A Dirichlet problem involving the divergence operator

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Contents

Abstract

We consider the problem

$$\begin{cases} \operatorname{div} u + \langle a; u \rangle = f & \text{in } \Omega \\ u = u_0 & \text{on } \partial\Omega. \end{cases}$$

We show that if

$$\operatorname{curl} a\,(x_0) \neq 0 \quad \text{for some } x_0 \in \Omega,$$

then the problem is solvable without restriction on f. We also discuss the regularity of the solution.
The present article is a detailed version of the one that will appear in Annales Institut Henri Poincaré.

1 Introduction

In this paper we study the existence of solutions for the problem

$$\begin{cases} \operatorname{div} u + \langle a; u \rangle = f & \text{in } \Omega \\ u = u_0 & \text{on } \partial\Omega \end{cases}$$

where a is a vector field and $\langle .; . \rangle$ stands for the scalar product in $\mathbb{R}^n$. The problem with $a \equiv 0$ has attracted considerable attention, notably by Bogovski [2], Borchers-Sohr [3], Csató-Dacorogna-Kneuss [4], Dacorogna [5], [6], Dacorogna-Moser [7], Dautray-Lions [8], Galdi [9], Girault-Raviart [11], Kapitanskii-Pileckas [12], Ladyzhenskaya [13], Ladyzhenskaya-Solonnikov [14], Necas [15], Tartar [16], Von Wahl [17], [18]. The following theorem is standard (cf. Theorem 9.2 in [4]).

Theorem 1 *Let $n \geq 1$, $r \geq 0$ be integers and $0 < s < 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open $C^{r+2,s}$ set with outward unit normal ν. The following conditions are then equivalent.*

(i) $f \in C^{r,s}\left(\overline{\Omega}\right)$ and $u_0 \in C^{r+1,s}\left(\overline{\Omega}; \mathbb{R}^n\right)$ satisfy

$$\int_\Omega f = \int_\Omega \operatorname{div} u_0 = \int_{\partial\Omega} \langle u_0; \nu \rangle .$$

(ii) There exists $u \in C^{r+1,s}\left(\overline{\Omega}; \mathbb{R}^n\right)$ verifying

$$\begin{cases} \operatorname{div} u = f & \textit{in } \Omega \\ u = u_0 & \textit{on } \partial\Omega. \end{cases}$$

If the vector field a is a gradient of a potential A, then the same result holds (just replace u by $e^A u$, u_0 by $e^A u_0$ and f by $e^A f$) and the problem

$$\begin{cases} \operatorname{div} u + \langle a; u \rangle = f & \text{in } \Omega \\ u = u_0 & \text{on } \partial\Omega \end{cases}$$

is solvable (with optimal regularity for a, f, u_0 and u) if and only if

$$\int_\Omega e^A f = \int_\Omega \operatorname{div}\left(e^A u_0\right) = \int_{\partial\Omega} e^A \langle u_0; \nu \rangle .$$

Our result will show that if

$$\operatorname{curl} a\,(x_0) \neq 0 \quad \text{for some } x_0 \in \Omega,$$

then (cf. Theorem 2) the problem is solvable without any integral restriction on f and u_0. Under slightly strengthening this last condition namely

$$\operatorname{curl} a \neq 0 \quad \text{on } \partial\Omega$$

we will also provide (cf. Theorem 4) a solution with optimal regularity. We should also point out an interesting point related to the topology of the domain Ω. In a special case (cf. Theorem 6) we will see that the right condition is not $\operatorname{curl} a \not\equiv 0$ but that a is not a gradient.

In Section 4 we will also study the kernel of the operator (this kernel will be used in a significant way in Theorem 4)

$$L_a(u) = \operatorname{div} u + \langle a; u \rangle .$$

When $a \equiv 0$ (or more generally when $a = \operatorname{grad} A$), the kernel is classically given (cf. Section 4) by $\operatorname{curl}^* w$ so that

$$\operatorname{div} \operatorname{curl}^* w = 0 \quad \text{for every } w \in C^2\left(\mathbb{R}^n; \mathbb{R}^{n(n-1)/2}\right).$$

When $\operatorname{curl} a \neq 0$, it is easily seen that the kernel operator cannot be a first order operator. We provide (cf. Proposition 11) the most general second order operator (which can be reduced to the operator curl^* when $a \equiv 0$) denoted by $N_{a,2}^{\alpha,\beta,\gamma}$, so that

$$L_a\left(N_{a,2}^{\alpha,\beta,\gamma}(w)\right) = 0 \quad \text{for every } w \in C^2\left(\mathbb{R}^n; \mathbb{R}^{n(n-1)/2}\right).$$

Finally in Sections 5 and 7 we discuss a Poincaré type lemma, both on the boundary (cf. Theorem 17) and in the interior (cf. Theorems 21 and 30), of the operator L_a. For example (cf. Theorem 30), we will find, if $L_a(u) = 0$, that there exists w such that

$$u = N_{a,2}^{\alpha,\beta,\gamma}(w) .$$

This is the exact analogue of the classical theorem which says that, if $\operatorname{div} u = 0$, there exists w such that $u = \operatorname{curl}^* w$.

2 The main theorems

2.1 The main theorems

In this paper we will adopt the notation

$$L_a(u) = \operatorname{div} u + \langle a; u \rangle .$$

Our first theorem is constructive and uses only elementary tools, notably the method of characteristics. It is sharp from the point of view of the condition on a (namely $\operatorname{curl} a \not\equiv 0$). However it is not sharp from the point of view of regularity.

Theorem 2 *Let $n \geq 2$, $r \geq 0$ be integers and $\Omega \subset \mathbb{R}^n$ a bounded open set, with $\overline{\Omega}$ C^{r+4} diffeomorphic to the unit closed ball. Let $f \in C^{r+3}\left(\overline{\Omega}\right)$, $u_0 \in C^{r+4}\left(\overline{\Omega}; \mathbb{R}^n\right)$ and $a \in C^{r+3}\left(\overline{\Omega}; \mathbb{R}^n\right)$ be such that*

$$\operatorname{curl} a(x_0) \neq 0 \quad \textit{for some } x_0 \in \Omega.$$

Then there exists $u \in C^{r+1}\left(\overline{\Omega}; \mathbb{R}^n\right)$ satisfying

$$\begin{cases} \operatorname{div} u + \langle a; u \rangle = f & \textit{in } \Omega \\ u = u_0 & \textit{on } \partial\Omega. \end{cases}$$

Remark 3 *The regularity can slightly be improved (see Theorem 10). We can assume that (letting $0 < s < 1$) $a \in C^{r+3,s}\left(\overline{\Omega}; \mathbb{R}^n\right)$, $f \in C^{r+2,s}\left(\overline{\Omega}\right)$, $u_0 \in C^{r+3,s}\left(\overline{\Omega}; \mathbb{R}^n\right)$ to obtain $u \in C^{r+1,s}\left(\overline{\Omega}; \mathbb{R}^n\right)$.*

Our second theorem is sharp from the point of view of regularity (except for the regularity of a), but the hypothesis $\operatorname{curl} a \not\equiv 0$ has to be strengthened.

Theorem 4 *Let $n \geq 2$, $r \geq 0$ be integers, $0 < s < 1$ and $\Omega \subset \mathbb{R}^n$ a bounded open smooth set. Let $f \in C^{r,s}\left(\overline{\Omega}\right)$, $u_0 \in C^{r+1,s}\left(\overline{\Omega}; \mathbb{R}^n\right)$ and $a \in C^{r+4,s}\left(\overline{\Omega}; \mathbb{R}^n\right)$ be such that*

$$\inf_{x \in \partial\Omega} \{|\operatorname{curl} a(x)|\} \geq \delta > 0.$$

Then there exists $u \in C^{r+1,s}\left(\overline{\Omega}; \mathbb{R}^n\right)$ satisfying

$$\begin{cases} \operatorname{div} u + \langle a; u \rangle = f & \textit{in } \Omega \\ u = u_0 & \textit{on } \partial\Omega. \end{cases}$$

Moreover the correspondence $(f, u_0) \to u$ can be chosen linear and there exists $K = K\left(r, s, \|a\|_{C^{r+4,s}}, \delta, \Omega\right)$ such that

$$\|u\|_{C^{r+1,s}} \leq K\left(\|f\|_{C^{r,s}} + \|u_0\|_{C^{r+1,s}}\right).$$

Remark 5 *(i) From the point of view of regularity the theorem is optimal for* f, u_0 *and* u, *but not for* a. *The optimal regularity should be* $a \in C^{r,s}$.

(ii) As already said the natural hypothesis is, in view of Theorem 2,

$$\operatorname{curl} a\,(x_0) \neq 0 \quad \textit{for some } x_0 \in \Omega$$

but, at the moment, in the present theorem we need a stronger hypothesis. It will be clear from the proof that we can replace the hypothesis $\operatorname{curl} a \neq 0$ *on* $\partial\Omega$, *by other hypotheses such as, for example, in addition to the natural condition,* $a = \operatorname{grad} A$ *near* $\partial\Omega$.

Finally when Ω is not simply connected we see, in a special case (including the case of harmonic fields), that the right condition is not $\operatorname{curl} a \not\equiv 0$ but that a is not a gradient.

Theorem 6 *Let* $n \geq 2$, $r \geq 0$ *be integers,* $0 < s < 1$ *and* $\Omega \subset \mathbb{R}^n$ *be a bounded open smooth set. Let* $f \in C^{r,s}\left(\overline{\Omega}\right)$, $u_0 \in C^{r+1,s}\left(\overline{\Omega}; \mathbb{R}^n\right)$ *and* $a \in C^{r,s}\left(\overline{\Omega}; \mathbb{R}^n\right)$ *be such that*

$$\operatorname{curl} a \equiv 0 \textit{ in } \overline{\Omega} \quad \textit{but} \quad \nexists\, A \in C^{r+1,s}\left(\overline{\Omega}\right) \textit{ with } a = \operatorname{grad} A.$$

Then there exists $u \in C^{r+1,s}\left(\overline{\Omega}; \mathbb{R}^n\right)$ *satisfying*

$$\begin{cases} \operatorname{div} u + \langle a; u \rangle = f & \textit{in } \Omega \\ u = u_0 & \textit{on } \partial\Omega. \end{cases}$$

Moreover the correspondence $(f, u_0) \to u$ *can be chosen linear and there exists* $K = K\left(r, s, \|a\|_{C^{r,s}}, \Omega\right)$ *such that*

$$\|u\|_{C^{r+1,s}} \leq K\left(\|f\|_{C^{r,s}} + \|u_0\|_{C^{r+1,s}}\right).$$

Remark 7 *(i) The hypothesis on* a *implies that* Ω *is not simply connected and* $a \not\equiv 0$. *Note that in Theorem 2 the set* Ω *is simply connected.*

(ii) Observe that the regularity in the theorem is optimal for all the data.

2.2 Some generalizations

We now show how a general operator of the form

$$L_{a,b}\,(u) = \sum_{1 \leq i,j \leq n} b_j^i\, u_{x_j}^i + \langle a; u \rangle = \langle B; \nabla u \rangle + \langle a; u \rangle$$

can be brought back to our analysis. But let us first introduce some notations. Let

$$B = \left(b_j^i\right)_{1 \leq j \leq n}^{1 \leq i \leq n} \in C^1\left(\overline{\Omega}; \mathbb{R}^{n \times n}\right)$$

and set

$$\operatorname{div} B = \left(\operatorname{div} B^1, \cdots, \operatorname{div} B^n\right) \quad \text{where} \quad \operatorname{div} B^i = \sum_{1 \leq j \leq n} \left(b_j^i\right)_{x_j}.$$

Let B be invertible and define

$$\widetilde{a} = B^{-t}\left(a - \operatorname{div} B^{t}\right).$$

Assume that Ω, f, u_0 and $\widetilde{a}$ verify the hypotheses of either Theorem 2 or Theorem 4. We then claim that there exists u (with the corresponding regularity) satisfying

$$\begin{cases} L_{a,b}(u) = f & \text{in } \Omega \\ u = u_0 & \text{on } \partial\Omega. \end{cases}$$

This is easily seen by setting $u = B^{-1}v$ where

$$\begin{cases} \operatorname{div} v + \langle \widetilde{a}; v \rangle = f & \text{in } \Omega \\ v = Bu_0 & \text{on } \partial\Omega. \end{cases}$$

Indeed it suffices to observe that

$$\operatorname{div} v + \langle \widetilde{a}; v \rangle = L_{a,b}(u).$$

3 Proof of Theorem 2

3.1 Notations

Although we will be dealing only with vector fields and the divergence and curl operators, it will be, sometimes, simpler to use the notations of differential geometry (we adopt the notations in [4]).

(i) We will often identify a vector field $u = \left(u^1, \cdots, u^n\right)$ with a $1-$form and write

$$u = \sum_{i=1}^{n} u^i \, dx^i$$

and thus the curl operator is identified with the operator d so that

$$du = \sum_{1 \le i < j \le n} \left[u^j_{x_i} - u^i_{x_j}\right] dx^i \wedge dx^j.$$

(ii) The divergence operator is seen as the δ operator acting on $1-$forms, namely

$$\delta u = \sum_{i=1}^{n} u^i_{x_i}.$$

(iii) Similarly the δ operator acting on $2-$forms $u = \sum_{1 \le i < j \le n} u^{ij} \, dx^i \wedge dx^j$ is defined through (where we set $u^{ij} = -u^{ji}$ if $i \ge j$)

$$\delta u = \sum_{j=1}^{n} \sum_{i=1}^{n} u^{ij}_{x_i} \, dx^j$$

and is also denoted

$$\operatorname{curl}^* u = \left((\operatorname{curl}^* u)^1 , \cdots , (\operatorname{curl}^* u)^n \right) \in \mathbb{R}^n$$

and

$$(\operatorname{curl}^* u)^i = \sum_{j=1}^{i-1} \frac{\partial u^{ji}}{\partial x_j} - \sum_{j=i+1}^{n} \frac{\partial u^{ij}}{\partial x_j} .$$

We therefore have, by abuse of notations,

$$\delta\delta u = \operatorname{div} \operatorname{curl}^* u = 0 \quad \text{for every } u \in C^2 \left(\mathbb{R}^n ; \mathbb{R}^{n(n-1)/2} \right) .$$

(iv) The exterior product of two 1−forms is defined as

$$u \wedge v = \sum_{1 \le i < j \le n} \left[u^i v^j - u^j v^i \right] dx^i \wedge dx^j .$$

Note that dv can be seen as $\nabla \wedge v$.

(v) The interior product of two 1−forms (or of two 2−forms) is just the ordinary scalar product, while the interior product of a 1−form u with a 2−form v is defined as

$$u \lrcorner v = \sum_{j=1}^{n} \sum_{i=1}^{n} u^i v^{ij} \, dx^j$$

Note that δv can be seen as $\nabla \lrcorner v$.

(vi) Finally for the Hodge * operation as well as for the definition and properties of the pullback operation we refer again to [4].

3.2 Preliminary lemmas

We start with a lemma that allows us to reduce the problem to the case where Ω is the unit ball B_1 .

Lemma 8 *Let $r \ge 1$ be an integer, $\Omega ; O \subset \mathbb{R}^n$ be two bounded open smooth sets. Let $\varphi \in \operatorname{Diff}^{r+1} \left(\overline{\Omega} ; \overline{O} \right)$, $a \in C^0 \left(\overline{O} ; \mathbb{R}^n \right)$ and $f \in C^0 \left(\overline{O} \right)$. Define*

$$b = \varphi^* (a) \quad \text{and} \quad g = (-1)^{n-1} * (\varphi^* (*f)) .$$

Then $u \in C^r \left(\overline{O} ; \mathbb{R}^n \right)$ solves the problem

$$\begin{cases} \operatorname{div} u + \langle a ; u \rangle = f & \text{in } O \\ u = 0 & \text{on } \partial O \end{cases} \tag{1}$$

if and only if $v = * (\varphi^* (*u)) \in C^r \left(\overline{\Omega} ; \mathbb{R}^n \right)$ solves

$$\begin{cases} \operatorname{div} v + \langle b ; v \rangle = g & \text{in } \Omega \\ v = 0 & \text{on } \partial \Omega . \end{cases} \tag{2}$$

Proof Clearly $u=0$ on ∂O if and only if $v=0$ on $\partial\Omega$. We rewrite (1), after composition with the $*$ operation as

$$*\delta u+*\left(a\lrcorner u\right)=*f\quad\Leftrightarrow\quad d\left(*u\right)+\left(a\wedge\left(*u\right)\right)=\left(*f\right).$$

Composing with φ, we find that the above is equivalent to

$$d\left(\varphi^{*}\left(*u\right)\right)+\left(\varphi^{*}\left(a\right)\wedge\varphi^{*}\left(*u\right)\right)=\varphi^{*}\left(*f\right)$$

which is in turn equivalent to (since $*v=\left(-1\right)^{n-1}\varphi^{*}\left(*u\right)$)

$$d\left(*v\right)+\left(\varphi^{*}\left(a\right)\wedge\left(*v\right)\right)=\left(-1\right)^{n-1}\varphi^{*}\left(*f\right)$$

and thus

$$*d\left(*v\right)+*\left(\varphi^{*}\left(a\right)\wedge\left(*v\right)\right)=\left(-1\right)^{n-1}*\varphi^{*}\left(*f\right)\quad\Leftrightarrow\quad\delta v+b\lrcorner v=g$$

which is exactly (2). ■

We next solve a first order equation on the unit ball by the classical method of characteristics.

Lemma 9 *Let $r\geq 1$ be an integer. Let $f,g\in C^{r}\left(\overline{B}_{1}\setminus\{0\}\right)$ and $v_{0}\in C^{r}\left(\partial B_{1}\right)$. Let $v\in C^{r}\left(\overline{B}_{1}\setminus\{0\}\right)$ be defined by*

$$v\left(x\right)=G\left(x\right)v_{0}\left(\frac{x}{|x|}\right)+V\left(x\right)$$

where

$$G\left(x\right)=\exp\left[\int_{|x|}^{1}g\left(\frac{s\,x}{|x|}\right)\frac{ds}{s}\right]$$

$$V\left(x\right)=\int_{1}^{|x|}\left\{\exp\left[\int_{|x|}^{r}g\left(\frac{s\,x}{|x|}\right)\frac{ds}{s}\right]f\left(\frac{r\,x}{|x|}\right)\frac{dr}{r}\right\}.$$

Then v satisfies

$$\begin{cases}\left\langle\nabla v\left(x\right);x\right\rangle+g\left(x\right)v\left(x\right)=f\left(x\right) & \text{if } x\in\overline{B}_{1}\setminus\{0\}\\ v\left(x\right)=v_{0}\left(x\right) & \text{if } x\in\partial B_{1}\,.\end{cases}$$

Proof The explicit solution is found through the method of characteristics. But let us verify, for the sake of completeness, that v is indeed a solution of our problem. Clearly the boundary condition is verified. Let us now check the equation.

(i) We recall that

$$\frac{\partial}{\partial x_{i}}\left(\frac{x_{j}}{|x|}\right)=\left(\frac{x_{j}}{|x|}\right)_{x_{i}}=\frac{\delta_{ij}\left|x\right|^{2}-x_{i}x_{j}}{\left|x\right|^{3}}\quad\Rightarrow\quad\sum_{i=1}^{n}x_{i}\left(\frac{x_{j}}{|x|}\right)_{x_{i}}=0.$$

(ii) We next compute

$$G_{x_i}(x) = G(x)\left\{-\frac{x_i}{|x|^2}\,g(x) + \int_{|x|}^{1}\sum_{j=1}^{n} g_{y_j}\left(\frac{s\,x}{|x|}\right)\left(\frac{x_j}{|x|}\right)_{x_i} ds\right\}$$

and thus

$$\langle \nabla G(x)\,; x\rangle = \sum_{i=1}^{n} x_i G_{x_i} = -G(x)\,g(x)\,.$$

(iii) Observe that

$$\left(v_0\left(\frac{x}{|x|}\right)\right)_{x_i} = \left\langle (\nabla v_0)\left(\frac{x}{|x|}\right); \left(\frac{x}{|x|}\right)_{x_i}\right\rangle = \sum_{j=1}^{n} (v_0)_{y_j}\left(\frac{x_j}{|x|}\right)_{x_i}$$

which implies

$$\left\langle \nabla\left(v_0\left(\frac{x}{|x|}\right)\right); x\right\rangle = 0.$$

(iv) Combining (ii) and (iii) we get

$$\langle \nabla(Gv_0)\,; x\rangle = -G(x)\,g(x)\,v_0\left(\frac{x}{|x|}\right)$$

giving

$$\begin{cases} \langle \nabla(Gv_0)(x)\,; x\rangle + g(x)(Gv_0)(x) = 0 & \text{if } x \in \overline{B}_1 \setminus \{0\} \\ (Gv_0)(x) = v_0(x) & \text{if } x \in \partial B_1\,. \end{cases}$$

(v) Let

$$H(r,x) = \exp\left[\int_{|x|}^{r} g\left(\frac{s\,x}{|x|}\right)\frac{ds}{s}\right]$$

so that

$$V(x) = \int_{1}^{|x|}\left\{H(r,x)\ f\left(\frac{r\,x}{|x|}\right)\frac{dr}{r}\right\}.$$

First compute

$$(H(r,x))_{x_i} = H(r,x)\left\{-\frac{x_i}{|x|^2}\,g(x) + \int_{|x|}^{r}\sum_{j=1}^{n} g_{y_j}\left(\frac{s\,x}{|x|}\right)\left(\frac{x_j}{|x|}\right)_{x_i} ds\right\}$$

and

$$\left(f\left(\frac{r\,x}{|x|}\right)\right)_{x_i} = \sum_{j=1}^{n} f_{y_j}\left(\frac{r\,x}{|x|}\right)\left(\frac{x_j}{|x|}\right)_{x_i} r$$

which lead to

$$\sum_{i=1}^{n} x_i \left(H(r,x)\ f\left(\frac{r\,x}{|x|}\right)\right)_{x_i} = \sum_{i=1}^{n} x_i \left(H(r,x)\right)_{x_i}\ f\left(\frac{r\,x}{|x|}\right) + \sum_{i=1}^{n} x_i\, H(r,x)\ \left(f\left(\frac{r\,x}{|x|}\right)\right)_{x_i}$$
$$= -H(r,x)\,g(x)\,f\left(\frac{r\,x}{|x|}\right).$$

We hence obtain

$$V_{x_i}(x) = \frac{x_i}{|x|^2}\,f(x) + \int_1^{|x|} \left(H(r,x)\ f\left(\frac{r\,x}{|x|}\right)\right)_{x_i} \frac{dr}{r}$$

implying that

$$\sum_{i=1}^{n} x_i V_{x_i}(x) = f(x) - g(x)\int_1^{|x|} H(r,x)\,f\left(\frac{r\,x}{|x|}\right)\frac{dr}{r}$$
$$= f(x) - g(x)\,V(x)\,.$$

(vi) We have therefore obtained

$$\begin{cases} \langle \nabla V(x)\,;x\rangle + g(x)\,V(x) = f(x) & \text{if } x \in \overline{B}_1 \setminus \{0\} \\ V(x) = 0 & \text{if } x \in \partial B_1\,. \end{cases}$$

(vii) The result follows from the combination of (iv) and (vi). ■

3.3 Proof of Theorem 2

We are now in a position to prove Theorem 2.

Proof *Step 1.* We start with some simplifications.

(i) Without loss of generality, we may assume that $u_0 = 0$; replacing u by $u - u_0$ and f by $f - \operatorname{div} u_0 - \langle a; u_0\rangle$.

(ii) Using Lemma 8, we find that we can take Ω to be the unit ball B_1. Combining Lemma 11.13 in [4] and Lemma 8 again, we can assume, without loss of generality, that $x_0 = 0$.

(iii) Finally we choose $\epsilon > 0$ sufficiently small so that $\operatorname{curl} a \neq 0$ in $B_{2\epsilon}$.

Step 2. We then search for solutions of the form

$$u = v + a \lrcorner w + \delta w$$

where $v \in C^{r+3}\left(\overline{B}_1; \mathbb{R}^n\right)$ (constructed in Step 3) and $w \in C^{r+2}\left(\overline{B}_1; \Lambda^2\right)$ (given in Step 4). The advantage of this decomposition is that it transforms the problem into (invoking Theorem 3.5 in [4])

$$L_a(u) = \operatorname{div} u + \langle a; u\rangle = \delta u + \langle a; u\rangle = \delta v + \langle a; v\rangle - da \lrcorner w = f.$$

So if, in addition to the above equation, we find that $v = 0$ on ∂B_1 and $w = 0$ in a neighborhood of ∂B_1 , we will have established the theorem.

Step 3. We first construct v on $\overline{B}_1 \setminus B_\epsilon$ as

$$v(x) = x\,V(x)$$

where $g(x) = n + \langle a(x); x\rangle$ and

$$V(x) = \int_1^{|x|} \left\{ \exp\left[\int_{|x|}^{r} g\left(\frac{s\,x}{|x|}\right) \frac{ds}{s} \right] f\left(\frac{r\,x}{|x|}\right) \frac{dr}{r} \right\}.$$

Observe that $v \in C^{r+3}\left(\overline{B}_1 \setminus B_\epsilon; \mathbb{R}^n\right)$. Applying Lemma 9, we find that V satisfies

$$\begin{cases} \langle \nabla V(x); x\rangle + g(x)\,V(x) = f(x) & \text{if } x \in \overline{B}_1 \setminus B_\epsilon \\ V(x) = 0 & \text{if } x \in \partial B_1 \end{cases}$$

and thus v verifies

$$\begin{cases} \operatorname{div} v + \langle a; v\rangle = \delta v + \langle a; v\rangle = f & \text{in } \overline{B}_1 \setminus B_\epsilon \\ v = 0 & \text{on } \partial B_1 . \end{cases}$$

We then extend v in any C^{r+3} way to B_ϵ .

Step 4. We finally construct $w \in C^{r+2}\left(\overline{B}_1; \Lambda^2\right)$ to be identically 0 in $\overline{B}_1 \setminus B_{2\epsilon}$ and, in $B_{2\epsilon}$, through the formula

$$w = \frac{\operatorname{curl} a}{|\operatorname{curl} a|^2} \left(\operatorname{div} v + \langle a; v\rangle - f\right).$$

The v and w have all the claimed properties. ■

3.4 Some improvement

Using standard elliptic estimates, we can slightly improve the regularity hypotheses.

Theorem 10 *Let $n \geq 2$, $r \geq 0$ be integers, $0 < s < 1$ and $\Omega \subset \mathbb{R}^n$ a bounded open set, with $\overline{\Omega}$ $C^{r+4,s}$ diffeomorphic to the closed unit ball. Let $f \in C^{r+2,s}\left(\overline{\Omega}\right)$, $u_0 \in C^{r+3,s}\left(\overline{\Omega}; \mathbb{R}^n\right)$ and $a \in C^{r+3,s}\left(\overline{\Omega}; \mathbb{R}^n\right)$ be such that*

$$\operatorname{curl} a(x_0) \neq 0 \quad \textit{for some } x_0 \in \Omega.$$

Then there exists $u \in C^{r+1,s}\left(\overline{\Omega}; \mathbb{R}^n\right)$ satisfying

$$\begin{cases} \operatorname{div} u + \langle a; u\rangle = f & \textit{in } \Omega \\ u = u_0 & \textit{on } \partial\Omega. \end{cases}$$

Proof *Step 1.* As in the proof of Theorem 2, we can assume, without loss of generality, that $u_0 = 0$, $\Omega = B_1$ and $x_0 = 0$. Moreover $\epsilon > 0$ is chosen sufficiently small so that $\operatorname{curl} a \neq 0$ in $B_{2\epsilon}$. We then find $\alpha \in C^{r+4,s}\left(\overline{\Omega}\right)$ a solution of

$$\begin{cases} \Delta\alpha + \langle a; \nabla\alpha \rangle = f & \text{in } B_1 \\ \alpha = 0 & \text{on } \partial B_1 . \end{cases}$$

Since $\alpha = 0$ on ∂B_1, we find that there exists $c = \langle \nu; \nabla\alpha \rangle \in C^{r+3,s}\left(\overline{B}_1\right)$ such that

$$\nabla\alpha = c\,\nu \quad \text{on } \partial B_1$$

where $\nu = \nu(x) = x$ is the outward unit normal to ∂B_1. We will then look for solutions u of the form

$$u = \nabla\alpha + \beta$$

where β satisfies

$$\begin{cases} \operatorname{div}\beta + \langle a; \beta \rangle = 0 & \text{in } B_1 \\ \beta = -c\,\nu & \text{on } \partial B_1 . \end{cases}$$

Step 2. We then continue as in the proof of Theorem 2 and search for solutions of the form

$$\beta = v + a \lrcorner w + \delta w$$

where $v \in C^{r+3,s}\left(\overline{B}_1; \mathbb{R}^n\right)$ (see Step 3 of Theorem 2 as well as Lemma 9 with $v_0 = c$ this time) satisfies

$$\begin{cases} \operatorname{div} v + \langle a; v \rangle = 0 & \text{in } \overline{B}_1 \setminus B_\epsilon \\ v = -c\,\nu & \text{on } \partial B_1 . \end{cases}$$

We then extend v in any $C^{r+3,s}$ way to B_ϵ. Finally we define $w \in C^{r+2,s}\left(\overline{B}_1; \Lambda^2\right)$ to be identically 0 in $\overline{B}_1 \setminus B_{2\epsilon}$ and, in $B_{2\epsilon}$, through the formula

$$w = \frac{\operatorname{curl} a}{|\operatorname{curl} a|^2} \left(\operatorname{div} v + \langle a; v \rangle - f\right).$$

The v and w have all the claimed properties. ■

4 The kernel of the operator

4.1 Definition of the kernel

We now study the kernel of

$$L_a(u) = \operatorname{div} u + \langle a; u \rangle .$$

Let us first examine the case $a \equiv 0$. We recall that for a C^1 vector field $w : \mathbb{R}^n \to \mathbb{R}^{n(n-1)/2}$ the kernel is given by $\operatorname{curl}^* w \in \mathbb{R}^n$ so that

$$\operatorname{div} \operatorname{curl}^* w = 0 \quad \text{for every } w \in C^2\left(\mathbb{R}^n; \mathbb{R}^{n(n-1)/2}\right).$$

We also know that if $\operatorname{div} u = 0$, then (by Poincaré lemma) there exists $w : \mathbb{R}^n \to \mathbb{R}^{n(n-1)/2}$ such that

$$u = \operatorname{curl}^* w.$$

We now turn to the general case where $a \not\equiv 0$. We will first define the most general second order kernel acting on functions and then extend this definition to kernels acting on $2-$forms. This extension to $2-$forms is motivated by extending the above result (when $a \equiv 0$). It will moreover turn out to be a crucial point in the proof of Theorem 4. Examples are discussed in Section 4.3.

Case 1: kernels acting on functions. Let $\alpha_{kl}^m, \beta_l^m, \gamma^m \in C^0\left(\overline{\Omega}\right)$. We define the operator, acting on C^2 functions w,

$$N_a^{\alpha,\beta,\gamma} : C^2\left(\overline{\Omega}\right) \to C^0\left(\overline{\Omega}; \mathbb{R}^n\right)$$

as, for $w \in C^2\left(\overline{\Omega}\right)$,

$$N_a^{\alpha,\beta,\gamma}(w) = \left(\left(N_a^{\alpha,\beta,\gamma}(w)\right)^1, \cdots, \left(N_a^{\alpha,\beta,\gamma}(w)\right)^n\right) = \sum_{m=1}^{n} \left(N_a^{\alpha,\beta,\gamma}(w)\right)^m dx^m$$

where

$$\left(N_a^{\alpha,\beta,\gamma}(w)\right)^m = \sum_{k \leq l} \alpha_{kl}^m \, w_{x_k x_l} + \sum_l \beta_l^m \, w_{x_l} + \gamma^m \, w.$$

Note that

$$\alpha = (\alpha_{kl}^m)_{1 \leq k \leq l \leq n}^{1 \leq m \leq n} \in \mathbb{R}^{(\binom{n}{2}+n)n},$$

$$\beta = (\beta_l^m)_{1 \leq l \leq n}^{1 \leq m \leq n} \in \mathbb{R}^{n^2} \quad \text{and} \quad \gamma = (\gamma^m)^{1 \leq m \leq n} \in \mathbb{R}^n.$$

It will also be convenient to write

$$\alpha_{kl} = \sum_m \alpha_{kl}^m \, dx^m, \quad \beta_l = \sum_m \beta_l^m \, dx^m \quad \text{and} \quad \gamma = \sum_m \gamma^m \, dx^m.$$

We should observe that the dependence on a in the kernel is only implicit. It is only when we want to determine α, β and γ so that

$$L_a\left(N_a^{\alpha,\beta,\gamma}(w)\right) = 0, \quad \forall\, w$$

that the a plays a role. So, sometimes, when we study the properties of the operator $N_a^{\alpha,\beta,\gamma}$ independently of the fact that $L_a\left(N_a^{\alpha,\beta,\gamma}(w)\right) = 0$, we will write only $N^{\alpha,\beta,\gamma}$.

Case 2: kernels acting on $2-$forms. For a $2-$form

$$w = \sum_{i<j} w^{ij} dx^i \wedge dx^j \in C^2\left(\overline{\Omega}; \Lambda^2\right)$$

let

$$N_{a,2}^{\alpha,\beta,\gamma}(w) = \sum_m \left[\sum_{i<j} \left(N_a^{\alpha_{ij},\beta_{ij},\gamma_{ij}}\left(w^{ij}\right)\right)^m\right] dx^m.$$

Note that here

$$\alpha = \left(\alpha_{ijkl}^{m}\right)_{1\le i<j\le n, 1\le k\le l\le n}^{1\le m\le n} \in \mathbb{R}^{\binom{n}{2}\left(\binom{n}{2}+n\right)n},$$

$$\beta = \left(\beta_{ijl}^{m}\right)_{1\le i<j\le n, 1\le l\le n}^{1\le m\le n} \in \mathbb{R}^{\binom{n}{2}n^2} \quad \text{and} \quad \gamma = \left(\gamma_{ij}^{m}\right)_{1\le i<j\le n,}^{1\le m\le n} \in \mathbb{R}^{\binom{n}{2}n}.$$

4.2 The necessary and sufficient condition

We then have the following proposition.

Proposition 11 *Let $\alpha_{kl}, \beta_l, \gamma \in C^1\left(\overline{\Omega}; \mathbb{R}^n\right)$. Then*

$$L_a\left(N_a^{\alpha,\beta,\gamma}(w)\right) = 0, \quad \forall\, w \in C^3\left(\overline{\Omega}\right)$$

if and only if

$$\begin{cases} \alpha_{mm}^{m} = 0 & \forall\, m \\ \alpha_{ll}^{m} + \alpha_{lm}^{l} = \alpha_{mm}^{l} + \alpha_{lm}^{m} = 0 & \forall\, l < m \\ \alpha_{lm}^{k} + \alpha_{km}^{l} + \alpha_{kl}^{m} = 0 & \forall\, k < l < m \end{cases} \tag{3}$$

$$\begin{cases} L_a\left(\alpha_{ll}\right) + \beta_l^l = 0 & \forall\, l \\ L_a\left(\alpha_{lm}\right) + \beta_l^m + \beta_m^l = 0 & \forall\, l < m \end{cases} \tag{4}$$

$$\begin{cases} L_a\left(\beta_l\right) + \gamma^l = 0 \quad \forall\, l \\ L_a\left(\gamma\right) = 0. \end{cases} \tag{5}$$

Remark 12 *A more condensed way of writing the three set of equations is*

$$A_{lm}^{k} + A_{km}^{l} + A_{kl}^{m} = 0, \quad \forall\, k \le l \le m$$

$$L_a\left(A_{lm}\right) + B_l^m + B_m^l = 0, \quad \forall\, l \le m$$

$$L_a\left(B_l\right) + \gamma^l = 0,\ \forall\, l \quad \text{and} \quad L_a\left(\gamma\right) = 0$$

where

$$A_{ll} = 2\alpha_{ll} \quad \text{and} \quad A_{lm} = \alpha_{lm}\,,\ \text{if } l < m$$

$$B_l^l = 2\beta_l^l \quad \text{and} \quad B_l^m = \beta_l^m\,,\ \text{if } l \neq m.$$

Proof We find

$$\begin{aligned} 0 = L_a\left(N_a^{\alpha,\beta,\gamma}(w)\right) &= \operatorname{div}\left(N_a^{\alpha,\beta,\gamma}(w)\right) + \left\langle a; N_a^{\alpha,\beta,\gamma}(w)\right\rangle \\ &= \sum_m \sum_{k\le l} \left(\alpha_{kl}^{m}\, w_{x_k x_l}\right)_{x_m} + \sum_m \sum_l \left(\beta_l^m\, w_{x_l}\right)_{x_m} + \sum_m \left(\gamma^m\, w\right)_{x_m} \\ &\quad + \sum_m \sum_{k\le l} a^m \left(\alpha_{kl}^{m}\, w_{x_k x_l}\right) + \sum_m \sum_l a^m \left(\beta_l^m\, w_{x_l}\right) + \sum_m a^m \left(\gamma^m\, w\right) \end{aligned}$$

and thus

$$
\begin{aligned}
0 &= \sum_m \sum_{k \le l} \left(\alpha_{kl}^m \, w_{x_k x_l x_m}\right) \\
&+ \sum_m \sum_{k \le l} \left(\left(\alpha_{kl}^m\right)_{x_m} + a^m \alpha_{kl}^m\right) w_{x_k x_l} + \sum_m \sum_l \left(\beta_l^m \, w_{x_l x_m}\right) \\
&+ \sum_m \sum_l \left(\left(\beta_l^m\right)_{x_m} + a^m \beta_l^m\right) w_{x_l} + \sum_m \left(\gamma^m \, w_{x_m}\right) \\
&+ \sum_m \left(\left(\gamma^m\right)_{x_m} + a^m \gamma^m\right) w
\end{aligned}
$$

which implies, with our notations,

$$
\begin{aligned}
0 = & \sum_m \sum_{k \le l} \left(\alpha_{kl}^m \, w_{x_k x_l x_m}\right) + \sum_{k \le l} \left(L_a\left(\alpha_{kl}\right)\right) w_{x_k x_l} + \sum_m \sum_l \left(\beta_l^m \, w_{x_l x_m}\right) \\
& + \sum_l \left(L_a\left(\beta_l\right)\right) w_{x_l} + \sum_m \left(\gamma^m \, w_{x_m}\right) + \left(L_a\left(\gamma\right)\right) w.
\end{aligned}
$$

Let us now examine all the terms depending on the order of derivatives of w.

Terms of order 0. We infer that

$$L_a\left(\gamma\right) = 0.$$

Terms of order 1. We rewrite all the terms with the same indices

$$\sum_l \left(L_a\left(\beta_l\right) + \gamma^l\right) w_{x_l} = 0$$

and hence

$$L_a\left(\beta_l\right) + \gamma^l = 0 \quad \forall\, l.$$

Terms of order 2. We find

$$\sum_{k \le l} + \sum_m \sum_l = \left[\sum_{l(k=l)} + \sum_{k<l}\right] + \left[\sum_{l<m} + \sum_{l(m=l)} + \sum_{m<l}\right]$$

and thus

$$
\begin{aligned}
& \sum_{k \le l} \left(L_a\left(\alpha_{kl}\right)\right) w_{x_k x_l} + \sum_m \sum_l \left(\beta_l^m \, w_{x_l x_m}\right) \\
&= \sum_{l(k=l)} \left(L_a\left(\alpha_{ll}\right)\right) w_{x_l x_l} + \sum_{k<l} \left(L_a\left(\alpha_{kl}\right)\right) w_{x_k x_l} \\
&+ \sum_{l<m} \left(\beta_l^m \, w_{x_l x_m}\right) + \sum_{l(m=l)} \left(\beta_l^l \, w_{x_l x_l}\right) + \sum_{m<l} \left(\beta_l^m \, w_{x_l x_m}\right).
\end{aligned}
$$

Rewriting with the the same indices, we get

$$\sum_{k\leq l}\left(L_a\left(\alpha_{kl}\right)\right)w_{x_kx_l}+\sum_m\sum_l\left(\beta_l^m\,w_{x_lx_m}\right)$$
$$=\sum_l\left(L_a\left(\alpha_{ll}\right)+\beta_l^l\right)w_{x_lx_l}+\sum_{l<m}\left(L_a\left(\alpha_{lm}\right)+\beta_l^m+\beta_m^l\right)w_{x_lx_m}\,.$$

We hence found

$$L_a\left(\alpha_{ll}\right)+\beta_l^l=0\quad\forall\,l$$
$$L_a\left(\alpha_{lm}\right)+\beta_l^m+\beta_m^l=0\quad\forall\,l<m.$$

Terms of order 3. Writing the same indices we obtain

$$\sum_m\sum_{k\leq l}=\left[\sum_m\sum_{l(k=l)}\right]+\left[\sum_m\sum_{k<l}\right]$$
$$=\left[\sum_{m(k=l=m)}+\sum_{m<l(k=l)}+\sum_{m>l(k=l)}\right]$$
$$+\left[\sum_{m<k<l}+\sum_{m<l(k=m)}+\sum_{k<m<l}+\sum_{k<m(l=m)}+\sum_{k<l<m}\right]$$

leading to

$$\sum_m\sum_{k\leq l}=\left[\sum_{m(k=l=m)}\right]+\left[\sum_{m<l(k=l)}+\sum_{m>l(k=l)}+\sum_{m<l(k=m)}+\sum_{k<m(l=m)}\right]$$
$$+\left[\sum_{m<k<l}+\sum_{k<m<l}+\sum_{k<l<m}\right].$$

We hence find

$$0=\sum_m\sum_{k\leq l}\left(\alpha_{kl}^m\,w_{x_kx_lx_m}\right)=\left[\sum_{m(k=l=m)}\alpha_{mm}^m\,w_{x_mx_mx_m}\right]$$
$$+\left[\sum_{m<l(k=l)}\alpha_{ll}^m\,w_{x_lx_lx_m}+\sum_{m>l(k=l)}\alpha_{ll}^m\,w_{x_lx_lx_m}\right]$$
$$+\left[\sum_{m<l(k=m)}\alpha_{ml}^m\,w_{x_lx_mx_m}+\sum_{k<m(l=m)}\alpha_{km}^m\,w_{x_kx_mx_m}\right]$$
$$+\left[\sum_{m<k<l}\alpha_{kl}^m\,w_{x_kx_lx_m}+\sum_{k<m<l}\alpha_{kl}^m\,w_{x_kx_lx_m}+\sum_{k<l<m}\alpha_{kl}^m\,w_{x_kx_lx_m}\right].$$

We find, after uniforming the indices,

$$\begin{aligned}0 = & \left[\sum_{m} \alpha_{mm}^{m}\, w_{x_m x_m x_m}\right] \\ & + \sum_{l<m} \left[\left(\alpha_{ll}^{m} + \alpha_{lm}^{l}\right) w_{x_l x_l x_m} + \left(\alpha_{mm}^{l} + \alpha_{lm}^{m}\right) w_{x_l x_m x_m}\right] \\ & + \sum_{k<l<m} \left[\alpha_{lm}^{k} + \alpha_{km}^{l} + \alpha_{kl}^{m}\right] w_{x_k x_l x_m}\,.\end{aligned}$$

We have therefore found that

$$\begin{cases} \alpha_{mm}^{m} = 0 & \forall\, m \\ \alpha_{ll}^{m} + \alpha_{lm}^{l} = \alpha_{mm}^{l} + \alpha_{lm}^{m} = 0 & \forall\, l < m \\ \alpha_{lm}^{k} + \alpha_{km}^{l} + \alpha_{kl}^{m} = 0 & \forall\, k < l < m. \end{cases}$$

The proof is therefore complete. ■

4.3 Some examples

We now give three examples.

Example 13 *When* $a = \operatorname{grad} A$, *we can choose*

$$N_{a,2}^{\alpha,\beta,\gamma}(w) = e^{-A} \operatorname{curl}^* w$$

for every $w \in C^2\left(\overline{\Omega}; \Lambda^2\right)$.

Proof Fix $i < j$ and define

$$\beta_{ijl}^{m} = e^{-A} \begin{cases} 1 & \text{if } l = i \text{ and } m = j \\ -1 & \text{if } m = i \text{ and } l = j \\ 0 & \text{otherwise.} \end{cases}$$

Choose $\alpha_{lm} = \gamma = 0$ so that

$$N_{a}^{\alpha,\beta_{ij},\gamma}(w) = e^{-A}\left[-w_{x_j}\, dx^i + w_{x_i}\, dx^j\right].$$

When applied to $2-$forms $w = \sum_{i<j} w^{ij} dx^i \wedge dx^j$ we can choose a linear combination of the above $N_{a}^{\alpha,\beta_{ij},\gamma}\left(w^{ij}\right)$. We thus find

$$\begin{aligned} e^{A}\left(N_{a,2}^{\alpha,\beta,\gamma}(w)\right)^{m} &= e^{A} \sum_{i<j} \left(N_{a}^{\alpha,\beta_{ij},\gamma}\left(w^{ij}\right)\right)^{m} = \sum_{i<j} \sum_{l} \beta_{ijl}^{m}\, w_{x_l}^{ij} \\ &= \sum_{l<m} \beta_{lml}^{m}\, w_{x_l}^{lm} + \sum_{l>m} \beta_{mll}^{m}\, w_{x_l}^{ml} = \sum_{l<m} w_{x_l}^{lm} - \sum_{l>m} w_{x_l}^{ml} \\ &= \left(\operatorname{curl}^* w\right)^{m} \end{aligned}$$

which is what had to be proved. ■

The next example is in some sense generic and will be used in Theorem 30.

Example 14 *Let* $0 \leq 2p \leq n$ *and*

$$\overline{a} = \sum_{i=1}^{p} x_{2i}\, dx^{2i-1}.$$

(i) We can then choose $\overline{\alpha}, \overline{\beta}, \overline{\gamma} \in C^\infty$ *such that, for every* $w \in C^2\left(\overline{\Omega}\right)$,

$$N_{\overline{a}}^{\overline{\alpha},\overline{\beta},\overline{\gamma}}(w) = e^{-x_1 x_2}\left(-w_{x_1 x_2} + x_1 w_{x_1} - w\right) dx^1 + e^{-x_1 x_2}\left(w_{x_1 x_1}\right) dx^2$$

and verifying

$$L_{\overline{a}}\left(N_{\overline{a}}^{\overline{\alpha},\overline{\beta},\overline{\gamma}}(w)\right) = 0, \quad \forall\, w \in C^3\left(\overline{\Omega}\right).$$

(ii) Similarly, for $2-$*forms* $w = \sum_{i<j} w^{ij} dx^i \wedge dx^j \in C^2\left(\overline{\Omega}; \Lambda^2\right)$, *we can find* $\overline{\alpha}, \overline{\beta}, \overline{\gamma} \in C^\infty$ *such that*

$$N_{\overline{a},2}^{\overline{\alpha},\overline{\beta},\overline{\gamma}}(w) = e^{-x_1 x_2}\left(-\sum_{j=2}^{n} w_{x_1 x_j}^{1j} + x_1\, w_{x_1}^{12} - \sum_{i=2}^{p} x_{2i}\, w_{x_1}^{1(2i-1)} - w^{12}\right) dx^1 + e^{-x_1 x_2} \sum_{j=2}^{n} w_{x_1 x_1}^{1j}\, dx^j$$

and satisfying

$$L_{\overline{a}}\left(N_{\overline{a},2}^{\overline{\alpha},\overline{\beta},\overline{\gamma}}(w)\right) = 0, \quad \forall\, w \in C^3\left(\overline{\Omega}; \Lambda^2\right).$$

It can, moreover, be rewritten as (where we let $\sigma = e^{-x_1 x_2} dx^1$*)*

$$N_{\overline{a},2}^{\overline{\alpha},\overline{\beta},\overline{\gamma}}(w) = -\left(\sigma \lrcorner\, d\overline{a}\right) \lrcorner\, w + \overline{a} \lrcorner \left(\sigma \wedge \delta w\right) + \delta\left(\sigma \wedge \delta w\right)$$

for every w *of the form*

$$w = \sum_{j=2}^{n} w^{1j} dx^1 \wedge dx^j.$$

(iii) If we now consider $a = \overline{a} + dS$, *where* S *is a function, we see that if*

$$(\alpha, \beta, \gamma) = e^{-S}\left(\overline{\alpha}, \overline{\beta}, \overline{\gamma}\right) \quad \Leftrightarrow \quad N_{a,2}^{\alpha,\beta,\gamma}(w) = e^{-S} N_{\overline{a},2}^{\overline{\alpha},\overline{\beta},\overline{\gamma}}(w),$$

then

$$L_a\left(N_{a,2}^{\alpha,\beta,\gamma}(w)\right) = 0, \quad \forall\, w \in C^3\left(\overline{\Omega}; \Lambda^2\right).$$

Proof *Step 1.* Let us first examine the case of functions. Indeed if we define

$$\overline{\alpha}_{kl}^{m} = e^{-x_1 x_2} \begin{cases} 1 & \text{if } k = l = 1 \text{ and } m = 2 \\ -1 & \text{if } k = m = 1 \text{ and } l = 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\overline{\beta}_{l}^{m}=e^{-x_1x_2}\begin{cases} x_1 & \text{if } l=m=1\\ 0 & \text{otherwise}\end{cases} \quad \text{and} \quad \overline{\gamma}^{m}=e^{-x_1x_2}\begin{cases} -1 & \text{if } m=1\\ 0 & \text{otherwise.}\end{cases}$$

Note that $\overline{\alpha}_{kl}\,,\overline{\beta}_{l},\overline{\gamma}$ satisfy (3), (4) and (5) of Proposition 11 and therefore

$$L_{\overline{a}}\left(N_{\overline{a}}^{\overline{\alpha},\overline{\beta},\overline{\gamma}}\left(w\right)\right)=0,\quad \forall\, w\in C^{3}\left(\overline{\Omega}\right).$$

Step 2. More generally if, for $i<j$,

$$\overline{\alpha}_{ijkl}^{m}=e^{-x_1x_2}\begin{cases} 1 & \text{if } i=k=l=1 \text{ and } j=m=2,\cdots,n\\ -1 & \text{if } i=k=m=1 \text{ and } j=l=2,\cdots,n\\ 0 & \text{otherwise}\end{cases}$$

$$\overline{\beta}_{ijl}^{m}=e^{-x_1x_2}\begin{cases} x_1 & \text{if } i=l=m=1 \text{ and } j=2\\ -x_{2s} & \text{if } i=l=m=1 \text{ and } j=2s-1 \text{ with } s=2,\cdots,p\\ 0 & \text{otherwise}\end{cases}$$

$$\overline{\gamma}_{ij}^{m}=e^{-x_1x_2}\begin{cases} -1 & \text{if } i=m=1 \text{ and } j=2\\ 0 & \text{otherwise.}\end{cases}$$

We therefore find, for every $w=\sum_{i<j}w^{ij}dx^{i}\wedge dx^{j}\in C^{2}\left(\overline{\Omega};\Lambda^{2}\right)$, that

$$\begin{aligned} N_{\overline{a},2}^{\overline{\alpha},\overline{\beta},\overline{\gamma}}\left(w\right)= \quad & e^{-x_1x_2}\left(-\textstyle\sum_{j=2}^{n}w_{x_1x_j}^{1j}+x_1\,w_{x_1}^{12}-\sum_{i=2}^{p}x_{2i}\,w_{x_1}^{1(2i-1)}-w^{12}\right)dx^{1}\\ & +\,e^{-x_1x_2}\textstyle\sum_{j=2}^{n}w_{x_1x_1}^{1j}\,dx^{j}. \end{aligned} \tag{6}$$

It is easy to see, either by direct computation or by Step 3 combined with Proposition 11, that $\overline{\alpha}_{ijkl}\,,\overline{\beta}_{ijl},\overline{\gamma}_{ij}$ satisfy (3), (4) and (5) of Proposition 11, we thus obtain that

$$L_{\overline{a}}\left(N_{\overline{a},2}^{\overline{\alpha},\overline{\beta},\overline{\gamma}}\left(w\right)\right)=0,\quad \forall\, w\in C^{3}\left(\overline{\Omega};\Lambda^{2}\right).$$

Step 3. For $w\in\Lambda^{2}$, let

$$\pi^{1}\left(w\right)=\sum_{j=2}^{n}w^{1j}dx^{1}\wedge dx^{j}$$

and observe that, for every $w\in\Lambda^{2}$,

$$N_{\overline{a},2}^{\overline{\alpha},\overline{\beta},\overline{\gamma}}\left(w\right)=N_{\overline{a},2}^{\overline{\alpha},\overline{\beta},\overline{\gamma}}\left(\pi^{1}\left(w\right)\right).$$

We then set, for $\sigma=e^{-x_1x_2}\,dx^{1}$,

$$M\left(w\right)=-\left(\sigma\lrcorner\, d\overline{a}\right)\lrcorner\, w+\overline{a}\lrcorner\left(\sigma\wedge\delta w\right)+\delta\left(\sigma\wedge\delta w\right).$$

The claim is that

$$N_{\overline{a},2}^{\overline{\alpha},\overline{\beta},\overline{\gamma}}\left(w\right)=N_{\overline{a},2}^{\overline{\alpha},\overline{\beta},\overline{\gamma}}\left(\pi^{1}\left(w\right)\right)=M\left(\pi^{1}\left(w\right)\right).$$

This will be established in Step 3.1; while in Step 3.2 we will show that

$$L_{\overline{a}}\left(M\left(\pi^{1}\left(w\right)\right)\right)=0,\quad \forall\, w\in C^{3}\left(\overline{\Omega};\Lambda^{2}\right).$$

Step 3.1. Let w be of the form

$$w=\pi^{1}\left(w\right)=\sum_{j=2}^{n}w^{1j}dx^{1}\wedge dx^{j}.$$

Let $\lambda=e^{-x_1x_2}$ so that $\sigma=\lambda\,dx^1$. Observe the following facts.

(i) Clearly

$$d\overline{a}=-\sum_{i=1}^{p}dx^{2i-1}\wedge dx^{2i}\quad\text{and}\quad \sigma\lrcorner\, d\overline{a}=-\lambda\,dx^{2}.$$

(ii) For w of the above form, we also have

$$\delta w=\left(\sum_{j=2}^{n}w_{x_j}^{j1}\right)dx^{1}+\sum_{j=2}^{n}w_{x_1}^{1j}\,dx^{j}=-\left(\sum_{j=2}^{n}w_{x_j}^{1j}\right)dx^{1}+\sum_{j=2}^{n}w_{x_1}^{1j}\,dx^{j}.$$

We hence obtain that

$$\sigma\wedge\delta w=\lambda\,dx^{1}\wedge\left(\left(\sum_{j=2}^{n}w_{x_j}^{j1}\right)dx^{1}+\sum_{j=2}^{n}w_{x_1}^{1j}\,dx^{j}\right)=\lambda\sum_{j=2}^{n}w_{x_1}^{1j}\,dx^{1}\wedge dx^{j}$$

and thus

$$\begin{aligned}\overline{a}\lrcorner\left(\sigma\wedge\delta w\right)&=\sum_{j=1}^{n}\left(\sum_{i=1}^{n}\left(\sigma\wedge\delta w\right)^{ij}\overline{a}^{i}\right)dx^{j}\\&=\left(\sum_{i=2}^{n}\left(\sigma\wedge\delta w\right)^{i1}\overline{a}^{i}\right)dx^{1}+\sum_{j=2}^{n}\left(\left(\sigma\wedge\delta w\right)^{1j}\overline{a}^{1}\right)dx^{j}\\&=-\lambda\left(\sum_{i=2}^{p}w_{x_1}^{1(2i-1)}\,x_{2i}\right)dx^{1}+\lambda\sum_{j=2}^{n}\left(x_2\,w_{x_1}^{1j}\right)dx^{j}\end{aligned}$$

while

$$\begin{aligned}\delta\left(\sigma\wedge\delta w\right)&=\sum_{j=1}^{n}\left(\sum_{i=1}^{n}\left(\sigma\wedge\delta w\right)_{x_i}^{ij}\right)dx^{j}\\&=\left(\sum_{i=2}^{n}\left(\sigma\wedge\delta w\right)_{x_i}^{i1}\right)dx^{1}+\sum_{j=2}^{n}\left(\left(\sigma\wedge\delta w\right)_{x_1}^{1j}\right)dx^{j}\\&=\left(-\sum_{i=2}^{n}\left(\lambda\,w_{x_1}^{1i}\right)_{x_i}\right)dx^{1}+\sum_{j=2}^{n}\left(\left(\lambda\,w_{x_1}^{1j}\right)_{x_1}\right)dx^{j}.\end{aligned}$$

(iii) Since we also have

$$-\left(\sigma \lrcorner d\overline{a}\right) \lrcorner w=-\sum_{j=1}^{n}\left(\sum_{i=1}^{n} w^{ij}\left(\sigma \lrcorner d\overline{a}\right)^{i}\right) dx^{j}=\lambda \sum_{j=1}^{n} w^{2j}\, dx^{j}=-\lambda\, w^{12}\, dx^{1},$$

we get, from (i) and (ii),

$$\begin{aligned} M\left(w\right) &= -\left(\sigma \lrcorner d\overline{a}\right) \lrcorner w+\overline{a} \lrcorner \left(\sigma \wedge \delta w\right)+\delta\left(\sigma \wedge \delta w\right) \\ &= \left[-\lambda w^{12}\, dx^{1}\right]+\left[-\lambda\left(\sum_{i=2}^{p} w_{x_1}^{1(2i-1)}\, x_{2i}\right) dx^{1}+\lambda \sum_{j=2}^{n}\left(x_{2}\, w_{x_1}^{1j}\right) dx^{j}\right] \\ &+\left[\left(-\sum_{i=2}^{n}\left(\lambda\, w_{x_1}^{1i}\right)_{x_i}\right) dx^{1}+\sum_{j=2}^{n}\left(\left(\lambda\, w_{x_1}^{1j}\right)_{x_1}\right) dx^{j}\right]. \end{aligned}$$

We have therefore found (recall that $\lambda_{x_1}=-x_2\,\lambda$, $\lambda_{x_2}=-x_1\,\lambda$ and $\lambda_{x_i}=0$ if $i\geq 3$)

$$\begin{aligned} M\left(w\right)= \quad & e^{-x_1 x_2}\left(-\textstyle\sum_{j=2}^{n} w_{x_1 x_j}^{1j}+x_1\, w_{x_1}^{12}-\sum_{i=2}^{p} x_{2i}\, w_{x_1}^{1(2i-1)}-w^{12}\right) dx^{1} \\ & + e^{-x_1 x_2} \textstyle\sum_{j=2}^{n} w_{x_1 x_1}^{1j}\, dx^{j}. \end{aligned}$$

which is, in view of (6), exactly our claim.

Step 3.2. (i) We start by proving that, for w of the form $w=\pi^{1}\left(w\right)$, we have

$$\left[d\left(\sigma \lrcorner d\overline{a}\right)+\overline{a}\wedge\left(\sigma \lrcorner d\overline{a}\right)\right] \lrcorner w=0.$$

By (i) of Step 3.1 we find that

$$d\left(\sigma \lrcorner d\overline{a}\right)=-d\lambda \wedge dx^{2}=-\lambda_{x_1}\, dx^{1}\wedge dx^{2}=\lambda\, x_2\, dx^{1}\wedge dx^{2}.$$

while

$$\overline{a}\wedge\left(\sigma \lrcorner d\overline{a}\right)=-\lambda \sum_{i=1}^{p} x_{2i}\, dx^{2i-1}\wedge dx^{2}=-\lambda\, x_2\, dx^{1}\wedge dx^{2}+\lambda \sum_{i=2}^{p} x_{2i}\, dx^{2}\wedge dx^{2i-1}.$$

We therefore have the claim since

$$\begin{aligned} \left[d\left(\sigma \lrcorner d\overline{a}\right)+\overline{a}\wedge\left(\sigma \lrcorner d\overline{a}\right)\right] \lrcorner w &= \left[\lambda \sum_{i=2}^{p} x_{2i}\, dx^{2}\wedge dx^{2i-1}\right] \lrcorner \left[\sum_{j=2}^{n} w^{1j}\, dx^{1}\wedge dx^{j}\right] \\ &= 0. \end{aligned}$$

(ii) The remaining part of the discussion does not rely neither on the special structure of σ nor on that of $\overline{a}$; it relies only on the fact that

$$\left[d\left(\sigma \lrcorner d\overline{a}\right)+\overline{a}\wedge\left(\sigma \lrcorner d\overline{a}\right)\right] \lrcorner w=0. \tag{7}$$

We first observe that

$$-\delta\left(\left(\sigma \lrcorner d\overline{a}\right) \lrcorner w\right) = \left[d\left(\sigma \lrcorner d\overline{a}\right)\right] \lrcorner w + \left(\sigma \lrcorner d\overline{a}\right) \lrcorner \delta w$$

$$\delta\left(\overline{a} \lrcorner \left(\sigma \wedge \delta w\right)\right) = -d\overline{a} \lrcorner \left(\sigma \wedge \delta w\right) - \overline{a} \lrcorner \delta\left(\sigma \wedge \delta w\right)$$

and thus

$$\begin{aligned}\delta\left(M\left(w\right)\right) &= \left\{\left[d\left(\sigma \lrcorner d\overline{a}\right)\right] \lrcorner w\right\} + \left\{\left(\sigma \lrcorner d\overline{a}\right) \lrcorner \delta w - d\overline{a} \lrcorner \left(\sigma \wedge \delta w\right)\right\} - \left\{\overline{a} \lrcorner \delta\left(\sigma \wedge \delta w\right)\right\} \\ &= \left\{\left[d\left(\sigma \lrcorner d\overline{a}\right)\right] \lrcorner w\right\} + \left\{\left\langle\left(\sigma \lrcorner d\overline{a}\right); \delta w\right\rangle - \left\langle d\overline{a}; \left(\sigma \wedge \delta w\right)\right\rangle\right\} - \left\{\overline{a} \lrcorner \delta\left(\sigma \wedge \delta w\right)\right\} \\ &= \left\{\left[d\left(\sigma \lrcorner d\overline{a}\right)\right] \lrcorner w\right\} - \left\{\overline{a} \lrcorner \delta\left(\sigma \wedge \delta w\right)\right\}.\end{aligned}$$

On the other hand we obtain

$$\begin{aligned}\overline{a} \lrcorner \left(M\left(w\right)\right) &= -\left\{\overline{a} \lrcorner \left(\left(\sigma \lrcorner d\overline{a}\right) \lrcorner w\right)\right\} + \left\{\overline{a} \lrcorner \delta\left(\sigma \wedge \delta w\right)\right\} \\ &= \left\{\left[\overline{a} \wedge \left(\sigma \lrcorner d\overline{a}\right)\right] \lrcorner w\right\} + \left\{\overline{a} \lrcorner \delta\left(\sigma \wedge \delta w\right)\right\}.\end{aligned}$$

We have therefore found, using (7), that

$$\begin{aligned}L_{\overline{a}}\left(M\left(w\right)\right) &= \delta\left(M\left(w\right)\right) + \overline{a} \lrcorner \left(M\left(w\right)\right) \\ &= \left[d\left(\sigma \lrcorner d\overline{a}\right) + \overline{a} \wedge \left(\sigma \lrcorner d\overline{a}\right)\right] \lrcorner w = 0\end{aligned}$$

which is what had to be proved.

Step 4. It now remains to show the last statement of the example. We have

$$\begin{aligned}L_a\left(N_{a,2}^{\alpha,\beta,\gamma}\left(w\right)\right) &= \operatorname{div}\left(e^{-S} N_{\overline{a},2}^{\overline{\alpha},\overline{\beta},\overline{\gamma}}\left(w\right)\right) + \left\langle a; e^{-S} N_{\overline{a},2}^{\overline{\alpha},\overline{\beta},\overline{\gamma}}\left(w\right)\right\rangle \\ &= e^{-S} \operatorname{div}\left(N_{\overline{a},2}^{\overline{\alpha},\overline{\beta},\overline{\gamma}}\left(w\right)\right) - e^{-S}\left\langle dS; e^{-S} N_{\overline{a},2}^{\overline{\alpha},\overline{\beta},\overline{\gamma}}\left(w\right)\right\rangle + \left\langle a; e^{-S} N_{\overline{a},2}^{\overline{\alpha},\overline{\beta},\overline{\gamma}}\left(w\right)\right\rangle \\ &= e^{-S}\left[\operatorname{div}\left(N_{\overline{a},2}^{\overline{\alpha},\overline{\beta},\overline{\gamma}}\left(w\right)\right) + \left\langle \overline{a}; N_{\overline{a},2}^{\overline{\alpha},\overline{\beta},\overline{\gamma}}\left(w\right)\right\rangle\right]\end{aligned}$$

and hence, for every $w \in C^3\left(\overline{\Omega}; \Lambda^2\right)$,

$$L_a\left(N_{a,2}^{\alpha,\beta,\gamma}\left(w\right)\right) = e^{-S} L_{\overline{a}}\left(N_{\overline{a},2}^{\overline{\alpha},\overline{\beta},\overline{\gamma}}\left(w\right)\right) = 0.$$

The proof is therefore complete. ■

We will not give the details for the next example, but it essentially follows from Step 2 in the proof of Theorem 2. This kernel acts on 1−forms, but it can be written as a linear combination of $N_a^{\alpha,\beta,\gamma}$, i.e. kernels acting on 0−forms.

Example 15 *Assume that* $\operatorname{curl} a \neq 0$ *in* $\overline{\Omega}$ *(recall that we identify* $\operatorname{curl} a$ *with* da*) and define, for* $w = \sum_i w^i dx^i \in C^2\left(\overline{\Omega}; \Lambda^1\right)$,

$$Q_a\left(w\right) = w + \left(\delta w + a \lrcorner w\right)\left(\frac{a \lrcorner da}{\left|da\right|^2}\right) + \delta\left(\left(\delta w + a \lrcorner w\right)\frac{da}{\left|da\right|^2}\right).$$

It can easily be proved that

$$L_a\left(Q_a\left(w\right)\right) = 0, \quad \forall\, w \in C^3\left(\overline{\Omega}; \Lambda^1\right).$$

4.4 A sufficient condition

We now show that if $\operatorname{curl} a \neq 0$, we can then find an operator $N_a^{\alpha,\beta,\gamma}$ satisfying the conditions of Proposition 11, with arbitrary α_{kl}.

Proposition 16 *Let $r \geq 1$ be an integer, $O \subset \mathbb{R}^n$ be a bounded open set, $\alpha_{kl}^m \in C^{r+4}\left(\overline{O}\right)$ be arbitrary and $a \in C^{r+3}\left(\overline{O}; \Lambda^1\right)$*

$$\inf_{x\in\overline{O}} \{|\operatorname{curl} a(x)|\} \geq \delta > 0.$$

Then there exist $\beta_l^m \in C^{r+1}\left(\overline{O}\right)$ and $\gamma^m \in C^r\left(\overline{O}\right)$ verifying

$$\begin{cases} L_a(\alpha_{ll}) + \beta_l^l = 0 & \forall\, l \\ L_a(\alpha_{lm}) + \beta_l^m + \beta_m^l = 0 & \forall\, l < m \end{cases}$$

$$\begin{cases} L_a(\beta_l) + \gamma^l = 0 & \forall\, l \\ L_a(\gamma) = 0. \end{cases}$$

Moreover it can be assumed that, for every l, m,

$$\operatorname{supp}\left[\beta_l^m\right], \operatorname{supp}\left[\gamma^m\right] \subset \bigcup_m \bigcup_{k\leq l} \operatorname{supp}\left[\alpha_{kl}^m\right]$$

and, for every $0 \leq s \leq 1$, there exists $K = K\left(\|a\|_{C^{r+3,s}}, \delta\right)$ such that

$$\|\beta_l^m\|_{C^{r+1,s}} + \|\gamma^m\|_{C^{r,s}} \leq K\,\|\alpha\|_{C^{r+4,s}}.$$

Proof (i) Define

$$\beta_l = -\sum_{s\geq l} L_a(\alpha_{ls})\, dx^s + \sum_s \lambda^{ls} dx^s = b_l + dx^l \lrcorner \lambda$$

where $\lambda^{ls} = -\lambda^{sl}$ (i.e. $\lambda \in \Lambda^2$) will be determined later and where we have let

$$b_l = -\sum_{s\geq l} L_a(\alpha_{ls})\, dx^s.$$

We therefore have

$$\beta_l^l = -L_a(\alpha_{ll})$$

and if $l < m$

$$\beta_m^l = \lambda^{ml} \quad \text{and} \quad \beta_l^m = -L_a(\alpha_{lm}) + \lambda^{lm}$$

which leads to

$$\beta_m^l + \beta_l^m = -L_a(\alpha_{lm}).$$

(ii) We let γ be defined by

$$\gamma = -\sum_l L_a(\beta_l)\, dx^l = -\sum_l L_a(b_l)\, dx^l - \sum_l L_a\left(dx^l \lrcorner \lambda\right) dx^l = c - e.$$

The condition $L_a(\gamma) = 0$ therefore becomes $L_a(e) = L_a(c)$. Let us calculate $L_a(e)$. We find

$$e = \sum_l L_a\left(dx^l \lrcorner \lambda\right) dx^l = \sum_l \left[\sum_s \left(\lambda^{ls}_{x_s} + a^s \lambda^{ls}\right)\right] dx^l$$

and thus

$$\begin{aligned} L_a(e) &= \sum_l \left[\sum_s \left(\lambda^{ls}_{x_s} + a^s\lambda^{ls}\right)_{x_l} + a^l \sum_s \left(\lambda^{ls}_{x_s} + a^s\lambda^{ls}\right)\right] \\ &= \sum_{l,s} \left[\lambda^{ls}_{x_s x_l} + a^s \lambda^{ls}_{x_l} + a^s_{x_l}\lambda^{ls} + a^l \lambda^{ls}_{x_s} + a^l a^s \lambda^{ls}\right] \\ &= \sum_{l<s} \left[\lambda^{ls}_{x_s x_l} + \left(a^s\lambda^{ls}_{x_l} + a^l \lambda^{ls}_{x_s}\right) + \left(a^s_{x_l} + a^l a^s\right)\lambda^{ls}\right] \\ &+ \sum_{l>s} \left[\lambda^{ls}_{x_s x_l} + \left(a^s\lambda^{ls}_{x_l} + a^l \lambda^{ls}_{x_s}\right) + \left(a^s_{x_l} + a^l a^s\right)\lambda^{ls}\right]. \end{aligned}$$

Appealing to the fact that $\lambda^{ls} = -\lambda^{sl}$, we obtain

$$\begin{aligned} L_a(e) &= \sum_{l<s} \left[\lambda^{ls}_{x_s x_l} + \left(a^s\lambda^{ls}_{x_l} + a^l \lambda^{ls}_{x_s}\right) + \left(a^s_{x_l} + a^l a^s\right)\lambda^{ls}\right] \\ &+ \sum_{l<s} \left[-\lambda^{ls}_{x_s x_l} - \left(a^s\lambda^{ls}_{x_l} + a^l \lambda^{ls}_{x_s}\right) - \left(a^l_{x_s} + a^l a^s\right)\lambda^{ls}\right] \\ &= \sum_{l<s} \left[\left(a^s_{x_l} - a^l_{x_s}\right)\lambda^{ls}\right] = \operatorname{curl} a \lrcorner \lambda. \end{aligned}$$

Since $\operatorname{curl} a \neq 0$, we can choose

$$\lambda^{ls} = \frac{\left(a^s_{x_l} - a^l_{x_s}\right)}{|\operatorname{curl} a|^2} L_a(c) \tag{8}$$

and thus the result.

(iii) The claim on the support and the estimate are obvious. Note that the dependence of K on δ follows from (8) and Proposition 16.29 in [4]. ■

5 A Poincaré type lemma on the boundary

We now consider our operator $N^{\alpha,\beta,\gamma}_{a,2}$ as acting on $2-$forms. We therefore have $\alpha^m_{ijkl}, \beta^m_{ijl}, \gamma^m_{ij} \in C^r\left(\overline{\Omega}\right)$ and $w \in C^{r+2}\left(\overline{\Omega}; \Lambda^2\right)$. The operator is then given by

$$N^{\alpha,\beta,\gamma}_{a,2}(w) = \sum_m \left(N^{\alpha,\beta,\gamma}_{a,2}(w)\right)^m dx^m$$

where

$$\left(N_{a,2}^{\alpha,\beta,\gamma}(w)\right)^m=\left(\sum_{k\leq l}\sum_{i<j}\alpha_{ijkl}^m\,w_{x_kx_l}^{ij}\right)+\left(\sum_{l}\sum_{i<j}\beta_{ijl}^m\,w_{x_l}^{ij}\right)+\left(\sum_{i<j}\gamma_{ij}^m\,w^{ij}\right)$$

and the α,β,γ are chosen (see Propositions 11 and 16) so that

$$L_a\left(N_{a,2}^{\alpha,\beta,\gamma}(w)\right)=0,\quad\forall\,w\in C^3\left(\overline{\Omega};\Lambda^2\right).$$

The first case that we study is a result on the boundary (in Section 7, we will study the case in the interior). More precisely given $c\in C^{r,s}\left(\overline{\Omega};\mathbb{R}^n\right)$ with

$$\nu\lrcorner\, c=\langle\nu;c\rangle=0\quad\text{on }\partial\Omega$$

(where ν is the outward unit normal to $\partial\Omega$) we will find $w\in C^{r+2,s}\left(\overline{\Omega};\Lambda^2\right)$ (and $\alpha_{ijkl}^m,\beta_{ijl}^m,\gamma_{ij}^m$) such that

$$N_{a,2}^{\alpha,\beta,\gamma}(w)=c\quad\text{on }\partial\Omega.$$

This is the exact analogue of solving, when $a\equiv 0$, $\operatorname{curl}^*w=c$ on $\partial\Omega$ (cf. Lemma 8.11 (ii) in [4]).

Theorem 17 *Let $r\geq 1$ be an integer, $0<s<1$ and $\Omega\subset\mathbb{R}^n$ a bounded open smooth set. Let $a\in C^{r+3,s}\left(\overline{\Omega};\mathbb{R}^n\right)$ be such that*

$$\inf_{x\in\partial\Omega}\{|\operatorname{curl}a(x)|\}\geq\delta>0.$$

There exist $\alpha_{ijkl}^m\in C^\infty\left(\overline{\Omega}\right)$, $\beta_{ijl}^m\in C^{r+1,s}\left(\overline{\Omega}\right)$, $\gamma_{ij}^m\in C^{r,s}\left(\overline{\Omega}\right)$ satisfying

$$L_a\left(N_{a,2}^{\alpha,\beta,\gamma}(w)\right)=0,\quad\forall\,w\in C^3\left(\overline{\Omega};\Lambda^2\right)$$

with the additional property that for any $c\in C^{r,s}\left(\overline{\Omega};\mathbb{R}^n\right)$ with

$$\nu\lrcorner\, c=\langle\nu;c\rangle=0\quad\textit{on }\partial\Omega$$

then there exists $w\in C^{r+2,s}\left(\overline{\Omega};\Lambda^2\right)$ verifying

$$N_{a,2}^{\alpha,\beta,\gamma}(w)=c\quad\textit{on }\partial\Omega.$$

Furthermore there exists $K=K\left(r,s,\|a\|_{C^{r+3,s}},\delta,\Omega\right)$ such that

$$\|w\|_{C^{r+2,s}}+\left\|N_{a,2}^{\alpha,\beta,\gamma}(w)\right\|_{C^{r,s}}\leq K\,\|c\|_{C^{r,s}}\,.$$

Remark 18 *The hypothesis $\operatorname{curl}a\neq 0$ on $\partial\Omega$ can be replaced, for example, by $a=\operatorname{grad}A$ near $\partial\Omega$. The result is then classical. More precisely first choose an* *appropriate cut off function χ so that $\chi \equiv 1$ near $\partial\Omega$. As in Example 13 we can write the kernel as*

$$N_{a,2}^{\alpha,\beta,\gamma}(w) = e^{-A}\delta(\chi w)$$

so that

$$L_a\left(N_{a,2}^{\alpha,\beta,\gamma}(w)\right) = 0, \quad \forall\, w \in C^3\left(\overline{\Omega};\Lambda^2\right).$$

Moreover, by Lemma 8.11 (ii) in [4], we can solve

$$\delta w = e^A c \text{ on } \partial\Omega \quad \Rightarrow \quad N_{a,2}^{\alpha,\beta,\gamma}(w) = c \text{ on } \partial\Omega.$$

Proof *Step 1.* We first find (using Lemma 19 below) $\alpha_{ijkl}^m \in C^\infty\left(\overline{\Omega}\right)$ satisfying (9) such that

$$N_{a,2}^{\alpha,\beta,\gamma}(w) = \nu \lrcorner \frac{\partial^2 w}{\partial \nu^2} \quad \text{on } \partial\Omega$$

for every $\beta_{ijl}^m, \gamma_{ij}^m \in C^0\left(\overline{\Omega}\right)$ and $w \in C^{r+3}\left(\overline{\Omega};\Lambda^2\right)$ verifying

$$\frac{\partial w}{\partial \nu} = w = 0 \quad \text{on } \partial\Omega.$$

It can also be ensured that α_{ijkl}^m has its support in a small neighborhood O of $\partial\Omega$ and $\operatorname{curl} a \neq 0$ in $\overline{O}$. Applying Proposition 16 on this O and extending β_{ijl}^m and γ_{ij}^m by 0 in $\Omega \setminus \overline{O}$, we then find (cf. Proposition 11) $\beta_{ijl}^m \in C^{r+1,s}\left(\overline{\Omega}\right)$ and $\gamma_{ij}^m \in C^{r,s}\left(\overline{\Omega}\right)$ so that $L_a\left(N_a^{\alpha,\beta,\gamma}(w)\right) = 0$ is verified for every $w \in C^3\left(\overline{\Omega};\Lambda^2\right)$.

Step 2. Observe that, since $\nu \lrcorner c = 0$ on $\partial\Omega$, we have

$$c = \nu \lrcorner (\nu \wedge c) \quad \text{on } \partial\Omega.$$

We then let

$$w = \sum_{1 \leq i < j \leq n} w^{ij} dx^i \wedge dx^j$$

satisfying (component by component)

$$\begin{cases} \Delta^3 w = 0 & \text{in } \Omega \\ \frac{\partial^2 w}{\partial \nu^2} = \nu \wedge c \quad \text{and} \quad \frac{\partial w}{\partial \nu} = w = 0 & \text{on } \partial\Omega. \end{cases}$$

The solution w is (cf. Theorems 7.3 and 12.10 in [1]) in $C^\infty\left(\Omega;\Lambda^2\right) \cap C^{r+2,s}\left(\overline{\Omega};\Lambda^2\right)$ and there exists $K = K(r,s,\Omega)$ such that

$$\|w\|_{C^{r+2,s}} \leq K \|c\|_{C^{r,s}}.$$

According to Lemma 19 w satisfies on $\partial\Omega$

$$N_{a,2}^{\alpha,\beta,\gamma}(w) = \nu \lrcorner \frac{\partial^2 w}{\partial \nu^2} = \nu \lrcorner (\nu \wedge c) = c$$

as wished. ■

In the proof of the theorem we used the following lemma.

Lemma 19 *Let $\Omega \subset \mathbb{R}^n$ a bounded open smooth set. Then there exist $\alpha_{ijkl}^m \in C^\infty\left(\overline{\Omega}\right)$ satisfying*

$$\left\{\begin{array}{ccc} \alpha_{ijmm}^m = 0 & \forall\, m & \forall\, i<j \\ \alpha_{ijll}^m + \alpha_{ijlm}^l = \alpha_{ijmm}^l + \alpha_{ijlm}^m = 0 & \forall\, l<m & \forall\, i<j \\ \alpha_{ijlm}^k + \alpha_{ijkm}^l + \alpha_{ijkl}^m = 0 & \forall\, k<l<m & \forall\, i<j \end{array}\right. \tag{9}$$

such that (for any $\beta_{ijl}^m, \gamma_{ij}^m \in C^0\left(\overline{\Omega}\right)$)

$$N_{a,2}^{\alpha,\beta,\gamma}(w) = \nu \lrcorner \frac{\partial^2 w}{\partial \nu^2} \quad \text{on } \partial\Omega$$

for any $w \in C^{r+2}\left(\overline{\Omega};\Lambda^2\right)$ verifying

$$\frac{\partial w}{\partial \nu} = w = 0 \quad \text{on } \partial\Omega$$

where ν is the outward unit normal to $\partial\Omega$. Furthermore α_{ijkl}^m can be chosen identically 0 outside an arbitrary small neighborhood of $\partial\Omega$.

In the proof of the lemma we will use the following easily proved result.

Lemma 20 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open smooth set. Let $w \in C^2\left(\overline{\Omega}\right)$ be such that*

$$\frac{\partial w}{\partial \nu} = w = 0 \quad \text{on } \partial\Omega.$$

Then, for every i, j,

$$\frac{\partial^2 w}{\partial x_i \partial x_j} = \frac{\partial^2 w}{\partial \nu^2}\nu_i \nu_j \quad \text{on } \partial\Omega.$$

Proof (Lemma 19). We provide two proofs. The first one is divided into three steps and the second proof is given in Step 4. In the sequel we have extended ν in a $C^\infty\left(\overline{\Omega};\mathbb{R}^n\right)$ way so that it is identically 0 outside an arbitrary small neighborhood of $\partial\Omega$.

Step 1. We define $\alpha_{ijkl}^m \in C^\infty\left(\overline{\Omega}\right)$ in the following way (recall that $i<j$ and $k \leq l$)

$$\alpha_{ijkl}^m = 0 \quad \text{if } i \neq m \text{ and } j \neq m$$

and if $i < m$

$$\alpha_{imkl}^m = \left\{\begin{array}{ll} 0 & \text{if } k \neq i \text{ and } l \neq i \\ \nu_l & \text{if } k = i \text{ and } l \neq i \\ \nu_k & \text{if } l = i \text{ and } k \neq i \\ \nu_i & \text{if } k = l = i \end{array}\right.$$

while if $j > m$

$$\alpha_{mjkl}^m = \left\{\begin{array}{ll} 0 & \text{if } k \neq j \text{ and } l \neq j \\ -\nu_l & \text{if } k = j \text{ and } l \neq j \\ -\nu_k & \text{if } l = j \text{ and } k \neq j \\ -\nu_j & \text{if } k = l = j. \end{array}\right.$$

Step 2. If we let $h = \partial^2 w / \partial \nu^2$, we find (on $\partial\Omega$), according to Lemma 20,

$$N_{a,2}^{\alpha,\beta,\gamma}(w) = \sum_m \left[\sum_{k \le l} \sum_{i<j} \alpha_{ijkl}^m \, h^{ij} \nu_k \nu_l \right] dx^m = \sum_{i<j} h^{ij} \left[\sum_m \left[\sum_{k \le l} \alpha_{ijkl}^m \, \nu_k \nu_l \right] dx^m \right]$$

and

$$\nu \lrcorner \frac{\partial^2 w}{\partial \nu^2} = \nu \lrcorner h = \sum_m \left[\sum_i h^{im} \nu_i \right] dx^m .$$

We therefore have to prove that, for every m,

$$\sum_{i<j} \sum_{k \le l} \alpha_{ijkl}^m \, h^{ij} \nu_k \nu_l = \sum_i h^{im} \nu_i \, .$$

It follows from the definition of α_{ijkl}^m that for $i < m$

$$\begin{aligned} \sum_{k \le l} \alpha_{imkl}^m \, \nu_k \nu_l &= \alpha_{imii}^m \, \nu_i^2 + \sum_{l(k=l \neq i)} \alpha_{imll}^m \, \nu_l^2 + \sum_{k<l(k,l \neq i)} \alpha_{imkl}^m \, \nu_k \nu_l \\ &+ \sum_{k<l(k=i)} \alpha_{imil}^m \, \nu_i \nu_l + \sum_{k<l(l=i)} \alpha_{imki}^m \, \nu_k \nu_i \\ &= \nu_i^3 + \nu_i \sum_{i<l} \nu_l^2 + \nu_i \sum_{k<i} \nu_k^2 = \nu_i \end{aligned}$$

and similarly for $j > m$

$$\sum_{k \le l} \alpha_{mjkl}^m \, \nu_k \nu_l = -\nu_j \, .$$

Combining these results we find

$$\sum_{i<j} h^{ij} \sum_{k \le l} \alpha_{ijkl}^m \, \nu_k \nu_l = \sum_{i<m} h^{im} \nu_i - \sum_{j>m} h^{mj} \nu_j = \sum_i h^{im} \nu_i$$

as wished.

Step 3. It remains to show that (9) is satisfied.

- We start with $\alpha_{ijmm}^m = 0$. We have

$$\alpha_{ijmm}^m = \begin{cases} 0 & \text{if } i, j \neq m \\ \alpha_{mjmm}^m = 0 & \text{if } i = m < j \\ \alpha_{immm}^m = 0 & \text{if } i < j = m. \end{cases}$$

- We then prove that $\alpha_{ijll}^m + \alpha_{ijlm}^l = \alpha_{ijmm}^l + \alpha_{ijlm}^m = 0$ (if $i < j$ and $l < m$). We get

$$\alpha_{ijll}^m = \begin{cases} 0 & \text{if } i, j \neq m \\ \alpha_{mjll}^m = 0 & \text{if } l < i = m < j \\ \alpha_{lmll}^m = \nu_l = \nu_i & \text{if } i = l < j = m \\ \alpha_{imll}^m = 0 & \text{if } i < j = m \text{ and } i \neq l \end{cases}$$

$$\alpha_{ijlm}^{l}=\begin{cases}0 & \text{if } i,j\neq l\\ \alpha_{ljlm}^{l}=0 & \text{if } i=l<j \text{ and } j\neq m\\ \alpha_{lmlm}^{l}=-\nu_l=-\nu_i & \text{if } i=l<j=m\\ \alpha_{illm}^{l}=0 & \text{if } i<j=l<m\end{cases}$$

and hence $\alpha_{ijll}^{m}+\alpha_{ijlm}^{l}=0$. We proceed in an analogous way for $\alpha_{ijmm}^{l}+\alpha_{ijlm}^{m}=0$. We indeed find

$$\alpha_{ijmm}^{l}=\begin{cases}0 & \text{if } i,j\neq l\\ \alpha_{ljmm}^{l}=0 & \text{if } i=l<j \text{ and } j\neq m\\ \alpha_{lmlm}^{l}=-\nu_l=-\nu_i & \text{if } i=l<j=m\\ \alpha_{ilmm}^{l}=0 & \text{if } i<j=l<m\end{cases}$$

$$\alpha_{ijlm}^{m}=\begin{cases}0 & \text{if } i,j\neq m\\ \alpha_{mjlm}^{m}=0 & \text{if } l<i=m<j\\ \alpha_{lmlm}^{m}=\nu_l=\nu_i & \text{if } i=l<j=m\\ \alpha_{imlm}^{m}=0 & \text{if } i<j=m \text{ and } i\neq l.\end{cases}$$

- We finally prove that, for $i<j$ and $k<l<m$,

$$\alpha_{ijlm}^{k}+\alpha_{ijkm}^{l}+\alpha_{ijkl}^{m}=0.$$

The only cases where there are nonzero entries are $i=k<j=l<m$, $i=k<l<j=m$ and $k<i=l<j=m$. We immediately obtain

$$\begin{aligned}&\alpha_{ijlm}^{k}+\alpha_{ijkm}^{l}+\alpha_{ijkl}^{m}\\ &=\begin{cases}\alpha_{illm}^{i}+\alpha_{ilim}^{l}+\alpha_{ilil}^{m}=-\nu_m+\nu_m+0=0 & \text{if } i=k<j=l<m\\ \alpha_{imlm}^{i}+\alpha_{imim}^{l}+\alpha_{imil}^{m}=-\nu_l+0+\nu_l=0 & \text{if } i=k<l<j=m\\ \alpha_{imim}^{k}+\alpha_{imkm}^{i}+\alpha_{imki}^{m}=0-\nu_k+\nu_k=0 & \text{if } k<i=l<j=m\end{cases}\end{aligned}$$

and thus the claim.

Step 4. We now provide the second proof. We define the α_{ijkl}^{m} as functions of ν implicitly by the equation

$$N_{a,2}^{\alpha,\beta,\gamma}(w)=\delta(\nu\wedge\delta w)+\text{ lower order terms in derivatives of } w.$$

Since $\delta\delta(\nu\wedge\delta w)=0$ for all $w\in C^{3}\left(\overline{\Omega};\mathbb{R}^{n(n-1)/2}\right)$, we obtain that $L_a\left(N_{a,2}^{\alpha,\beta,\gamma}(w)\right)$ contains only derivatives of w of order less or equal to 2. Therefore, as in the proof of Proposition 11, we must have that (9) is satisfied. Note that for any $w\in C^{r+2}\left(\overline{\Omega};\Lambda^{2}\right)$ verifying $w=\partial w/\partial\nu=0$ on $\partial\Omega$, we have

$$N_{a,2}^{\alpha,\beta,\gamma}(w)=\delta\left(\sum_{i,j,k=1}^{n}\nu_k w_{x_i}^{ij}dx^k\wedge dx^j\right)=\sum_{i,j,k=1}^{n}\nu_k w_{x_kx_i}^{ij}dx^j\quad\text{on }\partial\Omega.$$

From Lemma 20 and the fact that $|\nu|^2 = 1$ we get that

$$N_{a,2}^{\alpha,\beta,\gamma}(w) = \sum_{i,j,k=1}^{n} \nu_i \nu_k^2 \frac{\partial^2 w^{ij}}{\partial \nu^2} dx^j = \nu \lrcorner \frac{\partial^2 w}{\partial \nu^2} \quad \text{on } \partial\Omega.$$

This proves the lemma. ■

6 Proof of Theorems 4 and 6

We now turn to the proof of Theorem 4.

Proof *Step 1.* Let us first simplify the problem.

Step 1.1. We can assume, without loss of generality, that $u_0 = 0$. Indeed it suffices to set $v = u - u_0$ and we find (since $L_a(u_0) \in C^{r,s}(\overline{\Omega})$)

$$\begin{cases} L_a(v) = L_a(u) - L_a(u_0) = f - L_a(u_0) & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

So from now on we will assume that $u_0 = 0$.

Step 1.2. We next show that we can, without loss of generality, assume that

$$\operatorname{div} a = 0 \text{ in } \Omega \quad \text{and} \quad \langle a; \nu \rangle = 0 \text{ on } \partial\Omega \tag{10}$$

where ν is the outward unit normal to $\partial\Omega$. Indeed Theorem 6.12 (ii) in [4] implies that we can find

$$b \in C_N^{r+5,s}(\overline{\Omega}), \quad c \in C_N^{r+5,s}(\overline{\Omega}; \Lambda^2) \quad \text{and} \quad g \in \mathcal{H}_N(\Omega; \Lambda^1)$$

so that $a = db + \delta c + g$ and hence, in particular (recall that $\nu \lrcorner c = 0$ implies $\nu \lrcorner \delta c = 0$),

$$\operatorname{div}(\delta c + g) = \delta(\delta c + g) = 0 \text{ in } \Omega \quad \text{and} \quad \langle \delta c + g; \nu \rangle = \nu \lrcorner (\delta c + g) = 0 \text{ on } \partial\Omega.$$

Setting $v = e^b u$, we have $v = 0$ on $\partial\Omega$ and

$$\begin{aligned} \delta v + (\delta c + g) \lrcorner v &= \delta(e^b u) + (\delta c + g) \lrcorner (e^b u) = e^b [\delta u + db \lrcorner u + (\delta c + g) \lrcorner u] \\ &= e^b [\delta u + a \lrcorner u] = e^b f. \end{aligned}$$

Thus, up to replacing a by $\delta c + g$, f by $e^b f$ and u by $e^b u$, we are lead to solving our problem under the further hypotheses (10).

Step 2. Consider then the Neumann problem

$$\begin{cases} \Delta v - |a|^2 v = f & \text{in } \Omega \\ \frac{\partial v}{\partial \nu} = \langle \operatorname{grad} v; \nu \rangle = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $\operatorname{curl} a \not\equiv 0$ (here we do not need the full hypothesis on $\operatorname{curl} a$), we have that $a \not\equiv 0$. We can then find (cf. [10] page 130) a solution $v \in C^{r+2,s}(\overline{\Omega})$, without any restriction on f (note that here we only need $a \in C^{r,s}$). Moreover the solution is unique and (cf. [1] or Theorem 6.30 page 127 in [10]) there exists $K = K(r, s, \|a\|_{C^{r,s}}, \Omega)$ such that

$$\|v\|_{C^{r+2,s}} \leq K \|f\|_{C^{r,s}} .$$

Note that Step 2 is optimal both from the point of view of the regularity of a and for the condition $\operatorname{curl} a \not\equiv 0$.

Step 3. We then apply Theorem 17 to find $\alpha_{ijkl}^m \in C^\infty(\overline{\Omega})$, $\beta_{ijl}^m \in C^{r+2,s}(\overline{\Omega})$, $\gamma_{ij}^m \in C^{r+1,s}(\overline{\Omega})$ satisfying

$$L_a\left(N_{a,2}^{\alpha,\beta,\gamma}(w)\right) = 0, \quad \forall\, w \in C^3\left(\overline{\Omega}; \Lambda^2\right)$$

and $w \in C^{r+3,s}\left(\overline{\Omega}; \Lambda^2\right)$ verifying

$$N_{a,2}^{\alpha,\beta,\gamma}(w) = -\operatorname{grad} v + av \quad \text{on } \partial\Omega.$$

Note that this can be done, since $-\operatorname{grad} v + av \in C^{r+1,s}\left(\overline{\Omega}; \mathbb{R}^n\right)$ and

$$\langle \nu; -\operatorname{grad} v + av \rangle = 0.$$

We finally let

$$u = \operatorname{grad} v - av + N_{a,2}^{\alpha,\beta,\gamma}(w).$$

Clearly this is a solution of our problem. Indeed we have $u \in C^{r+1,s}\left(\overline{\Omega}; \mathbb{R}^n\right)$ and, by construction, $u = 0$ on $\partial\Omega$. Moreover (since $\operatorname{div} a = 0$) we have, in Ω, that

$$\begin{aligned} L_a(u) &= L_a(\operatorname{grad} v - av) + L_a\left(N_{a,2}^{\alpha,\beta,\gamma}(w)\right) \\ &= \operatorname{div}(\operatorname{grad} v - av) + \langle a; \operatorname{grad} v - av \rangle = \Delta v - |a|^2 v = f. \end{aligned}$$

The estimate easily follows and this concludes the proof of the theorem. ■

We now turn to the proof of Theorem 6.

Proof The proof is almost identical to the preceding one.

Step 1. As before we can assume that $u_0 = 0$. We next show that we can, without loss of generality, assume that $a \in C^\infty\left(\overline{\Omega}; \mathbb{R}^n\right)$,

$$\operatorname{curl} a = 0 \text{ in } \Omega, \quad \operatorname{div} a = 0 \text{ in } \Omega \quad \text{and} \quad \langle a; \nu \rangle = 0 \text{ on } \partial\Omega$$

where ν is the outward unit normal to $\partial\Omega$. As before we can find

$$b \in C_N^{r+1,s}\left(\overline{\Omega}\right), \quad c \in C_N^{r+1,s}\left(\overline{\Omega}; \Lambda^2\right) \quad \text{and} \quad g \in \mathcal{H}_N\left(\Omega; \Lambda^1\right)$$

so that $a = db + \delta c + g$. Observe that in the present case $\delta c \in \mathcal{H}_N\left(\Omega; \Lambda^1\right)$ and thus can be taken 0 (by incorporating it into g). Indeed since $da = 0$, we have $d(\delta c) = 0$, $\delta(\delta c) = 0$ and $\nu \lrcorner \delta c = 0$ (since $\nu \lrcorner c = 0$). Hence, in particular,

$$\operatorname{div} g = \delta g = 0 \text{ in } \Omega \quad \text{and} \quad \langle g; \nu \rangle = \nu \lrcorner g = 0 \text{ on } \partial\Omega.$$

The remaining part of the proof is then as in Step 1.2 of Theorem 4. Note that now, according to Theorem 6.3 in [4], $a \in C^{\infty}\left(\overline{\Omega}; \mathbb{R}^n\right)$.

Step 2. This step is identical to that of Theorem 4.

Step 3. We next observe that

$$N_{a,2}^{\alpha,\beta,\gamma}(w) = \delta w + a \lrcorner w$$

indeed satisfies

$$L_a\left(N_{a,2}^{\alpha,\beta,\gamma}(w)\right) = 0, \quad \forall\, w \in C^2\left(\overline{\Omega}; \Lambda^2\right).$$

We then invoke Lemma 8.11 (ii) in [4] to find $w \in C^{r+2,s}\left(\overline{\Omega}; \Lambda^2\right)$ verifying

$$N_{a,2}^{\alpha,\beta,\gamma}(w) = \delta w + a \lrcorner w = -\operatorname{grad} v + av \quad \text{on } \partial\Omega.$$

Note that this can be done, since $-\operatorname{grad} v + av \in C^{r+1,s}\left(\overline{\Omega}; \mathbb{R}^n\right)$ and

$$\langle \nu; -\operatorname{grad} v + av \rangle = 0.$$

We finally have that a solution is given by

$$u = \operatorname{grad} v - av + N_{a,2}^{\alpha,\beta,\gamma}(w) = \operatorname{grad} v - av + \delta w + a \lrcorner w.$$

The remaining part of the proof is as in Theorem 4. ■

7 Poincaré type lemma in the interior

The results of the present section are not used elsewhere in the article, but we give them for the sake of completeness. We have already studied the problem on the boundary, cf. Theorem 17. The second case that we want to discuss is the problem in the interior. Given u with

$$L_a(u) = \operatorname{div} u + \langle a; u \rangle = 0 \quad \text{in } \Omega$$

we will find w (and α, β, γ so that $L_a\left(N_{a,2}^{\alpha,\beta,\gamma}(v)\right) = 0$ for every v) such that

$$N_{a,2}^{\alpha,\beta,\gamma}(w) = u \quad \text{in } \Omega.$$

This is the exact analogue of finding, provided $\operatorname{div} u = 0$, a w such that $\operatorname{curl}^* w = u$ in Ω.

7.1 The case of the standard symplectic form

We recall the notations of Example 14. Let $0 \leq 2p \leq n$ and

$$\overline{a} = \sum_{i=1}^{p} x_{2i}\, dx^{2i-1}.$$

Furthermore there exist $\overline{\alpha}, \overline{\beta}, \overline{\gamma} \in C^{\infty}$ (as in Example 14) such that

$$\begin{aligned} N_{\overline{a},2}^{\overline{\alpha},\overline{\beta},\overline{\gamma}}(w) = \;& e^{-x_1 x_2}\left(-\textstyle\sum_{j=2}^{n} w_{x_1 x_j}^{1j} + x_1\, w_{x_1}^{12} - \sum_{i=2}^{p} x_{2i}\, w_{x_1}^{1(2i-1)} - w^{12}\right) dx^1 \\ & + e^{-x_1 x_2} \textstyle\sum_{j=2}^{n} w_{x_1 x_1}^{1j}\, dx^j \end{aligned} \tag{11}$$

and with

$$L_{\overline{a}}\left(N_{\overline{a},2}^{\overline{\alpha},\overline{\beta},\overline{\gamma}}(w) \right) = 0, \quad \forall\, w \in C^3\left(\overline{\Omega}; \Lambda^2\right). \tag{12}$$

We also recall that if $S \in C^{\infty}$, $a = \overline{a} + dS$,

$$(\alpha, \beta, \gamma) = e^{-S}\left(\overline{\alpha}, \overline{\beta}, \overline{\gamma}\right) \quad \Leftrightarrow \quad N_{a,2}^{\alpha,\beta,\gamma}(w) = e^{-S} N_{\overline{a},2}^{\overline{\alpha},\overline{\beta},\overline{\gamma}}(w), \tag{13}$$

then

$$L_a\left(N_{a,2}^{\alpha,\beta,\gamma}(w) \right) = 0, \quad \forall\, w \in C^3\left(\overline{\Omega}; \Lambda^2\right).$$

We start with a theorem that gives a global result.

Theorem 21 *Let $\Omega \subset \mathbb{R}^n$ a bounded open smooth set. Let $a = \overline{a} + dS \in C^{\infty}\left(\overline{\Omega}; \Lambda^1\right)$ and let α, β, γ be defined as in (13). Let $u \in C^{\infty}\left(\overline{\Omega}; \Lambda^1\right)$ satisfy $L_a(u) = 0$. Then there exists $w \in C^{\infty}\left(\overline{\Omega}; \Lambda^2\right)$ verifying*

$$u = N_{a,2}^{\alpha,\beta,\gamma}(w).$$

Proof We divide the proof into two steps.

Step 1. Let us first show that we can assume that $S = 0$. Indeed suppose that we already proved the lemma when $a = \overline{a}$, i.e. for every $U \in C^{\infty}\left(\overline{\Omega}; \Lambda^1\right)$ satisfying $L_{\overline{a}}(U) = 0$, we can find $w \in C^{\infty}\left(\overline{\Omega}; \Lambda^2\right)$ verifying

$$U = N_{\overline{a},2}^{\overline{\alpha},\overline{\beta},\overline{\gamma}}(w).$$

We then set $u = e^{-S} U$ and find that

$$u = e^{-S} N_{\overline{a},2}^{\overline{\alpha},\overline{\beta},\overline{\gamma}}(w) = N_{a,2}^{\alpha,\beta,\gamma}(w).$$

Observe also that $L_{\overline{a}}(U) = 0$ if and only if $L_a(u) = 0$, since

$$\begin{aligned} L_{\overline{a}}(U) &= \operatorname{div} U + \langle \overline{a}; U \rangle = \operatorname{div}\left(e^S u\right) + e^S \langle \overline{a}; u \rangle \\ &= e^S \operatorname{div} u + e^S \langle dS; u \rangle + e^S \langle \overline{a}; u \rangle \\ &= e^S \left[\operatorname{div} u + \langle \overline{a} + dS; u \rangle\right] = e^S L_a(u). \end{aligned}$$

So from now on we will assume that $a = \overline{a}$. Thus we may invoke (11) and (12).

Step 2. It remains to verify that with $a = \overline{a}$, we have the theorem. Namely if $L_{\overline{a}}(u) = 0$ in Ω, we have to find w such that

$$u = N_{\overline{a}}^{\overline{\alpha},\overline{\beta},\overline{\gamma}}(w) \quad \text{in } \Omega.$$

(i) We introduce the following notation

$$x = (x_1, \cdots, x_n) = (x_1, \widehat{x}).$$

We first extend u to $\mathbb{R}^n$ in an arbitrary way and then define

$$\overline{w}^{1j} = \int_0^{x_1} \left(\int_0^s e^{tx_2} u^j (t, \widehat{x})\, dt \right) ds \quad \text{for } j = 2, \cdots, n.$$

Observe that

$$\overline{w}^{1j}_{x_1 x_1} = e^{x_1 x_2} u^j \quad \text{in } \mathbb{R}^n \quad \text{for } j = 2, \cdots, n. \tag{14}$$

(ii) We next let

$$g(x) = -e^{x_1 x_2} u^1 - \sum_{j=2}^{n} \overline{w}^{1j}_{x_1 x_j} + x_1 \overline{w}^{12}_{x_1} - \sum_{i=2}^{p} x_{2i}\, \overline{w}^{1(2i-1)}_{x_1} - \overline{w}^{12}.$$

We claim that in Ω the function g is independent of x_1, that is $g(x) = g(\widehat{x})$. Indeed, using (14) and that $L_{\overline{a}}(u) = 0$ in Ω, we get

$$\begin{aligned} g_{x_1} &= -x_2 e^{x_1 x_2} u^1 - e^{x_1 x_2} u^1_{x_1} - \sum_{j=2}^{n} \left(e^{x_1 x_2} u^j \right)_{x_j} \\ &\quad + x_1 e^{x_1 x_2} u^2 - \sum_{i=2}^{p} x_{2i}\, e^{x_1 x_2} u^{2i-1} = -e^{x_1 x_2} L_{\overline{a}}(u) = 0. \end{aligned}$$

(iii) We finally choose

$$w^{12} = \overline{w}^{12} + g \quad \text{and} \quad w^{1j} = \overline{w}^{1j} \text{ for } j = 3, \cdots, n.$$

We therefore obtain from (14) and the definition of w that, in Ω,

$$u^j = e^{-x_1 x_2} \overline{w}^{1j}_{x_1 x_1} = e^{-x_1 x_2} w^{1j}_{x_1 x_1} = \left(N^{\overline{\alpha}, \overline{\beta}, \overline{\gamma}}_{\overline{a}, 2}(w) \right)^j \quad \text{for } j = 2, \cdots, n.$$

It follows from the definition of g and using that $g(x) = g(\widehat{x})$ in Ω, that we also have, in Ω,

$$\begin{aligned} u^1 &= e^{-x_1 x_2} \left(-\sum_{j=2}^{n} \overline{w}^{1j}_{x_1 x_j} + x_1 \overline{w}^{12}_{x_1} - \sum_{i=2}^{p} x_{2i}\, \overline{w}^{1(2i-1)}_{x_1} - \overline{w}^{12} - g \right) \\ &= e^{-x_1 x_2} \left(-\sum_{j=2}^{n} w^{1j}_{x_1 x_j} + x_1 w^{12}_{x_1} - \sum_{i=2}^{p} x_{2i}\, w^{1(2i-1)}_{x_1} - w^{12} \right) = \left(N^{\overline{\alpha}, \overline{\beta}, \overline{\gamma}}_{\overline{a}, 2}(w) \right)^1 \end{aligned}$$

which is what had to be established. ■

7.2 Some intermediate results: Part 1

We begin with a definition.

Definition 22 *Let $A \in \mathbb{R}^{n\times n}$ be a matrix with entries a^i_j, where the upper index i stands for the row and the lower index j stands for the column. Then we define $A^\sharp$ as the matrix*

$$A^\sharp = \left\{ \left(A^\sharp\right)^{kl}_{ij} \right\}^{1\le k\le l\le n}_{1\le i\le j\le n} \in \mathbb{R}^{\left(\binom{n}{2}+n\right)\times\left(\binom{n}{2}+n\right)}$$

by ordering the indices (k,l) standing for the rows, respectively (i,j) standing for the columns, in lexicographic order and

$$\left(A^\sharp\right)^{kl}_{ij} = \begin{cases} a^k_i a^k_j & \text{if } k = l \\ a^k_i a^l_j + a^k_j a^l_i & \text{if } k < l. \end{cases}$$

Example 23 *If $n = 2$, we have*

$$A = \begin{pmatrix} a^1_1 & a^1_2 \\ a^2_1 & a^2_2 \end{pmatrix} \quad \text{and} \quad A^\sharp = \begin{pmatrix} a^1_1 a^1_1 & a^1_1 a^1_2 & a^1_2 a^1_2 \\ 2a^1_1 a^2_1 & a^1_1 a^2_2 + a^1_2 a^2_1 & 2a^1_2 a^2_2 \\ a^2_1 a^2_1 & a^2_1 a^2_2 & a^2_2 a^2_2 \end{pmatrix}.$$

Lemma 24 *The following identity holds true*

$$(AB)^\sharp = A^\sharp B^\sharp$$

in particular

$$\left(A^\sharp\right)^{-1} = \left(A^{-1}\right)^\sharp.$$

Remark 25 *As it will be obvious from the proof, we do not need lexicographic ordering in the definition of $A^\sharp$ for this lemma to hold true.*

Proof Let $r \le t$ and $p \le q$ be given. We have to show that

$$\left((AB)^\sharp\right)^{rt}_{pq} = \left(A^\sharp B^\sharp\right)^{rt}_{pq}. \tag{15}$$

Case 1: $r < t$. In this case we get

$$\begin{aligned}\left((AB)^\sharp\right)^{rt}_{pq} &= (AB)^r_p (AB)^t_q + (AB)^r_q (AB)^t_p \\ &= \left(\sum_{i=1}^n a^r_i b^i_p\right)\left(\sum_{j=1}^n a^t_j b^j_q\right) + \left(\sum_{i=1}^n a^r_i b^i_q\right)\left(\sum_{j=1}^n a^t_j b^j_p\right).\end{aligned}$$

Thus we get that

$$\left((AB)^\sharp\right)^{rt}_{pq} = \sum_{i,j=1}^n a^r_i a^t_j b^i_p b^j_q + \sum_{i,j=1}^n a^r_i a^t_j b^i_q b^j_p. \tag{16}$$

On the other hand we have that the right hand side of (15) is

$$\left(A^\sharp B^\sharp\right)^{rt}_{pq} = \sum_{i\le j}\left(A^\sharp\right)^{rt}_{ij}\left(B^\sharp\right)^{ij}_{pq} = \sum_{i<j}\left(A^\sharp\right)^{rt}_{ij}\left(B^\sharp\right)^{ij}_{pq} + \sum_{i=1}^{n}\left(A^\sharp\right)^{rt}_{ii}\left(B^\sharp\right)^{ii}_{pq}$$

$$= \sum_{i<j}\left(a^r_i a^t_j + a^r_j a^t_i\right)\left(b^i_p b^j_q + b^i_q b^j_p\right) + \sum_{i=1}^{n}\left(a^r_i a^t_i + a^r_i a^t_i\right) b^i_p b^i_q\,.$$

If we expand the products in this last equation and split the sum in (16) as $\sum_{i,j} = \sum_{i<j} + \sum_{i>j} + \sum_{i=j}$, one easily confirms that (15) holds true.

Case 2: $r = t$. In this case we obtain that

$$\left((AB)^\sharp\right)^{rr}_{pq} = (AB)^r_p\,(AB)^r_q = \sum_{i,j=1}^{n} a^r_i a^r_j b^i_p b^j_q\,.$$

The right hand side of (15) is equal to

$$\left(A^\sharp B^\sharp\right)^{rr}_{pq} = \sum_{i\le j}\left(A^\sharp\right)^{rr}_{ij}\left(B^\sharp\right)^{ij}_{pq} = \sum_{i<j}\left(A^\sharp\right)^{rr}_{ij}\left(B^\sharp\right)^{ij}_{pq} + \sum_{i=1}^{n}\left(A^\sharp\right)^{rr}_{ii}\left(B^\sharp\right)^{ii}_{pq}$$

$$= \sum_{i<j} a^r_i a^r_j\left(b^i_p b^j_q + b^i_q b^j_p\right) + \sum_{i=1}^{n} a^r_i a^r_i b^i_p b^i_q\,.$$

We can now proceed as in Case 1 to conclude the proof of the lemma.

The last statement of the lemma follows from the first statement and the observation that

$$\left(I_n\right)^\sharp = I_{\binom{n}{2}+n}\,,$$

where $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix. ■

Lemma 26 *The following identity is valid*

$$\det A^\sharp = (\det A)^{n+1}\,.$$

Proof *Step 1.* Let us show that if A is an upper triangular matrix, i.e.

$$a^p_q = 0 \quad \text{if } q < p,$$

then $A^\sharp$ is also an upper triangular matrix. We have to show that for every $i \le j$ and $k \le l$

$$\left(A^\sharp\right)^{kl}_{ij} = 0 \quad \text{if } (ij) < (kl)\,.$$

In view of the lexicographic ordering, the inequality $(ij) < (kl)$ is equivalent to:

$$\text{either} \quad i < k \quad \text{or} \quad \{i = k \text{ and } j < l\}\,.$$

Case 1: $i < k$. By assumption we get that $a_i^k = 0$. Moreover if $l > k$, then also $a_i^l = 0$. We therefore have

$$\left(A^\sharp\right)_{ij}^{kl} = \begin{cases} a_i^k a_j^k = 0 & \text{if } k = l \\ a_i^k a_j^l + a_j^k a_i^l = 0 & \text{if } k < l. \end{cases}$$

Case 2: $i = k$ *and* $j < l$. If $k = l$, then we have by assumption that $a_j^k = a_j^l = 0$. Whereas if $k < l$, then $a_i^l = a_k^l = a_j^l = 0$. We can therefore conclude as in Case 1.

Step 2. Let us first assume that A is an upper triangular matrix. From Step 1 we therefore obtain

$$\begin{aligned} \det A^\sharp &= \prod_{i \leq j} \left(A^\sharp\right)_{ij}^{ij} = \prod_{i<j} \left(A^\sharp\right)_{ij}^{ij} \prod_{i=1}^{n} \left(A^\sharp\right)_{ii}^{ii} \\ &= \prod_{i<j} \left(a_i^i a_j^j + a_j^i a_i^j\right) \prod_{i=1}^{n} a_i^i a_i^i = \prod_{i<j} \left(a_i^i a_j^j\right) \prod_{i=1}^{n} a_i^i a_i^i . \end{aligned}$$

It can be easily shown by induction that

$$\prod_{i<j} a_i^i a_j^j = \left(\prod_{i=1}^{n} a_i^i\right)^{n-1} .$$

This leads to

$$\det A^\sharp = \left(\prod_{i=1}^{n} a_i^i\right)^{n-1} \left(\prod_{i=1}^{n} a_i^i\right)^{2} = \left(\prod_{i=1}^{n} a_i^i\right)^{n+1} = (\det A)^{n+1}.$$

This shows the lemma for upper triangular matrices. If A is an arbitrary matrix, then $PAP^{-1} = U$, where U is upper triangular. From Lemma 24 we obtain that

$$P^\sharp A^\sharp \left(P^\sharp\right)^{-1} = U^\sharp$$

which leads to

$$\det A^\sharp = \det U^\sharp = (\det U)^{n+1} = (\det A)^{n+1},$$

which was what had to be shown. ■

7.3 Some intermediate results: Part 2

We start with a lemma.

Lemma 27 *Let $O, \Omega \subset \mathbb{R}^n$ be two open sets and $\psi \in \mathrm{Diff}^\infty(\Omega; O)$. Then there exists a unique invertible map $\sigma = \sigma_\psi$, such that*

$$\sigma : C^\infty\left(\Omega; \mathbb{R}^{\left(\binom{n}{2}+n\right)n} \times \mathbb{R}^{n^2} \times \mathbb{R}^n\right) \to C^\infty\left(O; \mathbb{R}^{\left(\binom{n}{2}+n\right)n} \times \mathbb{R}^{n^2} \times \mathbb{R}^n\right),$$

satisfies

$$*\psi^* \left(*N^{\sigma(\alpha,\beta,\gamma)} (f) \right) = N^{\alpha,\beta,\gamma} (f \circ \psi) \quad \text{in } \Omega \tag{17}$$

for every $f \in C^\infty (O)$ *and every* $(\alpha, \beta, \gamma) \in C^\infty \left(\Omega; \mathbb{R}^{(\binom{n}{2}+n)n} \times \mathbb{R}^{n^2} \times \mathbb{R}^n \right)$.

Remark 28 *(i) Since in the statement of the lemma there is no dependence on* a*, we have dropped the subindex* a *in* $N_a^{\alpha,\beta,\gamma}$*.*

(ii) It follows from the proof that σ *is linear and can be written in the form*

$$\sigma (\alpha, \beta, \gamma) (x) = A_\psi (x) \cdot \left(\alpha \left(\psi^{-1} (x) \right), \beta \left(\psi^{-1} (x) \right), \gamma \left(\psi^{-1} (x) \right) \right),$$

where A_ψ *is an invertible matrix of dimension* $\left(\binom{n}{2} + n \right) n + n^2 + n$ *and* $\cdot$ *denotes the multiplication between matrices and vectors.*

Proof *Step 1.* Let us write explicitly the equation (17) in terms of $f (\psi), f_{x_s} (\psi)$ and $f_{x_s x_t} (\psi)$. Setting the identities

$$(f (\psi))_{x_l} = \sum_{s=1}^{n} f_{x_s} (\psi) \psi_{x_l}^s$$

$$(f (\psi))_{x_k x_l} = \sum_{s,t=1}^{n} f_{x_s x_t} (\psi) \psi_{x_l}^s \psi_{x_k}^t + \sum_{s=1}^{n} f_{x_s} (\psi) \psi_{x_l x_k}^s$$

into the right hand side of (17), given by

$$N^{\alpha,\beta,\gamma}(f (\psi)) = \sum_{m=1}^{n} \left[\sum_{k \le l} \alpha_{kl}^m (f (\psi))_{x_k x_l} + \sum_{l=1}^{n} \beta_l^m (f (\psi))_{x_l} + \gamma^m f (\psi) \right] dx^m,$$

and using that

$$\sum_{s,t=1}^{n} f_{x_s x_t} (\psi) \psi_{x_l}^s \psi_{x_k}^t = \sum_{s<t} f_{x_s x_t} (\psi) \left(\psi_{x_l}^s \psi_{x_k}^t + \psi_{x_l}^t \psi_{x_k}^s \right) + \sum_{s=1}^{n} f_{x_s x_s} (\psi) \psi_{x_l}^s \psi_{x_k}^s,$$

gives by a straightforward calculation

$$\begin{aligned} \left(N^{\alpha,\beta,\gamma}(f (\psi)) \right)^m = & \sum_{s<t} f_{x_s x_t} (\psi) \sum_{k \le l} \alpha_{kl}^m \left(\psi_{x_l}^s \psi_{x_k}^t + \psi_{x_l}^t \psi_{x_k}^s \right) \\ & + \sum_{s=1}^{n} f_{x_s x_s} (\psi) \sum_{k \le l} \alpha_{kl}^m \psi_{x_l}^s \psi_{x_k}^s \\ & + \sum_{s=1}^{n} f_{x_s} (\psi) \left(\sum_{k \le l} \alpha_{kl}^m \psi_{x_l x_k}^s + \sum_{l=1}^{n} \beta_l^m \psi_{x_l}^s \right) + \gamma^m f (\psi). \end{aligned} \tag{18}$$

Using the notation $\sigma(\alpha,\beta,\gamma) = \left(\overline{\alpha},\overline{\beta},\overline{\gamma}\right)$, we get by a direct calculation that the left hand side of (17) is of the form

$$\left(*\psi^*\left(*N^{\overline{\alpha},\overline{\beta},\overline{\gamma}}(f)\right)\right)^m = \sum_{s\leq t} A^m_{st} f_{x_s x_t}(\psi) + \sum_{s=1}^{m} B^m_s f_{x_s}(\psi) + C^m f(\psi),$$

where A^m_{st}, B^m_s and C^m are abbreviations for

$$\begin{aligned} A^m_{st} &= \sum_{q=1}^{n} (-1)^{q-1+n-m}\, \overline{\alpha}^q_{st}(\psi)\, \det(\nabla\psi)^{1\cdots\widehat{q}\cdots n}_{1\cdots\widehat{m}\cdots n} \\ B^m_s &= \sum_{q=1}^{n} (-1)^{q-1+n-m}\, \overline{\beta}^q_s(\psi)\, \det(\nabla\psi)^{1\cdots\widehat{q}\cdots n}_{1\cdots\widehat{m}\cdots n} \\ C^m &= \sum_{q=1}^{n} (-1)^{q-1+n-m}\, \overline{\gamma}^q(\psi)\, \det(\nabla\psi)^{1\cdots\widehat{q}\cdots n}_{1\cdots\widehat{m}\cdots n}, \end{aligned}$$

where $1\cdots\widehat{q}\cdots n$ means that the row q has been omitted in $\nabla\psi$, and $1\cdots\widehat{m}\cdots n$ means that the column m has been omitted. In view of (18), the equation (17) is valid for every f if and only if the following set of equations is satisfied for each $m = 1,\cdots,n$

$$\begin{cases} \displaystyle\sum_{k\leq l} \alpha^m_{kl}\left(\psi^s_{x_l}\psi^t_{x_k} + \psi^t_{x_l}\psi^s_{x_k}\right) = A^m_{st} & \text{for every } s<t \\ \displaystyle\sum_{k\leq l} \alpha^m_{kl}\psi^s_{x_l}\psi^s_{x_k} = A^m_{ss} & \text{for every } s = 1,\cdots,n \\ \displaystyle\sum_{k\leq l} \alpha^m_{kl}\psi^s_{x_k x_l} + \sum_{l=1}^{n}\beta^m_l\psi^s_{x_l} = B^m_s & \text{for every } s = 1,\cdots,n \\ \gamma^m = C^m \end{cases} \tag{19}$$

Step 2. In view of Step 1, we have to show that the linear system of equations (19) can be solved uniquely for $(\overline{\alpha},\overline{\beta},\overline{\gamma})$ in terms of (α,β,γ), and conversely.

Step 2.1. Let us first show that we can solve (19) for $(\overline{\alpha},\overline{\beta},\overline{\gamma})$. Note that entries of the adjoint matrix $\operatorname{adj}(\nabla\psi)$ of $\nabla\psi$ are precisely given by

$$(\operatorname{adj}(\nabla\psi))^q_m = (-1)^{q-m}\det(\nabla\psi)^{1\cdots\widehat{q}\cdots n}_{1\cdots\widehat{m}\cdots n} \quad \text{for every } q,m = 1,\cdots,n.$$

Moreover let us denote, for each $s<t$

$$\overline{\alpha}_{st} = \left(\overline{\alpha}^1_{st},\cdots,\overline{\alpha}^n_{st}\right) \quad \text{and} \quad A_{st} = \left(A^1_{st},\cdots,A^n_{st}\right).$$

From the definition of A^m_{st} we therefore obtain that

$$A_{st} = (-1)^{n-1}\left(\operatorname{adj}(\nabla\psi)\right)^t\,\overline{\alpha}_{st} = (-1)^{n-1}\det\nabla\psi\,(\nabla\psi)^{-t}\,\overline{\alpha}_{st}(\psi).$$

We therefore fix $s < t$ and obtain from the first system of equations in (19), that

$$\overline{\alpha}_{st}(\psi) = \frac{(-1)^{n-1}}{\det \nabla\psi} (\nabla\psi)^t \, A_{st} = \frac{(-1)^{n-1}}{\det \nabla\psi} (\nabla\psi)^t \left\{ \sum_{k \le l} \alpha_{kl}^m \left(\psi_{x_l}^s \psi_{x_k}^t + \psi_{x_l}^t \psi_{x_k}^t \right) \right\}_{m=1,\cdots,n} .$$

We can proceed exactly in the same way to solve for $\overline{\alpha}_{ss}$, $\overline{\beta}$ and γ. This proves the existence of σ.

Step 2.2. Let us now show that the system (19) can be solved for (α, β, γ) in terms of $(\overline{\alpha}, \overline{\beta}, \overline{\gamma})$. Let us denote $\tau = \sigma^{-1}$ and write τ as

$$\tau = \{(\tau_1^m, \tau_2^m, \tau_3^m)\}_{m=1,\cdots,n} ,$$

more precisely

$$\tau_1^m(\overline{\alpha}, \overline{\beta}, \overline{\gamma}) = \alpha^m, \quad \tau_2^m(\overline{\alpha}, \overline{\beta}, \overline{\gamma}) = \beta^m \quad \text{and} \quad \tau_3^m(\overline{\alpha}, \overline{\beta}, \overline{\gamma}) = \gamma^m.$$

From the last equation in (19) we immediately get that

$$\gamma^m = \tau_3^m(\overline{\alpha}, \overline{\beta}, \overline{\gamma}) = \tau_3^m(\overline{\gamma}) = C^m(\overline{\gamma})$$

is well defined. It remains to show that one can solve the first three equations in (19) for α and β. Note that for each fixed $m \in \{1, \cdots, n\}$ there are exactly $\binom{n}{2}+2n$ unknowns $(\alpha_{kl}^m, \beta_l^m)$. This is also exactly the number of linear equations. Thus one can write the first three equations of (19) in matrix form with some square matrix $M_\psi = M(\nabla\psi, \nabla^2\psi)$ as

$$M_\psi \cdot (\alpha^m, \beta^m) = G^m(\overline{\alpha}(\psi), \overline{\beta}(\psi), \nabla\psi),$$

for some vector G^m with $\binom{n}{2} + 2n$ entries depending on A_{st}^m and B_s^m. We will show that M_ψ is invertible. Using the lexicographic order to enumerate the $\binom{n}{2} + n$ entries of α^m and then the n entries of β^m, one observes that M can be written in block-matrix form

$$M_\psi = \left(\begin{array}{c|c} A & 0 \\ \hline B & C \end{array} \right),$$

where A, C are square matrices and (see Definition 22) $A = (\nabla\psi)^\sharp$, $C = \nabla\psi$ and $B = B(\nabla^2\psi)$ is a function of the second derivatives of ψ. From Lemma 26 we have

$$\det A = \det (\nabla\psi)^\sharp = (\det \nabla\psi)^{n+1}.$$

Hence we get that

$$\det M_\psi = (\det \nabla\psi)^{n+2}.$$

We can therefore define τ_1^m and τ_2^m as

$$(\alpha^m, \beta^m) = \left(\tau_1^m(\overline{\alpha}, \overline{\beta}, \overline{\gamma}), \tau_2^m(\overline{\alpha}, \overline{\beta}, \overline{\gamma})\right) = M_\psi^{-1} \cdot \left(G^m(\overline{\alpha}(\psi), \overline{\beta}(\psi), \nabla\psi)\right).$$

This shows the existence of τ. By construction, we obviously have that $\tau = \sigma^{-1}$. The lemma is therefore proved. ■

We use below the following notation

$$a_n = \binom{n}{2}\left(\binom{n}{2}+n\right)n, \quad b_n = \binom{n}{2}n^2 \quad \text{and} \quad c_n = \binom{n}{2}n.$$

Proposition 29 *Let $O, \Omega \subset \mathbb{R}^n$ be two open sets and $\psi \in \operatorname{Diff}^{\infty}(\Omega; O)$ and $\overline{a} \in C^{\infty}(O; \Lambda^1)$. Let $a = \psi^*(\overline{a})$. Then there exists*

$$(\overline{\alpha}, \overline{\beta}, \overline{\gamma}) \in C^{\infty}\left(O; \mathbb{R}^{a_n} \times \mathbb{R}^{b_n} \times \mathbb{R}^{c_n}\right)$$

such that

$$\operatorname{Ker} L_{\overline{a}} = \operatorname{Range} N_{\overline{a},2}^{\overline{\alpha},\overline{\beta},\overline{\gamma}}$$

if and only if there exists $(\alpha, \beta, \gamma) \in C^{\infty}\left(\Omega; \mathbb{R}^{a_n} \times \mathbb{R}^{b_n} \times \mathbb{R}^{c_n}\right)$ such that

$$\operatorname{Ker} L_a = \operatorname{Range} N_{a,2}^{\alpha,\beta,\gamma}$$

Proof We assume without loss of generality that (α, β, γ) exists such that $\operatorname{Ker} L_a = \operatorname{Range} N_{a,2}^{\alpha,\beta,\gamma}$. We have to show the existence of $\left(\overline{\alpha}, \overline{\beta}, \overline{\gamma}\right)$ such that $\operatorname{Ker} L_{\overline{a}} = \operatorname{Range} N_{\overline{a},2}^{\overline{\alpha},\overline{\beta},\overline{\gamma}}$. The reverse direction follows in the same way, by repeating the argument with ψ^{-1} instead of ψ.

Step 1. From Lemma 27, we know that for every $1 \leq i < j \leq n$ there exists a map σ_{ij} such that

$$\sigma_{ij} : C^{\infty}\left(\Omega; \mathbb{R}^{(\binom{n}{2}+n)n} \times \mathbb{R}^{n^2} \times \mathbb{R}^n\right) \to C^{\infty}\left(O; \mathbb{R}^{(\binom{n}{2}+n)n} \times \mathbb{R}^{n^2} \times \mathbb{R}^n\right),$$

satisfies, setting $\left(\overline{\alpha}_{ij}, \overline{\beta}_{ij}, \overline{\gamma}_{ij}\right) = \sigma_{ij}\left(\alpha_{ij}, \beta_{ij}, \gamma_{ij}\right)$,

$$*\psi^*\left(*N_{\overline{a}}^{\overline{\alpha}_{ij},\overline{\beta}_{ij},\overline{\gamma}_{ij}}(f)\right) = N_a^{\alpha_{ij},\beta_{ij},\gamma_{ij}}(f \circ \psi), \quad \text{for every } f \in C^{\infty}(O). \quad (20)$$

We define

$$\left(\overline{\alpha}, \overline{\beta}, \overline{\gamma}\right) = \left\{\overline{\alpha}_{ij}, \overline{\beta}_{ij}, \overline{\gamma}_{ij}\right\}_{1 \leq i < j \leq n}.$$

Let $\overline{w} = \sum_{i<j} \overline{w}^{ij} dx^i \wedge dx^j \in C^{\infty}\left(O; \Lambda^2\right)$, when we write $\overline{w} \circ \psi$ we mean that

$$\overline{w} \circ \psi(x) = \sum_{i<j} \overline{w}^{ij}(\psi(x))\, dx^i \wedge dx^j.$$

It follows from (20) and the definition of $N_{a,2}^{\alpha,\beta,\gamma}$, respectively $N_{\overline{a},2}^{\overline{\alpha},\overline{\beta},\overline{\gamma}}$, that

$$*\psi^*\left(*N_{\overline{a},2}^{\overline{\alpha},\overline{\beta},\overline{\gamma}}(\overline{w})\right) = N_{a,2}^{\alpha,\beta,\gamma}(\overline{w} \circ \psi) \quad \text{for every } \overline{w} \in C^{\infty}\left(O; \Lambda^2\right). \quad (21)$$

Step 2. We now show that with the above choice of $\left(\overline{\alpha}, \overline{\beta}, \overline{\gamma}\right)$ the following identity holds

$$\operatorname{Ker} L_{\overline{a}} = \operatorname{Range} N_{\overline{a},2}^{\overline{\alpha},\overline{\beta},\overline{\gamma}}.$$

Step 2.1: $\subset$. Let $u \in \operatorname{Ker} L_{\overline{a}}$. Define then v by

$$v = *\psi^* (*u) \in C^\infty \left(\Omega; \Lambda^1\right).$$

By Lemma 8 we have that

$$v \in \operatorname{Ker} L_{\psi^*(\overline{a})} = \operatorname{Ker} L_a$$

and so by assumption there exists a $w \in C^\infty \left(\Omega; \Lambda^2\right)$ such that

$$v = N_{a,2}^{\alpha,\beta,\gamma} (w).$$

Define $\overline{w} = w \circ \psi^{-1} \in C^\infty \left(O; \Lambda^2\right)$, this implies that

$$v = N_{a,2}^{\alpha,\beta,\gamma} (\overline{w} \circ \psi).$$

We then get from (21) that

$$v = N_{a,2}^{\alpha,\beta,\gamma} (\overline{w} \circ \psi) = *\psi^* \left(*N_{\overline{a},2}^{\overline{\alpha},\overline{\beta},\overline{\gamma}} (\overline{w})\right),$$

which implies that

$$u = N_{\overline{a},2}^{\overline{\alpha},\overline{\beta},\overline{\gamma}} (\overline{w})$$

and thus shows that $u \in \operatorname{Range} N_{\overline{a},2}^{\overline{\alpha},\overline{\beta},\overline{\gamma}}$.

Step 2.2: $\supset$. Let $u \in \operatorname{Range} N_{\overline{a},2}^{\overline{\alpha},\overline{\beta},\overline{\gamma}}$. Thus there exists $\overline{w} \in C^\infty \left(O; \Lambda^2\right)$ such that

$$u = N_{\overline{a},2}^{\overline{\alpha},\overline{\beta},\overline{\gamma}} (\overline{w}).$$

Define then

$$v = *\psi^* (*u) \in C^\infty \left(\Omega; \Lambda^1\right).$$

We obtain from (21) that

$$v = N_{a,2}^{\alpha,\beta,\gamma} (\overline{w} \circ \psi).$$

Therefore $v \in \operatorname{Range} N_{a,2}^{\alpha,\beta,\gamma}$ and it follows from the assumption that

$$L_a (v) = L_{\psi^*(\overline{a})} (v) = 0.$$

Lemma 8 implies then that

$$L_{\overline{a}} (u) = 0$$

which proves the proposition. ■

7.4 The local theorem

We now obtain a local result (in Theorem 21 we obtained a global result for the particular vector field $\overline{a}$ or $\overline{a} + dS$) for a general vector field a (recall that da is identified with $\operatorname{curl} a$).

Theorem 30 *Let $x_0 \in \mathbb{R}^n$, $0 \leq 2p \leq n$ and, in a neighborhood of x_0, $a \in C^\infty$ with*

$$\operatorname{rank}[da] = 2p.$$

Then there exist $\alpha, \beta, \gamma \in C^\infty$ such that, in a neighborhood of x_0,

$$L_a\left(N_{a,2}^{\alpha,\beta,\gamma}(w)\right) = 0, \quad \forall\, w \in C^3.$$

If furthermore $u \in C^\infty$ satisfies $L_a(u) = 0$, then there exists $w \in C^\infty$ such that, in a neighborhood of x_0,

$$u = N_{a,2}^{\alpha,\beta,\gamma}(w)$$

or equivalently

$$\operatorname{Ker} L_a = \operatorname{Range} N_{a,2}^{\alpha,\beta,\gamma}.$$

Proof From Theorem 13.6 in [4], we can find a diffeomorphism ψ such that

$$\psi^*(a) = \overline{a} + dS.$$

The theorem is therefore a consequence of Theorem 21 and Proposition 29. ■

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