Poincaré lemma in a star shaped domain

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We give an explicit formula for the Poincaré lemma in a star shaped domain. The formula is well known. We use the notations of [1].

Theorem 1 *Let $0 \leq k \leq n-1$, $\Omega \subset \mathbb{R}^n$ be a star shaped domain with respect to the origin and $f \in C^1\left(\Omega; \Lambda^{k+1}\right)$ be such that $df = 0$. Then $F \in C^1\left(\Omega; \Lambda^k\right)$ defined by*

$$F(x) = \int_0^1 \left[x \lrcorner f(t\,x)\right] t^k dt$$

verifies $dF = f$.

Remark 2 *(i)* The formula does not provide any gain in regularity (except, trivially, for $k = 0$). For the general Poincaré lemma for general domains and with optimal regularity, see Theorem 8.3 in [1].

(ii) Note that, trivially, the solution also satisfies $x \lrcorner F(x) = 0$ in Ω. Therefore if Ω is a ball centered at the origin (and ν denotes the outward unit normal), then our solution satisfies also

$$\begin{cases} dF = f & \text{in } \Omega \\ \nu \lrcorner F = 0 & \text{on } \partial\Omega. \end{cases}$$

(iii) The case $k = 0$ is the most classical and reads as

$$F(x) = \sum_{j=1}^{n} \int_0^1 \left[x_j\, f^j(t\,x)\right] dt \quad \Rightarrow \quad F_{x_i} = f^i \text{ (i.e. } \operatorname{grad} F = f\text{)}.$$

When $k = 1$ and $k = 2$ the formula becomes

$$F^j(x) = \sum_{i=1}^{n} \int_0^1 \left[x_i\, f^{ij}(t\,x)\right] t\, dt \quad \Rightarrow \quad F^j_{x_i} - F^i_{x_j} = f^{ij} \text{ (i.e. } \operatorname{curl} F = f\text{)}$$

$$F^{ij}(x) = \sum_{l=1}^{n} \int_0^1 \left[x_l\, f^{lij}(t\,x)\right] t^2\, dt \quad \Rightarrow \quad F^{jl}_{x_i} - F^{il}_{x_j} + F^{ij}_{x_l} = f^{ijl}.$$

(iv) Using the Hodge $*$ operator, we have the dual version, namely if $1 \leq k \leq n$ and $\varphi \in C^1\left(\Omega; \Lambda^{k-1}\right)$ is such that $\delta\varphi = 0$, then $\Phi \in C^1\left(\Omega; \Lambda^k\right)$ defined by

$$\Phi(x) = \int_0^1 \left[x \wedge \varphi(t\,x)\right] t^{n-k} dt$$

verifies $\delta\Phi = \varphi$. In particular when $k = 1$ the formula reads as

$$\Phi(x) = x \int_0^1 [\varphi(t\,x)]\, t^{n-1}\, dt \quad \Rightarrow \quad \operatorname{div} \Phi = \varphi.$$

Proof We have

$$F^{i_1 \cdots i_k} = \int_0^1 \left(\sum_{i=1}^n x_i\, f^{i i_1 \cdots i_k}(tx) \right) t^k dt$$

and hence

$$F^{i_1 \cdots i_k}_{x_{i_{k+1}}} = \int_0^1 \left(f^{i_{k+1} i_1 \cdots i_k}(tx) + t \sum_{i=1}^n x_i\, f^{i i_1 \cdots i_k}_{x_{i_{k+1}}}(tx) \right) t^k dt.$$

We therefore find that

$$\begin{aligned}(dF)^{i_1 \cdots i_{k+1}} &= \sum_{\gamma=1}^{k+1} (-1)^{\gamma-1}\, F^{i_1 \cdots i_{\gamma-1} i_{\gamma+1} \cdots i_{k+1}}_{x_{i_\gamma}} \\ &= \sum_{\gamma=1}^{k+1} (-1)^{\gamma-1} \int_0^1 \left(f^{i_\gamma i_1 \cdots i_{\gamma-1} i_{\gamma+1} \cdots i_{k+1}} + t \sum_{i=1}^n x_i\, f^{i i_1 \cdots i_{\gamma-1} i_{\gamma+1} \cdots i_{k+1}}_{x_{i_\gamma}} \right) t^k dt.\end{aligned}$$

Since (see below)

$$\sum_{\gamma=1}^{k+1} (-1)^{\gamma-1} \sum_{i=1}^n x_i\, f^{i i_1 \cdots i_{\gamma-1} i_{\gamma+1} \cdots i_{k+1}}_{x_{i_\gamma}} = \sum_{i=1}^n x_i\, f^{i_1 \cdots i_{k+1}}_{x_i} \tag{1}$$

we deduce that

$$\begin{aligned}(dF)^{i_1 \cdots i_{k+1}} &= \int_0^1 \left((k+1)\, f^{i_1 \cdots i_{k+1}} + t \sum_{i=1}^n x_i\, f^{i_1 \cdots i_{k+1}}_{x_i} \right) t^k dt \\ &= \int_0^1 \frac{d}{dt} \left(t^{k+1} f^{i_1 \cdots i_{k+1}}(tx) \right) dt = f^{i_1 \cdots i_{k+1}}(x).\end{aligned}$$

It remains to show (1). Since $df = 0$ we find that

$$\sum_{\epsilon=1}^{k+2} (-1)^{\epsilon-1}\, f^{i_1 \cdots i_{\epsilon-1} i_{\epsilon+1} \cdots i_{k+2}}_{x_{i_\epsilon}} = 0$$

and consequently

$$(-1)^k\, f^{i_1 \cdots i_{k+1}}_{x_{i_{k+2}}} = \sum_{\epsilon=1}^{k+1} (-1)^{\epsilon-1}\, f^{i_1 \cdots i_{\epsilon-1} i_{\epsilon+1} \cdots i_{k+1} i_{k+2}}_{x_{i_\epsilon}}$$

which implies (1). ■

References

[1] Csato G., Dacorogna B. and Kneuss O., *The pullback equation for differential forms*, Birkhaüser, 2012.