Calculus of Variations with Differential Forms

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Abstract

We study integrals of the form $\int_\Omega f(d\omega)$, where $1 \leq k \leq n$, $f : \Lambda^k \to \mathbb{R}$ is continuous and ω is a $(k-1)$-form. We introduce the appropriate notions of convexity, namely ext. one convexity, ext. quasiconvexity and ext. polyconvexity. We study their relations, give several examples and counterexamples. We finally conclude with an application to a minimization problem.

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1 Introduction

In this article, we study integrals of the form

$$\int_\Omega f(d\omega),$$

where $1 \leq k \leq n$ are integers, $f : \Lambda^k \to \mathbb{R}$ is a continuous function and ω is a $(k-1)$-form. When $k = 1$, by abuse of notations identifying Λ^1 with $\mathbb{R}^n$ and the operator d with the gradient, this is the classical problem of the calculus of variations where one studies integrals of the form

$$\int_\Omega f(\nabla\omega).$$

This is a *scalar* problem in the sense that there is only one function ω. It is well known that in this last case the *convexity* of f plays a crucial role. As soon as $k \geq 2$, the problem is more of a *vectorial* nature, since then ω has now several components. However, it has some special features that a general vectorial problem does not have. Before going further one should have two examples in mind.

1) If $k = 2$, with our usual abuse of notations,

$$\omega : \mathbb{R}^n \to \mathbb{R}^n \quad \text{and} \quad d\omega = \operatorname{curl} \omega.$$

2) If $k = n$, by abuse of notations and up to some changes of signs,

$$\omega : \mathbb{R}^n \to \mathbb{R}^n \quad \text{and} \quad d\omega = \operatorname{div} \omega.$$

So let us now discuss some specific features of our problem.

- The first important point is the lack of *coercivity*. Indeed, even if the function f grows at infinity as the norm to a certain power, this does not imply control on the full gradient but only on some combination of it, namely $d\omega$. So when looking at minimization problems, this fact requires special attention (see Theorem 31).

- From the point of view of convexity, the situation is, in some cases, simpler than in the general vectorial problem. Indeed consider the above two examples (with $n = 3$ for the first one). Although the problems are vectorial they behave as if they were scalar (cf. Theorem 8).

- One peculiarity (cf. Theorem 17) that particularly stands out is how the problem changes its behaviour with a change in the order of the form. When k is odd, or when $2k > n$ (in particular $k = n$), there is no nonlinear function which is ext. quasiaffine and therefore, the problem behaves as if it were scalar. However, the situation changes significantly when $k \leq 2n$ is even. It turns out that we have an ample supply of nonlinear functions that are ext. quasiaffine in this case. For example, the nonlinear function

$$f(\xi) = \langle c; \xi \wedge \xi \rangle,$$

where $c \in \Lambda^{2k}$, is ext. quasiaffine. See Theorem 17 for the complete characterization of ext. quasiaffine functions which, in turn, determines all weakly continuous functions with respect to the d-operator, see Bandyopadhyay-Sil [3] for detail.

Because of the special nature of our problem, we are led to introduce the following terminology: *ext. one convexity*, *ext. quasiconvexity* and *ext. polyconvexity*, which are the counterparts of the classical notions of the vectorial calculus of variations (see, in particular Dacorogna [7]), namely rank one convexity, quasiconvexity and polyconvexity. The relations between these notions (cf. Theorem 8) as well as their manifestations on the minimization problem are the subject of the present paper. Examples and counterexamples are also discussed in details, notably the case of ext. quasiaffine functions (see Theorem

17), the quadratic case (see Theorem 23) and a fundamental counterexample (see Theorem 26) similar to the famous example of Sverak [17].

Some of what has been done in this article may also be seen through classical vectorial calculus of variations. This connection is elaborated and pursued in detail in our forthcoming article, see Bandyopadhyay-Dacorogna-Sil [3]. However, the case of differential forms in the context of calculus of variations deserves a separate and independent treatment because of its special algebraic structure which renders much of the calculation intrinsic, natural and coordinate free.

We conclude this introduction by pointing out that the results discussed in this introduction may be interpreted very broadly in terms of the theory of compensated compactness introduced by Murat and Tartar, see [12], [18] and see also Dacorogna [6], Robbin-Rogers-Temple [13]. In particular, our notion of ext. one convexity is related to the so called condition of convexity in the directions of the wave cone Λ. Our definition of ext. quasiconvexity is related to those of A and $A-B$ quasiconvexity introduced by Dacorogna (cf. [5] and [6]), see also Fonseca-Müller [9].

2 Definitions and main properties

2.1 Definitions

We start with the different notions of convexity and affinity.

Definition 1 *Let* $1 \leq k \leq n$ *and* $f : \Lambda^k \to \mathbb{R}$.

(i) *We say that* f *is* ext. one convex, *if the function*

$$g : t \to g(t) = f(\xi + t\,\alpha \wedge \beta)$$

is convex for every $\xi \in \Lambda^k$, $\alpha \in \Lambda^{k-1}$ *and* $\beta \in \Lambda^1$. *If the function* g *is affine we say that* f *is* ext. one affine.

(ii) f *is said to be* ext. quasiconvex, *if* f *is Borel measurable, locally bounded and*

$$\int_\Omega f(\xi + d\omega) \geq f(\xi) \operatorname{meas} \Omega,$$

for every bounded open set $\Omega \subset \mathbb{R}^n$, $\xi \in \Lambda^k$ *and* $\omega \in W_0^{1,\infty}\left(\Omega; \Lambda^{k-1}\right)$. *If equality holds, we say that* f *is* ext. quasiaffine.

(iii) *We say that* f *is* ext. polyconvex, *if there exists a convex function*

$$F : \Lambda^k \times \Lambda^{2k} \times \cdots \times \Lambda^{[n/k]k} \to \mathbb{R}$$

such that

$$f(\xi) = F\left(\xi, \xi^2, \cdots, \xi^{[n/k]}\right), \text{ for all } \xi \in \Lambda^k.$$

If F *is affine, we say that* f *is* ext. polyaffine.

Remark 2 ***(i)*** *The ext. stands for exterior product in the first and third ones and for the exterior derivative for the second one.*

(ii) *When k is odd (since then $\xi^s = 0$ for every $s \geq 2$) or when $2k > n$ (in particular, when $k = n$ or $k = n - 1$), then ext. polyconvexity is equivalent to ordinary convexity (see Proposition 14).*

(iii) *When $k = 1$, all the above notions are equivalent to the classical notion of convexity (cf. Theorem 8).*

(iv) *As in Proposition 5.11 of [7], it can easily be shown that if the inequality of ext. quasiconvexity holds for a given bounded open set Ω, it holds for any bounded open set.*

(v) *The definition of ext. quasiconvexity is equivalent (as in Proposition 5.13 of [7]) to the following. Let $D = (0,1)^n$, the inequality*

$$\int_D f(\xi + d\omega) \geq f(\xi),$$

holds for every $\xi \in \Lambda^k$ and for every $\omega \in W_{per}^{1,\infty}\left(D; \Lambda^{k-1}\right)$, where

$$W_{per}^{1,\infty}\left(D; \Lambda^{k-1}\right) = \left\{\omega \in W^{1,\infty}\left(D; \Lambda^{k-1}\right) : \omega \text{ is 1-periodic in each variable}\right\}.$$

Definition 3 *Let $0 \leq k \leq n$ and $f : \Lambda^k \to \mathbb{R}$. The* Hodge transform *of f is the function $f_* : \Lambda^{n-k} \to \mathbb{R}$ defined as,*

$$f_*(\xi) = f(*\xi), \text{ for all } \xi \in \Lambda^{n-k}.$$

The notion of Hodge transform allows us to extend the notions of convexity with respect to the interior product and the δ-operator as follows.

Definition 4 *Let $0 \leq k \leq n - 1$ and $f : \Lambda^k \to \mathbb{R}$. We say that*

(i) f is int. one convex*, if f_* is* ext. one convex*.*

(ii) f is int. quasiconvex*, if f_* is* ext. quasiconvex*.*

(iii) f is int. polyconvex*, if f_* is* ext. polyconvex*.*

Remark 5 ***(i)*** *Statements similar to those in Remark 2 hold in the case of int. convexity as well.*

(ii) *It is easy to check that f is* int. one convex*, if and only if the function*

$$g : t \to g(t) = f(\xi + t\,\beta \lrcorner \alpha)$$

is convex for every $\xi \in \Lambda^k$, $\beta \in \Lambda^1$ and $\alpha \in \Lambda^{k+1}$. Furthermore, f is int. quasiconvex *if and only if f is Borel measurable, locally bounded and*

$$\int_\Omega f(\xi + \delta\omega) \geq f(\xi) \operatorname{meas}\Omega,$$

for every bounded open set $\Omega \subset \mathbb{R}^n$, $\xi \in \Lambda^k$ and $\omega \in W_0^{1,\infty}\left(\Omega; \Lambda^{k+1}\right)$. The third one is however a little more involved and we leave out the details.

In what follows, we will discuss the case of ext. convexity only. The case of int. convexity can be handled analogously.

2.2 Preliminary lemmas

In this subsection, we state two lemmas which will be used in sequel. See [15] for the proofs. We start with the following problem of prescribed differentials. Let us recall that $\alpha \in \Lambda^k$ is said to be *1-divisible* if $\alpha = a \wedge b$, for some $a \in \Lambda^{k-1}$, $b \in \Lambda^1$.

Lemma 6 *Let $1 \leq k \leq n$ and let $\omega_1, \omega_2 \in \Lambda^k$. Then, there exists $\omega \in W^{1,\infty}(\Omega; \Lambda^{k-1})$ satisfying*

$$d\omega \in \{\omega_1, \omega_2\}, \ \text{a.e. in } \Omega,$$

(and taking both values), if and only if $\omega_1 - \omega_2$ is 1-divisible.

Using Lemma 6, one can deduce the following approximation lemma for k-forms. See Lemma 3.11 of [7] for the case of the gradient.

Lemma 7 *Let $1 \leq k \leq n$, $t \in [0,1]$ and let $\alpha, \beta \in \Lambda^k$ be such that $\alpha \neq \beta$ and $\alpha - \beta$ is 1-divisible. Let $\Omega \subset \mathbb{R}^n$ be open, bounded and let $\omega : \overline{\Omega} \to \Lambda^{k-1}$ satisfy*

$$d\,\omega = t\alpha + (1-t)\beta, \ \text{in } \overline{\Omega}.$$

Then, for every $\epsilon > 0$, there exist $\omega_\epsilon \in \mathrm{Aff}_{piece}\left(\overline{\Omega}; \Lambda^{k-1}\right)$ and disjoint open sets $\Omega_\alpha, \Omega_\beta \subset \Omega$ such that

1. $|\operatorname{meas}(\Omega_\alpha) - t \operatorname{meas}(\Omega)| \leqslant \epsilon$ *and* $|\operatorname{meas}(\Omega_\beta) - (1-t) \operatorname{meas}(\Omega)| \leqslant \epsilon$,
2. $\omega_\epsilon = \omega$, *in a neighbourhood of* $\partial\Omega$,
3. $\|\omega_\epsilon - \omega\|_{L^\infty(\overline{\Omega})} \leqslant \epsilon$,
4. $d\omega_\epsilon(x) = \begin{cases} \alpha, & \text{if } x \in \Omega_\alpha, \\ \beta, & \text{if } x \in \Omega_\beta, \end{cases}$
5. $\operatorname{dist}(d\omega_\epsilon(x); \{t\alpha + (1-t)\beta : t \in [0,1]\}) \leqslant \epsilon$, *for all $x \in \Omega$ a.e.*

2.3 Main properties

The different notions of convexity are related as follows.

Theorem 8 *Let $1 \leq k \leq n$ and $f : \Lambda^k \to \mathbb{R}$.*

(i) *The following implications hold*

$$f \text{ convex} \Rightarrow f \text{ ext. polyconvex} \Rightarrow f \text{ ext. quasiconvex} \Rightarrow f \text{ ext. one convex.}$$

(ii) *If $k = 1, n-1, n$ or $k = n-2$ is odd, then*

$$f \text{ convex} \Leftrightarrow f \text{ ext. polyconvex} \Leftrightarrow f \text{ ext. quasiconvex} \Leftrightarrow f \text{ ext. one convex.}$$

Moreover, if k is odd or $2k > n$, then

$$f \text{ convex } \Leftrightarrow f \text{ ext. polyconvex.}$$

(iii) *If either $2 \leq k \leq n-3$ or $k = n-2 \geq 2$ is even, then*

$$f \text{ ext. polyconvex } \underset{\not\Leftarrow}{\Rightarrow} f \text{ ext. quasiconvex,}$$

while if $2 \leq k \leq n-3$ (and thus $n \geq k+3 \geq 5$), then

$$f \text{ ext. quasiconvex } \underset{\not\Leftarrow}{\Rightarrow} f \text{ ext. one convex.}$$

Remark 9 ***(i)*** *The last statement in (ii) for k even and $n \geq 2k$ is false, as the following simple example shows. Let $f : \Lambda^2\left(\mathbb{R}^4\right) \to \mathbb{R}$ be defined by*

$$f(\xi) = \left\langle e^1 \wedge e^2 \wedge e^3 \wedge e^4; \xi \wedge \xi \right\rangle.$$

The function f is clearly ext. polyconvex but not convex.

(ii) *The study of the implications and counter implications for ext. one convexity, ext. quasiconvexity and ext. polyconvexity is therefore complete, except for the last implication, namely*

$$f \text{ ext. quasiconvex } \underset{\not\Leftarrow}{\Rightarrow} f \text{ ext. one convex,}$$

only for the case $k = n-2 \geq 2$ even (including $k = 2$ and $n = 4$), which remains open.

(iii) *It is interesting to read the theorem when $k = 2$.*

- If $n = 2$ or $n = 3$, then

$$f \text{ convex } \Leftrightarrow f \text{ ext. polyconvex } \Leftrightarrow f \text{ ext. quasiconvex } \Leftrightarrow f \text{ ext. one convex.}$$

- If $n \geq 4$, then

$$f \text{ convex } \underset{\not\Leftarrow}{\Rightarrow} f \text{ ext. polyconvex } \underset{\not\Leftarrow}{\Rightarrow} f \text{ ext. quasiconvex.}$$

- If $n \geq 5$, then

$$f \text{ ext. quasiconvex } \underset{\not\Leftarrow}{\Rightarrow} f \text{ ext. one convex,}$$

while the case $n = 4$ remains open.

Proof (i) *Step 1.* The implication

$$f \text{ convex } \Rightarrow f \text{ ext. polyconvex}$$

is trivial.

Step 2. The statement

$$f \text{ ext. polyconvex } \Rightarrow f \text{ ext. quasiconvex}$$

is proved as follows. Observe first that if $\xi \in \Lambda^k$ and $\omega \in W_0^{1,\infty}\left(\Omega; \Lambda^{k-1}\right)$, then

$$\int_\Omega (\xi + d\omega)^s = \xi^s \operatorname{meas} \Omega, \quad \text{for every integer } s. \tag{1}$$

We proceed by induction on s. The case $s = 1$ is trivial, so we assume that the result has already been established for $s - 1$ and we prove it for s. Note that

$$\begin{aligned}(\xi + d\omega)^s &= \xi \wedge (\xi + d\omega)^{s-1} + d\omega \wedge (\xi + d\omega)^{s-1} \\ &= \xi \wedge (\xi + d\omega)^{s-1} + d\left[\omega \wedge (\xi + d\omega)^{s-1}\right].\end{aligned}$$

Integrating, using induction for the first integral and the fact that $\omega = 0$ on $\partial\Omega$ for the second one, we have indeed shown (1). We can now conclude. Since f is ext. polyconvex, we can find a convex function

$$F : \Lambda^k \times \Lambda^{2k} \times \cdots \times \Lambda^{[n/k]k} \to \mathbb{R}$$

such that

$$f(\xi) = F\left(\xi, \xi^2, \ldots, \xi^{[n/k]}\right).$$

Using Jensen inequality we find,

$$\frac{1}{\operatorname{meas}\Omega} \int_\Omega f(\xi + d\omega) \geq F\left(\frac{1}{\operatorname{meas}\Omega} \int_\Omega (\xi + d\omega), \ldots, \frac{1}{\operatorname{meas}\Omega} \int_\Omega (\xi + d\omega)^{[n/k]}\right).$$

Invoking (1), we have indeed obtained that

$$\int_\Omega f(\xi + d\omega) \geq f(\xi) \operatorname{meas}\Omega,$$

and the proof of Step 2 is complete.

Step 3. It follows from Lemma 7 that

$$f \text{ ext. quasiconvex } \Rightarrow f \text{ ext. one convex.}$$

With Lemma 7 at our disposal, the proof is very similar to that of the case of the gradient (cf. Theorem 5.3 in [7]) and is omitted. See [15] for the details. This concludes the proof of (i).

(ii) In all the cases under consideration any $\xi \in \Lambda^k$ is 1-divisible (cf. Proposition 2.43 in [4]). Hence, the result

$$f \text{ convex } \Leftrightarrow f \text{ ext. polyconvex } \Leftrightarrow f \text{ ext. quasiconvex } \Leftrightarrow f \text{ ext. one convex}$$

then follows at once. The extra statement (i.e. when k is odd or $2k > n$)

$$f \text{ convex } \Leftrightarrow f \text{ ext. polyconvex.}$$

is proved in Remark 2 (ii) and Proposition 14.

(iii) The statement that

$$f \text{ ext. polyconvex} \underset{\not\Leftarrow}{\Rightarrow} f \text{ ext. quasiconvex,}$$

when $3 \leq k \leq n-3$ or $k = n-2 \geq 4$ is even follows from Theorem 23 (v) and from Proposition 29 when $k = 2$ and $n \geq 4$ (for $k = 2$ and $n \geq 6$, we can also apply Theorem 23 (ii)).

The statement that if $2 \leq k \leq n-3$ (and thus $n \geq k+3 \geq 5$), then

$$f \text{ ext. quasiconvex} \underset{\not\Leftarrow}{\Rightarrow} f \text{ ext. one convex,}$$

follows from Theorem 26. ■

We also have the following elementary properties.

Proposition 10 *Let $1 \leq k \leq n$ and $f : \Lambda^k \to \mathbb{R}$.*

(i) *Any ext. one convex function is locally Lipschitz.*

(ii) *If f is ext. one convex and C^2, for every $\xi \in \Lambda^k$, $\alpha \in \Lambda^{k-1}$ and $\beta \in \Lambda^1$,*

$$\sum_{I,J \in \mathcal{T}_k^n} \frac{\partial^2 f(\xi)}{\partial \xi_I \partial \xi_J} (\alpha \wedge \beta)_I (\alpha \wedge \beta)_J \geq 0.$$

Proof (i) The fact that f is locally Lipschitz follows from the observation that any ext. one convex function is in fact separately convex. These last functions are known to be locally Lipschitz (cf. Theorem 2.31 in [7]).

(ii) We next assume that f is C^2. By definition the function

$$g : t \to g(t) = f(\xi + t\,\alpha \wedge \beta)$$

is convex for every $\xi \in \Lambda^k$, $\alpha \in \Lambda^{k-1}$ and $\beta \in \Lambda^1$. Since f is C^2, our claim follows from the fact that $g''(0) \geq 0$. ■

We now give an equivalent formulation of ext. quasiconvexity, but before that we need the following notation.

Notation 11 *Let $\Omega \subset \mathbb{R}^n$ be a smooth, open set. We define*

$$W_{\delta,T}^{1,\infty}\left(\Omega; \Lambda^k\right) = \left\{\omega \in W^{1,\infty}\left(\Omega; \Lambda^k\right) : \delta\omega = 0 \text{ in } \Omega \text{ and } \nu \wedge \omega = 0 \text{ on } \partial\Omega\right\},$$

where ν is the outward unit normal to $\partial\Omega$.

Proposition 12 *Let $f : \Lambda^k \to \mathbb{R}$ be continuous. The following statements are equivalent.*

(i) *f is* ext. quasiconvex.

(ii) *For every bounded smooth open set $\Omega \subset \mathbb{R}^n$, $\psi \in W_{\delta,T}^{1,\infty}\left(\Omega; \Lambda^{k-1}\right)$ and $\xi \in \Lambda^k$,*

$$\int_\Omega f(\xi + d\psi) \geq f(\xi) \operatorname{meas} \Omega.$$

Remark 13 *Given a function* $f:\Lambda^k\to\mathbb{R}$*, the* ext. quasiconvex envelope*, which is the largest ext quasiconvex function below* f*, is given by (as in Theorem 6.9 of [7])*

$$\begin{aligned} Q_{ext}f(\xi) &= \inf\left\{\frac{1}{\operatorname{meas}\Omega}\int_\Omega f(\xi+d\omega):\omega\in W_0^{1,\infty}\left(\Omega;\Lambda^{k-1}\right)\right\} \\ &= \inf\left\{\frac{1}{\operatorname{meas}\Omega}\int_\Omega f(\xi+d\psi):\psi\in W_{\delta,T}^{1,\infty}\left(\Omega;\Lambda^{k-1}\right)\right\}. \end{aligned}$$

Proof (ii) $\Rightarrow$ **(i):** Let $\Omega\subset\mathbb{R}^n$ be a bounded smooth open set, $\xi\in\Lambda^k$ and let $\omega\in W_0^{1,\infty}\left(\Omega;\Lambda^{k-1}\right)$. Using density, we find $\omega_\epsilon\in C_0^\infty\left(\Omega;\Lambda^{k-1}\right)$ such that

$$\sup_{\epsilon>0}\|\nabla\omega_\epsilon\|_{L^\infty}<\infty \text{ and } \omega_\epsilon\to\omega,\ \text{ in } W^{1,2}\left(\Omega;\Lambda^{k-1}\right). \tag{2}$$

Appealing to Theorem 7.2 in [4], we now find $\psi_\epsilon\in C_{\delta,T}^\infty\left(\Omega;\Lambda^{k-1}\right)$ such that

$$\begin{cases} d\psi_\epsilon=d\omega_\epsilon & \text{in } \Omega \\ \delta\psi_\epsilon=0 & \text{in } \Omega \\ \nu\wedge\psi_\epsilon=0 & \text{on } \partial\Omega. \end{cases}$$

We use (2) to apply dominated convergence theorem to obtain

$$\int_\Omega f(\xi+d\omega)=\lim_{\epsilon\to0}\int_\Omega f(\xi+d\omega_\epsilon)=\lim_{\epsilon\to0}\int_\Omega f(\xi+d\psi_\epsilon)\geq f(\xi)\operatorname{meas}\Omega,$$

where we have used **(ii)** in the last step. Therefore, f is ext. quasiconvex.

(i) $\Rightarrow$ **(ii):** Let $\psi\in W_{\delta,T}^{1,\infty}\left(\Omega;\Lambda^{k-1}\right)$. Then, by Theorem 8.16 in [4], we can find $\omega\in W_0^{1,2}\left(\Omega;\Lambda^{k-1}\right)$ such that

$$\begin{cases} d\omega=d\psi & \text{in } \Omega \\ \omega=0 & \text{on } \partial\Omega. \end{cases}$$

With a similar argument as above, we infer that

$$\int_\Omega f(\xi+d\psi)=\int_\Omega f(\xi+d\omega)=\lim_{\epsilon\to0}\int_\Omega f(\xi+d\omega_\epsilon)\geq f(\xi)\operatorname{meas}\Omega,$$

as claimed. ■

We finally have also another formulation of ext. polyconvexity.

Proposition 14 *Let* $f:\Lambda^k\to\mathbb{R}$*. The following statements are then equivalent.*

(i) *The function* f *is ext. polyconvex.*

(ii) *For every* $\xi\in\Lambda^k$*, there exist* $c_s=c_s(\xi)\in\Lambda^{ks}$*,* $1\leq s\leq[n/k]$*, such that*

$$f(\eta)\geq f(\xi)+\sum_{s=1}^{[n/k]}\langle c_s(\xi);\eta^s-\xi^s\rangle,\quad \textit{for every } \eta\in\Lambda^k.$$

(iii) *Let*

$$N=\sum_{s=1}^{[n/k]}\binom{n}{sk}.$$

For every $t_i\geq 0$ *with* $\sum_{i=1}^{N+1}t_i=1$ *and every* $\xi_i\in\Lambda^k$ *such that*

$$\sum_{i=1}^{N+1}t_i\xi_i^s=\left(\sum_{i=1}^{N+1}t_i\xi_i\right)^s,\quad \text{for every } 1\leq s\leq [n/k]\,,$$

the following inequality holds

$$f\left(\sum_{i=1}^{N+1}t_i\xi_i\right)\leq\sum_{i=1}^{N+1}t_if\left(\xi_i\right).$$

Proof (i) ⇒ (ii): Since f is ext. polyconvex, there exists a convex function F such that

$$f\left(\xi\right)=F\left(\xi,\xi^2,\ldots,\xi^{[n/k]}\right).$$

F being convex, for every $\xi\in\Lambda^k$, there exists $c_s=c_s\left(\xi\right)\in\Lambda^{ks}$, $1\leq s\leq[n/k]$, such that, for all $\eta\in\Lambda^k$,

$$f\left(\eta\right)-f\left(\xi\right)=F\left(\eta,\ldots,\eta^{[n/k]}\right)-F\left(\xi,\ldots,\xi^{[n/k]}\right)\geq\sum_{s=1}^{[n/k]}\left\langle c_s;\eta^s-\xi^s\right\rangle,$$

as claimed.

(ii) ⇒ (i): Conversely assume that the inequality is valid and, for $\theta=\left(\theta_1,\ldots,\theta_{[n/k]}\right)\in\Lambda^k\times\cdots\times\Lambda^{[n/k]k}$, we define

$$F\left(\theta\right)=\sup_{\xi\in\Lambda^k}\left\{f\left(\xi\right)+\textstyle\sum_{s=1}^{[n/k]}\left\langle c_s\left(\xi\right);\theta_s-\xi^s\right\rangle\right\}.$$

Clearly F is convex as a supremum of affine functions. Moreover if

$$\theta=\left(\eta,\ldots,\eta^{[n/k]}\right),$$

then, in view of the inequality, the supremum is attained by $\xi=\eta$, i.e.

$$f\left(\eta\right)=F\left(\eta,\ldots,\eta^{[n/k]}\right),$$

and thus f is ext. polyconvex.

(i) ⇒ (iii): Since f is ext. polyconvex, there exists a convex function F such that

$$f\left(\xi\right)=F\left(\xi,\xi^2,\ldots,\xi^{[n/k]}\right).$$

The convexity of F implies that

$$\begin{aligned} f\left(\sum_{i=1}^{N+1} t_i\xi_i\right) &= F\left(\left(\sum_{i=1}^{N+1} t_i\xi_i\right),\ldots,\left(\sum_{i=1}^{N+1} t_i\xi_i\right)^{[n/k]}\right) \\ &= F\left(\sum_{i=1}^{N+1} t_i\left(\xi_i,\ldots,\xi_i^{[n/k]}\right)\right) \le \sum_{i=1}^{N+1} t_i F\left(\xi_i,\ldots,\xi_i^{[n/k]}\right) \\ &= \sum_{i=1}^{N+1} t_i f\left(\xi_i\right). \end{aligned}$$

(iii) $\Rightarrow$ (i): The proof is based on Carathéodory theorem and follows exactly as in Theorem 5.6 in [7]. ■

3 The quasiaffine case

3.1 Some preliminary results

We start with two elementary results.

Lemma 15 *Let $f:\Lambda^k\to\mathbb{R}$ be ext. one affine with $1\le k\le n$. Then*

$$f\left(\xi+\sum_{i=1}^{N} t_i\,\alpha_i\wedge a\right)=f\left(\xi\right)+\sum_{i=1}^{N} t_i\left[f\left(\xi+\alpha_i\wedge a\right)-f\left(\xi\right)\right],$$

for every $t_i\in\mathbb{R}$, $\xi\in\Lambda^k$, $\alpha_i\in\Lambda^{k-1}$, $a\in\Lambda^1$.

Proof *Step 1.* Since f is ext. one affine,

$$f\left(\xi+t\alpha\wedge a\right)=f\left(\xi\right)+t\left[f\left(\xi+\alpha\wedge a\right)-f\left(\xi\right)\right].$$

Step 2. Let us first prove that

$$f\left(\xi+\alpha\wedge a+\beta\wedge a\right)+f\left(\xi\right)=f\left(\xi+\alpha\wedge a\right)+f\left(\xi+\beta\wedge a\right).$$

First assume that $s\neq 0$. We have, using Step 1, that

$$\begin{aligned} f\left(\xi+s\,\alpha\wedge a+\beta\wedge a\right)=&f\left(\xi+s\,\left(\alpha+\frac{1}{s}\,\beta\right)\wedge a\right) \\ =&f\left(\xi\right)+s\left[f\left(\xi+\left(\alpha+\frac{1}{s}\,\beta\right)\wedge a\right)-f\left(\xi\right)\right], \end{aligned}$$

and hence, using Step 1 again,

$$\begin{aligned} &f\left(\xi+s\,\alpha\wedge a+\beta\wedge a\right) \\ &=f\left(\xi\right)+s\left\{f\left(\xi+\alpha\wedge a\right)+\frac{1}{s}\left[f\left(\xi+\alpha\wedge a+\beta\wedge a\right)-f\left(\xi+\alpha\wedge a\right)\right]-f\left(\xi\right)\right\} \\ &=f\left(\xi\right)+s\left[f\left(\xi+\alpha\wedge a\right)-f\left(\xi\right)\right]+\left[f\left(\xi+\alpha\wedge a+\beta\wedge a\right)-f\left(\xi+\alpha\wedge a\right)\right]. \end{aligned}$$

Since f is continuous, we have the result by letting $s \to 0$.

Step 3. We now prove the claim. We proceed by induction. The case $N = 1$ is just Step 1. We first use the induction hypothesis to write

$$f\left(\xi + \sum_{i=1}^{N} t_i\, \alpha_i \wedge a\right) = f\left(\xi + t_N\, \alpha_N \wedge a + \sum_{i=1}^{N-1} t_i\, \alpha_i \wedge a\right)$$
$$= f\left(\xi + t_N\, \alpha_N \wedge a\right) + \sum_{i=1}^{N-1} t_i \left[f\left(\xi + t_N\, \alpha_N \wedge a + \alpha_i \wedge a\right) - f\left(\xi + t_N\, \alpha_N \wedge a\right)\right].$$

We then appeal to Step 1 to get

$$f\left(\xi + \textstyle\sum_{i=1}^{N} t_i\, \alpha_i \wedge a\right) = f(\xi) + t_N\left[f\left(\xi + \alpha_N \wedge a\right) - f(\xi)\right]$$
$$+ \sum_{i=1}^{N-1} t_i \left\{ \begin{array}{c} f\left(\xi + \alpha_i \wedge a\right) + t_N\left[f\left(\xi + \alpha_i \wedge a + \alpha_N \wedge a\right) - f\left(\xi + \alpha_i \wedge a\right)\right] \\ -f(\xi) - t_N\left[f\left(\xi + \alpha_N \wedge a\right) - f(\xi)\right] \end{array} \right\},$$

and thus

$$f\left(\xi + \textstyle\sum_{i=1}^{N} t_i\, \alpha_i \wedge a\right) = f(\xi) + \displaystyle\sum_{i=1}^{N} t_i\left[f\left(\xi + \alpha_i \wedge a\right) - f(\xi)\right]$$
$$+ t_N \sum_{i=1}^{N-1} t_i \left\{f\left(\xi + \alpha_i \wedge a + \alpha_N \wedge a\right) - f\left(\xi + \alpha_i \wedge a\right) - f\left(\xi + \alpha_N \wedge a\right) + f(\xi)\right\}.$$

Appealing to Step 2, we see that the last term vanishes and therefore the induction reasoning is complete and this achieves the proof of the lemma. ■

We have as an immediate consequence the following result.

Corollary 16 *Let $f : \Lambda^k \to \mathbb{R}$ be ext. one affine with $1 \leq k \leq n$. Then*

$$\begin{aligned} &\left[f\left(\xi + \alpha \wedge a + \beta \wedge b\right) - f(\xi)\right] + \left[f\left(\xi + \beta \wedge a + \alpha \wedge b\right) - f(\xi)\right] \\ &= \left[f\left(\xi + \alpha \wedge a\right) - f(\xi)\right] + \left[f\left(\xi + \beta \wedge a\right) - f(\xi)\right] \\ &+ \left[f\left(\xi + \alpha \wedge b\right) - f(\xi)\right] + \left[f\left(\xi + \beta \wedge b\right) - f(\xi)\right], \end{aligned}$$

for every $\xi \in \Lambda^k$, $\alpha, \beta \in \Lambda^{k-1}$, $a, b \in \Lambda^1$.

3.2 The main theorem

Theorem 17 *Let $1 \leq k \leq n$ and $f : \Lambda^k \to \mathbb{R}$. The following statements are then equivalent.*

(i) *f is ext. polyaffine.*

(ii) *f is ext. quasiaffine.*

(iii) *f is ext. one affine.*

(iv) *For every $0 \leq s \leq [n/k]$, there exist $c_s \in \Lambda^{ks}$ such that,*

$$f(\xi) = \sum_{s=0}^{[n/k]} \langle c_s; \xi^s \rangle, \text{ for every } \xi \in \Lambda^k.$$

Remark 18 *When k is odd (since then $\xi^s = 0$ for every $s \geq 2$) or when $2k > n$ (in particular when $k = n$ or $k = n-1$), then all the statements are equivalent to f being affine.*

Proof The statements

$$(i) \Rightarrow (ii) \Rightarrow (iii)$$

follow at once from Theorem 8. The statement

$$(iv) \Rightarrow (i)$$

is a direct consequence of the definition of ext. polyconvexity. So it only remains to prove

$$(iii) \Rightarrow (iv).$$

We divide the proof into three steps.

Step 1. We first prove that f is a polynomial of degree at most n and is of the form

$$f(\xi) = \sum_{s=0}^{n} f_s(\xi), \tag{3}$$

where, for each $s = 0, \ldots, n$, f_s is a homogeneous polynomial of degree s and is ext. one affine. To prove (3), let us proceed by induction on the dimension n. The case $n = 1$ is trivial to check. For each $\xi \in \Lambda^k$, we write

$$\xi = \sum_{I \in \mathcal{T}_k^n, 1 \in I} \xi_I e^I + \xi_N, \text{ where } \xi_N = \sum_{I \in \mathcal{T}_k^n, 1 \notin I} \xi_I e^I.$$

Note that, $\xi_N \in \Lambda^k\left(\{e^1\}^\perp\right)$. Invoking Lemma 15, we obtain that

$$f(\xi) = f(\xi_N) + \sum_{I \in \mathcal{T}_k^n, 1 \in I} \xi_I \left[f\left(\xi_N + e^I\right) - f(\xi_N) \right].$$

Since f is ext. one affine on $\Lambda^k\left(\{e^1\}^\perp\right)$, applying the induction hypothesis to $f|_{\Lambda^k(\{e^1\}^\perp)}$ and $f\left(e^I + \cdot\right)|_{\Lambda^k(\{e^1\}^\perp)}$, we deduce that both are polynomials of degree at most $(n-1)$. Hence, f is a polynomial of degree at most n. This proves the claim by induction. That each of f_s is ext. one affine, follows from the fact that each f_s has a different degree of homogeneity.

Step 2. We now show that f is, in fact, a polynomial of degree at most $[n/k]$, which is equivalent to proving that each f_s in (3) is a polynomial of degree at most $[n/k]$. Since f_s is a homogeneous polynomial of degree s, we can write

$$f_s(\xi) = \sum_{I^1, \ldots, I^s \in \mathcal{T}_k^n} d_{I^1 \ldots I^s} \, \xi_{I^1} \cdots \xi_{I^s}, \tag{4}$$

where $d_{I^1 \dots I^s} \in \mathbb{R}$. It is enough to prove that, for some $I^1, \dots, I^s \in \mathcal{T}_k^n$, whenever $d_{I^1 \dots I^s} \neq 0$, we have,

$$I^p \cap I^q = \emptyset, \text{ for all } p, q = 1, \dots, s; \ p \neq q.$$

Let us suppose to the contrary that, for some p, q, $p \neq q$, we have $I^p \cap I^q \neq \emptyset$. For $t \in \mathbb{R}$, let us define

$$\xi(t) = t\left(e^{I^p} + e^{I^q}\right) + \sum_{\substack{a=1 \\ a \neq p,q}}^{s} e^{I^a}.$$

We, therefore, have according to (4) that, for all $t \in \mathbb{R}$,

$$f_s(\xi(t)) = t^2 d_{I^1 \dots I^s}, \text{ for all } t \in \mathbb{R}.$$

On the other hand, using Lemma 15, it follows that

$$f_s(\xi(t)) = f_s\left(\sum_{\substack{a=1 \\ a \neq p,q}}^{s} e^{I^a} + t\left(e^{I^p} + e^{I^q}\right)\right) = f_s(\xi(0)) + t\left[f_s(\xi(1)) - f_s(\xi(0))\right],$$

which is an affine function of t. This proves the claim by contradiction.

Step 3. Henceforth, to avoid any ambiguity, let us fix an order in which multiindices $I^1, \cdots, I^s$ are considered to appear in Equation (4) and we choose the order to that

$$i_1^1 < \cdots < i_1^s,$$

where i_1^j is the first element of I^j, for all $j = 1, \dots, s$. With the aforementioned order in force, we re-arrange Equation (4) to have

$$f(\xi) = \sum_{r=0}^{[n/k]} f_s(\xi), \quad \text{where} \quad f_s(\xi) = \sum_{I^1, \dots, I^s} c_{I^1 \dots I^s} \, \xi_{I^1} \cdots \xi_{I^s}, \tag{5}$$

with $c_{I^1 \dots I^s} \in \mathbb{R} \setminus \{0\}$, and the ordered multiindices $I^1, \dots, I^s \in \mathcal{T}_k^n$ with $i_1^1 < \cdots < i_1^s$. Note that, the theorem is proved once we show that

$$f_s(\xi) = \langle c_s; \xi^s \rangle,$$

which is equivalent to proving that,

$$c_{J^1 \dots J^s} = \operatorname{sgn}(\sigma) \, c_{I^1 \dots I^s}, \tag{6}$$

for all $I^1, \dots, I^s, J^1, \dots, J^s \in \mathcal{T}_k^n$ satisfying $J^1 \cup \cdots \cup J^s = I^1 \cup \cdots \cup I^s$, $\sigma\left(J^1 J^2 \cdots J^s\right) = \left(I^1 I^2 \cdots I^s\right)$, where $\sigma \in \mathcal{S}^{sk}$ is the permutation of indices that respects the aforementioned order.

Step 3.1. Before concluding the proof, we observe that, for all $s = 2, \dots, n$,

$$f_s\left(\sum_{i=1}^{s-1} t_i\, \alpha_i\right) = 0,$$

where $t_i \in \mathbb{R}$ and α_i is an element of the standard basis of Λ^k. This is a direct consequence of the fact that f_s is homogeneous of degree s and that

$$\xi = \sum_{i=1}^{s-1} t_i\, \alpha_i$$

has at most $(s-1)$ coefficients that are non-zero.

Step 3.2. We finally establish Equation (6). Let $I^1, \dots, I^s, J^1, \dots, J^s \in \mathcal{T}_k^n$ satisfy

$$J^1 \cup \cdots \cup J^s = I^1 \cup \cdots \cup I^s, \text{ and } \sigma\left(J^1 J^2 \cdots J^s\right) = \left(I^1 I^2 \cdots I^s\right),$$

where $\sigma \in \mathcal{S}^{sk}$ is the permutation of indices that respects the order. We claim that

$$c_{J^1 \dots J^s} = \operatorname{sgn}(\sigma)\, c_{I^1 \dots I^s}. \tag{7}$$

Since any permutation that respects the ordering scheme is a product (unique up to parity) of transpositions each of which respects the ordering, it is enough to prove the result for the case where σ is a transposition that respects the ordering. Hence, Equation (7) reduces to proving that

$$c_{I^1 \dots I^s} = -c_{J^1 \dots J^s}\,. \tag{8}$$

Let us write

$$I^1 = \left(i_1^1, \dots, i_k^1\right), \dots, I^s = \left(i_1^s, \dots, i_k^s\right),$$

and

$$J^1 = \left(j_1^1, \dots, j_k^1\right), \dots, J^s = \left(j_1^s, \dots, j_k^s\right).$$

Then, $i_1^1 < \cdots < i_1^s$ and $j_1^1 < \cdots < j_1^s$. Since σ respects the ordering scheme, σ flips two indices $i_{r_1}^{q_1}$ and $i_{r_2}^{q_2}$, with $q_1 \neq q_2$ and leaves other indices fixed. Note that, from (5), we have

$$c_{I^1 \dots I^s} = f_s\left(\sum_{m=1}^{s} e^{I^m}\right) = f_s\left(\sum_{m=1}^{s} e^{i_1^m} \wedge \cdots \wedge e^{i_k^m}\right), \tag{9}$$

and

$$c_{J^1 \dots J^s} = f_s\left(\sum_{m=1}^{s} e^{J^m}\right) = f_s\left(\sum_{m=1}^{s} e^{j_1^m} \wedge \cdots \wedge e^{j_k^m}\right). \tag{10}$$

Since f_s is ext. one affine, setting

$$a = e^{i_{r_1}^{q_1}}, \quad b = e^{i_{r_2}^{q_2}}, \quad \xi = \sum_{\substack{m=1 \\ m \neq q_1, q_2}}^{s} e^{i_1^m} \wedge \cdots \wedge e^{i_k^m} = \sum_{\substack{m=1 \\ m \neq q_1, q_2}}^{s} e^{I^m},$$

$$\alpha = \pm e^{i_1^{q_1}} \wedge \cdots \wedge \widehat{e^{i_{r_1}^{q_1}}} \wedge \cdots \wedge e^{i_k^{q_1}} \quad \text{and} \quad \beta = \pm e^{i_1^{q_2}} \wedge \cdots \wedge \widehat{e^{i_{r_2}^{q_2}}} \wedge \cdots \wedge e^{i_k^{q_2}},$$

with signs being chosen appropriately so that

$$\alpha \wedge a = e^{I^{q_1}} = e^{i_1^{q_1}} \wedge \cdots \wedge e^{i_k^{q_1}} \quad \text{and} \quad \beta \wedge b = e^{I^{q_2}} = e^{i_1^{q_2}} \wedge \cdots \wedge e^{i_k^{q_2}},$$

we can apply Corollary 16 to f_s to obtain

$$\begin{aligned}
&[f_s(\xi + \alpha \wedge a + \beta \wedge b) - f_s(\xi)] + [f_s(\xi + \beta \wedge a + \alpha \wedge b) - f_s(\xi)] \\
&= [f_s(\xi + \alpha \wedge a) - f_s(\xi)] + [f_s(\xi + \beta \wedge b) - f_s(\xi)] \\
&+ [f_s(\xi + \beta \wedge a) - f_s(\xi)] + [f_s(\xi + \alpha \wedge b) - f_s(\xi)] .
\end{aligned}$$

Using Step 3.1, all except $f_s(\xi + \alpha \wedge a + \beta \wedge b)$ and $f_s(\xi + \beta \wedge a + \alpha \wedge b)$ are 0. Hence, we deduce

$$f_s(\xi + \alpha \wedge a + \beta \wedge b) = -f_s(\xi + \beta \wedge a + \alpha \wedge b),$$

which, together with (9) and (10) proves (8). This conclude the proof of Step 3.2 and thus of the theorem. ■

4 Some examples

4.1 The quadratic case

4.1.1 Some preliminary results

Before stating the main theorem on quadratic forms, we need a lemma whose proof is straightforward.

Lemma 19 *Let $1 \leq k \leq n$, $M : \Lambda^k \to \Lambda^k$ be a symmetric linear operator and $f : \Lambda^k \to \mathbb{R}$ be such that, for every $\xi \in \Lambda^k$,*

$$f(\xi) = \langle M\xi; \xi \rangle .$$

The following statements then hold true.

(i) *f is ext. polyconvex if and only if there exists $\beta \in \Lambda^{2k}$ so that,*

$$f(\xi) \geq \langle \beta; \xi \wedge \xi \rangle, \text{ for every } \xi \in \Lambda^k.$$

(ii) *f is ext. quasiconvex if and only if*

$$\int_\Omega f(d\omega) \geq 0,$$

for every bounded open set $\Omega \subset \mathbb{R}^n$ and $\omega \in W_0^{1,\infty}(\Omega; \Lambda^{k-1})$.

(iii) *f is ext. one convex if and only if, for every $a \in \Lambda^{k-1}$ and $b \in \Lambda^1$,*

$$f(a \wedge b) \geq 0,$$

4.1.2 Some examples

We start with the following example that will be used in Theorem 23 below.

Proposition 20 *Let $2 \leq k \leq n-2$. Let $\alpha \in \Lambda^k$ be not 1-divisible, then there exists $c > 0$ such that*

$$f(\xi) = |\xi|^2 - c\left(\langle \alpha; \xi \rangle\right)^2,$$

is ext. quasiconvex but not convex. If, in addition $\alpha \wedge \alpha = 0$, then the above f, for an appropriate c, is ext. quasiconvex but not ext. polyconvex.

Remark 21 ***(i)*** *It is easy to see that α is not 1-divisible if and only if*

$$\operatorname{rank}[*\alpha] = n.$$

This results from Remark 2.44 (iv) (with the help of Proposition 2.33 (iii)) in [4]. Such an α always exists if either of the following holds (see Propositions 2.37 (ii) and 2.43 in [4])

- *$k = n-2 \geq 2$ is even,*
- *$2 \leq k \leq n-3$.*

For example,

$$\alpha = e^1 \wedge e^2 \wedge e^3 + e^4 \wedge e^5 \wedge e^6 \in \Lambda^3\left(\mathbb{R}^6\right),$$

is not 1-divisible.

(ii) *Note that, when $k = 2$ every form α such that $\alpha \wedge \alpha = 0$ is necessarily 1-divisible. While, as soon as k is even and $4 \leq k \leq n-2$, there exists α that is not 1-divisible, but $\alpha \wedge \alpha = 0$. For example, when $k = 4$,*

$$\alpha = e^1 \wedge e^2 \wedge e^3 \wedge e^4 + e^1 \wedge e^2 \wedge e^5 \wedge e^6 + e^3 \wedge e^4 \wedge e^5 \wedge e^6 \in \Lambda^4\left(\mathbb{R}^6\right).$$

Proof Since the function is quadratic, the notions of ext. one convexity and ext. quasiconvexity are equivalent (see Theorem 23). We therefore only need to discuss the ext. one convexity. We divide the proof into two steps.

Step 1. We first show that if

$$\frac{1}{c} = \sup\left\{\left(\langle \alpha; a \wedge b \rangle\right)^2 : a \in \Lambda^{k-1},\, b \in \Lambda^1,\, |a \wedge b| = 1\right\},$$

then

$$\frac{1}{c} < |\alpha|^2.$$

We prove this statement as follows. Let $a_s \in \Lambda^{k-1}$, $b_s \in \Lambda^1$ be a maximizing sequence. Up to a subsequence that we do not relabel, we find that there exists $\lambda \in \Lambda^k$ so that

$$a_s \wedge b_s \to \lambda \quad \text{with} \quad |\lambda| = 1.$$

Similarly, up to a subsequence that we do not relabel, we have that there exists $\overline{b} \in \Lambda^1$ so that

$$\frac{b_s}{|b_s|} \to \overline{b}.$$

Since

$$a_s \wedge b_s \wedge \frac{b_s}{|b_s|} = 0,$$

we deduce that

$$\lambda \wedge \overline{b} = 0.$$

Appealing to Cartan lemma (see Theorem 2.42 in [4]), we find that there exists $\overline{a} \in \Lambda^{k-1}$ such that

$$\lambda = \overline{a} \wedge \overline{b} \quad \text{with} \quad \left| \overline{a} \wedge \overline{b} \right| = 1.$$

We therefore have found that

$$\frac{1}{c} = \left(\langle \alpha; \overline{a} \wedge \overline{b} \rangle \right)^2 .$$

Note that $\frac{1}{c} < |\alpha|^2$, otherwise $\overline{a} \wedge \overline{b}$ would be parallel to α and thus α would be 1-divisible which contradicts the hypothesis.

Step 2. So let

$$f(\xi) = |\xi|^2 - c \left(\langle \alpha; \xi \rangle \right)^2 .$$

(i) Observe that f is not convex since $c |\alpha|^2 > 1$ (by Step 1). Indeed,

$$f\left(\frac{1}{2}\alpha + \frac{1}{2}(-\alpha) \right) = f(0) = 0 > |\alpha|^2 \left(1 - c |\alpha|^2 \right) = f(\alpha) = \frac{1}{2} f(\alpha) + \frac{1}{2} f(-\alpha).$$

(ii) However, f is ext. one convex (and thus, invoking Theorem 23, f is ext. quasiconvex). Indeed, let

$$g(t) = f(\xi + t\, a \wedge b) = |\xi + t\, a \wedge b|^2 - c \left(\langle \alpha; \xi + t\, a \wedge b \rangle \right)^2 .$$

Note that

$$g''(t) = 2 \left[|a \wedge b|^2 - c \left(\langle \alpha; a \wedge b \rangle \right)^2 \right],$$

which is non-negative by Step 1. Thus g is convex.

(iii) Let $\alpha \wedge \alpha = 0$ and assume, for the sake of contradiction, that f is ext. polyconvex. Then there should exist (cf. Lemma 19) $\beta \in \Lambda^{2k}$ so that, for every $\xi \in \Lambda^k$,

$$f(\xi) \geq \langle \beta; \xi \wedge \xi \rangle .$$

This is clearly impossible, in view of the fact that $c |\alpha|^2 > 1$, since choosing $\xi = \alpha$, we get

$$f(\alpha) = |\alpha|^2 \left(1 - c |\alpha|^2 \right) < 0 = \langle \beta; \alpha \wedge \alpha \rangle .$$

The proof is therefore complete. ■

We conclude with another example.

Proposition 22 *Let $1 \leq k \leq n$, $T : \mathbb{R}^n \to \mathbb{R}^n$ be a symmetric linear operator and $T^* : \Lambda^k \to \Lambda^k$ be the pullback of T. Let $f : \Lambda^k \to \mathbb{R}$ be defined as*

$$f(\xi) = \langle T^*(\xi); \xi \rangle , \text{ for every } \xi \in \Lambda^k.$$

Then f is ext. one convex if and only if f is convex.

Proof Since convexity implies ext. one convexity, we only have to prove the reverse implication.

Step 1. Since T is symmetric, we can find eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ (not necessarily distinct) of T with a corresponding set of orthonormal eigenvectors $\{\varepsilon^1, \ldots, \varepsilon^n\}$. Let $\{e^1, \ldots, e^n\}$ be the standard basis of $\mathbb{R}^n$ and let $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ and Q be the orthogonal matrix so that

$$Q^*\left(\varepsilon^i\right) = e^i, \quad \text{for } i = 1, \ldots, n.$$

In terms of matrices what we have written just means that

$$T = Q\Lambda Q^t.$$

Observe that, for every $i = 1, \ldots, n$,

$$T^*\left(\varepsilon^i\right) = \left(Q\Lambda Q^t\right)^*\left(\varepsilon^i\right) = \left(Q^t\right)^*\left(\Lambda^*\left(Q^*\left(\varepsilon^i\right)\right)\right) = \left(Q^t\right)^*\left(\Lambda^*\left(e^i\right)\right) = \lambda_i \varepsilon^i.$$

This implies, for every $1 \leq k \leq n$ and $I \in \mathcal{T}_k^n$,

$$T^*\left(\varepsilon^I\right) = T^*\left(\varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_k}\right) = T^*\left(\varepsilon^{i_1}\right) \wedge \cdots \wedge T^*\left(\varepsilon^{i_k}\right) = \left(\prod_{j=1}^{k} \lambda_{i_j}\right) \varepsilon^I.$$

Step 2. Since f is ext one convex and in view of Lemma 19 (iii), we have

$$f\left(\varepsilon^I\right) = \left\langle \left(T^*\left(\varepsilon^I\right)\right); \varepsilon^I \right\rangle \geq 0$$

and thus

$$\prod_{j=1}^{k} \lambda_{i_j} = \prod_{i \in I} \lambda_i \geq 0. \tag{11}$$

Writing ξ in the basis $\{\varepsilon^1, \ldots, \varepsilon^n\}$, we get

$$f(\xi) = \langle T^*(\xi); \xi \rangle = \left\langle T^*\left(\sum_{I \in \mathcal{T}_k^n} \xi_I \varepsilon^I\right); \sum_{I \in \mathcal{T}_k^n} \xi_I \varepsilon^I \right\rangle = \sum_{I \in \mathcal{T}_k^n} \left(\prod_{i \in I} \lambda_i\right) (\xi_I)^2,$$

which according to (11) is non negative. This shows that f is convex as wished. ■

4.1.3 The main result

We now turn to the main theorem.

Theorem 23 *Let $1 \leq k \leq n$, $M : \Lambda^k \to \Lambda^k$ be a symmetric linear operator and $f : \Lambda^k \to \mathbb{R}$ be such that, for every $\xi \in \Lambda^k$,*

$$f(\xi) = \langle M\xi; \xi \rangle.$$

(i) *The following equivalence holds in all cases*

$$f \text{ ext. quasiconvex} \Leftrightarrow f \text{ ext. one convex.}$$

(ii) *Let $k = 2$. If $n = 2$ or $n = 3$, then*

$$f \text{ convex} \Leftrightarrow f \text{ ext. polyconvex} \Leftrightarrow f \text{ ext. quasiconvex} \Leftrightarrow f \text{ ext. one convex.}$$

If $n = 4$, then

$$f \text{ convex} \underset{\not\Leftarrow}{\Rightarrow} f \text{ ext. polyconvex} \Leftrightarrow f \text{ ext. quasiconvex} \Leftrightarrow f \text{ ext. one convex,}$$

while if $n \geq 6$, then

$$f \text{ ext. polyconvex} \underset{\not\Leftarrow}{\Rightarrow} f \text{ ext. quasiconvex} \Leftrightarrow f \text{ ext. one convex.}$$

(iii) *If k is odd or if $2k > n$, then*

$$f \text{ convex} \Leftrightarrow f \text{ ext. polyconvex.}$$

(iv) *If k is even and $2k \leq n$, then*

$$f \text{ convex} \underset{\not\Leftarrow}{\Rightarrow} f \text{ ext. polyconvex.}$$

(v) *If either $3 \leq k \leq n - 3$ or $k = n - 2 \geq 4$ is even, then*

$$f \text{ ext. polyconvex} \underset{\not\Leftarrow}{\Rightarrow} f \text{ ext. quasiconvex} \Leftrightarrow f \text{ ext. one convex.}$$

Remark 24 ***(i)*** *We recall that when $k = 1$ all notions of convexity are equivalent.*

(ii) *When $k = 2$ and $n = 5$, the equivalence between ext. polyconvexity and ext. quasiconvexity remains open.*

Proof (i) The result follows from Lemma 19 and Plancherel formula. The proof is similar to that of the case of the gradient (cf. Theorem 5.25 and Lemma 5.28 in [7]).

(ii) If $n = 2$ or $n = 3$, the result follows from Theorem 8 (ii). If $n \geq 6$, see Theorem 25. So we now assume that $n = 4$ (for the counter implication see (iv) below). We only have to prove that

$$f \text{ ext. one convex} \Rightarrow f \text{ ext. polyconvex.}$$

We know (by ext. one convexity) that, for every $a, b \in \Lambda^1\left(\mathbb{R}^4\right)$

$$f\left(a \wedge b\right) \geq 0,$$

and we wish to show (cf. Lemma 19) that we can find $\alpha \in \Lambda^4\left(\mathbb{R}^4\right)$ so that

$$f\left(\xi\right) \geq \left\langle \alpha; \xi \wedge \xi \right\rangle.$$

Step 1. Let us change slightly the notations and write $\xi \in \Lambda^2\left(\mathbb{R}^4\right)$ as a vector of $\mathbb{R}^6$ in the following manner

$$\xi = (\xi_{12}, \xi_{13}, \xi_{14}, \xi_{23}, \xi_{24}, \xi_{34}),$$

and therefore f can be seen as a quadratic form over $\mathbb{R}^6$ which is non-negative whenever the quadratic form (note also that g is indefinite)

$$g(\xi) = \left\langle e^1 \wedge e^2 \wedge e^3 \wedge e^4; \xi \wedge \xi \right\rangle = 2\left(\xi_{12}\xi_{34} - \xi_{13}\xi_{24} + \xi_{14}\xi_{23}\right)$$

vanishes. Indeed note that, by Proposition 2.37 in [4],

$$g(\xi) = 0 \;\Leftrightarrow\; \xi \wedge \xi = 0 \;\Leftrightarrow\; \operatorname{rank}[\xi] \in \{0, 2\}.$$

By Proposition 2.43 in [4], this last condition is equivalent to the existence of $a, b \in \Lambda^1\left(\mathbb{R}^4\right)$ so that

$$\xi = a \wedge b,$$

and by ext. one convexity we know that $f(a \wedge b) \geq 0$.

Step 2. We now invoke Theorem 2 in [10] to find $\lambda \in \mathbb{R}$ such that

$$f(\xi) - \lambda g(\xi) \geq 0.$$

But this is exactly what we had to prove.

(iii) This is a general fact. (See Remark 2 (ii) and Theorem 8)

(iv) The counterexample is just

$$f(\xi) = \langle \alpha; \xi \wedge \xi \rangle,$$

for any $\alpha \in \Lambda^{2k}$, $\alpha \neq 0$.

(v) This is just Proposition 20 and the remark following it. Indeed we consider the two following cases.

- If k is odd (and since $3 \leq k \leq n-3$, then $n \geq 6$), we know from (iii) that f is ext. polyconvex if and only if f is convex and we also know that there exists an exterior k-form which is not 1-divisible. Proposition 20 gives therefore the result.

- If k is even and $4 \leq k \leq n-2$ (which implies again $n \geq 6$), then there exists an exterior k-form α which is not 1-divisible, but $\alpha \wedge \alpha = 0$. The result thus follows again by Proposition 20. ■

4.1.4 A counterexample for $k = 2$

We now turn to a counterexample that has been mentioned in Theorem 23.

Theorem 25 *Let $n \geq 6$. Then there exists a quadratic form $f : \Lambda^2 \to \mathbb{R}$ which is ext. one convex but not ext. polyconvex.*

Proof It is enough to establish the theorem for $n = 6$. Our counterexample is inspired by Serre [14] and Terpstra [19] (see Theorem 5.25 (iii) in [7]). It is more convenient to write here $\xi \in \Lambda^2(\mathbb{R}^6)$ as

$$\xi = \sum_{1 \leq i < j \leq 6} \xi_j^i \, e^i \wedge e^j .$$

So let

$$g(\xi) = (\xi_2^1)^2 + (\xi_3^1)^2 + (\xi_3^2)^2 + (\xi_5^4)^2 + (\xi_6^4)^2 + (\xi_6^5)^2 + h(\xi),$$

where

$$h(\xi) = (\xi_4^1 - \xi_5^3 - \xi_6^2)^2 + (\xi_5^1 - \xi_4^3 + \xi_6^1)^2 + (\xi_4^2 - \xi_4^3 - \xi_6^1)^2 + (\xi_5^2)^2 + (\xi_6^3)^2 .$$

Note that $g \geq 0$. We claim that there exists $\gamma > 0$ so that

$$f(\xi) = g(\xi) - \gamma |\xi|^2$$

is ext. one convex (cf. Step 1) but not ext. polyconvex (cf. Step 2).

Step 1. Define

$$\gamma := \inf \left\{ g(a \wedge b) : a, b \in \Lambda^1(\mathbb{R}^6), \; |a \wedge b| = 1 \right\}. \tag{12}$$

Note that $\gamma \geq 0$, and it follows from Lemma 19 that f is ext. one convex.
Let us claim that, in fact $\gamma > 0$, which will imply in *Step 2* that f is not ext. polyconvex. To show that $\gamma > 0$, we proceed by contradiction and assume that $\gamma = 0$. This implies that we can find $a, b \in \Lambda^1(\mathbb{R}^6)$ with $|a \wedge b| = 1$ such that

$$\begin{cases} a^1 b_2 - a^2 b_1 = 0 \\ a^1 b_3 - a^3 b_1 = 0 \\ a^2 b_3 - a^3 b_2 = 0 \end{cases} \quad \begin{cases} a^4 b_5 - a^5 b_4 = 0 \\ a^4 b_6 - a^6 b_4 = 0 \\ a^5 b_6 - a^6 b_5 = 0 \end{cases} \quad \begin{cases} a^2 b_5 - a^5 b_2 = 0 \\ a^3 b_6 - a^6 b_3 = 0 \end{cases}$$

$$\begin{cases} (a^1 b_4 - a^4 b_1) - (a^3 b_5 - a^5 b_3) - (a^2 b_6 - a^6 b_2) = 0 \\ (a^1 b_5 - a^5 b_1) - (a^3 b_4 - a^4 b_3) + (a^1 b_6 - a^6 b_1) = 0 \\ (a^2 b_4 - a^4 b_2) - (a^3 b_4 - a^4 b_3) - (a^1 b_6 - a^6 b_1) = 0. \end{cases}$$

Let us introduce some notation, we write

$$\underline{a} = \begin{pmatrix} a^1 \\ a^2 \\ a^3 \end{pmatrix}, \quad \underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \quad \overline{a} = \begin{pmatrix} a^4 \\ a^5 \\ a^6 \end{pmatrix}, \quad \overline{b} = \begin{pmatrix} b_4 \\ b_5 \\ b_6 \end{pmatrix}.$$

Note that the first and second sets of equations lead to

$$\underline{a} \| \underline{b} \quad \text{and} \quad \overline{a} \| \overline{b}.$$

We consider two cases starting with the generic case.

Case 1: there exist $\lambda, \mu \in \mathbb{R}$ such that

$$\underline{a} = \lambda \, \underline{b} \quad \text{and} \quad \overline{a} = \mu \, \overline{b}.$$

(The same reasoning applies to all other cases, for example $\underline{b} = \lambda \, \underline{a}$ and $\overline{b} = \mu \, \overline{a}$). Note that $\lambda \neq \mu$, otherwise we would have $a = \lambda \, b$ and thus $a \wedge b = 0$ contradicting the fact that $|a \wedge b| = 1$. Inserting this in the third and fourth sets of equations we get

$$\begin{cases} (\lambda - \mu) \, b_2 b_5 = 0 \\ (\lambda - \mu) \, b_3 b_6 = 0 \end{cases} \qquad \begin{cases} (\lambda - \mu) \, [b_1 b_4 - b_3 b_5 - b_2 b_6] = 0 \\ (\lambda - \mu) \, [b_1 b_5 - b_3 b_4 + b_1 b_6] = 0 \\ (\lambda - \mu) \, [b_2 b_4 - b_3 b_4 - b_1 b_6] = 0 \end{cases}$$

and thus, since $\lambda \neq \mu$,

$$b_2 b_5 = b_3 b_6 = b_1 b_4 - b_3 b_5 - b_2 b_6 = b_1 b_5 - b_3 b_4 + b_1 b_6 = b_2 b_4 - b_3 b_4 - b_1 b_6 = 0.$$

We have to consider separately the cases $b_2 = b_3 = 0$, $b_5 = b_6 = 0$, $b_2 = b_6 = 0$ and $b_3 = b_5 = 0$. We do only the first case. Other cases can be handled similarly. Let us assume that $b_2 = b_3 = 0$. We thus have

$$b_1 b_4 = b_1 b_5 + b_1 b_6 = b_1 b_6 = 0.$$

So either $b_1 = 0$ and thus $\underline{b} = 0$ and hence $\underline{a} = 0$ and again this implies that $a = \mu \, b$ which contradicts the fact that $|a \wedge b| = 1$. Or $b_4 = b_5 = b_6 = 0$ and thus $\overline{b} = \overline{a} = 0$ which as before contradicts the fact that $|a \wedge b| = 1$.

Case 2: $\underline{b} = 0$ and $\overline{a} = 0$ (or $\underline{a} = 0$ and $\overline{b} = 0$ which is handled similarly). This means that $a^4 = a^5 = a^6 = 0$ and $b_1 = b_2 = b_3 = 0$. We therefore have

$$a^2 b_5 = a^3 b_6 = a^1 b_4 - a^3 b_5 - a^2 b_6 = a^1 b_5 - a^3 b_4 + a^1 b_6 = a^2 b_4 - a^3 b_4 - a^1 b_6 = 0.$$

Four cases can happen $a^2 = a^3 = 0$, $a^2 = b_6 = 0$, $a^3 = b_5 = 0$ and $b_5 = b_6 = 0$. Again, we handle only the first case. Other cases can be treated similarly. Assuming that $a^2 = a^3 = 0$, we have

$$a^1 b_4 = a^1 b_5 + a^1 b_6 = a^1 b_6 = 0.$$

So either $a^1 = 0$ and thus $a = 0$ which is impossible. Or $b_4 = b_5 = b_6 = 0$ and thus $b = 0$ which again cannot happen. Hence, we have proved that γ defined in Equation (12) is positive.

Step 2. We now show that f is not ext. polyconvex. In view of Lemma 19 (i), it is sufficient to show that for every $\alpha \in \Lambda^4 \left(\mathbb{R}^6 \right)$, there exists $\xi \in \Lambda^2 \left(\mathbb{R}^6 \right)$ such that

$$f \left(\xi \right) + \frac{1}{2} \left\langle \alpha; \xi \wedge \xi \right\rangle < 0.$$

We prove that the above inequality holds for forms ξ of the following form

$$\begin{aligned} \xi = {} & (b+d) \, e^1 \wedge e^4 + (c-a) \, e^1 \wedge e^5 + a \, e^1 \wedge e^6 \\ & + (c+a) \, e^2 \wedge e^4 + b \, e^2 \wedge e^6 + c \, e^3 \wedge e^4 + d \, e^3 \wedge e^5. \end{aligned}$$

Note that

$$\frac{1}{2}\xi\wedge\xi=\left(c^{2}-a^{2}\right)e^{1}\wedge e^{2}\wedge e^{4}\wedge e^{5}+\left(ac+a^{2}-b^{2}-bd\right)e^{1}\wedge e^{2}\wedge e^{4}\wedge e^{6}$$
$$+\left(ab-bc\right)e^{1}\wedge e^{2}\wedge e^{5}\wedge e^{6}+\left(c^{2}-ac-bd-d^{2}\right)e^{1}\wedge e^{3}\wedge e^{4}\wedge e^{5}$$
$$+ac\,e^{1}\wedge e^{3}\wedge e^{4}\wedge e^{6}+ad\,e^{1}\wedge e^{3}\wedge e^{5}\wedge e^{6}$$
$$+\left(-cd-ad\right)e^{2}\wedge e^{3}\wedge e^{4}\wedge e^{5}+bc\,e^{2}\wedge e^{3}\wedge e^{4}\wedge e^{6}$$
$$+bd\,e^{2}\wedge e^{3}\wedge e^{5}\wedge e^{6}$$

For such forms we have $g\left(\xi\right)=0$ and therefore

$$f\left(\xi\right)=-\gamma\left|\xi\right|^{2}=-\gamma\left[\left(b+d\right)^{2}+\left(c-a\right)^{2}+a^{2}+\left(c+a\right)^{2}+b^{2}+c^{2}+d^{2}\right].$$

Moreover,

$$\frac{1}{2}\left\langle\alpha;\xi\wedge\xi\right\rangle=\alpha_{1245}\left(c^{2}-a^{2}\right)+\alpha_{1246}\left(ac+a^{2}-b^{2}-bd\right)$$
$$+\alpha_{1256}\left(ab-bc\right)+\alpha_{1345}\left(c^{2}-ac-bd-d^{2}\right)+\alpha_{1346}\,ac$$
$$+\alpha_{1356}\,ad+\alpha_{2345}\left(-cd-ad\right)+\alpha_{2346}\,bc+\alpha_{2356}\,bd.$$

We consider three cases.

Case 1. If $\alpha_{1246}>0$, then take $a=c=d=0$ and $b\neq 0$, to get

$$f\left(\xi\right)+\frac{1}{2}\left\langle\alpha;\xi\wedge\xi\right\rangle=-\gamma\left(2b^{2}\right)-\alpha_{1246}b^{2}<0.$$

Case 2. If $\alpha_{1345}>0$, then take $a=b=c=0$ and $d\neq 0$, to get

$$f\left(\xi\right)+\frac{1}{2}\left\langle\alpha;\xi\wedge\xi\right\rangle=-\gamma\left(2d^{2}\right)-\alpha_{1345}d^{2}<0.$$

We therefore can assume that $\alpha_{1246}\leq 0$ and $\alpha_{1345}\leq 0$.

Case 3. If $\alpha_{1245}+\alpha_{1345}<0$ $\left(\alpha_{1246}\leq 0,\ \alpha_{1345}\leq 0\right)$, then take $a=b=d=0$ and $c\neq 0$ to get

$$f\left(\xi\right)+\frac{1}{2}\left\langle\alpha;\xi\wedge\xi\right\rangle=-\gamma\left(3c^{2}\right)+\left(\alpha_{1245}+\alpha_{1345}\right)c^{2}<0.$$

We therefore assume $\alpha_{1246}\leq 0$, $\alpha_{1345}\leq 0$ and $\alpha_{1245}+\alpha_{1345}\geq 0$. From these three inequalities we deduce that $\alpha_{1246}-\alpha_{1245}\leq 0$, and then taking $b=c=d=0$ and $a\neq 0$, we get

$$f\left(\xi\right)+\frac{1}{2}\left\langle\alpha;\xi\wedge\xi\right\rangle=-\gamma\left(3a^{2}\right)+\left(\alpha_{1246}-\alpha_{1245}\right)a^{2}<0.$$

This concludes the proof of the theorem. ■

4.2 Ext. one convexity does not imply ext. quasiconvexity

We now give an important counterexample for any $k \geq 2$. It is an adaptation of the fundamental result of Sverak [17] (see also Theorem 5.50 in [7]).

Theorem 26 *Let $2 \leq k \leq n-3$. Then there exists $f : \Lambda^k \to \mathbb{R}$ ext. one convex but not ext. quasiconvex.*

Remark 27 *We know that when $k = 1, n-1, n$ or $k = n-2$ is odd, then*

$$f \text{ convex } \Leftrightarrow f \text{ ext. polyconvex } \Leftrightarrow f \text{ ext. quasiconvex } \Leftrightarrow f \text{ ext. one convex.}$$

Therefore only the case $k = n-2 \geq 2$ even (including $k = 2$ and $n = 4$) remains open.

The main algebraic tool in order to adapt Sverak example is given in the following lemma.

Lemma 28 *Let $k \geq 2$ and $n = k+3$. There exist*

$$\alpha, \beta, \gamma \in \operatorname{span}\left\{e^{i_1} \wedge \cdots \wedge e^{i_{k-1}}, \quad 3 \leq i_1 < \cdots < i_{k-1} \leq k+3\right\} \subset \Lambda^{k-1}\left(\mathbb{R}^{k+3}\right),$$

such that if

$$L = \operatorname{span}\left\{e^1 \wedge \alpha, e^2 \wedge \beta, \left(e^1 + e^2\right) \wedge \gamma\right\},$$

then any 1-divisible $\xi = (x, y, z) = xe^1 \wedge \alpha + ye^2 \wedge \beta + z\left(e^1 + e^2\right) \wedge \gamma \in L$ necessarily verifies

$$xy = xz = yz = 0.$$

Proof *Step 1.* We choose, recall that $n = k+3$,

$$\alpha = \begin{cases} \sum_{i=2}^{l+1} \left(\widehat{e^{2i}} \wedge \widehat{e^{2i+1}}\right) & \text{if } k = 2l, \\ \sum_{i=2}^{l+2} \left(\widehat{e^{2i-1}} \wedge \widehat{e^{2i}}\right) & \text{if } k = 2l+1. \end{cases}$$

$$\beta = \begin{cases} \widehat{e^3} \wedge \widehat{e^{2l+3}} & \text{if } k = 2l, \\ \left(\widehat{e^3} \wedge \widehat{e^5}\right) + \left(\widehat{e^4} \wedge \widehat{e^6}\right) & \text{if } k = 3, \\ \sum_{i=2}^{l} \left(\widehat{e^{2i-1}} \wedge \widehat{e^{2i}}\right) & \text{if } k = 2l+1 \text{ and } k \geq 5. \end{cases}$$

$$\gamma = \begin{cases} \sum_{i=2}^{l+1} \left(\widehat{e^{2i-1}} \wedge \widehat{e^{2i}}\right) & \text{if } k = 2l, \\ \left(\widehat{e^{2l+1}} \wedge \widehat{e^{2l+4}}\right) + \left(\widehat{e^{2l+2}} \wedge \widehat{e^{2l+3}}\right) & \text{if } k = 2l+1, \end{cases}$$

where we write, by abuse of notations, for $3 \leq i < j \leq k+3$,

$$\widehat{e^i} \wedge \widehat{e^j} = e^3 \wedge \cdots \wedge \widehat{e^i} \wedge \cdots \wedge \widehat{e^j} \wedge \cdots \wedge e^{k+3}.$$

Observe that $\{\alpha, \beta, \gamma\}$ are linearly independent.

Step 2. We now prove the statement, namely that if $\xi = (x, y, z) \in L$ is 1-divisible (i.e. $\xi = b \wedge a$ for $a \in \Lambda^1$ and $b \in \Lambda^{k-1}$), then necessarily

$$xy = xz = yz = 0.$$

Assume that $\xi \neq 0$ (otherwise the result is trivial) and thus $a \neq 0$. Note that if $\xi = b \wedge a$, then $a \wedge \xi = 0$. We write

$$a = \sum_{i=1}^{k+3} a_i \, e^i \neq 0.$$

Step 2.1. Since $a \wedge \xi = 0$ we deduce that the term involving $e^1 \wedge e^2$ must be 0 and thus

$$-a_2 x \, \alpha + a_1 y \, \beta + (a_1 - a_2) \, z \, \gamma = 0.$$

Since $\{\alpha, \beta, \gamma\}$ are linearly independent, we deduce that

$$a_2 x = a_1 y = (a_1 - a_2) \, z = 0.$$

From there we infer that $xy = xz = yz = 0$, as soon as either $a_1 \neq 0$ or $a_2 \neq 0$. So in order to establish the lemma it is enough to consider a of the form

$$a = \sum_{i=3}^{k+3} a_i \, e^i \neq 0.$$

We therefore have

$$\sum_{i=3}^{k+3} a_i \, e^i \wedge \left[e^1 \wedge (x \, \alpha + z \, \gamma) + e^2 \wedge (y \, \beta + z \, \gamma) \right] = 0,$$

which implies that

$$\begin{cases} a \wedge (x \, \alpha + z \, \gamma) = \sum_{i=3}^{k+3} a_i \, e^i \wedge (x \, \alpha + z \, \gamma) = 0, \\ a \wedge (y \, \beta + z \, \gamma) = \sum_{i=3}^{k+3} a_i \, e^i \wedge (y \, \beta + z \, \gamma) = 0. \end{cases} \tag{13}$$

We continue the discussion considering separately the cases k even, $k = 3$ and $k \geq 5$ odd. They are all treated in the same way and we prove it only in the even case.

Step 2.2: $k = 2l \geq 2$. We have to prove that if

$$a = \sum_{i=3}^{2l+3} a_i e^i \neq 0$$

satisfies (13), then necessarily

$$xy = xz = yz = 0.$$

We find (up to a $+$ or $-$ sign but here it is immaterial)

$$\begin{aligned}
a\wedge\alpha =& \sum_{i=2}^{l+1}\left(a_{2i+1}\widehat{e^{2i}}\right)+\sum_{i=2}^{l}\left(a_{2i}\widehat{e^{2i+1}}\right)+a_{2l+2}\widehat{e^{2l+3}},\\
a\wedge\beta =& a_{2l+3}\widehat{e^{3}}+a_{3}\widehat{e^{2l+3}},\\
a\wedge\gamma =& a_{4}\widehat{e^{3}}+\sum_{i=2}^{l+1}\left(a_{2i-1}\widehat{e^{2i}}\right)+\sum_{i=2}^{l}\left(a_{2i+2}\widehat{e^{2i+1}}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
a\wedge(x\,\alpha+z\,\gamma) =& z\,a_{4}\widehat{e^{3}}+\sum_{i=2}^{l+1}\left(x\,a_{2i+1}+z\,a_{2i-1}\right)\widehat{e^{2i}}\\
&+\sum_{i=2}^{l}\left(x\,a_{2i}+z\,a_{2i+2}\right)\widehat{e^{2i+1}}+x\,a_{2l+2}\widehat{e^{2l+3}},\\
a\wedge(y\,\beta+z\,\gamma) =& \left(y\,a_{2l+3}+z\,a_{4}\right)\widehat{e^{3}}+z\left\{\sum_{i=2}^{l+1}\left(a_{2i-1}\widehat{e^{2i}}\right)+\sum_{i=2}^{l}\left(a_{2i+2}\widehat{e^{2i+1}}\right)\right\}\\
&+y\,a_{3}\widehat{e^{2l+3}}.
\end{aligned}$$

Case 1 : $x=z=0$. This is our claim.

Case 2 : $z=0$ and $x\neq 0$. We can also assume that $y\neq 0$ otherwise we have the claim $y=z=0$. From the first equation we obtain

$$a_{2i}=a_{2i+1}=0,\quad i=2,\cdots,l+1.$$

So only a_3 might be non-zero. However since $y\neq 0$ we deduce from the second equation that $a_3=0$ and thus $a=0$ which is impossible.

Case 3 : $x=0$ and $z\neq 0$. We can also assume that $y\neq 0$ otherwise we have the claim $x=y=0$. From the first equation we obtain

$$a_{2i}=a_{2i-1}=0,\quad i=2,\cdots,l+1$$

So only a_{2l+3} might be non-zero. However since $y\neq 0$ we deduce, appealing to the second equation, that $a_{2l+3}=0$ and thus $a=0$ which is again impossible.

Case 4 : $xz\neq 0$. From the first equation we deduce that

$$a_{2i}=0,\quad i=2,\cdots,l+1$$

Inserting this in the second equation we get

$$a\wedge(y\,\beta+z\,\gamma)=y\,a_{2l+3}\widehat{e^{3}}+z\sum_{i=2}^{l+1}\left(a_{2i-1}\widehat{e^{2i}}\right)+y\,a_{3}\widehat{e^{2l+3}}.$$

Since $z \neq 0$, we infer that

$$a_{2i-1} = 0, \quad i = 2, \cdots, l+1.$$

So only a_{2l+3} might be non-zero. However returning to the first equation we have

$$x\, a_{2l+3} = 0.$$

But since $x \neq 0$, we deduce that $a_{2l+3} = 0$ and thus $a = 0$ which is again impossible. This settles the case k even. The odd case is handled in a very similar manner and we leave out the details ■

We may now conclude with the proof of Theorem 26, which is, once the above lemma is established, almost identical to the proof of Sverak.

Proof It is enough to prove the theorem for $n = k + 3$.

Step 1. We start with some notations. Let L be as in Lemma 28. An element ξ of L is, when convenient, denoted by $\xi = (x, y, z) \in L$. Recall that if $\xi = (x, y, z) \in L$ is 1-divisible, meaning that $\xi = b \wedge a$ for a certain $a \in \Lambda^1$ and $b \in \Lambda^{k-1}$, then necessarily

$$xy = xz = yz = 0.$$

We next let $P : \Lambda^k \left(\mathbb{R}^{k+3}\right) \to L$ be the projection map; in particular $P(\xi) = \xi$ if $\xi \in L$.

Step 2. Let $g : L \subset \Lambda^k \left(\mathbb{R}^{k+3}\right) \to \mathbb{R}$ be defined by

$$g(\xi) = -x\, y\, z.$$

Observe that, g is ext. one affine when restricted to L. Indeed, if $\xi = (x, y, z) \in L$ and $\eta = (a, b, c) \in L$ is 1-divisible (which implies that $ab = ac = bc = 0$), then

$$g(\xi + t\eta) = -(x + ta)(y + tb)(z + tc) = -x\, y\, z - t\,[x\, y\, c + x\, z\, b + y\, z\, a]\,.$$

We therefore have that, for every $\xi, \eta \in L$ with η being 1-divisible,

$$L_g(\xi, \eta) = \left.\frac{d^2}{dt^2} g(\xi + t\eta)\right|_{t=0} = 0.$$

Step 3. Let $\omega \in C^{\infty}_{per}\left((0, 2\pi)^{k+3}; \Lambda^{k-1}\right)$ be defined by

$$\omega = (\sin x_1)\,\alpha + (\sin x_2)\,\beta + (\sin(x_1 + x_2))\,\gamma,$$

so that

$$d\omega = (\cos x_1)\, dx^1 \wedge \alpha + (\cos x_2)\, dx^2 \wedge \beta + (\cos(x_1 + x_2))\left(dx^1 + dx^2\right) \wedge \gamma,$$

and hence $d\omega \in L$. Note that

$$\int_0^{2\pi}\int_0^{2\pi} g(d\omega)\, dx_1\, dx_2 = -\int_0^{2\pi}\int_0^{2\pi} (\cos x_1)^2 (\cos x_2)^2\, dx_1\, dx_2 < 0.$$

Step 4. Assume, cf. Step 5, that we have shown that for every $\epsilon > 0$, we can find $\gamma = \gamma(\epsilon) > 0$ such that

$$f_\epsilon(\xi) = g(P(\xi)) + \epsilon |\xi|^2 + \epsilon |\xi|^4 + \gamma |\xi - P(\xi)|^2$$

is ext. one convex. Then noting that

$$f_\epsilon(d\omega) = g(d\omega) + \epsilon |d\omega|^2 + \epsilon |d\omega|^4,$$

we deduce from Step 3 that, for $\epsilon > 0$ small enough

$$\int_{(0,2\pi)^{k+3}} f_\epsilon(d\omega)\, dx < 0.$$

This shows that f_ϵ is not ext. quasiconvex. The proposition is therefore proved.

Step 5. It remains to prove that for every $\epsilon > 0$ we can find $\gamma = \gamma(\epsilon) > 0$ such that

$$f_\epsilon(\xi) = g(P(\xi)) + \epsilon |\xi|^2 + \epsilon |\xi|^4 + \gamma |\xi - P(\xi)|^2$$

is ext. one convex. This is equivalent to showing that, for every $\xi, \eta \in \Lambda^k$ with η being 1-divisible,

$$\begin{aligned} L_{f_\epsilon}(\xi,\eta) &= \left.\frac{d^2}{dt^2} f_\epsilon(\xi + t\eta)\right|_{t=0} \\ &= L_g(P(\xi), P(\eta)) + 2\epsilon |\eta|^2 + 4\epsilon |\xi|^2 |\eta|^2 + 8\epsilon (\langle \xi; \eta \rangle)^2 + 2\gamma |\eta - P(\eta)|^2 \\ &\geq 0. \end{aligned}$$

The proof follows in the standard way, see [7] and [15] for more detail. ■

4.3 Some further examples

We here give another counterexample for $k = 2$.

Proposition 29 *Let $n \geq 4$. Then there exists an ext. quasiconvex function $f : \Lambda^2(\mathbb{R}^n) \to \mathbb{R}$ which is not ext. polyconvex.*

Remark 30 *This example is mostly interesting when $n = 4$ or 5. Since when $n \geq 6$, we already have such a counterexample (cf. Theorem 25).*

Proof As in previous theorems, it is easy to see that it is enough to establish the theorem for $n = 4$. Let $1 < p < 2$, $\alpha = e^1 \wedge e^2 + e^3 \wedge e^4$ and $g : \Lambda^2(\mathbb{R}^4) \to \mathbb{R}$ be given by

$$g(\xi) = \left(|\xi|^2 - 2|\langle \alpha; \xi \rangle| + |\alpha|^2\right)^{p/2} = \min\{|\xi - \alpha|^p, |\xi + \alpha|^p\}.$$

The claim is that $f = Q_{ext} g$ has all the desired properties (the proof is inspired by the one of Sverak [16], see also Theorem 5.54 in [7]). Indeed f is by construction ext. quasiconvex and if we can show (cf. Step 2) that f is not convex (note that f is subquadratic and using Proposition 14, any subquadratic ext. polyconvex function is convex) we will have established the proposition.

Step 1. First observe that a direct computation gives

$$|\xi|^2 - 2|\langle \alpha; \xi\rangle| + |\alpha|^2 = \min\left\{ |\xi - \alpha|^2, |\xi + \alpha|^2 \right\} \geq \frac{1}{2}\left[|\xi|^2 - \frac{1}{2}\langle \alpha \wedge \alpha; \xi \wedge \xi\rangle \right] \geq 0.$$

We therefore get that there exists a constant $c_1 > 0$ such that

$$\begin{aligned} g(\xi) &\geq \left(\frac{1}{2}\right)^{\frac{p}{2}} \left[|\xi|^2 - \frac{1}{2}\langle \alpha \wedge \alpha; \xi \wedge \xi\rangle \right]^{p/2} \\ &\geq c_1 \left[|\xi_{12} - \xi_{34}|^p + |\xi_{13} + \xi_{24}|^p + |\xi_{14} - \xi_{23}|^p \right]. \end{aligned}$$

Call h the right hand side, namely

$$h(\xi) = c_1 \left[|\xi_{12} - \xi_{34}|^p + |\xi_{13} + \xi_{24}|^p + |\xi_{14} - \xi_{23}|^p \right].$$

Step 2. To prove that f is not convex, we proceed by contradiction. Let us suppose, on the contrary, that f is convex. This implies that $f(0) = 0$, because

$$0 \leq f(0) = f\left(\frac{1}{2}\alpha + \frac{1}{2}(-\alpha) \right) \leq \frac{1}{2} f(\alpha) + \frac{1}{2} f(-\alpha) = 0.$$

Use Remark 13 to find a sequence of $\omega_s \in W^{1,\infty}_{\delta,T}\left(\Omega; \Lambda^1\right)$ (we can choose an Ω with smooth boundary and by density we can also assume that $\omega_s \in C^{\infty}_{\delta,T}\left(\overline{\Omega}; \Lambda^1\right)$) such that

$$0 \leq \frac{1}{\operatorname{meas}\Omega} \int_\Omega g(d\omega_s) \leq Q_{ext} g(0) + \frac{1}{s} = f(0) + \frac{1}{s} = \frac{1}{s},$$

which implies that

$$\lim_{s \to \infty} \frac{1}{\operatorname{meas}\Omega} \int_\Omega g(d\omega_s) = 0. \tag{14}$$

On the other hand, from Step 1, we deduce that

$$0 \leq \int_\Omega h(d\omega_s) \leq \frac{\operatorname{meas}\Omega}{s} \to 0.$$

We now invoke Step 3 to find a constant $c_2 > 0$ such that

$$c_2 \|\nabla \omega_s\|^p_{L^p} \leq \int_\Omega h(d\omega_s).$$

Thus $\|d\omega_s\|_{L^p} \to 0$ and hence, up to the extraction of a subsequence,

$$\frac{1}{\operatorname{meas}\Omega} \int_\Omega g(d\omega_s) \to g(0) = |\alpha|^p \neq 0,$$

which contradicts Equation (14). Therefore, f is not convex.

Step 3. It remains to prove that there exists a constant $\lambda > 0$ such that

$$\lambda \left\|\nabla\omega\right\|_{L^p}^p \leq \int_\Omega h\left(d\omega\right) = \left\|\left[h\left(d\omega\right)\right]^{1/p}\right\|_{L^p}^p , \quad \text{for every } \omega \in C_{\delta,T}^\infty\left(\overline{\Omega};\Lambda^1\right).$$

To establish the estimate we proceed as follows. Let $\omega \in C_{\delta,T}^\infty\left(\overline{\Omega};\Lambda^1\right)$, $\alpha,\beta,\gamma \in C^\infty\left(\overline{\Omega}\right)$ be such that

$$\begin{array}{rcccl} \alpha & = & (d\omega)_{12} - (d\omega)_{34} & = & -\omega^1_{x_2} + \omega^2_{x_1} + \omega^3_{x_4} - \omega^4_{x_3}, \\ \beta & = & (d\omega)_{13} + (d\omega)_{24} & = & -\omega^1_{x_3} + \omega^3_{x_1} - \omega^2_{x_4} + \omega^4_{x_2}, \\ \gamma & = & (d\omega)_{14} - (d\omega)_{23} & = & -\omega^1_{x_4} + \omega^4_{x_1} + \omega^2_{x_3} - \omega^3_{x_2}, \\ 0 & = & \delta\omega & = & \omega^1_{x_1} + \omega^2_{x_2} + \omega^3_{x_3} + \omega^4_{x_4}. \end{array}$$

Note that

$$h\left(d\omega\right) = c_1\left[|\alpha|^p + |\beta|^p + |\gamma|^p\right].$$

Differentiating appropriately the four equations we find

$$\left\{\begin{array}{llll} \Delta\omega^1 & = -\alpha_{x_2} - \beta_{x_3} - \gamma_{x_4}, & \Delta\omega^2 & = \alpha_{x_1} - \beta_{x_4} + \gamma_{x_3}, \\ \Delta\omega^3 & = \alpha_{x_4} + \beta_{x_1} - \gamma_{x_2}, & \Delta\omega^4 & = -\alpha_{x_3} + \beta_{x_2} + \gamma_{x_1}. \end{array}\right.$$

Letting

$$\begin{aligned} \psi := &-\left(\alpha_{x_2} + \beta_{x_3} + \gamma_{x_4}\right)dx^1 + \left(\alpha_{x_1} - \beta_{x_4} + \gamma_{x_3}\right)dx^2 + \left(\alpha_{x_4} + \beta_{x_1} - \gamma_{x_2}\right)dx^3 \\ &+ \left(-\alpha_{x_3} + \beta_{x_2} + \gamma_{x_1}\right)dx^4, \end{aligned}$$

we get

$$\left\{\begin{array}{ll} \Delta\omega = \psi & \text{in } \Omega \\ \nu\wedge\delta\omega = 0,\ \nu\wedge\omega = 0 & \text{on } \partial\Omega. \end{array}\right.$$

Using classical elliptic regularity theory, see, for example, Theorem 6.3.7 of [11], we deduce that

$$\left\|\omega\right\|_{W^{1,p}} \leq \lambda_2 \left\|\psi\right\|_{W^{-1,p}}.$$

In other words,

$$\left\|\nabla\omega\right\|_{L^p} \leq \lambda_2\left\|\psi\right\|_{W^{-1,p}} \leq \lambda_3\left\|(\alpha,\beta,\gamma)\right\|_{L^p} \leq \lambda_4\left\|\left[h\left(d\omega\right)\right]^{1/p}\right\|_{L^p}.$$

This is exactly what had to be proved. ■

5 Application to a minimization problem

Theorem 31 *Let $1 \leq k \leq n$, $p > 1$, $\Omega \subset \mathbb{R}^n$ be a bounded smooth open set, $\omega_0 \in W^{1,p}\left(\Omega;\Lambda^{k-1}\right)$ and $f : \Lambda^k\left(\mathbb{R}^n\right) \to \mathbb{R}$ be ext. quasiconvex verifying,*

$$c_1\left(|\xi|^p - 1\right) \leq f\left(\xi\right) \leq c_2\left(|\xi|^p + 1\right), \ \textit{for every } \xi \in \Lambda^k,$$

for some $c_1, c_2 > 0$. *Let*

$$(\mathcal{P}_0) \qquad \inf\left\{\int_\Omega f(d\omega) : \omega \in \omega_0 + W_0^{1,p}\left(\Omega;\Lambda^{k-1}\right)\right\} = m.$$

Then the problem $(\mathcal{P}_0)$ *has a minimizer.*

Remark 32 ***(i)*** *If*

$$(\mathcal{P}_{\delta,T}) \qquad \inf\left\{\int_\Omega f(d\omega) : \omega \in \omega_0 + W_{\delta,T}^{1,p}\left(\Omega;\Lambda^{k-1}\right)\right\} = m_{\delta,T},$$

where $\omega_0 + W_{\delta,T}^{1,p}\left(\Omega;\Lambda^{k-1}\right)$ *stands for the set of all* $\omega \in W^{1,p}\left(\Omega;\Lambda^{k-1}\right)$ *such that*

$$\delta\omega = 0 \text{ in } \Omega \quad \text{and} \quad \nu \wedge \omega = \nu \wedge \omega_0 \text{ on } \partial\Omega,$$

the proof of the theorem will show that $(\mathcal{P}_{\delta,T})$ *also has a minimizer and that* $m_{\delta,T} = m$.

(ii) *When the function* f *is not ext. quasiconvex, in general the problem will not have a solution. However, in many cases it does have one, but the argument is of a different nature and uses results on differential inclusions, see [1], [2] and [8].*

Proof *Step 1.* Using a variant of the classical result, see [3] and [15], we note that if

$$\alpha_s \rightharpoonup \alpha \quad \text{in } W^{1,p}\left(\Omega;\Lambda^{k-1}\right),$$

then

$$\liminf_{s\to\infty} \int_\Omega f(d\alpha_s) \geq \int_\Omega f(d\alpha).$$

Step 2. Let ω_s be a minimizing sequence of $(\mathcal{P}_0)$, i.e.

$$\int_\Omega f(d\omega_s) \to m.$$

In view of the coercivity condition, we find that there exists a constant $c_3 > 0$ such that

$$\|d\omega_s\|_{L^p} \leq c_3.$$

(i) According to Theorem 7.2 in [4] (when $p \geq 2$) and [15] (when $p > 1$), we can find $\alpha_s \in \omega_0 + W_{\delta,T}^{1,p}\left(\Omega;\Lambda^{k-1}\right)$ such that

$$\begin{cases} d\alpha_s = d\omega_s & \text{in } \Omega \\ \delta\alpha_s = 0 & \text{in } \Omega \\ \nu \wedge \alpha_s = \nu \wedge \omega_s = \nu \wedge \omega_0 & \text{on } \partial\Omega \end{cases}$$

and there exist constants $c_4, c_5 > 0$ such that

$$\|\alpha_s\|_{W^{1,p}} \leq c_4\left[\|d\omega_s\|_{L^p} + \|\omega_0\|_{W^{1,p}}\right] \leq c_5.$$

(ii) Therefore, up to the extraction of a subsequence that we do not relabel, there exists $\alpha \in \omega_0 + W_{\delta,T}^{1,p}\left(\Omega;\Lambda^{k-1}\right)$ such that

$$\alpha_s \rightharpoonup \alpha, \quad \text{in } W^{1,p}\left(\Omega;\Lambda^{k-1}\right).$$

(iii) We then use Theorem 8.16 in [4] (when $p \geq 2$) and [15] (when $p > 1$), to find $\omega \in \omega_0 + W_0^{1,p}\left(\Omega;\Lambda^{k-1}\right)$ such that

$$\begin{cases} d\omega = d\alpha & \text{in } \Omega \\ \omega = \omega_0 & \text{on } \partial\Omega. \end{cases}$$

Step 3. We combine the two steps to get

$$m = \liminf_{s\to\infty} \int_\Omega f\left(d\omega_s\right) = \liminf_{s\to\infty} \int_\Omega f\left(d\alpha_s\right) \geq \int_\Omega f\left(d\alpha\right) = \int_\Omega f\left(d\omega\right) \geq m.$$

This concludes the proof of the theorem. ■

6 Notations

We gather here the notations which we will use throughout this article. For more details on exterior algebra and differential forms, see [4] and for the notions of convexity used in the calculus of variations, see [7].

1. Let k, n be two integers.

   - We write $\Lambda^k\left(\mathbb{R}^n\right)$ (or simply Λ^k) to denote the vector space of all alternating k-linear maps $f : \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{k-\text{times}} \to \mathbb{R}$. For $k = 0$, we set $\Lambda^0\left(\mathbb{R}^n\right) = \mathbb{R}$. Note that $\Lambda^k\left(\mathbb{R}^n\right) = \{0\}$ for $k > n$ and, for $k \leq n$, $\dim\left(\Lambda^k\left(\mathbb{R}^n\right)\right) = \binom{n}{k}$.
   - $\wedge$, $\lrcorner$, $\langle;\rangle$ and $*$ denote the exterior product, the interior product, the scalar product and the Hodge star operator respectively.
   - If $\left\{e^1, \cdots, e^n\right\}$ is a basis of $\mathbb{R}^n$, then, identifying Λ^1 with $\mathbb{R}^n$,

     $$\left\{e^{i_1} \wedge \cdots \wedge e^{i_k} : 1 \leq i_1 < \cdots < i_k \leq n\right\}$$

     is a basis of Λ^k. An element $\xi \in \Lambda^k\left(\mathbb{R}^n\right)$ will therefore be written as

     $$\xi = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \xi_{i_1 i_2 \cdots i_k}\, e^{i_1} \wedge \cdots \wedge e^{i_k} = \sum_{I \in \mathcal{T}_k^n} \xi_I\, e^I,$$

     where

     $$\mathcal{T}_k^n = \left\{I = \left(i_1, \cdots, i_k\right) \in \mathbb{N}^k : 1 \leq i_1 < \cdots < i_k \leq n\right\}.$$

- We write

$$e^{i_1} \wedge \cdots \wedge \widehat{e^{i_s}} \wedge \cdots \wedge e^{i_k} = e^{i_1} \wedge \cdots \wedge e^{i_{s-1}} \wedge e^{i_{s+1}} \wedge \cdots \wedge e^{i_k}.$$

2. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set.

- The spaces $C^1\left(\Omega; \Lambda^k\right)$, $W^{1,p}\left(\Omega; \Lambda^k\right)$ and $W_0^{1,p}\left(\Omega; \Lambda^k\right)$, $1 \leq p \leq \infty$ are defined in the usual way.
- For $\omega \in W^{1,p}\left(\Omega; \Lambda^k\right)$, the exterior derivative $d\omega$ belongs to $L^p(\Omega; \Lambda^{k+1})$ and is defined by

$$(d\omega)_{i_1 \cdots i_{k+1}} = \sum_{j=1}^{k+1} (-1)^{j+1} \frac{\partial \omega_{i_1 \cdots i_{j-1} i_{j+1} \cdots i_{k+1}}}{\partial x_{i_j}},$$

for $1 \leq i_1 < \cdots < i_{k+1} \leq n$. If $k = 0$, then $d\omega \simeq \operatorname{grad} \omega$. If $k = 1$, for $1 \leq i < j \leq n$,

$$(d\omega)_{ij} = \frac{\partial \omega_j}{\partial x_i} - \frac{\partial \omega_i}{\partial x_j},$$

i.e. $d\omega \simeq \operatorname{curl} \omega$.

- The interior derivative (or codifferential) of $\omega \in W^{1,p}\left(\Omega; \Lambda^k\right)$, denoted $\delta\omega$, belongs to $L^p(\Omega; \Lambda^{k-1})$ and is defined as

$$\delta\omega = (-1)^{n(k-1)} * (d(*\omega)).$$

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