*

$$\|f - \omega_m\|_{C^{0,\alpha/2}} \leq \epsilon,$$

then there exists a unique $\Phi \in C^{r+2,\alpha}(\overline{\Omega}; \Lambda^2)$ satisfying the elliptic system

$$\begin{cases} (\delta\Phi \lrcorner \omega_m)^*(\omega_m) = f & \text{and } d\Phi = 0 & \text{in } \Omega \\ \nabla(\delta\Phi \lrcorner \omega_m) + (\nabla(\delta\Phi \lrcorner \omega_m))^t > 0 & & \text{in } \Omega \\ \nu \lrcorner \Phi = -\nu \lrcorner H & & \text{on } \partial\Omega. \end{cases}$$

Here H is such that $\delta H = \text{id} \lrcorner \omega_m$.

Remark 69 (i) *Writing*

$$\Phi = \sum_{i < j} \Phi^{ij} dx^i \wedge dx^j$$

and similarly for f we have that $(\delta\Phi \lrcorner \omega_m)^*(\omega_m) = f$ reads as (recalling that $\Phi^{ij} = -\Phi^{ji}$)

$$\sum_{l=1}^m \sum_{s,t=1}^{2m} \left[\Phi_{x_s x_i}^{s(2l-1)} \Phi_{x_t x_j}^{t(2l)} - \Phi_{x_s x_j}^{s(2l-1)} \Phi_{x_t x_i}^{t(2l)} \right] = f^{ij}, \quad 1 \leq i < j \leq n \quad (40)$$

while $d\Phi = 0$ means that

$$\Phi_{x_k}^{ij} - \Phi_{x_j}^{ik} + \Phi_{x_i}^{jk} = 0, \quad 1 \leq i < j < k \leq n.$$

Note that when $n = 2$ the equation $d\Phi = 0$ is trivially fulfilled, while (40) is exactly Monge-Ampère equation.

(ii) The form H can be taken, for example, as

$$H = \sum_{i=1}^m \frac{(x_{2i-1})^2 + (x_{2i})^2}{2} dx^{2i-1} \wedge dx^{2i}.$$

9 Hölder spaces

We now present fine properties of Hölder continuous functions. Most of the results are "standard", but they are scattered in the literature. There does not exist such a huge literature as the one for Sobolev spaces. The most complete reference for this chapter is [17]

9.1 Definition and extension of Hölder functions

We give here the definition of Hölder continuous functions.

Definition 70 Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $f : \overline{\Omega} \rightarrow \mathbb{R}$ and $0 < \alpha \leq 1$. Let

$$[f]_{C^{0,\alpha}(\overline{\Omega})} = \sup_{\substack{x,y \in \overline{\Omega} \\ x \neq y}} \left\{ \frac{|f(x) - f(y)|}{|x - y|^\alpha} \right\}.$$

(i) The set $C^{0,\alpha}(\overline{\Omega})$ is the set of $f \in C^0(\overline{\Omega})$ so that

$$\|f\|_{C^{0,\alpha}(\overline{\Omega})} = \|f\|_{C^0(\overline{\Omega})} + [f]_{C^{0,\alpha}(\overline{\Omega})} < \infty$$

where

$$\|f\|_{C^0(\overline{\Omega})} = \sup_{x \in \overline{\Omega}} \{|f(x)|\}.$$

If there is no ambiguity we drop the dependence on the set $\bar{\Omega}$ and write simply

$$\|f\|_{C^{0,\alpha}} = \|f\|_{C^0} + [f]_{C^{0,\alpha}}.$$

(ii) If $r \geq 1$ is an integer, then the set $C^{r,\alpha}(\bar{\Omega})$ is the set of functions $f \in C^r(\bar{\Omega})$ so that

$$[\nabla^r f]_{C^{0,\alpha}(\bar{\Omega})} < \infty.$$

We equip $C^{r,\alpha}(\bar{\Omega})$ with the following norm

$$\|f\|_{C^{r,\alpha}(\bar{\Omega})} = \|f\|_{C^r(\bar{\Omega})} + [\nabla^r f]_{C^{0,\alpha}(\bar{\Omega})}$$

where

$$\|f\|_{C^r(\bar{\Omega})} = \sum_{m=0}^r \|\nabla^m f\|_{C^0(\bar{\Omega})}.$$

Remark 71 (i) $C^{r,\alpha}(\bar{\Omega})$ with its norm $\|\cdot\|_{C^{r,\alpha}}$ is a Banach space.

(ii) If $\alpha = 0$, we set

$$\|f\|_{C^{r,0}} = \|f\|_{C^r}.$$

(iii) If we assume that the domain is Lipschitz, then the following norms

$$\|f\|_{C^{r,\alpha}} = \sum_{m=0}^r \|\nabla^m f\|_{C^{0,\alpha}}$$

and

$$\|f\|_{C^{r,\alpha}} = \begin{cases} \|f\|_{C^0} + [\nabla^r f]_{C^{0,\alpha}} & \text{if } 0 < \alpha \leq 1 \\ \|f\|_{C^0} + \|\nabla^r f\|_{C^0} & \text{if } \alpha = 0. \end{cases}$$

are equivalent to the one defined above. We should, however, point out that these norms are, in general, not equivalent for very wild domains.

(iv) When $\alpha = 1$, we note that $C^{0,1}(\bar{\Omega})$ is in fact the set of Lipschitz continuous functions.

The following extension result is due to Calderon and Stein.

Theorem 72 Let $\Omega \subset \mathbb{R}^n$ be a bounded open Lipschitz set. Then there exists a continuous linear extension operator

$$E : C^{r,\alpha}(\bar{\Omega}) \rightarrow C_0^{r,\alpha}(\mathbb{R}^n)$$

for any integer $r \geq 0$ and any $0 \leq \alpha \leq 1$. More precisely there exists a constant $C = C(r, \Omega) > 0$ such that, for every $f \in C^{r,\alpha}(\bar{\Omega})$,

$$E(f)|_{\bar{\Omega}} = f, \quad \text{supp}[E(f)] \text{ is compact,}$$

$$\|E(f)\|_{C^{r,\alpha}(\mathbb{R}^n)} \leq C \|f\|_{C^{r,\alpha}(\bar{\Omega})}.$$

Remark 73 The extension is universal, in the sense that the same extension also leads to

$$\|E(f)\|_{C^{s,\beta}(\mathbb{R}^n)} \leq C \|f\|_{C^{s,\beta}(\bar{\Omega})}$$

for any integer s and any $0 \leq \beta \leq 1$, with, of course, $C = C(s, \Omega)$ as far as $f \in C^{s,\beta}(\bar{\Omega})$. The same extension is also valid for Sobolev spaces.

9.2 Product, composition and inverse

We start with a result on products of Hölder continuous functions.

Theorem 74 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open Lipschitz set, $r \geq 0$ an integer and $0 \leq \alpha \leq 1$. Then there exists a constant $C = C(r, \Omega) > 0$ such that*

$$\|fg\|_{C^{r,\alpha}} \leq C (\|f\|_{C^{r,\alpha}} \|g\|_{C^0} + \|f\|_{C^0} \|g\|_{C^{r,\alpha}}).$$

The next theorem has also been intensively used.

Theorem 75 *Let $\Omega \subset \mathbb{R}^n$, $O \subset \mathbb{R}^m$ be bounded open Lipschitz sets, $r \geq 0$ an integer and $0 \leq \alpha \leq 1$. Let $g \in C^{r,\alpha}(\overline{O})$ and $f \in C^{r,\alpha}(\overline{\Omega}; \overline{O}) \cap C^1(\overline{\Omega}; \overline{O})$. Then*

$$\|g \circ f\|_{C^{0,\alpha}(\overline{\Omega})} \leq \|g\|_{C^{0,\alpha}(\overline{O})} \|f\|_{C^1(\overline{\Omega})}^\alpha + \|g\|_{C^0(\overline{O})}$$

while if $r \geq 1$, there exists a constant $C = C(r, \Omega, O) > 0$ such that

$$\|g \circ f\|_{C^{r,\alpha}(\overline{\Omega})} \leq C \left[\|g\|_{C^{r,\alpha}(\overline{O})} \|f\|_{C^1(\overline{\Omega})}^{r+\alpha} + \|g\|_{C^1(\overline{O})} \|f\|_{C^{r,\alpha}(\overline{\Omega})} + \|g\|_{C^0(\overline{O})} \right].$$

We easily deduce, from the previous results, an estimate on the inverse.

Theorem 76 *Let $\Omega, O \subset \mathbb{R}^n$ be bounded open Lipschitz sets, $r \geq 1$ an integer and $0 \leq \alpha \leq 1$. Let $c > 0$. Let $f \in C^{r,\alpha}(\overline{\Omega}; \overline{O})$ and $g \in C^{r,\alpha}(\overline{O}; \overline{\Omega})$ be such that*

$$g \circ f = \text{id} \quad \text{and} \quad \|g\|_{C^1(\overline{O})}, \|f\|_{C^1(\overline{\Omega})} \leq c.$$

Then there exists a constant $C = C(c, r, \Omega, O) > 0$ such that

$$\|f\|_{C^{r,\alpha}(\overline{O})} \leq C \|g\|_{C^{r,\alpha}(\overline{\Omega})}.$$

9.3 Smoothing operator

The next theorem is about smoothing C^r or $C^{r,\alpha}$ functions. We should draw the attention that, in order to get the conclusions of the theorem, one proceeds, as usual, by convolution. However the kernel has to be chosen carefully.

Theorem 77 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open Lipschitz set. Let $s \geq r \geq t \geq 0$ be integers and $0 \leq \alpha, \beta, \gamma \leq 1$ be such that*

$$t + \gamma \leq r + \alpha \leq s + \beta.$$

Let $f \in C^{r,\alpha}(\overline{\Omega})$. Then, for every $0 < \epsilon \leq 1$, there exist a constant $C = C(s, \Omega) > 0$ and $f_\epsilon \in C^\infty(\overline{\Omega})$ such that

$$\|f_\epsilon\|_{C^{s,\beta}} \leq \frac{C}{\epsilon^{(s+\beta)-(r+\alpha)}} \|f\|_{C^{r,\alpha}}$$

$$\|f - f_\epsilon\|_{C^{t,\gamma}} \leq C \epsilon^{(r+\alpha)-(t+\gamma)} \|f\|_{C^{r,\alpha}}.$$

We also need to approximate closed forms in $C^{r,\alpha}(\bar{\Omega}; \Lambda^k)$ by smooth closed forms in a precise way.

Theorem 78 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open smooth set and ν be the exterior unit normal. Let $s \geq r \geq t \geq 0$ with $s \geq 1$ and $1 \leq k \leq n-1$ be integers. Let $0 < \alpha, \beta, \gamma < 1$ be such that*

$$t + \gamma \leq r + \alpha \leq s + \beta.$$

Let $g \in C^{r,\alpha}(\bar{\Omega}; \Lambda^k)$ with

$$dg = 0 \text{ in } \Omega \quad \text{and} \quad \nu \wedge g \in C^{s,\beta}(\partial\Omega; \Lambda^{k+1}).$$

Then for every $\epsilon \in (0, 1]$, there exist $g_\epsilon \in C^\infty(\Omega; \Lambda^k) \cap C^{s,\beta}(\bar{\Omega}; \Lambda^k)$ and a constant $C = C(s, \alpha, \beta, \gamma, \Omega) > 0$ such that

$$\begin{aligned} dg_\epsilon &= 0 \text{ in } \Omega, \quad \nu \wedge g_\epsilon = \nu \wedge g \text{ on } \partial\Omega \\ \int_\Omega \langle g_\epsilon; \psi \rangle &= \int_\Omega \langle g; \psi \rangle, \quad \text{for every } \psi \in \mathcal{H}_T(\Omega; \Lambda^k) \\ \|g_\epsilon\|_{C^{s,\beta}(\bar{\Omega})} &\leq \frac{C}{\epsilon^{(s+\beta)-(r+\alpha)}} \|g\|_{C^{r,\alpha}(\bar{\Omega})} + C \|\nu \wedge g\|_{C^{s,\beta}(\partial\Omega)} \\ \|g_\epsilon - g\|_{C^{t,\gamma}(\bar{\Omega})} &\leq C \epsilon^{(r+\alpha)-(t+\gamma)} \|g\|_{C^{r,\alpha}(\bar{\Omega})}. \end{aligned}$$

Remark 79 *We recall that if Ω is contractible and since $1 \leq k \leq n-1$, then*

$$\mathcal{H}_T(\Omega; \Lambda^k) = \{0\}.$$

10 An abstract fixed point theorem

The following theorem is particularly useful when dealing with non-linear problems, once good estimates are known for the linearized problem (see [5] for some applications). We give it under a general form (still a more sophisticated version can be found in [17]), because we have used it this way in Theorems 44 and 52. However, in many instances, Corollary 81 is amply sufficient. Our theorem will lean on the following hypotheses.

(H_{XY}) Let $X_1 \supset X_2$ be Banach spaces and $Y_1 \supset Y_2$ be normed spaces such that the following property holds: if

$$u_\nu \xrightarrow{X_1} u \quad \text{and} \quad \|u_\nu\|_{X_2} \leq r$$

then $u \in X_2$ and

$$\|u\|_{X_2} \leq r.$$

(H_L) Let $L : X_2 \rightarrow Y_2$ be such that there exists a linear right inverse operator $L^{-1} : Y_2 \rightarrow X_2$ (namely $LL^{-1} = \text{id}$ on Y_2). Moreover there exist $k > 0$ such that for every $f \in Y_2$

$$\|L^{-1}f\|_{X_i} \leq k\|f\|_{Y_i} \quad i = 1, 2.$$

(H_Q) There exists $\rho > 0$ such that

$$Q : B_\rho = \{u \in X_2 : \|u\|_{X_1} \leq \rho\} \rightarrow Y_2$$

$Q(0) = 0$ and for every $u, v \in B_\rho$, the following two inequalities hold

$$\|Q(u) - Q(v)\|_{Y_1} \leq c(\|u\|_{X_1} + \|v\|_{X_1})\|u - v\|_{X_1} \quad (41)$$

$$\|Q(v)\|_{Y_2} \leq c\|v\|_{X_1}\|v\|_{X_2} \quad (42)$$

where $c > 0$ is a constant.

Theorem 80 (Fixed point theorem) *Let X_1, X_2, Y_1, Y_2, L, Q satisfy the hypotheses (H_{XY}), (H_L) and (H_Q). Then, for every $f \in Y_2$ verifying*

$$\|f\|_{Y_1} \leq \min \left\{ \frac{\rho}{2k}, \frac{1}{8k^2c} \right\} \quad (43)$$

there exists $u \in B_\rho \subset X_2$ such that

$$Lu = Q(u) + f \quad \text{and} \quad \|u\|_{X_i} \leq 2k\|f\|_{Y_i}, \quad i = 1, 2. \quad (44)$$

We have as an immediate consequence of the theorem the following result.

Corollary 81 *Let X be a Banach space and Y a normed space. Let $L : X \rightarrow Y$ be such that there exists a linear right inverse operator $L^{-1} : Y \rightarrow X$ (namely $LL^{-1} = \text{id}$ on Y) and there exists $k > 0$ such that*

$$\|L^{-1}f\|_X \leq k\|f\|_Y.$$

Let $\rho > 0$ and

$$Q : B_\rho = \{u \in X : \|u\|_X \leq \rho\} \rightarrow Y$$

with $Q(0) = 0$ and, for every $u, v \in B_\rho$,

$$\|Q(u) - Q(v)\|_Y \leq c(\|u\|_X + \|v\|_X)\|u - v\|_X$$

and where $c > 0$. If

$$\|f\|_{Y_1} \leq \min \left\{ \frac{\rho}{2k}, \frac{1}{8k^2c} \right\}$$

then there exists $u \in B_\rho \subset X$ such that

$$Lu = Q(u) + f \quad \text{and} \quad \|u\|_X \leq 2k\|f\|_Y.$$

We now turn to the proof of Theorem 80.

Proof We set

$$N(u) = Q(u) + f.$$

We next define

$$B = \{u \in X_2 : \|u\|_{X_i} \leq 2k\|f\|_{Y_i} \quad i = 1, 2\}.$$

We endow B with $\|\cdot\|_{X_1}$ norm; the property (H_{XY}) ensures that B is closed. We now want to show that $L^{-1}N : B \rightarrow B$ is a contraction mapping (cf. Claim 1 and 2 below). Applying Banach fixed point theorem we will have indeed found a solution verifying (44), since $LL^{-1} = \text{id}$.

Claim 1. Let us first show that $L^{-1}N$ is a contraction on B . To show this, let $u, v \in B$ and use (41), (43) to get that

$$\begin{aligned} \|L^{-1}N(u) - L^{-1}N(v)\|_{X_1} &\leq k\|N(u) - N(v)\|_{Y_1} = k\|Q(u) - Q(v)\|_{Y_1} \\ &\leq kc(\|u\|_{X_1} + \|v\|_{X_1})\|u - v\|_{X_1} \\ &\leq kc(2k\|f\|_{Y_1} + 2k\|f\|_{Y_1})\|u - v\|_{X_1} \\ &\leq \frac{1}{2}\|u - v\|_{X_1}. \end{aligned}$$

Claim 2. We next show $L^{-1}N : B \rightarrow B$ is well-defined. First, note that

$$\|L^{-1}N(0)\|_{X_1} \leq k\|N(0)\|_{Y_1} = k\|f\|_{Y_1}.$$

Therefore, using Claim 1, we obtain

$$\begin{aligned} \|L^{-1}N(u)\|_{X_1} &\leq \|L^{-1}N(u) - L^{-1}N(0)\|_{X_1} + \|L^{-1}N(0)\|_{X_1} \\ &\leq \frac{1}{2}\|u\|_{X_1} + k\|f\|_{Y_1} \leq 2k\|f\|_{Y_1}. \end{aligned}$$

It remains to show that

$$\|L^{-1}N(u)\|_{X_2} \leq 2k\|f\|_{Y_2}.$$

Using (42), we have

$$\begin{aligned} \|L^{-1}N(u)\|_{X_2} &\leq k\|N(u)\|_{Y_2} \leq k\|Q(u)\|_{Y_2} + k\|f\|_{Y_2} \\ &\leq kc\|u\|_{X_1}\|u\|_{X_2} + k\|f\|_{Y_2} \\ &\leq k[c(2k\|f\|_{Y_1} \cdot 2k\|f\|_{Y_2}) + \|f\|_{Y_2}] \\ &\leq k[4k^2c\|f\|_{Y_1} + 1]\|f\|_{Y_2} \end{aligned}$$

and hence, appealing once more to (43),

$$\|L^{-1}N(u)\|_{X_2} \leq 2k\|f\|_{Y_2}.$$

This concludes the proof of Claim 2 and thus of the theorem. ■

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