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SOME NEW RESULTS ON DIFFERENTIAL INCLUSIONS FOR DIFFERENTIAL FORMS

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ABSTRACT. In this article we study some necessary and sufficient conditions for the existence of solutions in $W_0^{1,\infty}(\Omega;\Lambda^k)$ of the differential inclusion

$$d\omega \in E \quad \text{a.e. in } \Omega$$

where $E \subset \Lambda^{k+1}$ is a prescribed set.

In this article we discuss the existence of a $k-$form ω, $0 \leq k \leq n-1$, verifying

$$\begin{cases} d\omega \in E & \text{in } \Omega \\ \omega = 0 & \text{on } \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set and $E \subset \Lambda^{k+1}$ is a given set of $(k+1)-$forms. For the precise notations we refer to the following section.

This problem has been mostly studied in the case $k = 0$ ($d\omega$ can then be identified with $\operatorname{grad}\omega$), see [3], [4], [5], [9], [10], [11] and for an extensive bibliography on the subject see [7].

The case $k = 1$ ($d\omega$ is then identified with $\operatorname{curl}\omega$) has also received some attention, see [1], [2], [8], [12].

The general case, $0 \leq k \leq n-1$, has been first considered in [1].

We here improve on [1] in two directions. The first result concerns the existence part (cf. Theorem 3.7).

Theorem 0.1. *Let $0 \leq k \leq n-1$ be two integers, $\Omega \subset \mathbb{R}^n$ be a bounded open set, $b \in \Lambda^k \setminus \{0\}$ and $E \subset \Lambda^{k+1}$. Then the following statements are equivalent.*

(i) There exists $\omega \in W_0^{1,\infty}\left(\Omega;\Lambda^k\right)$ of the form $\omega(x) = u(x)\, b$ where $u \in W_0^{1,\infty}(\Omega)$ such that

$$d\omega = (\operatorname{grad} u) \wedge b \in E \text{ a.e. in } \Omega \quad \text{and} \quad \int_\Omega \omega \neq 0.$$

(ii) The following holds

$$0 \in \operatorname{int}_{\mathbb{R}^n \wedge b} \operatorname{co}[E \cap (\mathbb{R}^n \wedge b)].$$

This result was already obtained in [1] but only for b of the form

$$b = b^1 \wedge \cdots \wedge b^k$$

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where $b^1, \cdots, b^k \in \Lambda^1$. Our present theorem allows us (cf. Corollary 3.9) to get a complete picture when

$$\dim \operatorname{span} E = n - k.$$

Our second contribution concerns necessary conditions (cf. Theorem 2.5).

Theorem 0.2. *Let $0 \leq k \leq n-1$ be two integers, $\Omega \subset \mathbb{R}^n$ be a bounded open set, $E \subset \Lambda^{k+1}$ and $\omega \in W_0^{1,\infty}\left(\Omega; \Lambda^k\right)$ be such that*

$$d\omega \in E \text{ a.e. in } \Omega \quad \text{and} \quad \int_\Omega \omega \neq 0.$$

Then

$$\dim \operatorname{span} E \geq n - k$$

and more precisely

$$\mathbb{R}^n \wedge \left(\int_\Omega \omega \right) \subset \operatorname{span} E.$$

Moreover if

$$\dim \operatorname{span} E = n - k$$

then

$$\mathbb{R}^n \wedge \left(\int_\Omega \omega \right) = \operatorname{span} E \quad \text{and} \quad \int_\Omega \omega = b^1 \wedge \cdots \wedge b^k$$

for some $b^1, \cdots, b^k \in \Lambda^1$.

1. Notations

We gather here the notations which we will use throughout this article. For more details on exterior algebra and differential forms see [6] and for convex analysis see [7] or [13].

(1) Let k, n be two integers.
- We write $\Lambda^k(\mathbb{R}^n)$ (or simply Λ^k) to denote the vector space of all alternating $k-$linear maps $f : \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{k-\text{times}} \to \mathbb{R}$. For $k = 0$, we set $\Lambda^0(\mathbb{R}^n) = \mathbb{R}$. Note that $\Lambda^k(\mathbb{R}^n) = \{0\}$ for $k > n$ and, for $k \leq n$, $\dim\left(\Lambda^k(\mathbb{R}^n)\right) = \binom{n}{k}$.
- $\wedge$, $\lrcorner$, $\langle ; \rangle$ and, respectively, $*$ denote the exterior product, the interior product, the scalar product and, respectively, the Hodge star operator.
- For $b \in \Lambda^k$, rank$[b]$ denotes the rank of the exterior $k-$form b.
- If $\left\{e^1, \cdots, e^n\right\}$ is a basis of $\mathbb{R}^n$, then, identifying Λ^1 with $\mathbb{R}^n$,
$$\left\{e^{i_1} \wedge \cdots \wedge e^{i_k} : 1 \leq i_1 < \cdots < i_k \leq n\right\}$$
is a basis of Λ^k.
- For $E \subset \Lambda^k$, $\operatorname{span} E$ denotes the subspace spanned by E.
- Let W be a subspace of Λ^k. We write $\dim W$ to denote the dimension of W and $W^\perp$ to denote the orthogonal complement of W.
- For $b \in \Lambda^k$, we write, identifying again Λ^1 with $\mathbb{R}^n$,
$$\mathbb{R}^n \wedge b = \Lambda^1 \wedge b = \left\{x \wedge b : x \in \Lambda^1\right\} \subset \Lambda^{k+1}.$$

(2) Let $\Omega \subset \mathbb{R}^n$ be a bounded open set.
- The spaces $C^1\left(\Omega; \Lambda^k\right)$, $W^{1,p}\left(\Omega; \Lambda^k\right)$ and $W_0^{1,p}\left(\Omega; \Lambda^k\right)$, $1 \leq p \leq \infty$ are defined in the usual way.

- For $\omega \in W^{1,p}\left(\Omega;\Lambda^k\right)$, $\int_\Omega \omega$ denotes the exterior $k-$form obtained integrating componentwise the differential form ω. Explicitly, for $1 \leq i_1 < \cdots < i_k \leq n$,
$$\left(\int_\Omega \omega\right)_{i_1\cdots i_k} = \int_\Omega \omega_{i_1\cdots i_k}.$$
- For $\omega \in W^{1,p}\left(\Omega;\Lambda^k\right)$, the exterior derivative $d\omega$ belongs to $L^p(\Omega;\Lambda^{k+1})$ and is defined by
$$(d\omega)_{i_1\cdots i_{k+1}} = \sum_{j=1}^{k+1}(-1)^{j+1}\frac{\partial\omega_{i_1\cdots i_{j-1}i_{j+1}\cdots i_{k+1}}}{\partial x_{i_j}},$$
for $1 \leq i_1 < \cdots < i_{k+1} \leq n$. If $k=0$, then $d\omega \simeq \operatorname{grad}\omega$. If $k=1$, for $1 \leq i < j \leq n$,
$$(d\omega)_{ij} = \frac{\partial\omega_j}{\partial x_i} - \frac{\partial\omega_i}{\partial x_j},$$
i.e. $d\omega \simeq \operatorname{curl}\omega$.

(3) For subsets $C, V \subset \Lambda^k$,
- $\operatorname{co} C$ denotes the convex hull of C;
- $\operatorname{int}_V C$ denotes the interior of C with respect to the topology relative to V.

(4) For a convex set $C \subset \Lambda^k$,
- $\operatorname{aff} C$ denotes the affine hull of C which is the intersection of all affine subsets of Λ^k containing C;
- $\operatorname{ri} C$ denotes the relative interior of C which is the interior of C with respect to the topology relative to the affine hull of C. Equivalently $\operatorname{ri} C = \operatorname{int}_{\operatorname{aff} C} C$;
- $\operatorname{rbd} C$ denotes the relative boundary of C which is $\overline{C} \setminus \operatorname{ri} C$.

2. Necessary conditions

2.1. **Preliminaries.**

Lemma 2.1. *Let $0 \leq k \leq n-1$ and let $b \in \Lambda^k \setminus \{0\}$. Then*
$$\dim\left(\mathbb{R}^n \wedge b\right) \geq n-k.$$
Furthermore
$$\dim\left(\mathbb{R}^n \wedge b\right) = n-k \quad \Leftrightarrow \quad b = b^1 \wedge \cdots \wedge b^k \textit{ for some } b^i \in \Lambda^1.$$

Proof. Step 1. We prove the first part. By definition of the interior product and the Hodge star operator (cf. Definition 2.11 in [6]), we have that
$$\mathbb{R}^n \lrcorner (*b) = (-1)^{k^2} * \left(\mathbb{R}^n \wedge b\right).$$
Hence, since (cf. Proposition 2.32 (i) in [6])
$$\dim\left(\mathbb{R}^n \lrcorner (*b)\right) = \operatorname{rank}[*b]$$
and since (cf. Proposition 2.37 (ii) of [6])
$$\operatorname{rank}[*b] \geq n-k$$
we have proved the first part of the lemma.

Step 2. We prove the second part. First note that if $b = b^1 \wedge \cdots \wedge b^k$ where $b^i \in \Lambda^1$, it is elementary to see that

$$\dim (\mathbb{R}^n \wedge b) = n - k.$$

We prove the converse. In this case

$$n - k = \dim (\mathbb{R}^n \wedge b) = \dim (\mathbb{R}^n \lrcorner (*b)),$$

and so, as in Step 1, rank $[*b] = n - k$ and hence (cf. Proposition 2.43 (ii) in [6]) there exist $c^1, \cdots, c^{n-k} \in \Lambda^1$ such that

$$*b = c^1 \wedge \cdots \wedge c^{n-k}.$$

Using Proposition 2.19 in [6], it is not difficult to see that

$$b = b^1 \wedge \cdots \wedge b^k$$

for some $b^i \in \Lambda^1$. The lemma is therefore proved. □

Proposition 2.2. *Let $0 \leq k \leq n-1$ be two integers and $f : \mathbb{R}^n \to \Lambda^k$ be continuous at 0 and such that $f(0) \neq 0$. Then*

$$\mathbb{R}^n \wedge f(0) \subset \operatorname{span} \{x \wedge f(x) : x \in \mathbb{R}^n\}$$

and therefore

$$\dim \operatorname{span} \{x \wedge f(x) : x \in \mathbb{R}^n\} \geq n - k.$$

Moreover, if

$$\dim \operatorname{span} \{x \wedge f(x) : x \in \mathbb{R}^n\} = n - k,$$

then

$$\mathbb{R}^n \wedge f(0) = \operatorname{span} \{x \wedge f(x) : x \in \mathbb{R}^n\} \quad \text{and} \quad f(0) = b^1 \wedge \cdots \wedge b^k$$

for some $b^i \in \Lambda^1$.

Proof. Step 1. From Lemma 2.1, since $f(0) \in \Lambda^k$ and $f(0) \neq 0$, we have

$$\dim \operatorname{span}\{x \wedge f(0) : x \in \mathbb{R}^n\} \geq n - k.$$

Moreover if $\dim \operatorname{span} \{x \wedge f(0) : x \in \mathbb{R}^n\} = n - k$, then

$$f(0) = b^1 \wedge \cdots \wedge b^k$$

for some $b^i \in \Lambda^1$.

Step 2. We prove the first assertion. Let $x \in \mathbb{R}^n \setminus \{0\}$ be fixed. Note that, for every $\lambda \in \mathbb{R} \setminus \{0\}$,

$$x \wedge f(\lambda x) = \frac{1}{\lambda} [\lambda x \wedge f(\lambda x)] \in \operatorname{span} \{y \wedge f(y) : y \in \mathbb{R}^n\}.$$

Since $\operatorname{span} \{y \wedge f(y) : y \in \mathbb{R}^n\}$ is closed and f is continuous at 0, it follows, letting $\lambda \to 0$,

$$x \wedge f(0) \in \operatorname{span} \{y \wedge f(y) : y \in \mathbb{R}^n\}.$$

Therefore

$$\mathbb{R}^n \wedge f(0) \subset \operatorname{span} \{y \wedge f(y) : y \in \mathbb{R}^n\}$$

which directly shows that (cf. Step 1)

$$\dim \operatorname{span} \{y \wedge f(y) : y \in \mathbb{R}^n\} \geq \dim (\mathbb{R}^n \wedge f(0)) \geq n - k.$$

Step 3. We finally prove the extra assertion. We already know (cf. Step 2) that

$$\mathbb{R}^n \wedge f(0) \subset \operatorname{span} \{x \wedge f(x) : x \in \mathbb{R}^n\}.$$

Since, by hypothesis,

$$\dim \operatorname{span}\{x \wedge f(x) : x \in \mathbb{R}^n\} = n - k,$$

we directly deduce from Step 1 that

$$\mathbb{R}^n \wedge f(0) = \operatorname{span}\{x \wedge f(x) : x \in \mathbb{R}^n\}$$

and that

$$f(0) = b^1 \wedge \cdots \wedge b^k$$

for some $b^i \in \Lambda^1$. The proof is therefore complete. □

Remark 2.3. (i) With a very similar proof one can show that if $f : \mathbb{R}^n \to \Lambda^k$ is differentiable at some point $x_0 \in \mathbb{R}^n$ then, for every $x \in \mathbb{R}^n$,

$$x_0 \wedge Df(x_0; x) + x \wedge f(x_0) \subset \operatorname{span}\{y \wedge f(y) : y \in \mathbb{R}^n\}$$

where $Df(x_0; x)$ denotes the directional derivative of f at x_0 in the direction of x. In particular if $Df(x_0) = 0$ then

$$\mathbb{R}^n \wedge f(x_0) \subset \operatorname{span}\{y \wedge f(y) : y \in \mathbb{R}^n\}.$$

(ii) It can also be proved that if $f : \mathbb{R}^n \to \Lambda^k$ is continuous and $x_0 \in \mathbb{R}^n$ is such that

$$x_0 \wedge f(x_0) \neq 0$$

then, necessarily (even if $f(0) = 0$),

$$\dim \operatorname{span}\{x \wedge f(x) : x \in \mathbb{R}^n\} \geq n - k.$$

The following lemma will be used in the proofs of Theorems 2.6 and 3.7 and Corollary 3.9.

Lemma 2.4. *Let $0 \leq k \leq n-1$ be two integers, $\Omega \subset \mathbb{R}^n$ be a bounded open set, $E \subset \Lambda^{k+1}$ and $\omega \in W_0^{1,\infty}(\Omega; \Lambda^k)$ be such that*

$$d\omega \in E \quad \text{a.e. in } \Omega.$$

Then

$$0 \in \overline{\operatorname{co} E}.$$

Moreover, if $0 \notin \operatorname{ri}\operatorname{co}(E)$, then there exists $D \subset \Omega$ such that $\operatorname{meas}(\Omega \setminus D) = 0$, *$d\omega(D) \subset E$ and*

$$\dim \operatorname{span}(d\omega(D)) < \dim \operatorname{span} E.$$

Proof. Since $\omega \in W_0^{1,\infty}(\Omega; \Lambda^k)$ and hence

$$\int_\Omega d\omega = 0,$$

we deduce, using Proposition 2.36 in [7] and Jensen inequality, that

$$0 \in \overline{\operatorname{co} E}.$$

Suppose that $0 \notin \operatorname{ri}\operatorname{co}(E)$ and hence (using the previous observation)

$$0 \in \operatorname{rbd}\operatorname{co} E.$$

Using Separation Theorem (cf. Theorem 2.10 in [7]), we easily deduce from the previous equation that there exists $b \in (\operatorname{span} E) \setminus \{0\}$ such that

$$\langle h; b \rangle \geq 0 \quad \text{for every } h \in \operatorname{co} E.$$

In particular $\langle d\omega ; b\rangle \geq 0$ a.e. in Ω. But, as $\omega \in W_0^{1,\infty}\left(\Omega ; \Lambda^k\right)$ and

$$\int_\Omega \langle d\omega ; b\rangle = 0$$

we therefore obtain that

$$\langle d\omega ; b\rangle = 0 \quad \text{a.e. in } \Omega.$$

Let $D \subset \Omega$ be the set where the previous equation holds. Taking D smaller if necessary we can assume without loss of generality that $d\omega(D) \subset E$ (and of course $\operatorname{meas}(\Omega \setminus D) = 0$) and hence

$$\operatorname{span} d\omega(D) \subset \operatorname{span} E.$$

As $\langle d\omega(x) ; b\rangle = 0$ for every $x \in D$ and $b \in (\operatorname{span} E) \setminus \{0\}$, we deduce that

$$\operatorname{span} d\omega(D) \underset{\neq}{\subset} \operatorname{span} E$$

and thus

$$\dim \operatorname{span} d\omega(D) < \dim \operatorname{span} E$$

as wished. □

2.2. **The main results.** We first state the main results of this section.

Theorem 2.5. *Let $0 \leq k \leq n-1$ be two integers, $\Omega \subset \mathbb{R}^n$ be a bounded open set, $E \subset \Lambda^{k+1}$ and $\omega \in W_0^{1,\infty}\left(\Omega ; \Lambda^k\right)$ be such that*

$$d\omega \in E \text{ a.e. in } \Omega \quad \text{and} \quad \int_\Omega \omega \neq 0.$$

Then

$$\mathbb{R}^n \wedge \left(\int_\Omega \omega\right) \subset \operatorname{span} E \quad \text{and} \quad \dim \operatorname{span} E \geq n-k.$$

Moreover if

$$\dim \operatorname{span} E = n-k$$

then

$$\mathbb{R}^n \wedge \left(\int_\Omega \omega\right) = \operatorname{span} E \quad \text{and} \quad \int_\Omega \omega = b^1 \wedge \cdots \wedge b^k$$

for some $b^i \in \Lambda^1$.

The following result improves Theorem 3.6 of [1], since E is not assumed to be finite and F is given explicitly.

Theorem 2.6. *Let $0 \leq k \leq n-1$ be integers, $\Omega \subset \mathbb{R}^n$ be a bounded open set and $E \subset \Lambda^{k+1} \setminus \{0\}$. Let $\omega \in W_0^{1,\infty}\left(\Omega ; \Lambda^k\right)$ be such that*

$$d\omega \in E \quad \text{a.e. in } \Omega.$$

Then there exists $F \subset E$ such that

$$0 \in \operatorname{ri} \operatorname{co} F.$$

More precisely F can be taken as $d\omega(D)$ for some $D \subset \Omega$ with $\operatorname{meas}(\Omega \setminus D) = 0$.

We start with the proof of Theorem 2.5.

Proof. Let $\mathbb{P} : \Lambda^{k+1} \to \Lambda^{k+1}$ denote the projection onto the orthogonal complement of span E. Since $\omega \in W_0^{1,\infty}\left(\Omega; \Lambda^k\right)$, extending ω by 0 to $\mathbb{R}^n$, it follows that

$$\mathbb{P}(d\omega) = 0 \quad \text{a.e. in } \mathbb{R}^n.$$

Applying Fourier transform we obtain (recalling that the Fourier transform of ω is continuous)

$$\mathbb{P}\left(x \wedge \left[\int_{\mathbb{R}^n} \omega(y) \cos\left(2\pi\langle x; y\rangle\right) dy\right]\right) = 0 \quad \text{for every } x \in \mathbb{R}^n$$

which is equivalent to

$$x \wedge \left[\int_{\mathbb{R}^n} \omega(y) \cos\left(2\pi\langle x; y\rangle\right) dy\right] \in \operatorname{span} E \quad \text{for every } x \in \mathbb{R}^n.$$

Letting

$$f\left(x\right) = \int_{\mathbb{R}^n} \omega(y) \cos\left(2\pi\langle x; y\rangle\right) dy$$

we get

$$f\left(0\right) = \int_{\Omega} \omega \neq 0.$$

Applying then Proposition 2.2 to the above f we have indeed established the theorem. □

We next prove Theorem 2.6.

Proof. Let $D \subset \Omega$ (not necessarily unique) be such that

$$\operatorname{meas}(\Omega \setminus D) = 0, \quad d\omega\left(D\right) \subset E$$

and, for every $D_1 \subset D$ with $\operatorname{meas}(\Omega \setminus D_1) = 0$, then

$$\dim \operatorname{span} d\omega\left(D_1\right) = \dim \operatorname{span} d\omega\left(D\right). \tag{2.1}$$

If such a D did not exist, we would find, after a finite induction on the dimension, that

$$d\omega = 0 \quad \text{a.e. in } \Omega$$

which contradicts the fact that $0 \notin E$. Letting $F = d\omega(D) \subset E$, it remains to show that

$$0 \in \operatorname{ri}\operatorname{co} F.$$

For the sake of contradiction suppose that this is not the case. Hence using Lemma 2.4 (with E replaced by F) there exists a set $D_1 \subset D$ (in the conclusion of Lemma 2.4 the set D_1 is only contained in Ω but taking it smaller, if necessary, we can assume that $D_1 \subset D$) such that $\operatorname{meas}(\Omega \setminus D_1) = 0$ and

$$\dim \operatorname{span} d\omega(D_1) < \dim \operatorname{span} F.$$

This is the desired contradiction since, using (2.1),

$$\dim \operatorname{span} d\omega(D_1) = \dim \operatorname{span} F.$$

The proof is therefore complete. □

2.3. **Further remarks.** In this section we want to discuss the hypothesis

$$\int_\Omega \omega \neq 0$$

that has been made in Theorem 2.5. We will concentrate on the case $k = 1$. We start with an elementary lemma.

Lemma 2.7. *Let $E \subset \Lambda^2(\mathbb{R}^n)$ be such that there exists a non-degenerate $g \in \Lambda^2$ (i.e. $h \lrcorner g \neq 0$ for every $h \in \Lambda^1 \setminus \{0\}$) and*

$$g \in (\operatorname{span} E)^\perp .$$

Then the two following statements hold true.

(i) There exists no $b \in \Lambda^1(\mathbb{R}^n) \setminus \{0\}$ such that

$$\mathbb{R}^n \wedge b \subset \operatorname{span} E.$$

(ii) There exists no $\omega \in W_0^{1,\infty}(\Omega; \Lambda^1)$ such that

$$d\omega \in E \text{ a.e. in } \Omega \quad \text{and} \quad \int_\Omega \omega \neq 0.$$

Proof. (i) We proceed by contradiction and assume that $\mathbb{R}^n \wedge b \subset \operatorname{span} E$ with $b \neq 0$. Since $g \in (\operatorname{span} E)^\perp$, we deduce that

$$g \in (\mathbb{R}^n \wedge b)^\perp$$

which is equivalent to

$$\langle g; a \wedge b \rangle = 0 \quad \text{for every } a \in \Lambda^1.$$

This in turn is equivalent to (cf. Proposition 2.16 in [6])

$$\langle b \lrcorner g; a \rangle = 0 \quad \text{for every } a \in \Lambda^1.$$

Since the previous equation is the same as

$$b \lrcorner g = 0,$$

we deduce, appealing to the fact that g is non-degenerate, that $b = 0$. This is the desired contradiction.

(ii) Indeed, if such a solution exists, then (cf. Theorem 2.5)

$$\mathbb{R}^n \wedge \left(\int_\Omega \omega \right) \subset \operatorname{span} E.$$

Define $b = \int_\Omega \omega$ and apply the previous point (i) to get the result. □

The next proposition shows that the hypothesis $\int_\Omega \omega \neq 0$ cannot be removed, in general, in Theorem 2.5.

Proposition 2.8. *Let $B \subset \mathbb{R}^4$ be the unit ball centered at 0. Then there exists $E \subset \Lambda^2(\mathbb{R}^4) \setminus \{0\}$ with the following properties.*

(i) There exists $\omega_0 \in W_0^{1,\infty}(B; \Lambda^1)$ such that

$$d\omega_0 \in E \quad \text{a.e. in } B.$$

(ii) There exists no $b \in \Lambda^1(\mathbb{R}^4) \setminus \{0\}$ such that

$$\mathbb{R}^4 \wedge b \subset \operatorname{span} E.$$

(iii) For every $\omega \in W_0^{1,\infty}\left(B;\Lambda^1\right)$ *such that* $d\omega \in \operatorname{span} E$ *a.e. in* B *then, necessarily,*

$$\int_B \omega = 0.$$

Remark 2.9. (i) Using Vitali covering theorem (see [10]), the same set E also works for any open set Ω (instead of B).

(ii) This phenomenon only occurs where $n \geq 4$. Indeed if $n \leq 3$ we always can find (cf. Theorem 3.11) a solution with a non-zero average (as far as there exists a non-trivial solution).

(iii) The previous proposition has an interesting implication for differential inclusions where we seek solutions $u \in W_0^{1,\infty}\left(\Omega;\mathbb{R}^N\right)$, and $\Omega \subset \mathbb{R}^n$, verifying

$$\nabla u \in F \quad \text{a.e. in } \Omega.$$

In the scalar case $N = 1$, we can always ensure (cf. Theorem 3.5) that if there is a non-trivial solution u of the differential inclusion, then there are some solutions with non-zero average. This is no longer the case in the vectorial context when $n, N > 1$. Indeed let $n = N = 4$ and define $F \subset \mathbb{R}^{4\times 4} \approx \mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R}^4$ by

$$F = \left\{(u_1,u_2,u_3,u_4) \in \mathbb{R}^{4\times 4} : u_1 \wedge dx^1 + u_2 \wedge dx^2 + u_3 \wedge dx^3 + u_4 \wedge dx^4 \in E\right\}.$$

where E is as in the proposition. Noticing that, for $\omega = u_1 dx^1 + \cdots + u_4 dx^4$,

$$d\omega \in E \quad \Leftrightarrow \quad \nabla u = (\nabla u_1, \nabla u_2, \nabla u_3, \nabla u_4) \in F$$

we have the result.

(iv) If $\dim \operatorname{span} E = n-1$ then, as far as there exists a non-trivial solution, (cf. Theorem 4.13 [1]) we always have (without assuming the existence of a solution with non-zero average) $\operatorname{span} E = \mathbb{R}^n \wedge b$ for some $b \in \Lambda^1 \setminus \{0\}$.

Proof. (i) Let

$$\omega_0(x) = \left(|x|^2 - 1\right)\left(x_1\, dx^1 + x_2\, dx^2 + 2x_3\, dx^3 + 2x_4\, dx^4\right).$$

Obviously $\omega_0 \in W_0^{1,\infty}(B;\Lambda^1)$ and

$$d\omega_0(x) = 2x_1x_3\, dx^1 \wedge dx^3 + 2x_1x_4\, dx^1 \wedge dx^4 + 2x_2x_3\, dx^2 \wedge dx^3 + 2x_2x_4\, dx^2 \wedge dx^4.$$

Let

$$\Sigma = \{x_1 = 0\} \cup \{x_2 = 0\} \cup \{x_3 = 0\} \cup \{x_4 = 0\}.$$

Note in particular that

$$d\omega_0(x) \neq 0 \quad \text{for every } x \in \mathbb{R}^4 \setminus \Sigma.$$

Observe that ω_0 and

$$E = d\omega_0\left(B \setminus \Sigma\right)$$

satisfy all the requirements of the first statement of the proposition, since, trivially, $0 \notin E$, $\omega_0 \in W_0^{1,\infty}\left(B;\Lambda^1\right)$ and $d\omega_0 \in E$ a.e. in B.

(ii) We now prove the second statement. First observe that for every $h \in E$ we have $h_{12} = h_{34} = 0$. We thus deduce that for every $h \in \operatorname{span} E$ we also have $h_{12} = h_{34} = 0$. In other words $dx^1 \wedge dx^2, dx^3 \wedge dx^4 \in (\operatorname{span} E)^{\perp}$ and hence, in particular,

$$g = dx^1 \wedge dx^2 + dx^3 \wedge dx^4 \in (\operatorname{span} E)^{\perp}.$$

Note that g is non-degenerate. We therefore can invoke Lemma 2.7 (i) to get the claim.

(iii) The last statement of the proposition is a direct consequence of Theorem 2.5 and of the previous point (ii). $\square$

3. Necessary and sufficient conditions

3.1. **Preliminaries.** We start with an elementary lemma whose proof is straightforward.

Lemma 3.1. *Let X be a normed linear space, $Y \subset X$ be a subspace and $A \subset X$. Then $0 \in \operatorname{int}_Y \operatorname{co}(A \cap Y)$ if and only if*

$$\operatorname{span}(A \cap Y) = Y \quad \textit{and} \quad 0 \in \operatorname{ri}\operatorname{co}(A \cap Y).$$

We next recall Caratheodory theorem (see Corollary 2.16 in [7]).

Lemma 3.2 (Caratheodory theorem)**.** *Let X be a normed linear space and $E \subset X$. Then $0 \in \operatorname{ri}\operatorname{co} E$ if and only if there exist $m \in \mathbb{N}$, $m \geq \dim \operatorname{span} E + 1$, $z^i \in E$, $t^i > 0$ for every $1 \leq i \leq m$, such that*

$$\sum_{i=1}^{m} t^i z^i = 0, \quad \sum_{i=1}^{m} t^i = 1 \quad \textit{and} \quad \operatorname{span}\left\{z^1, \cdots, z^m\right\} = \operatorname{span} E.$$

In what follows we will constantly identify Λ^1 with $\mathbb{R}^n$.

Lemma 3.3. *Let $0 \leq k \leq n-1$ be two integers, $b \in \Lambda^k \setminus \{0\}$ and $E \subset \Lambda^{k+1}$ be such that*

$$0 \in \operatorname{int}_{\mathbb{R}^n \wedge b} \operatorname{co}\left[E \cap (\mathbb{R}^n \wedge b)\right].$$

Then there exists $F \subset \mathbb{R}^n$ such that

$$E \cap (\mathbb{R}^n \wedge b) = F \wedge b \quad \textit{and} \quad 0 \in \operatorname{int}\operatorname{co} F.$$

Proof. Step 1. Let us define

$$F = \{x \in \mathbb{R}^n : x \wedge b \in E\}.$$

Evidently, $E \cap (\mathbb{R}^n \wedge b) = F \wedge b$. Since $0 \in \operatorname{int}_{\mathbb{R}^n \wedge b} \operatorname{co}\left[E \cap (\mathbb{R}^n \wedge b)\right]$, it follows from Lemma 3.1 that

$$\operatorname{span}\left[E \cap (\mathbb{R}^n \wedge b)\right] = \mathbb{R}^n \wedge b \quad \text{and} \quad 0 \in \operatorname{ri}\operatorname{co}\left[E \cap (\mathbb{R}^n \wedge b)\right]. \tag{3.1}$$

Using Lemma 3.2 we find $m \in \mathbb{N}$, $m \geq \dim \operatorname{span}\left[E \cap (\mathbb{R}^n \wedge b)\right] + 1$, $z^i \in E \cap (\mathbb{R}^n \wedge b)$, $t^i > 0$ for every $1 \leq i \leq m$, such that

$$\sum_{i=1}^{m} t^i z^i = 0, \quad \sum_{i=1}^{m} t^i = 1 \tag{3.2}$$

and

$$\operatorname{span}\left\{z^1, \cdots, z^m\right\} = \operatorname{span}\left[E \cap (\mathbb{R}^n \wedge b)\right] = \mathbb{R}^n \wedge b. \tag{3.3}$$

As $z^i \in E \cap (\mathbb{R}^n \wedge b)$, for every $1 \leq i \leq m$, there exists $a^i \in F$ such that

$$z^i = a^i \wedge b \quad \text{for every } 1 \leq i \leq m. \tag{3.4}$$

It therefore follows from (3.2) that

$$\left(\sum_{i=1}^{m} t^i a^i\right) \wedge b = 0. \tag{3.5}$$

We next define

$$\mathcal{W} = \{x \in \mathbb{R}^n : x \wedge b = 0\}.$$

Note that it follows from Lemma 2.1 that $0 \leq \dim \mathcal{W} \leq k$. We now consider two cases (cf. Steps 2 and 3).

Step 2. We prove the lemma when $\dim \mathcal{W} = 0$. Note that in this case $\dim (\mathbb{R}^n \wedge b) = n$ and hence $m \geq n+1$. Since $\dim \mathcal{W} = 0$, we deduce from (3.5) that

$$\sum_{i=1}^{m} t^i a^i = 0.$$

Observe that $\operatorname{span}\{a^1, \cdots, a^m\} = \mathbb{R}^n$. Indeed if $\operatorname{span}\{a^1, \cdots, a^m\}$ was a proper subspace of $\mathbb{R}^n$, so would $\operatorname{span}\{z^1, \cdots, z^m\}$ be a proper subspace of $\mathbb{R}^n \wedge b$ which contradicts (3.3). Therefore, using Lemma 3.2, we have that $0 \in \operatorname{int} \operatorname{co} F$. This proves the lemma when $\dim \mathcal{W} = 0$ (or equivalently when b does not have any $1-$form as a factor).

Step 3. We prove the lemma when $\dim \mathcal{W} = r$, where $1 \leq r \leq k$.

Step 3.1. In this case, we have that

$$\dim (\mathbb{R}^n \wedge b) = n - r$$

and thus $m \geq n - r + 1$. Let $\{b^1, \cdots, b^r\}$ be an orthonormal basis of $\mathcal{W}$. Without loss of generality, we can assume that

$$b^j = e^j \quad \text{for every } 1 \leq j \leq r$$

where $\{e^1, \cdots, e^n\}$ is the standard basis of $\mathbb{R}^n$. Since $e^i \wedge b = 0$ for every $1 \leq i \leq r$, we have, invoking Cartan Lemma (cf. Theorem 2.42 in [6]), that b can be written as

$$b = e^1 \wedge \cdots \wedge e^r \wedge \overline{b}$$

where $\overline{b} \in \Lambda^{k-r}$ does not have any $1-$form as a factor i.e. we have that

$$x \wedge \overline{b} \neq 0 \quad \text{for every } x \in \mathbb{R}^n \setminus \{0\}\,.$$

Note that

$$z^i = a^i \wedge b = a^i \wedge e^1 \wedge \cdots \wedge e^r \wedge \overline{b} = \left(\sum_{j=r+1}^{n} a^i_j e^j \right) \wedge e^1 \wedge \cdots \wedge e^r \wedge \overline{b}.$$

Hence, without loss of generality, we assume that $\{a^1, \cdots, a^m\}$ in (3.4) also satisfy

$$a^i_j = 0 \quad \text{for every } 1 \leq i \leq m \text{ and } 1 \leq j \leq r. \tag{3.6}$$

In addition, since $x \wedge \overline{b} \neq 0$ for every $x \in \mathbb{R}^n \setminus \{0\}$ and using (3.6), we deduce that

$$\dim \big(\operatorname{span}\{a^1, \cdots, a^m\}\big) = \dim \big(\operatorname{span}\{z^1, \cdots, z^m\}\big) = n - r. \tag{3.7}$$

Step 3.2. To show that $0 \in \operatorname{int} \operatorname{co} F$, let us define $v^{i,j} \in \mathbb{R}^n$, for $1 \leq i \leq m$ and $1 \leq j \leq 2^r$, as

$$v^{i,1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \\ 1 \\ a^i_{r+1} \\ \vdots \\ a^i_n \end{pmatrix}, \quad v^{i,2} = \begin{pmatrix} 1 \\ \vdots \\ 1 \\ -1 \\ a^i_{r+1} \\ \vdots \\ a^i_n \end{pmatrix}, \cdots, \; v^{i,2^r} = \begin{pmatrix} -1 \\ \vdots \\ -1 \\ -1 \\ a^i_{r+1} \\ \vdots \\ a^i_n \end{pmatrix}. \tag{3.8}$$

Clearly $v^{i,j} \in F$, for every $1 \leq i \leq m$ and $1 \leq j \leq 2^r$, since

$$v^{i,j} \wedge b = a^i \wedge b \in E \quad \text{for every } 1 \leq i \leq m \text{ and } 1 \leq j \leq 2^r.$$

Furthermore defining

$$t^{i,j} = \frac{t^i}{2^r} \quad \text{for every } 1 \leq i \leq m \text{ and } 1 \leq j \leq 2^r$$

it is easy check that $m2^r \geq n+1$ for every $1 \leq r \leq n-1$ and

$$\sum_{i=1}^{m} \sum_{j=1}^{2^r} t^{i,j} v^{i,j} = 0$$

and

$$\sum_{i=1}^{m} \sum_{j=1}^{2^r} t^{i,j} = 1.$$

If we show (cf. Step 3.3) that

$$\operatorname{span}\{v^{i,j} : 1 \leq i \leq m;\ 1 \leq j \leq 2^r\} = \mathbb{R}^n \tag{3.9}$$

then, using Lemma 3.2, we will have proved that $0 \in \operatorname{int} \operatorname{co} F$ and the proof will be finished.

Step 3.3. We show (3.9). Let $y \in \mathbb{R}^n$ be such that $\langle y; v^{i,j} \rangle = 0$, for every $1 \leq i \leq m$ and $1 \leq j \leq 2^r$. It is enough to prove that $y = 0$ to have the claim. Let $1 \leq i \leq m$ be fixed. For every $1 \leq p \leq r$, we choose $1 \leq j, l \leq 2^r$ such that

$$v^{i,j} - v^{i,l} = 2e^p.$$

Then

$$0 = \langle y; v^{i,j} - v^{i,l} \rangle = 2\langle y; e^p \rangle \quad \text{for every } p \in \{1, \ldots, r\}$$

which shows that $y_p = 0$, for every $1 \leq p \leq r$. The system

$$\langle y; v^{i,j} \rangle = 0 \quad \text{for every } 1 \leq i \leq m \text{ and } 1 \leq j \leq 2^r,$$

is therefore equivalent to the following system

$$\sum_{j=r+1}^{n} a^i_j y_j = 0 \quad \text{for every } 1 \leq i \leq m. \tag{3.10}$$

We now define a matrix $A \in \mathbb{R}^{m \times (n-r)}$ as

$$A^i_j = a^i_{j+r} \quad \text{for every } 1 \leq i \leq m \text{ and } 1 \leq j \leq n-r.$$

Then (3.10) can be written as

$$A\bar{y} = 0$$

where $\bar{y} = (y_{r+1}, \cdots, y_n)$.

To show that $\bar{y} = 0$, it is enough to prove that $\operatorname{rank} A = n - r$ (recall that $m \geq n - r + 1$) which directly follows from (3.6) and (3.7). Therefore, $\bar{y} = 0$ and hence, $y = 0$. This proves the claim and concludes the proof. □

Definition 3.4. A map $\omega : \Omega \to \Lambda^k(\mathbb{R}^n)$ is said to be (locally finite) piecewise affine if there exist $A_i \in \mathbb{R}^{\binom{n}{k} \times n}$, $\alpha_i \in \mathbb{R}^{\binom{n}{k}}$ and $\Omega_i \subset \Omega$ open sets where i runs through an at most countable set I such that

(1) $\omega(x) = A_i x + \alpha_i$ in Ω_i for every $i \in I$,
(2) $\operatorname{meas}(\Omega \setminus \cup_{i \in I} \Omega_i) = 0$ and $\Omega_i \cap \Omega_j = \emptyset$ for $i, j \in I$ with $i \neq j$
(3) for every compact $K \subset \Omega$
$$\{i \in I : K \cap \Omega_i \neq \emptyset\} \quad \text{is finite.}$$

We recall a result concerning the scalar gradient case (see Lemma 2.11 in [10] and its proof).

Theorem 3.5 (Gradient case)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and $F \subset \mathbb{R}^n$. Then there exists $u \in W_0^{1,\infty}(\Omega)$ such that*
$$\operatorname{grad} u \in F \quad \textit{a.e. in } \Omega$$
if and only if
$$0 \in F \cup \operatorname{int} \operatorname{co} F.$$
If moreover $0 \in \operatorname{int} \operatorname{co} F$, then u can be taken piecewise affine, $u \geq 0$ and
$$\int_\Omega u > 0.$$

Finally we give the following elementary lemma.

Lemma 3.6. *Let $0 \leq k \leq n$ be two integers and $\omega \in C^\infty(\mathbb{R}^n; \Lambda^k)$ be such that*
$$\omega(x) = Ax + a \quad \textit{for every } x \in \mathbb{R}^n$$
where $a \in \mathbb{R}^{\binom{n}{k}}$ and $A \in \mathbb{R}^{\binom{n}{k} \times n}$ is such that
$$\operatorname{rank} A \leq 1.$$
Then there exists $b \in \Lambda^1 \setminus \{0\}$ such that
$$b \wedge d\omega = 0.$$

Proof. Since we will use the lemma only when $k = 1$, we prove it in this context; the general case is similar. The result being trivial if $\operatorname{rank} A = 0$ we assume that $\operatorname{rank} A = 1$. Hence there exist $\alpha \in \mathbb{R}^n \setminus \{0\}$ and $b \in \mathbb{R}^n \setminus \{0\}$ such that
$$A_j^i = \alpha_i b_j \quad \text{for every } 1 \leq i, j \leq n.$$
We claim that
$$b \wedge d\omega = 0$$
which will prove the lemma. Since, for $1 \leq j_1 < j_2 \leq n$,
$$(d\omega)_{j_1 j_2} = \alpha_{j_1} b_{j_2} - \alpha_{j_2} b_{j_1},$$
we deduce that, for every $1 \leq r_1 < r_2 < r_3 \leq n$,
$$\begin{aligned}(b \wedge d\omega)_{r_1 r_2 r_3} &= (\alpha_{r_1} b_{r_2} - \alpha_{r_2} b_{r_1}) b_{r_3} - (\alpha_{r_1} b_{r_3} - \alpha_{r_3} b_{r_1}) b_{r_2} \\ &\quad + (\alpha_{r_2} b_{r_3} - \alpha_{r_3} b_{r_2}) b_{r_1} \\ &= 0.\end{aligned}$$

This proves the claim and concludes the lemma. □

3.2. Statement of the main results.

Theorem 3.7 (Necessary and sufficient condition)**.** *Let $0 \leq k \leq n-1$ be two integers, $\Omega \subset \mathbb{R}^n$ be a bounded open set, $b \in \Lambda^k \setminus \{0\}$ and $E \subset \Lambda^{k+1}$. Then the following statements are equivalent.*

(i) There exists $u \in W_0^{1,\infty}(\Omega)$ such that

$$(\operatorname{grad} u) \wedge b \in E \textit{ a.e. in } \Omega \quad \textit{and} \quad \int_\Omega u \neq 0.$$

(ii) The following holds

$$0 \in \operatorname{int}_{\mathbb{R}^n \wedge b} \operatorname{co}[E \cap (\mathbb{R}^n \wedge b)].$$

Remark 3.8. (i) The u will be constructed piecewise affine with $u \geq 0$.

(ii) Letting $\omega(x) = u(x)\, b$, we get that $\omega \in W_0^{1,\infty}(\Omega; \Lambda^k)$,

$$d\omega \in E \text{ a.e. in } \Omega \quad \text{and} \quad \int_\Omega \omega \neq 0.$$

Corollary 3.9. *Let $0 \leq k \leq n-1$ be two integers, $\Omega \subset \mathbb{R}^n$ be a bounded open set and $E \subset \Lambda^{k+1}$ be such that*

$$\dim \operatorname{span} E = n - k.$$

Then the following statements are equivalent.

(i) There exists $\omega \in W_0^{1,\infty}(\Omega; \Lambda^k)$ such that

$$d\omega \in E \textit{ a.e. in } \Omega \quad \textit{and} \quad \int_\Omega \omega \neq 0.$$

(ii) There exist $b = b^1 \wedge \cdots \wedge b^k \neq 0$ where $b^i \in \Lambda^1$ such that

$$0 \in \operatorname{ri} \operatorname{co} E \quad \textit{and} \quad \operatorname{span} E = \mathbb{R}^n \wedge b. \tag{3.11}$$

Remark 3.10. Using Lemma 3.1, note that (3.11) implies (since in this case $E = E \cap (\mathbb{R}^n \wedge b)$)

$$0 \in \operatorname{int}_{\mathbb{R}^n \wedge b} \operatorname{co} E.$$

In the case $n = 3$ the result, obtained in Theorem 4.15 of [1], takes the following optimal form. Note that in this case we can identify $d\omega$ with $\operatorname{curl} \omega$ and $\Lambda^2(\mathbb{R}^3)$ with $\mathbb{R}^3$.

Theorem 3.11. *Let $\Omega \subset \mathbb{R}^3$ be a bounded open set and $E \subset \mathbb{R}^3$. Then the three following assertions are equivalent.*

(i) There exists $\omega \in W_0^{1,\infty}(\Omega; \mathbb{R}^3)$ such that

$$\operatorname{curl} \omega \in E \quad \textit{a.e. } \Omega.$$

(ii) There exists a piecewise affine $\omega \in W_0^{1,\infty}(\Omega; \mathbb{R}^3)$ such that

$$\operatorname{curl} \omega \in E \textit{ a.e. } \Omega \quad \textit{and} \quad \int_\Omega \omega \neq 0.$$

(iii) There exists $F \subset E$ such that

$$\dim \operatorname{span} F \geq 2 \quad \textit{and} \quad 0 \in \operatorname{ri} \operatorname{co} F.$$

It is interesting to compare Theorems 3.7 and 3.11. Indeed one should not infer from the first theorem that any solution of $d\omega \in E$ is of the form $\omega = u\, b$, as the following proposition shows.

Proposition 3.12. *Let $\Omega \subset \mathbb{R}^3$ be a bounded open set. Then there exists a set $E \subset \Lambda^2 \setminus \{0\}$ such that*

$$0 \in \operatorname{int} \operatorname{co} E$$

with the following properties.

(i) There exists no $b \in \Lambda^1 \setminus \{0\}$ such that

$$0 \in \operatorname{int}_{\mathbb{R}^3 \wedge b} \operatorname{co} \left[E \cap \left(\mathbb{R}^3 \wedge b \right) \right]$$

and therefore there is no $u \in W_0^{1,\infty}(\Omega)$ such that

$$(\operatorname{grad} u) \wedge b \in E \quad a.e. \ in \ \Omega.$$

(ii) There exists $\omega \in W_0^{1,\infty}\left(\Omega; \Lambda^2\right)$ such that

$$d\omega \in E \quad a.e. \ in \ \Omega.$$

The general situation when $n \geq 4$ is considerably harder in view of the following considerations. The situation in Theorem 3.11 is very specific to the dimension $n = 3$. In the next proposition we will exhibit a set $E \subset \Lambda^2\left(\mathbb{R}^4\right)$ with

$$0 \in \operatorname{int}_{\mathbb{R}^4 \wedge e^1 + \mathbb{R}^4 \wedge e^2} \operatorname{co} \left[E \cap \left(\mathbb{R}^4 \wedge e^1 + \mathbb{R}^4 \wedge e^2 \right) \right]$$

for which there cannot exist a piecewise affine solution ω of $d\omega \in E$. However when $n = 3$, since $\mathbb{R}^3 \wedge e^1 + \mathbb{R}^3 \wedge e^2 = \Lambda^2\left(\mathbb{R}^3\right)$, we always have

$$\operatorname{int}_{\mathbb{R}^3 \wedge e^1 + \mathbb{R}^3 \wedge e^2} \operatorname{co} \left[E \cap \left(\mathbb{R}^3 \wedge e^1 + \mathbb{R}^3 \wedge e^2 \right) \right] = \operatorname{int} \operatorname{co} E.$$

Therefore, appealing to Theorem 3.11, if

$$0 \in \operatorname{int}_{\mathbb{R}^3 \wedge e^1 + \mathbb{R}^3 \wedge e^2} \operatorname{co} \left[E \cap \left(\mathbb{R}^3 \wedge e^1 + \mathbb{R}^3 \wedge e^2 \right) \right]$$

we can find a piecewise affine solution ω of $d\omega \in E$.

Proposition 3.13. *There exists a set $E \subset \Lambda^2\left(\mathbb{R}^4\right)$ with*

$$0 \in \operatorname{int}_{\mathbb{R}^4 \wedge e^1 + \mathbb{R}^4 \wedge e^2} \operatorname{co} \left[E \cap \left(\mathbb{R}^4 \wedge e^1 + \mathbb{R}^4 \wedge e^2 \right) \right]$$

with the following property: for every bounded open set $\Omega \subset \mathbb{R}^4$ there exists no piecewise affine $\omega \in W_0^{1,\infty}\left(\Omega; \Lambda^1\right)$ such that

$$d\omega \in E \quad a.e. \ in \ \Omega.$$

3.3. **Proof of the main results.** We start by showing Theorem 3.7.

Proof. Part 1. $(i) \Rightarrow (ii)$. Using Lemma 3.1 we have to show that

$$\mathbb{R}^n \wedge b = \operatorname{span} \left[E \cap \left(\mathbb{R}^n \wedge b \right) \right] \tag{3.12}$$

and

$$0 \in \operatorname{ri} \operatorname{co} \left[E \cap \left(\mathbb{R}^n \wedge b \right) \right]. \tag{3.13}$$

Let $\omega = u\, b \in W_0^{1,\infty}\left(\Omega; \Lambda^k\right)$. Obviously

$$d\omega \in E \cap \left(\mathbb{R}^n \wedge b \right) \text{ a.e. in } \Omega \quad \text{and} \quad \int_\Omega \omega = \left(\int_\Omega u \right) b \neq 0.$$

Hence using Theorem 2.5 (with E replaced by $E\cap(\mathbb{R}^n\wedge b)$), we obtain that

$$\mathbb{R}^n\wedge b\subset\operatorname{span}\left[E\cap(\mathbb{R}^n\wedge b)\right]$$

which obviously implies (3.12). We now show (3.13). For the sake of contradiction suppose that (3.13) does not hold. Then using Lemma 2.4 (with E replaced by $E\cap(\mathbb{R}^n\wedge b)$) there exists a set $D\subset\Omega$ such that $\operatorname{meas}(\Omega\setminus D)=0$, $d\omega(D)\subset E\cap(\mathbb{R}^n\wedge b)$ and

$$\dim\operatorname{span} d\omega(D)<\dim\operatorname{span}\left[E\cap(\mathbb{R}^n\wedge b)\right]. \tag{3.14}$$

But, since $d\omega\in d\omega(D)$ a.e. in Ω, we deduce, using again Theorem 2.5, that

$$\mathbb{R}^n\wedge b\subset\operatorname{span}\left(d\omega(D)\right).$$

This is the desired contradiction since, using (3.12) and (3.14), we have

$$\dim(\mathbb{R}^n\wedge b)\leq\dim\operatorname{span}\left(d\omega(D)\right)<\dim\operatorname{span}\left[E\cap(\mathbb{R}^n\wedge b)\right]=\dim(\mathbb{R}^n\wedge b).$$

Part 2. $(ii)\Rightarrow(i)$. Using Lemma 3.3, we find $F\subset\mathbb{R}^n$ such that

$$E\cap(\mathbb{R}^n\wedge b)=F\wedge b\quad\text{and}\quad 0\in\operatorname{int}\operatorname{co}F. \tag{3.15}$$

Appealing to Theorem 3.5, we find $u\in W_0^{1,\infty}(\Omega)$ such that

$$\operatorname{grad}u\in F\quad\text{a.e. in }\Omega. \tag{3.16}$$

Moreover u can be chosen piecewise affine and such that $\int_\Omega u\neq 0$ (and also such that $u\geq 0$). Let us now define $\omega\in W_0^{1,\infty}\left(\Omega;\Lambda^k\right)$ by

$$\omega(x)=u(x)\,b\quad\text{for every }x\in\Omega.$$

It is easy to check that

$$d\omega=(\operatorname{grad}u)\wedge b,\ \text{ a.e. in }\Omega.$$

It therefore follows from (3.15) and (3.16) that

$$d\omega\in E\cap(\mathbb{R}^n\wedge b)\subset E\quad\text{a.e. in }\Omega.$$

This finishes Part (ii) and the proof. ☐

We now prove Corollary 3.9.

Proof. Part 1: $(i)\Rightarrow(ii)$. Since $\dim\operatorname{span}E=n-k$, using Theorem 2.5, there exist $b^1,\cdots,b^k\in\Lambda^1$ such that

$$\mathbb{R}^n\wedge\left(\int_\Omega\omega\right)=\operatorname{span}E\quad\text{and}\quad\int_\Omega\omega=b^1\wedge\cdots\wedge b^k$$

which shows the second part of (3.11). Proceeding exactly as in Part 1 of the proof of Theorem 3.7, we can prove that

$$0\in\operatorname{ri}\operatorname{co}E.$$

This concludes Part 1.

Part 2: $(ii)\Rightarrow(i)$. Using Lemma 3.1, we have that (3.11) implies (since $E=E\cap(\mathbb{R}^n\wedge b)$)

$$0\in\operatorname{int}_{\mathbb{R}^n\wedge b}\operatorname{co}\left[E\cap(\mathbb{R}^n\wedge b)\right].$$

Hence, by Theorem 3.7, there exists $u\in W_0^{1,\infty}(\Omega)$ such that

$$(\operatorname{grad}u)\wedge b\in E\text{ a.e. in }\Omega\quad\text{and}\quad\int_\Omega u\neq 0.$$

Thus $\omega = u\, b$ has the desired properties. This concludes Part 2 and the proof. □

We next turn to the proof of Proposition 3.12.

Proof. Let

$$E = \left\{ e^1 \wedge e^2, -e^1 \wedge e^2 + e^1 \wedge e^3, -e^1 \wedge e^2 - e^1 \wedge e^3 + e^2 \wedge e^3, -e^1 \wedge e^2 - e^1 \wedge e^3 - e^2 \wedge e^3 \right\}.$$

It is easy to see that

$$0 \notin \operatorname{int}_{\mathbb{R}^3 \wedge b} \operatorname{co}\left[E \cap \left(\mathbb{R}^3 \wedge b \right) \right] \quad \text{for every } b \in \Lambda^1$$

but that

$$0 \in \operatorname{int} \operatorname{co} E.$$

(i) The first statement combined with Theorem 3.7 shows that there is no $u \in W_0^{1,\infty}(\Omega)$ such that

$$(\operatorname{grad} u) \wedge b \in E \quad \text{a.e. in } \Omega.$$

(ii) The second statement implies (cf. Theorem 3.11) the existence of $\omega \in W_0^{1,\infty}(\Omega; \Lambda^1)$ such that

$$d\omega \in E \quad \text{a.e. in } \Omega.$$

The proof of the proposition is therefore complete. □

Finally we establish Proposition 3.13.

Proof. Step 1. Let

$$a^1 = e^1 \wedge e^2, \quad a^2 = \left(e^1 + e^2\right) \wedge e^3, \quad a^3 = \left(e^1 + 2e^2\right) \wedge e^3,$$

$$a^4 = \left(e^1 + 3e^2\right) \wedge e^4, \quad a^5 = \left(e^1 + 4e^2\right) \wedge e^4,$$

$$a^6 = -\sum_{j=1}^{5} a^i = -\left(e^1 \wedge e^2 + 2e^1 \wedge e^3 + 2e^1 \wedge e^4 + 3e^2 \wedge e^3 + 7e^2 \wedge e^4 \right)$$

and finally let

$$E = \left\{ a^1, \cdots, a^6 \right\}.$$

We will show that E has all the desired properties (cf. Steps 2 and 3).

Step 2. Note that

$$\operatorname{span} E = \mathbb{R}^4 \wedge e^1 + \mathbb{R}^4 \wedge e^2.$$

Since obviously

$$\sum_{j=1}^{6} \frac{1}{6}\, a^j = 0,$$

it follows directly from Lemmas 3.1 and 3.2 that

$$0 \in \operatorname{int}_{\mathbb{R}^4 \wedge e^1 + \mathbb{R}^4 \wedge e^2} \operatorname{co}\left[E \cap \left(\mathbb{R}^4 \wedge e^1 + \mathbb{R}^4 \wedge e^2 \right) \right].$$

A simple calculation shows that, for every $1 \leq j \leq 5$,

$$\left(a^j - a^6\right) \wedge \left(a^j - a^6\right) \neq 0.$$

This says that

$$\operatorname{rank}\left[a^j - a^6\right] = 4 \quad \text{for every } 1 \leq j \leq 5$$

which is equivalent, in view of Cartan Lemma (cf. Theorem 2.42 in [6]), to saying that, for every $1 \leq j \leq 5$,

$$b \wedge \left(a^j - a^6\right) \neq 0 \quad \text{for every } b \in \Lambda^1 \setminus \{0\}. \tag{3.17}$$

Step 3. Let $\Omega \subset \mathbb{R}^4$ be a bounded open set. We claim that there does not exist $\omega \in W_0^{1,\infty}(\Omega;\Lambda^1)$ piecewise affine such that

$$d\omega \in E = \{a^1, \cdots, a^6\} \quad \text{a.e. in } \Omega.$$

By contradiction assume that such a map exists. Therefore (cf. Definition 3.4) there exist $A_i \in \mathbb{R}^{4\times 4}$, $\alpha_i \in \mathbb{R}^4$ and $\Omega_i \subset \Omega$ open sets where i runs through an at most countable set I such that

(1) $\omega(x) = A_i x + \alpha_i$ in Ω_i for every $i \in I$
(2) $\operatorname{meas}(\Omega \setminus \cup_{i\in I}\Omega_i) = 0$ and $\Omega_i \cap \Omega_j = \emptyset$ for every $i, j \in I$ with $i \neq j$
(3) for every compact $K \subset \Omega$

$$\{i \in I : K \cap \Omega_i \neq \emptyset\} \quad \text{is finite.}$$

It is well known and easy to see that if $\partial\Omega_i \cap \partial\Omega_j$ contains a 3 dimensional subset of a hyperplane then necessarily

$$\operatorname{rank}[A_i - A_j] \leq 1. \tag{3.18}$$

From now on we assume that Ω is connected, otherwise we reason separately on every connected component of Ω. Since the partition of Ω is locally finite (cf. Point 3 above) it is easy to see that for every $i, j \in I$ with $i \neq j$ there exist $l_1 = i$, $l_2, \cdots, l_N = j$ such that, for every $2 \leq m \leq N$, either $A_{l_{m-1}} = A_{l_m}$ or

$$\partial\Omega_{l_{m-1}} \cap \partial\Omega_{l_m} \quad \text{contains a 3 dimensional subset of a hyperplane.} \tag{3.19}$$

Since (cf. Lemma 2.4) $0 \in \overline{\operatorname{co} E} = \operatorname{co} E$ and since

$$\{a^1, \cdots, a^5\} \quad \text{are linearly independent,}$$

we deduce that there exists $\bar{i}$ such that

$$d\omega = a^6 \quad \text{in } \Omega_{\bar{i}}\,.$$

Let $I_1 \subset I$ be defined by

$$\{i \in I : d\omega \in \{a^6\} \text{ in } \Omega_i\}.$$

We claim that $I_1 = I$ which implies that

$$d\omega \in \{a^6\} \quad \text{a.e. in } \Omega$$

and this is the desired contradiction. Choose $i \in I$ with $i \neq \bar{i}$. Combining (3.18) and (3.19), there exist $l_1 = i$, $l_2, \cdots, l_N = \bar{i}$ such that, for every $2 \leq j \leq N$,

$$\operatorname{rank}\left[A_{l_{j-1}} - A_{l_j}\right] \leq 1.$$

For every $1 \leq j \leq N$, let $r_j \in \{1, \cdots, 6\}$ be such that

$$d\omega = a^{r_j} \quad \text{in } \Omega_{l_j}\,.$$

Note that in particular, $r_N = 6$. Using Lemma 3.6 we find that, for every $2 \leq j \leq N$, there exists $b^j \in \Lambda^1 \setminus \{0\}$ such that

$$b^j \wedge (a^{r_{j-1}} - a^{r_j}) = 0.$$

Combining the previous equation with (3.17) we immediately deduce that

$$r_1 = \cdots = r_N = 6$$

and hence $i \in I_1$. This shows that $I_1 = I$ and proves the proposition. □

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