

Some remarks on the Lie derivative and the pullback equation for contact forms

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Abstract

This article is dedicated to Ireneo Peral for his seventieth birthday and will appear in a special issue of *Advanced Nonlinear Studies*

Given f and g contact forms and h a 1-form, we discuss the existence of a vector field u verifying

$$\mathcal{L}_u(f) = d(u \lrcorner f) + u \lrcorner df = h.$$

This is closely related to the *pullback equation* where we seek for a diffeomorphism φ satisfying

$$\varphi^*(f) = g.$$

1 Introduction

Let $\Omega \subset \mathbb{R}^{2m+1}$, $m \geq 1$, be an open set. Let $f \in C^1(\Omega; \Lambda^1)$ and $h \in C^0(\Omega; \Lambda^1)$, we wish to find $u \in C^1(\Omega; \mathbb{R}^{2m+1})$ verifying

$$\mathcal{L}_u(f) = h,$$

where \mathcal{L}_u is the *Lie derivative*. By Cartan formula this reads as

$$\mathcal{L}_u(f) = d(u \lrcorner f) + u \lrcorner df = h.$$

This question is intimately related to the *pullback equation* where, given $f, g \in C^1(\Omega; \Lambda^1)$, we want to find a diffeomorphism φ satisfying

$$\varphi^*(f) = g.$$

We restrict our attention to forms f which are *contact forms* over the open set $\Omega \subset \mathbb{R}^{2m+1}$ meaning that

$$f \wedge (df)^m \neq 0 \quad \text{in } \Omega.$$

We will give some local (Theorem 2 for the Lie derivative) as well as global (Theorem 6 and Corollary 8 for the Lie derivative and Theorem 9 for the pullback equation) results.

In the first appendix (Theorem 11) we reprove the famous result of Gray [4] (see also [5] or [6]) which asserts that given two contact forms f and g there exist a scalar function r and a diffeomorphism φ satisfying

$$\varphi^*(rf) = g.$$

Our Theorem 6 gives some cases where one can choose $r \equiv 1$.

In the second appendix we recall some well known facts about first order partial differential equations that we use in the core of the article.

Throughout the article we use the notations of [2]. We should also point out that, by abuse of notations, we do not distinguish between a vector field φ and its associated 1-form φ^\flat that we still denote φ .

2 The Lie derivative for contact forms

2.1 An algebraic lemma

We start with an algebraic lemma that will be used on several occasions.

Lemma 1 *Let $n = 2m + 1$ and $F \in \Lambda^2(\mathbb{R}^n)$. Then*

$$*(F^m) \lrcorner F = 0.$$

Moreover, if $f \in \Lambda^1(\mathbb{R}^n)$ is such that

$$f \wedge F^m \neq 0,$$

then the following statements hold.

(i) *If $u \in \Lambda^1(\mathbb{R}^n)$ satisfies*

$$u \wedge f \wedge F^{m-1} = 0 \quad \text{and} \quad u \wedge F^m = 0,$$

then $u = 0$.

(ii) *If $u \in \Lambda^1(\mathbb{R}^n)$ satisfies*

$$u \lrcorner f = \langle u; f \rangle = 0 \quad \text{and} \quad u \lrcorner F = 0,$$

then $u = 0$.

(iii) *Given $z \in \Lambda^1(\mathbb{R}^n)$, there exists a unique $u \in \Lambda^1(\mathbb{R}^n)$ verifying*

$$u \lrcorner f = \langle u; f \rangle = 0 \quad \text{and} \quad u \lrcorner F = z$$

if, and only if,

$$z \wedge F^m = 0.$$

Proof We first find, if $F^m \neq 0$, (cf. Proposition 2.24 (ii) in [2]) an invertible matrix A such that

$$A^*(F) = \omega_m = \sum_{i=1}^m e^{2i-1} \wedge e^{2i}.$$

Step 1. We start by proving that $*(F^m) \lrcorner F = 0$. If $F^m = 0$, nothing is to be proved, so we assume that $F^m \neq 0$. By Proposition 2.19 in [2], we have that, noticing that $*(\omega_m)^m = m! e^{2m+1}$,

$$A^*[* (F^m) \lrcorner F] = \det \left[(A)^{-1} \right] (*(\omega_m)^m) \lrcorner \omega_m = m! \det \left[(A)^{-1} \right] e^{2m+1} \lrcorner \omega_m = 0,$$

which establishes the claim.

Step 2. We now discuss the statement (i). We then write

$$A^*(f) = g = \sum_{i=1}^{2m+1} g^i e^i \quad \text{and} \quad A^*(u) = \alpha = \sum_{i=1}^{2m+1} \alpha^i e^i.$$

The condition $f \wedge F^m \neq 0$ implies that $g^{2m+1} \neq 0$. We prove (i) which is equivalent to showing that, if

$$\alpha \wedge g \wedge (\omega_m)^{m-1} = 0 \quad \text{and} \quad \alpha \wedge (\omega_m)^m = 0,$$

then $\alpha = 0$, provided $g \wedge (\omega_m)^m \neq 0$. The condition $\alpha \wedge (\omega_m)^m = 0$ implies that $\alpha^{2m+1} = 0$. Let us show that $\alpha^1 = \dots = \alpha^{2m} = 0$, which will prove that $\omega = 0$. Indeed from the equation $\alpha \wedge g \wedge (\omega_m)^{m-1} = 0$ we infer, in particular, that the only term in $e^1 \wedge (e^3 \wedge \dots \wedge e^{2m}) \wedge e^{2m+1}$ is

$$(\alpha^1 e^1) \wedge (g^{2m+1} e^{2m+1}) \wedge (e^3 \wedge \dots \wedge e^{2m}) = 0$$

and thus $\alpha^1 = 0$ and similarly for the other α^i .

Step 3. We establish (ii) and we set

$$A^*(f) = g = \sum_{i=1}^{2m+1} g^i e^i \quad \text{and} \quad (A^{-t})^*(u) = \beta = \sum_{i=1}^{2m+1} \beta^i e^i.$$

The statement is equivalent (cf. Proposition 2.19 in [2]) to showing that

$$\beta \lrcorner g = 0 \quad \text{and} \quad \beta \lrcorner \omega_m = 0$$

imply $\beta = 0$, provided $g \wedge (\omega_m)^m \neq 0$. The equation $\beta \lrcorner \omega_m = 0$ implies that $\beta^1 = \dots = \beta^{2m} = 0$ and hence, since $g \wedge (\omega_m)^m \neq 0$,

$$\beta \lrcorner g = 0 \quad \Rightarrow \quad \beta^{2m+1} = 0.$$

This concludes the proof of (ii).

Step 4. We finally prove (iii). Let $z \in \Lambda^1$. By Proposition 2.50 in [2], we know that there exists $\tilde{u} \in \Lambda^1$ such that

$$\tilde{u} \lrcorner F = z \quad \Leftrightarrow \quad z \wedge F^m = 0.$$

Since by hypothesis $f \wedge F^m \neq 0$ or, equivalently, $\langle f; *(F^m) \rangle \neq 0$, we obtain (iii) of the lemma by simply letting

$$u = \tilde{u} - \frac{\langle f; \tilde{u} \rangle}{\langle f; *(F^m) \rangle} (*(F^m)).$$

This concludes the existence part. The uniqueness is established in (ii). ■

2.2 The local case

In the sequel we let

$$\alpha_m = \sum_{i=1}^m x_{2i-1} dx^{2i} + dx^{2m+1} \quad \text{and} \quad \omega_m = d\alpha_m = \sum_{i=1}^m dx^{2i-1} \wedge dx^{2i}.$$

The main theorem in the local case is the following.

Theorem 2 Let $n = 2m + 1$, $r \geq 1$ an integer and $\bar{x} \in \Omega \subset \mathbb{R}^n$ an open set. Let $h \in C^{r+1}(\Omega; \Lambda^1)$ and $f \in C^{r+2}(\Omega; \Lambda^1)$ verify

$$f \wedge (df)^m \neq 0 \quad \text{in } \Omega.$$

Then there exist an open ball $U \subset \Omega$, with $\bar{x} \in U$, and $u \in C^r(U; \Lambda^1)$ solving

$$\mathcal{L}_u(f) = d(u \lrcorner f) + u \lrcorner df = h \quad \text{in } U.$$

In the proof of the theorem we will need the following lemma, which, under further assumptions, gives a stronger result.

Lemma 3 Let $n = 2m + 1$, $r \geq 1$ an integer and $\Omega \subset \mathbb{R}^n$ be an open set. Let $f \in C^{r+2}(\Omega; \Lambda^1)$ verify

$$f \wedge (df)^m \neq 0 \quad \text{in } \Omega.$$

Let $h \in C^{r+1}(\Omega; \Lambda^1)$ and $H \in C^{r+1}(\Omega)$ satisfy

$$dH \wedge (df)^m = h \wedge (df)^m \quad \text{in } \Omega. \quad (1)$$

Then there exists a unique $u \in C^r(\Omega; \Lambda^1)$ verifying in Ω

$$u \lrcorner f = H \quad \text{and} \quad u \lrcorner df = h - dH \quad (2)$$

and it is given by the formula

$$u \lrcorner (f \wedge (df)^m) = H (df)^m + m \left((h - dH) \wedge f \wedge (df)^{m-1} \right). \quad (3)$$

Remark 4 (i) The result of the lemma is sharper than the one in Theorem 2 in the sense that (2) implies

$$d(u \lrcorner f) + u \lrcorner df = h.$$

(ii) One should not infer from the lemma that there is uniqueness in Theorem 2. For example if

$$v = c \frac{*((df)^m)}{*(f \wedge (df)^m)} \in \Lambda^1$$

where $c \in \mathbb{R}$ is arbitrary, we have

$$\mathcal{L}_{u+v}(f) = \mathcal{L}_u(f) = h.$$

Indeed observe that v satisfies

$$v \lrcorner f = f \lrcorner v = *(f \wedge (*v)) = * \left(c \frac{f \wedge (df)^m}{*(f \wedge (df)^m)} \right) = c.$$

Moreover since $*((df)^m) \lrcorner df = 0$ (cf. Lemma 1), we get $v \lrcorner df = 0$ and hence, if u is as in (3), we obtain

$$(u + v) \lrcorner f = H + c \quad \text{and} \quad (u + v) \lrcorner df = h - dH = h - d(H + c)$$

thus

$$d((u + v) \lrcorner f) + (u + v) \lrcorner df = h.$$

Proof (Lemma 3) *Step 1 (existence)*. Let us show that u given by (3) satisfies (2).

(i) We first observe that

$$\begin{aligned} u \lrcorner (f \wedge (df)^m) &= (u \lrcorner f) \wedge (df)^m + (u \lrcorner (df)^m) \wedge f \\ &= (u \lrcorner f) \wedge (df)^m + m \left[(u \lrcorner df) \wedge (df)^{m-1} \right] \wedge f \\ &= (u \lrcorner f) \wedge (df)^m + m \left[(u \lrcorner df) \wedge f \wedge (df)^{m-1} \right], \end{aligned}$$

where we have used Proposition 2.16 in [2]. Combining this computation and the definition of u given in (3), we obtain

$$[(u \lrcorner f) - H] \wedge (df)^m + m \left[(u \lrcorner df) - (h - dH) \right] \wedge f \wedge (df)^{m-1} = 0. \quad (4)$$

(ii) Multiplying (4) by f , we infer that

$$[(u \lrcorner f) - H] \wedge f \wedge (df)^m = 0$$

and, since $f \wedge (df)^m \neq 0$, we deduce that the first equation in (2) is satisfied, namely

$$u \lrcorner f = H. \quad (5)$$

(iii) Combining (4) and (5), we find

$$[(u \lrcorner df) - (h - dH)] \wedge f \wedge (df)^{m-1} = 0. \quad (6)$$

From (1) and Proposition 2.16 in [2], we obtain

$$(h - dH) \wedge (df)^m = 0 \quad \text{and} \quad (u \lrcorner df) \wedge (df)^m = \frac{u \lrcorner (df)^{m+1}}{m+1} = 0$$

and thus

$$[(u \lrcorner df) - (h - dH)] \wedge (df)^m = 0. \quad (7)$$

Combining (6), (7) and Lemma 1 (i) it follows that the second equation, $u \lrcorner df = h - dH$, in (2) is verified. The proof of the existence part is therefore complete.

Step 2 (uniqueness). The uniqueness follows from Lemma 1 (ii). This completes the proof of the lemma. ■

We finally turn to the proof of the theorem.

Proof (Theorem 2) We first find, by the method of characteristics (Theorem 12), a local solution $H \in C^{r+1}(U)$ satisfying (1) i.e.

$$dH \wedge (df)^m = h \wedge (df)^m \quad \text{in } U.$$

We then define $u \in C^r(U; \Lambda^1)$ as in Lemma 3, namely

$$u \lrcorner (f \wedge (df)^m) = H (df)^m + m \left((h - dH) \wedge f \wedge (df)^{m-1} \right)$$

which solves

$$u \lrcorner f = H \quad \text{and} \quad u \lrcorner df = h - dH.$$

This immediately implies that

$$\mathcal{L}_u(f) = d(u \lrcorner f) + u \lrcorner df = h$$

and concludes the proof of the theorem. ■

2.3 The global case

We now give a global version of the above theorem. For this we need to introduce the following notation

Notation 5 Let, for $\varphi \in C^1(\mathbb{R}^n)$,

$$L_f(\varphi) = * [d\varphi \wedge (df)^m].$$

We are now in a position to state the theorem.

Theorem 6 Let $n = 2m + 1$, $r \geq 1$ an integer, $h \in C^{r+1}(\mathbb{R}^n; \Lambda^1)$ and let $f \in C^{r+2}(\mathbb{R}^n; \Lambda^1)$ be a contact form. Let $\varphi \in C^\infty(\mathbb{R}^n)$ be such that

$$L_f^2(\varphi) > 0 \quad \text{in } \mathbb{R}^n$$

and the set $\{x \in \mathbb{R}^n : \varphi(x) \leq c\}$ is compact for every $c \in \mathbb{R}$. Then there exists $u \in C^r(\mathbb{R}^n; \Lambda^1)$ solving

$$\mathcal{L}_u(f) = d(u \lrcorner f) + u \lrcorner df = h \quad \text{in } \mathbb{R}^n.$$

Remark 7 As seen in Theorem 16, if instead of \mathbb{R}^n we deal with $\overline{\Omega}$ where Ω is a bounded open set, we can equivalently write the hypothesis $L_f^2(\varphi) > 0$, as: there exists $\psi \in C^\infty(\overline{\Omega})$ such that $L_f(\psi) \neq 0$ or, in other words,

$$d\psi \wedge (df)^m \neq 0 \quad \text{in } \overline{\Omega}.$$

Proof The theorem follows from Lemma 3 once we have found $H \in C^{r+1}(\mathbb{R}^n)$ satisfying

$$dH \wedge (df)^m = h \wedge (df)^m \quad \text{in } \mathbb{R}^n.$$

But this is possible in view of Theorem 19 and the hypothesis $L_f^2(\varphi) > 0$ and the compactness of the level sets of φ . ■

We finally point out an interesting corollary where we can get global existence.

Corollary 8 Let $n = 2m + 1$, $r \geq 1$ an integer, $h \in C^{r+1}(\mathbb{R}^n; \Lambda^1)$ and

$$\alpha_m = \sum_{i=1}^m x_{2i-1} dx^{2i} + dx^{2m+1}.$$

Then there exists $\epsilon > 0$ such that if $f \in C^{r+2}(\mathbb{R}^n; \Lambda^1)$ with

$$\|f - \alpha_m\|_{C^1} \leq \epsilon,$$

then there exists $u \in C^r(\mathbb{R}^n; \Lambda^1)$ solving

$$\mathcal{L}_u(f) = d(u \lrcorner f) + u \lrcorner df = h \quad \text{in } \mathbb{R}^n.$$

If, moreover, $\text{supp}(f - \alpha_m)$ and $\text{supp} h$ are compact, then u can be taken globally Lipschitz.

Proof Step 1. Observe first that

$$\mathcal{L}_u(\alpha_m) = d(u \lrcorner \alpha_m) + u \lrcorner d\alpha_m = h \quad \text{in } \mathbb{R}^n.$$

is solvable by the same method as in Theorem 6, since

$$dH \wedge (d\alpha_m)^m = h \wedge (d\alpha_m)^m \quad \Leftrightarrow \quad H_{x_{2m+1}} = h^{2m+1}$$

is clearly globally solvable. Since $\|f - \alpha_m\|_{C^1} \leq \epsilon$, we have that the coefficient in front of $H_{x_{2m+1}}$ in the equation $dH \wedge (df)^m = h \wedge (df)^m$ is strictly positive. We can therefore invoke Proposition 20 to find $H \in C^{r+1}(\mathbb{R}^n)$ such that

$$dH \wedge (df)^m = h \wedge (df)^m \quad \text{in } \mathbb{R}^n.$$

This settles the existence part.

Step 2. We now prove the extra statement. Since, by hypothesis, the coefficients in front of H_{x_j} , $1 \leq j \leq 2m$, in the equation $dH \wedge (df)^m = h \wedge (df)^m$ have compact support and the support of h is also compact, we deduce, in view of Proposition 20, that H can be chosen so $H, \nabla H, \nabla^2 H \in L^\infty(\mathbb{R}^n)$ and

$$H(x) \equiv 0 \quad \text{for every } |(x_1, \dots, x_{2m})| \text{ large enough.}$$

Therefore, since u given by (3) is a solution of $\mathcal{L}_u(f) = h$, h has compact support, $f \wedge (df)^m$ and $(df)^m$ are constant outside of a compact set and since

$$f \wedge (df)^{m-1}(x) = O\left(\sum_{i=1}^{2m} |x_i|\right) \quad \text{outside of a compact set}$$

we conclude that u is globally Lipschitz. ■

3 The pullback equation for contact forms

The above considerations on the Lie derivative lead immediately the following results for the pullback equation. We discuss only the global case.

Theorem 9 *Let $n = 2m + 1$, $r \geq 1$ an integer and*

$$\alpha_m = \sum_{i=1}^m x_{2i-1} dx^{2i} + dx^{2m+1}.$$

There exists $\epsilon > 0$ such that, if $f \in C^{r+2}(\mathbb{R}^n; \Lambda^1)$ is such that

$$\text{supp}(f - \alpha_m) \text{ is compact and } \|f - \alpha_m\|_{C^1} \leq \epsilon,$$

then there exists $\varphi \in \text{Diff}^r(\mathbb{R}^n, \mathbb{R}^n)$ satisfying

$$\varphi^*(f) = \alpha_m \quad \text{in } \mathbb{R}^n.$$

Proof We set, for $t \in [0, 1]$, $f_t = \alpha_m + t(f - \alpha_m)$ and observe that, for ϵ small enough,

$$\text{supp}(f_t - \alpha_m) \text{ is compact and } f_t \wedge (df_t)^m \neq 0 \text{ in } \mathbb{R}^n.$$

We then apply Corollary 8 to find $u_t \in C^r(\mathbb{R}^n; \Lambda^1)$ and globally Lipschitz solving

$$\mathcal{L}_{u_t}(f_t) = d(u_t \lrcorner f_t) + u_t \lrcorner df_t = -\partial_t(f_t) = \alpha_m - f_t \quad \text{in } \mathbb{R}^n.$$

The flow method of Moser (see Theorem 12.5 in [2]) shows that the solution (which globally exists since u_t is globally Lipschitz) of

$$\begin{cases} \partial_t(\varphi_t) = u_t(\varphi_t) & t \in [0, 1] \\ \varphi_0 = \text{id} \end{cases}$$

satisfies

$$\varphi_t^*(f_t) = \alpha_m \quad \text{in } \mathbb{R}^n.$$

Taking $t = 1$ we have our claim. ■

We have as an immediate corollary a result for closed 2-forms.

Corollary 10 *Let $n = 2m + 1$, $r \geq 1$ an integer and*

$$\omega_m = \sum_{i=1}^m dx^{2i-1} \wedge dx^{2i}.$$

There exists $\epsilon > 0$ such that, if $F \in C^{r+2}(\mathbb{R}^n; \Lambda^2)$ is closed and such that

$$\text{supp}(F - \omega_m) \text{ is compact and } \|F - \omega_m\|_{C^1} \leq \epsilon,$$

then there exists $\varphi \in \text{Diff}^r(\mathbb{R}^n, \mathbb{R}^n)$ satisfying

$$\varphi^*(F) = \omega_m \quad \text{in } \mathbb{R}^n.$$

Proof It is enough to find $f \in C^{r+2}(\mathbb{R}^n; \Lambda^1)$ such that $df = F$ verifying

$$\text{supp}(f - \alpha_m) \text{ is compact and } \|f - \alpha_m\|_{C^1} \leq \epsilon$$

and then apply the theorem. The existence of such an f is standard (cf., for example, Theorem 8.1 in [2]), what is, however, less standard is the assertion on the support (see [7]). ■

4 Appendix I: contact structures

The following theorem is standard and was first proved by Gray [4]. Since we did not fully understand the proofs that we have seen in the literature, we give here a self contained one. We adopt the following notation. For functions or 1-forms g defined on $[0, 1] \times \mathbb{R}^n$ we will constantly use the notation

$$g(t, x) = g_t(x) \quad \text{and} \quad \partial_t g_t = \frac{\partial}{\partial t} [g(t, x)].$$

Theorem 11 *Let $n = 2m + 1$, $K \subset \mathbb{R}^n$ be compact and $f \in C^\infty([0, 1] \times \mathbb{R}^n; \Lambda^1)$ be such that, for every $t \in [0, 1]$,*

$$f_t \wedge (df_t)^m \neq 0 \text{ in } \mathbb{R}^n \quad \text{and} \quad \text{supp}(\partial_t f_t) \subset K.$$

Then there exist $\varphi_t \in \text{Diff}^\infty(\mathbb{R}^n; \mathbb{R}^n)$ and $r_t \in C^\infty(\mathbb{R}^n; (0, \infty))$ such that $\varphi_0 = \text{id}$, $r_0 = 1$ and, for every $t \in [0, 1]$,

$$\text{supp}(r_t - 1), \text{supp}(\varphi_t - \text{id}) \subset K \quad \text{and} \quad \varphi_t^*(r_t f_t) = f_t.$$

Proof *Step 1 (construction of r_t).* We claim that exists $r \in C^\infty([0, 1] \times \mathbb{R}^n; (0, \infty))$ such that $r_0 \equiv 1$, $\text{supp}(r_t - 1) \subset K$ and

$$\partial_t (r_t f_t) \wedge [d(r_t f_t)]^m = 0 \quad \text{in } \mathbb{R}^n \text{ for every } t \in [0, 1]. \quad (8)$$

First note that, since $\alpha \wedge \alpha = 0$ for every 1-form α ,

$$[d(r_t f_t)]^m = [dr_t \wedge f_t + r_t df_t]^m = r_t^m (df_t)^m + m r_t^{m-1} dr_t \wedge f_t \wedge (df_t)^{m-1}.$$

Thus (8) reads as

$$[(\partial_t r_t) f_t + r_t (\partial_t f_t)] \wedge [r_t^m (df_t)^m + m r_t^{m-1} dr_t \wedge f_t \wedge (df_t)^{m-1}] = 0$$

and hence

$$\partial_t (r_t^{m+1}) \frac{f_t \wedge (df_t)^m}{m+1} + d(r_t^{m+1}) \wedge \frac{m f_t \wedge \partial_t f_t \wedge (df_t)^{m-1}}{m+1} + r_t^{m+1} \partial_t f_t \wedge (df_t)^m = 0.$$

Recalling that $f_t \wedge (df_t)^m \neq 0$ in \mathbb{R}^n and letting $\mu_t = r_t^{m+1}$, (8) takes the form

$$\partial_t \mu_t + \langle \nabla_x \mu_t; g_t \rangle + \mu_t \alpha_t = 0, \quad (9)$$

for appropriate $g \in C^\infty([0, 1] \times \mathbb{R}^n; \mathbb{R}^n)$ and $\alpha \in C^\infty([0, 1] \times \mathbb{R}^n)$ with

$$\text{supp } g_t, \text{supp } \alpha_t \subset K \quad \text{for every } t \in [0, 1].$$

Hence to show the claim (8) it is enough to prove that there exists

$$\mu \in C^\infty([0, 1] \times \mathbb{R}^n; (0, \infty))$$

satisfying (9), $\text{supp}(\mu_t - 1) \subset K$ for every $t \in [0, 1]$ and $\mu_0 = 1$; indeed $r = \mu^{1/(m+1)}$ has all the desired properties. We show, by the method of characteristics, how to construct such a (unique) μ . First solve

$$\begin{cases} \partial_\tau X(\tau, s) = (1, g) \circ X(\tau, s) & X(0, s) = (0, s) \\ \partial_\tau Z(\tau, s) = -\alpha(X(\tau, s)) Z(\tau, s) & Z(0, s) = 1. \end{cases}$$

Note that $X(\tau, s) = (\tau, \tilde{X}(\tau, s))$ where $\tilde{X}(\tau, s)$ satisfies

$$\partial_\tau \tilde{X}(\tau, s) = g(\tau, \tilde{X}(\tau, s)) \quad \text{and} \quad \tilde{X}(0, s) = s.$$

Hence, for every fixed $\tau \in [0, 1]$, $\tilde{X}(\tau, \cdot)$ is a C^∞ diffeomorphism from \mathbb{R}^n onto \mathbb{R}^n with

$$\text{supp}(\tilde{X}(\tau, \cdot) - \text{id}) \subset K \quad \text{for every } \tau \in [0, 1].$$

This trivially implies that X is a smooth diffeomorphism from $[0, 1] \times \mathbb{R}^n$ onto $[0, 1] \times \mathbb{R}^n$. Obviously

$$Z \in C^\infty([0, 1] \times \mathbb{R}^n; (0, \infty)).$$

Finally, by the method of characteristics (see Theorem 12), we know that

$$\mu = Z \circ X^{-1}$$

solves (9) and that $\mu_0 \equiv 1$. Note also that $\text{supp}(\mu_t - 1) \subset K$ for every $t \in [0, 1]$.

Step 2. By (8) and Lemma 1 (iii) there exists a unique $u \in C^\infty([0, 1] \times \mathbb{R}^n; \Lambda^1)$ verifying, for every $t \in [0, 1]$,

$$\langle u_t; r_t f_t \rangle = 0 \quad \text{and} \quad u_t \lrcorner d(r_t f_t) = -\partial_t(r_t f_t) \quad \text{in } \mathbb{R}^n.$$

Since $r_t f_t$ is independent of t outside K we additionally deduce that

$$\text{supp}(u_t) \subset K \quad \text{for every } t \in [0, 1].$$

Let φ be the solution of

$$\partial_t \varphi_t = u_t(\varphi_t) \quad \text{and} \quad \varphi_0 = \text{id}.$$

Classical results show that $\varphi \in C^\infty([0, 1] \times \mathbb{R}^n; \mathbb{R}^n)$ and, for every $t \in [0, 1]$, $\varphi_t \in \text{Diff}^\infty(\mathbb{R}^n; \mathbb{R}^n)$ with $\text{supp}(\varphi_t - \text{id}) \subset K$. We claim that, for every $t \in [0, 1]$,

$$\varphi_t^*(r_t f_t) = f_0 \quad \text{in } \mathbb{R}^n.$$

Since the previous equation is trivially fulfilled for $t = 0$, it is enough to check that

$$\partial_t(\varphi_t^*(r_t f_t)) = 0$$

to have the claim. By Theorem 12.5 in [2]

$$\partial_t (\varphi_t^* (r_t f_t)) = \varphi_t^* (\partial_t (r_t f_t) + d(u_t \lrcorner r_t f_t) + u_t \lrcorner d(r_t f_t))$$

and thus we have to check that

$$\partial_t (r_t f_t) + d(u_t \lrcorner r_t f_t) + u_t \lrcorner d(r_t f_t) = 0,$$

which is satisfied since, by construction,

$$u_t \lrcorner r_t f_t = \langle u_t; r_t f_t \rangle = 0 \quad \text{and} \quad \partial_t (r_t f_t) + u_t \lrcorner d(r_t f_t) = 0.$$

This concludes the proof. ■

5 Appendix II: first order pdes

Let $r, n \geq 1$ be two integers and let $a \in C^r(\mathbb{R}^n; \mathbb{R}^n)$. In this appendix we study the first order linear operator

$$Lu = \langle a; \nabla u \rangle.$$

5.1 Local existence

We first recall the elementary classical method of characteristics, whose proof is standard.

Theorem 12 (Method of characteristics) *Let $a \in C^r(\mathbb{R}^n; \mathbb{R}^n)$ and $U \subset \mathbb{R}^n$ be an open set such that for every $(t, s) \in U$ (where $t \in \mathbb{R}$ and $s \in \mathbb{R}^{n-1}$)*

$$\{t \in \mathbb{R} : (t, s) \in U\} \quad \text{is an open interval containing } 0.$$

Suppose that there exists $X \in \text{Diff}^r(U; X(U))$ a solution of

$$\partial_t X(t, s) = a(X(t, s)) \quad \text{for every } (t, s) \in U.$$

Let $f \in C^r(X(U))$ and $Z \in C^r(U)$ be such that

$$\partial_t Z = f(X) \quad \text{in } U.$$

(i) Existence. *The function $u = Z \circ X^{-1} \in C^r(X(U))$ is a solution of*

$$Lu = f \quad \text{in } X(U).$$

(ii) Uniqueness of the Cauchy problem. *Let*

$$\widehat{U} = \{s \in \mathbb{R}^{n-1} : (0, s) \in U\},$$

$g \in C^r(\widehat{U}; \mathbb{R}^{n-1})$ and $h \in C^r(\widehat{U})$. If, moreover,

$$X(0, \cdot) = g(\cdot) \quad \text{and} \quad Z(0, \cdot) = h(\cdot) \quad \text{in } \widehat{U}$$

then $u = Z \circ X^{-1}$ is the unique solution of

$$Lu = f \text{ in } X(U) \quad \text{and} \quad u \circ g = h \text{ on } \widehat{U}. \tag{10}$$

Remark 13 (i) As soon as the characteristics X stop to be invertible no uniqueness has to be expected in general, as shows the following example. Consider the problem

$$\begin{cases} u_{x_1} + x_2 u_{x_2} = 0 \\ u(x_1, -1) = 0. \end{cases}$$

Then, for every $w \in C_0^\infty(\mathbb{R})$,

$$u(x_1, x_2) = \begin{cases} 0 & \text{if } x_2 \leq 0 \\ w(x_1 - \log x_2) & \text{if } x_2 > 0 \end{cases}$$

is a solution. Note that

$$\partial_t X = (1, X_2) \quad \text{and} \quad X(0, s) = (s, -1)$$

are

$$X(t, s) = (t + s, -e^t)$$

and thus X is invertible only when $x_2 < 0$ (with $X^{-1}(x_1, x_2) = (x_1 - \log(-x_2), \log(-x_2))$); and so, in particular, any solution is necessarily 0 for $x_2 < 0$.

(ii) No solution with compact support to $Lu = f$ is to be expected in general when f has compact support. Indeed let $f(x_1, x_2) = \chi(x_1)\chi(x_2)$ where $\chi \in C_0^\infty(-1, 1)$, $\chi \geq 0$ and $\chi(0) = \int_{-1}^1 \chi(t) dt = 1$. Consider the equation

$$u_{x_2} = \chi(x_1)\chi(x_2) \quad \Rightarrow \quad u(x_1, x_2) = \alpha(x_1) + \chi(x_1) \int_{-1}^{x_2} \chi(t) dt.$$

Clearly u cannot have compact support, since

$$u(0, x_2) = \begin{cases} \alpha(0) + 1 & \text{if } x_2 \geq 1 \\ \alpha(0) & \text{if } x_2 \leq -1. \end{cases}$$

Theorem 14 (Local existence) Let $a \in C^r(\mathbb{R}^n; \mathbb{R}^n)$, $f \in C^r(\mathbb{R}^n)$, $g \in C^r(\mathbb{R}^{n-1}; \mathbb{R}^n)$, $h \in C^r(\mathbb{R}^{n-1})$ and $\bar{s} \in \mathbb{R}^{n-1}$ be such that

$$g_{s_1}(\bar{s}) \wedge \cdots \wedge g_{s_{n-1}}(\bar{s}) \wedge a(g(\bar{s})) \neq 0. \quad (11)$$

Then, there exists a neighborhood $V \subset \mathbb{R}^n$ of $g(\bar{s})$ and a unique $u \in C^r(V)$ such that,

$$\begin{cases} Lu = f & \text{in } V \\ u(g(s)) = h(s) & \text{for } |s - \bar{s}| \text{ small.} \end{cases} \quad (12)$$

Proof The theorem follows immediately from Theorem 12 combined, appealing to (11), with the inverse function theorem. ■

5.2 Global existence

In this subsection we deal with global existence of solutions to $Lu = f$ for arbitrary f and follow the results of Duistermaat-Hörmander [3]. We start with some notations.

Notation 15 For every $x \in \mathbb{R}^n$ let γ_x be the integral curve of a , which passes through x at time 0 defined on its maximal interval $(t_-(x), t_+(x))$, i.e.

$$\begin{cases} \gamma'_x(t) = a(\gamma_x(t)), & t \in (t_-(x), t_+(x)) \\ \gamma_x(0) = x. \end{cases}$$

The orbit of x is the set $\gamma_x(t_-(x), t_+(x))$. Recall that if a is sublinear, i.e. there exist $c_1, c_2 > 0$ with

$$|a(x)| \leq c_1|x| + c_2 \quad \text{for every } x \in \mathbb{R}^n,$$

then $t_-(x) = -\infty$ and $t_+(x) = \infty$ for every $x \in \mathbb{R}^n$.

We first give some equivalent conditions for the solvability of L in bounded sets established in [3].

Theorem 16 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and $a \in C^r(\mathbb{R}^n; \mathbb{R}^n)$. The following conditions are then equivalent.*

(i) $\forall f \in C^r(\overline{\Omega})$, there exists $u \in C^r(\overline{\Omega})$ satisfying

$$Lu = f \quad \text{in } \Omega.$$

(ii) $\forall f, g \in C^r(\overline{\Omega})$, there exists $u \in C^r(\overline{\Omega})$ satisfying

$$Lu + gu = f \quad \text{in } \Omega.$$

(iii) There exists $u \in C^\infty(\overline{\Omega})$ satisfying

$$Lu \neq 0 \quad \text{in } \overline{\Omega}.$$

(iv) There exists $v \in C^\infty(\overline{\Omega})$ such that

$$L^2v \neq 0 \quad \text{in } \overline{\Omega}.$$

(v) No orbit of a is contained in $\overline{\Omega}$, i.e. for every $x \in \overline{\Omega}$

$$\gamma_x(t_-(x), t_+(x)) \cap (\overline{\Omega})^c \neq \emptyset.$$

Remark 17 Note that

$$L^2\varphi = \langle a; \nabla^2\varphi \cdot a \rangle + \langle a; \nabla a \cdot \nabla\varphi \rangle$$

and hence L^2 is degenerate elliptic since

$$\sum_{i,j} a^j a^i \xi^i \xi^j = (\langle a; \xi \rangle)^2 \geq 0.$$

Proof The equivalence between (i), (ii), (iv) and (v) when $r = \infty$ is established in Theorem 6.4.1 in [3]. The same proof gives in fact equivalence between (i), (ii), (iv) and (v) when $r \geq 1$.

We finally show the equivalence between (iii) and (iv). First note that (iv) \Rightarrow (iii) is elementary, simply take u to be any C^∞ function such that $\|u - Lv\|_{C^1(\overline{\Omega})}$ is small enough where v is given by (iv). Let us show now that (iii) \Rightarrow (iv). Changing u to $-u$ we can assume, with no loss of generality, that there exists $u \in C^\infty(\overline{\Omega})$ such that $Lu > 0$ in $\overline{\Omega}$. Since $\overline{\Omega}$ is compact, there exists $c > 0$ such that

$$u + c > 0 \quad \text{in } \overline{\Omega}.$$

Using again the compactness of $\overline{\Omega}$ and the fact that $Lu > 0$, we deduce the existence of $m \in \mathbb{N}$ big enough so that

$$(u + c)^{m-2} \left[(m-1)(Lu)^2 + (u+c)L^2u \right] > 0 \quad \text{in } \overline{\Omega}.$$

We then define

$$v(x) = \frac{(u(x) + c)^m}{m}.$$

We deduce

$$\begin{aligned} L^2 v &= L^2 \left[\frac{(u+c)^m}{m} \right] = L \left[(u+c)^{m-1} Lu \right] \\ &= (m-1)(u+c)^{m-2} (Lu)^2 + (u+c)^{m-1} L^2 u > 0. \end{aligned}$$

This concludes the proof of the theorem. ■

Note that the condition $a \neq 0$ everywhere in $\bar{\Omega}$ is clearly necessary for the equivalent conditions of the previous theorem to be fulfilled (indeed if $a(x_0) = 0$ then it is obviously not possible to solve $Lu = f$ for f such that $f(x_0) \neq 0$; note also that in that case $\gamma_{x_0} \equiv x_0 \in \bar{\Omega}$). However this condition is sufficient in dimension 2 if $\bar{\Omega}$ is simply connected (and not, in general, otherwise) as shown in the next result.

Proposition 18 *Let $n = 2$, $\Omega \subset \mathbb{R}^n$ be a bounded simply connected open set with Lipschitz boundary. Let $a \in C^r(\bar{\Omega}; \mathbb{R}^n)$ be such that $a(x) \neq 0$ for every $x \in \bar{\Omega}$. Then for every $f \in C^r(\bar{\Omega})$ there exists $u \in C^r(\bar{\Omega})$ solving*

$$Lu = f \quad \text{in } \Omega.$$

Furthermore, if $n = 2$ and Ω is not simply connected or if $n \geq 3$, then the above conclusion is false in general.

Proof Step 1. We prove the first part of the proposition. Let $a \in C^r(\bar{\Omega}; \mathbb{R}^n)$ be non vanishing. Suppose, for the sake of contradiction, that $Lu = f$ is not solvable in Ω for some $f \in C^r(\bar{\Omega})$. Then by Theorem 16 (v) there exists $x_0 \in \bar{\Omega}$ such that γ_{x_0} remains in $\bar{\Omega}$. By Poincaré-Bendixon Theorem (cf., for example, Theorem 2.1 and Corollary page 394 of Chapter 16 in [1]), since a never vanishes, we conclude that there exists $y_0 \in \bar{\Omega}$ whose orbit is periodic and contained in $\bar{\Omega}$. Since Ω is simply connected we deduce that the interior of the above periodic orbit (which is well defined by Jordan Theorem) is contained in Ω . Hence, applying Corollary 1 page 400 of Chapter 16 in [1], we deduce that a must vanish in $\bar{\Omega}$ which is our desired contradiction.

Step 2. We show the second part of the proposition when $n = 2$. Let $\Omega = B_1 \setminus \overline{B_{1/2}}$ and $a(x_1, x_2) = (-x_2, x_1)$. Then, trivially, for $x_0 = (3/4, 0) \in \Omega$,

$$\gamma_{x_0}(\theta) = 3/4(\cos \theta, \sin \theta) \in \Omega.$$

We claim that $Lu = 1$ cannot be solved in Ω . Indeed from

$$(u(\gamma_{x_0}(\theta)))' = \langle \nabla u(\gamma_{x_0}(\theta)); a(\gamma_{x_0}(\theta)) \rangle = 1$$

we obtain a contradiction since

$$\int_0^{2\pi} (u(\gamma_{x_0}(\theta)))' d\theta = u(\gamma_{x_0}(2\pi)) - u(\gamma_{x_0}(0)) = 0.$$

Note that we could have obtained the claim by using Theorem 16 (v) directly.

Step 3. We finally show the second part of the proposition when $n \geq 3$. Let $\Omega \subset \mathbb{R}^n$ be an open set. With no loss of generality we can assume that $0 \in \Omega$. Let $\epsilon > 0$ be small enough so that $\overline{B_\epsilon(0)} \subset \Omega$ and define $a \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ by

$$a(x) = (-x_2, x_1, \rho(x_1^2 + x_2^2), 0, \dots, 0)$$

where ρ is a smooth function such that

$$\rho \equiv 1 \quad \text{on } \left[0, (\epsilon/2)^2\right] \quad \text{and} \quad \rho \equiv 0 \quad \text{on } [\epsilon^2, \infty).$$

Then a does not vanish in \mathbb{R}^n and its integral curve starting at $x_0 = (\epsilon, 0, \dots, 0)$ is

$$\gamma_{x_0}(\theta) = (\epsilon \cos \theta, \epsilon \sin \theta, 0, \dots, 0)$$

which is periodic and contained in Ω . Exactly as in Step 2 we conclude that $Lu = 1$ has no solution in Ω . ■

We now give equivalent condition for the solvability of L in \mathbb{R}^n . We refer to Theorem 6.4.2 in [3] for a proof as well as for more equivalent conditions.

Theorem 19 *Let $a \in C^r(\mathbb{R}^n; \mathbb{R}^n)$. The following conditions are then equivalent.*

(i) $\forall f \in C^r(\mathbb{R}^n)$, there exists $u \in C^r(\mathbb{R}^n)$ satisfying

$$Lu = f \quad \text{in } \mathbb{R}^n.$$

(ii) $\forall f, g \in C^r(\mathbb{R}^n)$, there exists $u \in C^r(\mathbb{R}^n)$ satisfying

$$Lu + gu = f \quad \text{in } \mathbb{R}^n.$$

(iii) There exists $\varphi \in C^\infty(\mathbb{R}^n)$ be such that

$$L^2\varphi > 0 \quad \text{in } \mathbb{R}^n$$

and the set $\{x \in \mathbb{R}^n : \varphi(x) \leq c\}$ is compact for every $c \in \mathbb{R}$.

(iv) No orbit of a is bounded (i.e. $\gamma_x(t_-(x), t_+(x))$ is unbounded for every $x \in \mathbb{R}^n$) and for every compact set $K_1 \subset \mathbb{R}^n$ there exists a compact set $K_2 \subset \mathbb{R}^n$ with the property that

$$\gamma_x([a, b]) \subset K_2$$

for every $t_-(x) < a < b < t_+(x)$ with $\gamma_x(a), \gamma_x(b) \in K_1$.

We now give a useful sufficient condition on a ensuring global existence in \mathbb{R}^n .

Proposition 20 *Let $a \in C^r(\mathbb{R}^n; \mathbb{R}^n)$ be such that*

$$|a(x)| \leq c_1|x| + c_2 \quad \text{for some } c_1, c_2 > 0 \text{ and for every } x \in \mathbb{R}^n$$

with

$$\inf_{\mathbb{R}^n} |a_i| > 0, \quad \text{for some } 1 \leq i \leq n.$$

Then, for every $f \in C^r(\mathbb{R}^n)$, there exists $u \in C^r(\mathbb{R}^n)$ such that

$$Lu = f \quad \text{in } \mathbb{R}^n.$$

Furthermore if, additionally, $\text{supp } f$ is compact and

$$\text{supp } a_j \quad \text{is compact for every } j \neq i,$$

then u can be chosen so that, additionally,

$$u, \nabla u, \dots, \nabla^r u \in L^\infty(\mathbb{R}^n)$$

and

$$u(x) = 0 \quad \text{for every } |(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)| \text{ large enough.}$$

Proof Step 1 (simplification). With no loss of generality we can assume that $i = 1$. We may also assume that $a_1 = 1$; indeed, since

$$\langle a; \nabla u \rangle = f \quad \Leftrightarrow \quad \left\langle \frac{a}{a_1}; \nabla u \right\rangle = \frac{f}{a_1},$$

it suffices to replace f by f/a_1 and a by a/a_1 which is still sublinear and C^r .

Step 2. Let us establish the global existence of u . For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ we write $x = (x_1, \hat{x})$, with $\hat{x} \in \mathbb{R}^{n-1}$. Let $X = X(t, s)$ be the solution of

$$\frac{dX}{dt} = a(X) \quad \text{with } X(0, s) = (0, s).$$

By the methods of characteristics it suffices to prove that $X \in \text{Diff}^r(\mathbb{R}^n; \mathbb{R}^n)$; $u = Z \circ X^{-1}$ being then a solution where Z solves

$$\partial_t Z = f(X).$$

Since obviously $X_1(t, s) = t$, we infer that

$$\frac{d\hat{X}}{dt} = \hat{a}(t, \hat{X}(t, s)), \quad \text{with } \hat{X}(0, s) = s.$$

Since \hat{a} is sublinear and \hat{X} has a flow structure, we deduce that $\hat{X}(t, \cdot) \in \text{Diff}^r(\mathbb{R}^{n-1}; \mathbb{R}^{n-1})$ for every fixed t . We therefore have that

$$X(t, s) = (t, \hat{X}(t, s))$$

is a C^r diffeomorphism from \mathbb{R}^n onto \mathbb{R}^n .

Step 3. We finally prove the extra statement. With the simplification made in Step 1 we can assume that $a_1 \equiv 1$ and that $\text{supp } \hat{a}$ is compact. Let $u = Z \circ X^{-1}$ where

$$Z(t, s) = \int_0^t f(X(\tau, s)) d\tau$$

and $X(t, s) = (t, \hat{X}(t, s))$ is as before. Hence

$$u(x) = \int_0^{x_1} f\left(\tau, \hat{X}\left(\tau, \hat{X}^{-1}(x_1, \hat{x})\right)\right) d\tau \quad (13)$$

where $\hat{X}^{-1}(x_1, \hat{x})$ is the preimage of \hat{x} by $\hat{X}(x_1, \cdot)$. Since $\hat{a} = \hat{a}(t, s)$ has compact support, one directly deduces that there exists $M > 0$ so that

$$\hat{X}(t, s) = s, \quad \text{for every } |s| \geq M \text{ and for every } t \quad (14)$$

$$\hat{X}(t, s) = \hat{X}\left(\frac{t}{|t|}M, s\right), \quad \text{for every } |t| \geq M \text{ and for every } s. \quad (15)$$

Combining (13) and (14) we get

$$u(x) = \int_0^{x_1} f(\tau, \hat{x}) d\tau, \quad \text{for every } |\hat{x}| \geq M.$$

Hence, taking M larger if necessary, the compactness of the support of f yields that

$$u(x) \equiv 0, \quad \text{for every } |\hat{x}| \geq M, \quad (16)$$

which is exactly the last statement of the proposition. Next from (13) and (15) we get that

$$u(x) = \int_0^{x_1} f \left(\tau, \widehat{X} \left(\tau, \widehat{X}^{-1} \left(\frac{x_1}{|x_1|} M, \widehat{x} \right) \right) \right) d\tau, \quad \text{for every } |x_1| \geq M.$$

Therefore, since $\text{supp } f$ is compact, taking M again larger if necessary, we get that

$$u(x) = \int_0^{\frac{x_1}{|x_1|} M} f \left(\tau, \widehat{X} \left(\tau, \widehat{X}^{-1} \left(\frac{x_1}{|x_1|} M, \widehat{x} \right) \right) \right) d\tau, \quad \text{for every } |x_1| \geq M. \quad (17)$$

Since $f, \widehat{X}, \widehat{X}^{-1}$ and all their derivatives up to order r are bounded on every compact set, we directly deduce from (16) and (17) that

$$u, \nabla u, \dots, \nabla^r u \in L^\infty(\mathbb{R}^n)$$

as wished. ■

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