An identity involving exterior derivatives and applications to Gaffney inequality

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Abstract

Given two $k-$forms α and β we derive an identity relating

$$\int_\Omega (\langle d\alpha ; d\beta\rangle + \langle \delta\alpha ; \delta\beta\rangle - \langle \nabla\alpha ; \nabla\beta\rangle)$$

to an integral on the boundary of the domain and involving only the tangential and the normal components of α and β. We use this identity to deduce in a very simple way the classical Gaffney inequality.

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1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary and ν be the outward unit normal to the boundary. Let

$$\omega = \sum_I \omega_I dx^I$$

be a differential $k-$form. Denote by $d\omega$, $\delta\omega$ and $\nabla\omega$ respectively the exterior derivative, the codifferential and the usual gradient (componentwise) of the form ω. Finally let $\nu \wedge \omega$ and $\nu \lrcorner \omega$ be respectively the tangential and the normal component of ω on the boundary of Ω.

Our main result (cf. Theorem 3 for a more general version) is the following identity

$$\begin{aligned}\int_\Omega \left(|d\omega|^2 + |\delta\omega|^2 - |\nabla\omega|^2\right) \\ = -\int_{\partial\Omega} (\langle \nu \wedge d(\nu \lrcorner \omega); \nu \wedge \omega\rangle + \langle \nu \lrcorner \delta(\nu \wedge \omega); \nu \lrcorner \omega\rangle) \\ +\int_{\partial\Omega} (\langle L^\nu(\nu \wedge \omega); \nu \wedge \omega\rangle + \langle K^\nu(\nu \lrcorner \omega); \nu \lrcorner \omega\rangle)\end{aligned} \tag{1}$$

where $\langle .;. \rangle$ denotes scalar product and L^{ν} and K^{ν} are matrices acting on $(k+1)-$form and $(k-1)-$form respectively (where we have identified $k-$forms with $\binom{n}{k}-$vectors). They depend only on the geometry of Ω and on the degree k of the form. They can easily be calculated explicitly for general $k-$form and when Ω is a ball of radius R (cf. Corollary 5 for a more general version), it turns out that

$$L^{\nu}(\nu \wedge \omega) = \frac{k}{R}\, \nu \wedge \omega \quad \text{and} \quad K^{\nu}(\nu \lrcorner \omega) = \frac{n-k}{R}\, \nu \lrcorner \omega.$$

We therefore have

$$\langle L^{\nu}(\nu \wedge \omega); \nu \wedge \omega \rangle = \frac{k}{R}\, |\nu \wedge \omega|^2 \quad \text{and} \quad \langle K^{\nu}(\nu \lrcorner \omega); \nu \lrcorner \omega \rangle = \frac{n-k}{R}\, |\nu \lrcorner \omega|^2.$$

We will also give general formulas (cf. Proposition 7 and Corollary 8) in the case of $1-$form and for general domains Ω; in this case K^{ν} is a scalar and it is a multiple of κ, the mean curvature of the hypersurface $\partial\Omega$, namely

$$K^{\nu} = (n-1)\, \kappa.$$

The advantages of this formula, besides its generality and elegancy, are the following.

1) The right hand side of the identity is expressed solely in terms of the tangential and normal components of ω. Therefore if either $\nu \wedge \omega = 0$ or $\nu \lrcorner \omega = 0$, then the right hand side of (1) does not depend on derivatives of ω. It hence leads to an elementary proof of the classical *Gaffney inequality* (cf. Theorem 12 below). This inequality states that there exists a constant $C = C(\Omega) > 0$ such that, for every $k-$form ω with either $\nu \wedge \omega = 0$ or $\nu \lrcorner \omega = 0$,

$$C \int_{\Omega} |\nabla \omega|^2 \leq \int_{\Omega} |d\omega|^2 + \int_{\Omega} |\delta\omega|^2 + \int_{\Omega} |\omega|^2. \tag{2}$$

The classical proof of (2) by Morrey [8], [9] (see also, for example, Iwaniec-Scott-Stroffolini [6]), generalizing results of Gaffney [4], [5], is much more complicated. It requires the use of local rectification of the boundary, partition of unity and some lengthy estimates concerning $d\omega$, $\delta\omega$ and $\nabla\omega$.

2) The formula is valid with no restriction on the behavior of ω on $\partial\Omega$. This observation will allow us to obtain (cf. Theorem 14) Gaffney type inequalities for much more general boundary conditions than the classical ones which are $\nu \wedge \omega = 0$ or $\nu \lrcorner \omega = 0$. If one assumes $\nu \wedge \omega = 0$ (and similarly if $\nu \lrcorner \omega = 0$), then an identity in the same spirit as (1) can be found in Duvaut-Lions [3] (cf. the proof of Theorem 6.1 in Chapter 7) for the special case of $1-$form in $\mathbb{R}^3$ and in Schwarz [11] (cf. Theorem 2.1.5). However, in this last book, the actual K^{ν} is very implicitly defined.

The proof of our formula goes as follows. We start, as in classical proofs of Gaffney inequality, by expressing the left hand side of (1) by a boundary integral through several quite simple integrations by parts, together with the formula $d\delta + \delta d = \Delta$, obtaining that

$$\int_\Omega \left(|d\omega|^2 + |\delta\omega|^2 - |\nabla\omega|^2\right) = \int_{\partial\Omega} \left(\langle d\omega;\nu\wedge\omega\rangle + \langle\delta\omega;\nu\lrcorner\,\omega\rangle - \textstyle\sum_I \langle\nabla\omega_I;\nu\rangle\omega_I\right).$$

We then transform the right hand side through algebraic manipulations only, and no more integration by parts, so as to get our formula.

It goes without saying that our identity and its proof can be carried over without difficulty to manifolds with boundary.

2 Notation and Preliminaries

2.1 Notation

We will denote a differential form $\omega : U \subset \mathbb{R}^n \to \Lambda^k(\mathbb{R}^n)$, $0 \le k \le n$, by

$$\omega = \sum_{1\le i_1<\cdots<i_k\le n} \omega_{i_1\cdots i_k}(x)dx^{i_1}\wedge\cdots\wedge dx^{i_k} = \sum_I \omega_I dx^I \in S(U;\Lambda^k),$$

where I runs over all elements of

$$\mathcal{T}_k = \{I = (i_1,\cdots,i_k) : 1 \le i_1 < \cdots < i_k \le n\},$$

the set of strictly increasing $k-$indices. S will denote the function space to which the ω_I belong, for instance $S = C^1$, C^∞, $W^{k,p}$. We will sometimes use the notations

$$dx^{i_1}\wedge\cdots\wedge dx^{i_{s-1}}\wedge dx^{i_{s+1}}\wedge\cdots\wedge dx^{i_k} = dx^{i_1}\wedge\cdots\wedge\widehat{dx^{i_s}}\wedge\cdots\wedge dx^{i_k} = dx^{I\setminus\{i_s\}}$$

for denoting that dx^{i_s} has been omitted. The scalar product between two $k-$forms, is defined by

$$\langle\omega;\alpha\rangle = \sum_I \omega_I\,\alpha_I$$

and $|\omega|^2$ will abbreviate $\langle\omega;\omega\rangle$. The exterior product between forms is denoted by $\wedge$. We have, in particular, that if $\alpha\in\Lambda^k$ and $\beta\in\Lambda^l$, then

$$\alpha\wedge\beta = (-1)^{kl}\,\beta\wedge\alpha.$$

If $\omega\in\Lambda^k$ its Hodge dual $*\omega\in\Lambda^{n-k}$ is given by

$$\omega\wedge\beta = \langle *\omega;\beta\rangle dV, \quad \text{for every } \beta\in\Lambda^{n-k}$$

where $dV = dx^1\wedge\cdots\wedge dx^n$ denotes the standard volume form. If $I\in\mathcal{T}_k$ then $I^c\in\mathcal{T}_{n-k}$ will denote its complement in the set $\{1,\cdots,n\}$. The Hodge star operator satisfies

$$\begin{array}{ll} \text{(i)} & *\,dV = 1 \quad\text{and}\quad *\,1 = dV \\ \text{(ii)} & \alpha\wedge*\beta = \langle\alpha;\beta\rangle dV = \beta\wedge*\alpha \\ \text{(iii)} & **=(-1)^{k(n-k)}\,\mathrm{Id}_{\Lambda^k} \\ \text{(iv)} & *\,dx^I = (-1)^{\sigma(I,I^c)}dx^{I^c} \end{array} \tag{3}$$

where σ is given by $dx^I \wedge dx^{I^c} = (-1)^{\sigma(I,I^c)} dV$ and Id_{Λ^k} denotes the identity map on $k-$form. If $\alpha \in \Lambda^l$ and $\omega \in \Lambda^k$ we define the interior product of α with ω by

$$\alpha \lrcorner \omega = (-1)^{n(k-l)} * (\alpha \wedge *\omega).$$

We will, sometimes and by abuse of notations, identify in a natural way a $k-$form with a vector in $\mathbb{R}^{\binom{n}{k}}$. By means of this identification one can easily verify that $\alpha \lrcorner \omega = i_\alpha \omega$ if $\alpha \in \Lambda^1$, a notation used by some other authors. The following identities hold true, for every $\alpha \in \Lambda^1$, $\beta \in \Lambda^l$, $\gamma \in \Lambda^{k+1}$ and $\omega \in \Lambda^k$,

$$\begin{array}{ll} \text{(i)} & \alpha \wedge \alpha = 0 \quad \text{and} \quad \alpha \lrcorner (\alpha \lrcorner \omega) = 0 \\ \text{(ii)} & |\alpha|^2 \omega = \alpha \wedge (\alpha \lrcorner \omega) + \alpha \lrcorner (\alpha \wedge \omega) \\ \text{(iii)} & \langle \alpha \wedge \omega; \gamma \rangle = \langle \omega; \alpha \lrcorner \gamma \rangle \\ \text{(iv)} & \alpha \lrcorner (\omega \wedge \beta) = (\alpha \lrcorner \omega) \wedge \beta + (-1)^k \omega \wedge (\alpha \lrcorner \beta). \end{array} \tag{4}$$

The exterior derivative d and the codifferential δ (also called interior derivative) are defined by

$$d\omega = \sum_{i_1 < \cdots < i_k} \sum_{m=1}^{n} \frac{\partial \omega_{i_1 \cdots i_k}}{\partial x_m} dx^m \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

$$\delta \omega = (-1)^{n(k-1)} * (d(*\omega)).$$

They satisfy

$$\begin{array}{ll} \text{(i)} & dd\omega = 0 \quad \text{and} \quad \delta\delta\omega = 0 \\ \text{(ii)} & d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta \quad \text{if } \alpha \text{ is a } k-\text{form} \\ \text{(iii)} & d\delta\omega + \delta d\omega = \Delta\omega = \sum_I \Delta\omega_I \, dx^I. \end{array}$$

If $\phi = (\phi^1, \cdots, \phi^n) : O \to V \subset \mathbb{R}^m$ is a C^1 map, then the pullback $\phi^* \omega$ is a $k-$form on O given by

$$\phi^*(\omega) = \sum_{i_1 < \cdots < i_k} (\omega_{i_1 \cdots i_k} \circ \phi) d\phi^{i_1} \wedge \cdots \wedge d\phi^{i_k} = \sum_I (\omega_I \circ \phi) d\phi^I.$$

Let $U \subset \mathbb{R}^n$ be open, $\mu \in C^1(U; \Lambda^1)$ (with the usual identification between vectors and $1-$forms) and $\omega \in C^1(U; \Lambda^k)$. The Lie derivative $\mathcal{L}_\mu \omega$ is given by

$$\mathcal{L}_\mu \omega = \frac{\partial}{\partial t}\Big|_{t=0} \phi_t^*(\omega)$$

where ϕ_t is the flow associated to the vector field μ, namely

$$\begin{cases} \frac{d}{dt} \phi_t = \mu(\phi_t) & \text{if } t \geq 0 \\ \phi_0(x) = x & \text{for every } x \in U. \end{cases}$$

Calculating explicitly $\mathcal{L}_\mu \omega$ gives

$$\mathcal{L}_\mu \omega = \frac{\partial}{\partial t}\Big|_{t=0} \sum_I \omega_I(\phi_t(x))\, d\phi_t^I = \sum_I \langle \nabla \omega_I ; \mu \rangle dx^I + L^\mu(\omega). \tag{5}$$

where $L^\mu(\omega)$ abbreviates the term containing the time derivative of $d\phi_t^I$, i.e. for $k \geq 1$

$$L^\mu\,(\omega) = \sum_I \omega_I(x) \frac{\partial}{\partial t}\Big|_{t=0} d\phi_t^I(x) \tag{6}$$

while for $k = 0$

$$L^\mu\,(\omega) = 0.$$

The dual versions of the two preceding equations are

$$(-1)^{k(n-k)} * \mathcal{L}_\mu(*\omega) = \sum_I \langle \nabla \omega_I ; \mu \rangle dx^I \; + K^\mu(\omega), \tag{7}$$

where $K^\mu(\omega)$ is given by, if $k \leq n-1$,

$$K^\mu(\omega) = (-1)^{k(n-k)} * \Big(\sum_I \omega_I(x) \frac{\partial}{\partial t}\Big|_{t=0} d\phi_t^{I^c}(x)(-1)^{\sigma(I,I^c)} \Big) \tag{8}$$

and if $k = n$

$$K^\mu(\omega) = 0.$$

The following identities can be easily verified

$$K^\mu(\omega) = (-1)^{k(n-k)} * L^\mu(*\omega) \quad \text{and} \quad L^\mu(\omega) = (-1)^{k(n-k)} * K^\mu(*\omega) \tag{9}$$

$$\langle \mu \lrcorner K^\mu(\omega); \mu \lrcorner \omega \rangle = \langle \mu \wedge L^\mu(*\omega); \mu \wedge *\omega \rangle. \tag{10}$$

We recall Cartan formula and its dual version

$$\begin{gathered} \mathcal{L}_\mu \omega = \mu \lrcorner d\omega + d(\mu \lrcorner \omega) \\ (-1)^{k(n-k)} * \mathcal{L}_\mu(*\omega) = \mu \wedge \delta\omega + \delta(\mu \wedge \omega). \end{gathered} \tag{11}$$

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^2 boundary and unit outward normal ν. Let $\omega \in C(\overline{\Omega}; \Lambda^k)$. The *tangential component* and the *normal component* of ω on $\partial\Omega$ are, the $(k+1)-$form and $(k-1)-$form, defined respectively by

$$\nu \wedge \omega \in C(\partial\Omega; \Lambda^{k+1}) \quad \text{and} \quad \nu \lrcorner \omega \in C(\partial\Omega; \Lambda^{k-1}).$$

Some authors call $t\omega = \nu \lrcorner (\nu \wedge \omega)$ the tangential component and $n\omega = \nu \wedge (\nu \lrcorner \omega)$ the normal component of ω. It is easily seen that, using (4),

$$t\omega = 0 \;\Leftrightarrow\; \nu \wedge \omega = 0 \quad \text{and} \quad n\omega = 0 \;\Leftrightarrow\; \nu \lrcorner \omega = 0.$$

Furthermore the following implications hold true

$$\begin{gathered} \nu \wedge \omega = 0 \text{ on } \partial\Omega \;\Rightarrow\; \nu \wedge d\omega = 0 \text{ on } \partial\Omega \\ \nu \lrcorner \omega = 0 \text{ on } \partial\Omega \;\Rightarrow\; \nu \lrcorner \delta\omega = 0 \text{ on } \partial\Omega. \end{gathered} \tag{12}$$

Suppose $\alpha \in C^1(\overline{\Omega}; \Lambda^{k-1})$ and $\beta \in C^1(\overline{\Omega}; \Lambda^k)$, $1 \le k \le n$. Then the following integration by parts formula (Gauss-Green theorem) holds

$$\int_\Omega \langle d\alpha; \beta\rangle + \int_\Omega \langle \alpha; \delta\beta\rangle = \int_{\partial\Omega} \langle \nu \wedge \alpha; \beta\rangle = \int_{\partial\Omega} \langle \alpha; \nu \lrcorner \beta\rangle. \tag{13}$$

The scalar product of the gradients is defined by

$$\langle \nabla\alpha; \nabla\beta\rangle = \sum_I \sum_{m=1}^n \frac{\partial \alpha_I}{\partial x_m} \frac{\partial \beta_I}{\partial x_m}.$$

2.2 Two preliminary Lemmas

Lemma 1 *Let $0 \le k \le n$, U be an open subset of $\mathbb{R}^n$, $\mu \in C^1(U; \Lambda^1)$, α, $\beta \in C^1(U; \Lambda^k)$ and let $x \in U$ be such that $|\mu(x)| = 1$. Then the following equation holds true, for every such x,*

$$\begin{aligned}\langle d\alpha; \mu \wedge \beta\rangle + \langle \delta\alpha; \mu \lrcorner \beta\rangle - \textstyle\sum_I \langle \nabla\alpha_I; \mu\rangle \beta_I \\ = - \langle \mu \wedge d(\mu \lrcorner \alpha); \mu \wedge \beta\rangle - \langle \mu \lrcorner \delta(\mu \wedge \alpha); \mu \lrcorner \beta\rangle \\ + \langle \mu \wedge L^\mu(\alpha); \mu \wedge \beta\rangle + \langle \mu \lrcorner K^\mu(\alpha); \mu \lrcorner \beta\rangle.\end{aligned} \tag{14}$$

Proof First note that (5), (7) and (11) imply

$$\begin{aligned}\textstyle\sum_I \langle \nabla\alpha_I; \mu\rangle dx^I &= \mu \lrcorner d\alpha + d(\mu \lrcorner \alpha) - L^\mu(\alpha) \\ &= \mu \wedge \delta\alpha + \delta(\mu \wedge \alpha) - K^\mu(\alpha).\end{aligned} \tag{15}$$

We now use equations (i)-(iii) of (4) and $|\mu(x)| = 1$. We split $\sum \langle \nabla\alpha_I; \mu\rangle \beta_I$ in the following way

$$\begin{aligned}\textstyle\sum_I \langle \nabla\alpha_I; \mu\rangle \beta_I &= \langle \textstyle\sum_I \langle \nabla\alpha_I; \mu\rangle dx^I; \beta\rangle \\ &= \langle \textstyle\sum_I \langle \nabla\alpha_I; \mu\rangle dx^I; \mu \wedge (\mu \lrcorner \beta)\rangle + \langle \textstyle\sum_I \langle \nabla\alpha_I; \mu\rangle dx^I; \mu \lrcorner (\mu \wedge \beta)\rangle.\end{aligned}$$

And similarly

$$\begin{aligned}\langle d\alpha; \mu \wedge \beta\rangle &= \langle \mu \wedge (\mu \lrcorner d\alpha); \mu \wedge \beta\rangle + \langle \mu \lrcorner (\mu \wedge d\alpha); \mu \wedge \beta\rangle \\ &= \langle \mu \wedge (\mu \lrcorner d\alpha); \mu \wedge \beta\rangle.\end{aligned}$$

Using these identities and (15) we obtain

$$\begin{aligned}&\langle d\alpha; \mu \wedge \beta\rangle - \langle \textstyle\sum_I \langle \nabla\alpha_I; \mu\rangle dx^I; \mu \lrcorner (\mu \wedge \beta)\rangle \\ &\quad = \langle \mu \wedge (\mu \lrcorner d\alpha); \mu \wedge \beta\rangle - \langle \mu \wedge \left(\textstyle\sum_I \langle \nabla\alpha_I; \mu\rangle dx^I\right); \mu \wedge \beta\rangle \\ &\quad = -\langle \mu \wedge d(\mu \lrcorner \alpha); \mu \wedge \beta\rangle + \langle \mu \wedge L^\mu(\alpha); \mu \wedge \beta\rangle.\end{aligned}$$

We now carry out the analogous computations for $\langle \delta\alpha; \mu \lrcorner \beta\rangle$, which yield in a similar way

$$\begin{aligned}
&\langle \delta\alpha; \mu \lrcorner \beta\rangle - \langle \textstyle\sum_I \langle \nabla \alpha_I; \mu\rangle dx^I; \mu \wedge (\mu \lrcorner \beta)\rangle \\
&\quad = \langle \mu \lrcorner (\mu \wedge \delta\alpha); \mu \lrcorner \beta\rangle - \langle \mu \lrcorner \Big(\textstyle\sum_I \langle \nabla \alpha_I; \mu\rangle dx^I\Big); \mu \lrcorner \beta\rangle \\
&\quad = -\langle \mu \lrcorner \delta(\mu \wedge \alpha); \mu \lrcorner \beta\rangle + \langle \mu \lrcorner K^\mu(\alpha); \mu \lrcorner \beta\rangle .
\end{aligned}$$

Adding the previous two equalities concludes the proof of the lemma. ■

Lemma 2 *Let $0 \le k \le n$, U be an open subset of $\mathbb{R}^n$, $\alpha \in C^1(U; \Lambda^k)$, $\mu \in C^1(U; \Lambda^1)$ such that $|\mu| = 1$ in U, then the following identities hold true in U*

$$L^\mu(\mu \wedge \alpha) = \mu \wedge L^\mu(\alpha) \quad \textit{and} \quad K^\mu(\mu \lrcorner \alpha) = \mu \lrcorner K^\mu(\alpha).$$

Proof Let $I \in \mathcal{T}_k$ and $J \in \mathcal{T}_{k+1}$. We use equation (15) for expressing both $L^\mu(\alpha)$ and $L^\mu(\mu \wedge \alpha)$. This gives

$$\mu \wedge L^\mu(\alpha) = \mu \wedge (\mu \lrcorner d\alpha) + \mu \wedge d(\mu \lrcorner \alpha) - \mu \wedge \sum_I \langle \nabla \alpha_I; \mu\rangle dx^I$$

$$L^\mu(\mu \wedge \alpha) = \mu \lrcorner d(\mu \wedge \alpha) + d(\mu \lrcorner (\mu \wedge \alpha)) - \sum_J \langle \nabla(\mu \wedge \alpha)_J; \mu\rangle dx^J .$$

In the first one of these two equations, we use the identity

$$\mu \wedge (\mu \lrcorner d\alpha) = d\alpha - \mu \lrcorner (\mu \wedge d\alpha)$$

to get

$$\mu \wedge L^\mu(\alpha) = d\alpha - \mu \lrcorner (\mu \wedge d\alpha) + \mu \wedge d(\mu \lrcorner \alpha) - \mu \wedge \sum_I \langle \nabla \alpha_I; \mu\rangle dx^I .$$

In the second one we use the following relations

$$d(\mu \wedge \alpha) = d\mu \wedge \alpha - \mu \wedge d\alpha$$

$$\begin{aligned}
d(\mu \lrcorner (\mu \wedge \alpha)) &= d\alpha - d(\mu \wedge (\mu \lrcorner \alpha)) \\
&= d\alpha - d\mu \wedge (\mu \lrcorner \alpha) + \mu \wedge d(\mu \lrcorner \alpha)
\end{aligned}$$

to obtain, after using (iv) of (4),

$$\begin{aligned}
L^\mu(\mu \wedge \alpha) = {} & \mu \lrcorner d(\mu \wedge \alpha) + d\alpha - d\mu \wedge (\mu \lrcorner \alpha) + \mu \wedge d(\mu \lrcorner \alpha) \\
& - \sum_J \langle \nabla(\mu \wedge \alpha)_J; \mu\rangle dx^J .
\end{aligned}$$

Setting now $\mu \wedge L^\mu(\alpha)$ equal to $L^\mu(\mu \wedge \alpha)$, we see that the first statement of the lemma is equivalent to

$$\sum_J \langle \nabla(\mu \wedge \alpha)_J; \mu\rangle dx^J = \mu \wedge \sum_I \langle \nabla \alpha_I; \mu\rangle \, dx^I + (\mu \lrcorner d\mu) \wedge \alpha. \tag{16}$$

The hypothesis $|\mu| = 1$ in an open set implies for any $l = 1, \cdots, n$ that

$$\sum_{k=1}^{n} \frac{\partial \mu_k}{\partial x_l} \mu_k = 0.$$

Using this fact one can easily verify that

$$\mu \lrcorner d\mu = \sum_{s=1}^{n} \langle \nabla \mu_s ; \mu \rangle dx^s. \tag{17}$$

We can now prove (16) by a straightforward calculation. Due to the linearity in α we can suppose that α is of the form $\alpha = f dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ for some function f. We therefore have

$$\begin{aligned}
&\sum_{J} \langle \nabla (\mu \wedge \alpha)_J ; \mu \rangle dx^J = \sum_{s=1}^{n} \langle \nabla (f \mu_s) ; \mu \rangle \, dx^s \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \\
=&\sum_{s=1}^{n} \langle \nabla f ; \mu \rangle \mu_s \, dx^s \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} + \sum_{s=1}^{n} \langle \nabla \mu_s ; \mu \rangle f \, dx^s \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \\
=&\mu \wedge \sum_{I} \langle \nabla \alpha_I ; \mu \rangle dx^I + \sum_{s=1}^{n} \langle \nabla \mu_s ; \mu \rangle f \, dx^s \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.
\end{aligned}$$

The last term in the last equation is equal to $(\mu \lrcorner d\mu) \wedge \alpha$. This follows from equation (17) by taking the wedge product with α. The second result concerning K^μ follows now from the first one by duality, see (9). ■

3 The Main Theorem

Theorem 3 (Main Theorem) *Let $0 \le k \le n$ and let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^2 boundary, with unit outward normal ν. Then every α, $\beta \in C^1(\overline{\Omega}; \Lambda^k)$ satisfy the equation*

$$\begin{aligned}
&\int_\Omega \left(\langle d\alpha ; d\beta \rangle + \langle \delta\alpha ; \delta\beta \rangle - \langle \nabla \alpha ; \nabla \beta \rangle \right) \\
&\quad = -\int_{\partial\Omega} \left(\langle \nu \wedge d(\nu \lrcorner \alpha) ; \nu \wedge \beta \rangle + \langle \nu \lrcorner \delta(\nu \wedge \alpha) ; \nu \lrcorner \beta \rangle \right) \\
&\quad\quad + \int_{\partial\Omega} \left(\langle L^\nu (\nu \wedge \alpha) ; \nu \wedge \beta \rangle + \langle K^\nu (\nu \lrcorner \alpha) ; \nu \lrcorner \beta \rangle \right).
\end{aligned}$$

Remark 4 (i) In the above theorem and in the sequel, we have always assumed that the outward unit normal ν has been extended to $\mathbb{R}^n$ in a C^1 way with $|\nu| = 1$ in a neighborhood of $\partial\Omega$. This is, of course, always possible. The formulas here and below will be seen to be independent of the extension.

(ii) An alternative version to formulate the theorem would be (see Lemma 2)

$$\begin{aligned}\int_{\Omega} & (\langle d\alpha; d\beta\rangle + \langle \delta\alpha; \delta\beta\rangle - \langle \nabla\alpha; \nabla\beta\rangle) \\ & = -\int_{\partial\Omega} (\langle \nu \wedge d(\nu \lrcorner \alpha); \nu \wedge \beta\rangle + \langle \nu \lrcorner \delta(\nu \wedge \alpha); \nu \lrcorner \beta\rangle) \\ & \quad + \int_{\partial\Omega} (\langle \nu \wedge L^{\nu}(\alpha); \nu \wedge \beta\rangle + \langle \nu \lrcorner K^{\nu}(\alpha); \nu \lrcorner \beta\rangle) .\end{aligned} \tag{18}$$

In that case we do not need to extend ν, since all four terms in the boundary integral depend only on the values of ν on the boundary. This is obvious for $\nu \wedge d(\nu \lrcorner \alpha)$ and $\nu \lrcorner \delta(\nu \wedge \alpha)$ due to (12). For the other terms see Proposition 6.

(iii) If $\alpha = \beta$ the first boundary integral could be expressed more compactly, since by taking an arbitrary extension of ν onto the whole Ω, we obtain, appealing to (13),

$$\begin{aligned}\int_{\Omega} \langle d(\nu \lrcorner \alpha); \delta(\nu \wedge \alpha)\rangle & = \int_{\partial\Omega} \langle \nu \lrcorner \delta(\nu \wedge \alpha); \nu \lrcorner \alpha\rangle \\ & = \int_{\partial\Omega} \langle \nu \wedge d(\nu \lrcorner \alpha); \nu \wedge \alpha\rangle.\end{aligned}$$

(iv) In the special cases $k = 0$ or $k = n$, the proof is much more immediate than the one we will provide below.

As an example we first present the following corollary.

Corollary 5 (Ball of radius R) *Let $\Omega = B_R(a)$ be the ball of radius R centered at a with unit outward normal ν. Then*

$$L^{\nu}(\nu \wedge \alpha) = \frac{k}{R}\, \nu \wedge \alpha \quad \text{and} \quad K^{\nu}(\nu \lrcorner \alpha) = \frac{n-k}{R}\, \nu \lrcorner \alpha$$

and thus every α, $\beta \in C^1(\overline{\Omega}; \Lambda^k)$ satisfy the equation

$$\begin{aligned}\int_{\Omega} & (\langle d\alpha; d\beta\rangle + \langle \delta\alpha; \delta\beta\rangle - \langle \nabla\alpha; \nabla\beta\rangle) \\ & = -\int_{\partial\Omega} (\langle \nu \wedge d(\nu \lrcorner \alpha); \nu \wedge \beta\rangle + \langle \nu \lrcorner \delta(\nu \wedge \alpha); \nu \lrcorner \beta\rangle) \\ & + \int_{\partial\Omega} \left(\frac{k}{R} \langle \nu \wedge \alpha; \nu \wedge \beta\rangle + \frac{n-k}{R} \langle \nu \lrcorner \alpha; \nu \lrcorner \beta\rangle \right).\end{aligned}$$

Proof of the corollary Without loss of generality we can assume $a = 0$. We use Lemma 2 and Proposition 6 below to obtain

$$L^{\nu}(\nu \wedge \alpha) = \nu \wedge L^{\nu}(\alpha) = \mu \wedge L^{\mu}(\alpha),$$

where $\mu(x) = x/R$. Let ϕ_t be the flow associated to this μ, namely

$$\phi_t = \phi_t(x) = e^{\frac{1}{R}t} x.$$

We therefore obtain

$$d\phi^I = e^{\frac{k}{R}t} dx^I \quad \Rightarrow \quad \frac{\partial}{\partial t}\Big|_{t=0} d\phi^I = \frac{k}{R}\, dx^I$$

and thus

$$L^\mu(\alpha) = \frac{k}{R}\,\alpha \quad \text{and} \quad \mu \wedge L^\mu(\alpha) = \frac{k}{R}\,\mu \wedge \alpha.$$

It now follows from identity (9) that

$$K^\mu(\alpha) = \frac{n-k}{R}\,\alpha \quad \text{and} \quad \mu \lrcorner K^\mu(\alpha) = \frac{n-k}{R}\,\mu \lrcorner \alpha.$$

The corollary is therefore proved. ■

We now continue with the proof of the main theorem.

Proof of the main theorem We assume $\alpha,\, \beta \in C^2(\overline{\Omega}; \Lambda^k)$, since the result for $\alpha,\, \beta \in C^1(\overline{\Omega}; \Lambda^k)$ follows by a density argument. Integrating several times partially yields

$$\begin{aligned}
&\int_\Omega \langle d\alpha; d\beta\rangle + \int_\Omega \langle \delta\alpha; \delta\beta\rangle \\
&= -\int_\Omega \langle \delta d\alpha + d\delta\alpha; \beta\rangle + \int_{\partial\Omega} \langle d\alpha; \nu \wedge \beta\rangle + \int_{\partial\Omega} \langle \delta\alpha; \nu \lrcorner \beta\rangle \\
&= -\int_\Omega \langle \Delta\alpha; \beta\rangle + \int_{\partial\Omega} (\langle d\alpha; \nu \wedge \beta\rangle + \langle \delta\alpha; \nu \lrcorner \beta\rangle) \\
&= -\int_\Omega \sum_I \Delta\alpha_I\, \beta_I + \int_{\partial\Omega} (\langle d\alpha; \nu \wedge \beta\rangle + \langle \delta\alpha; \nu \lrcorner \beta\rangle) \\
&= \int_\Omega \sum_I \langle \nabla\alpha_I; \nabla\beta_I\rangle - \int_{\partial\Omega} \sum_I \beta_I \langle \nabla\alpha_I; \nu\rangle + \int_{\partial\Omega} \langle d\alpha; \nu \wedge \beta\rangle + \int_{\partial\Omega} \langle \delta\alpha; \nu \lrcorner \beta\rangle.
\end{aligned}$$

We now apply Lemma 1, which proves (18). The theorem then follows from Lemma 2. ■

Consider the tangent vectors $E_{ij} = (E_{ij}^1, \cdots, E_{ij}^n)$ at $x \in \partial\Omega$ defined, for every $1 \le i, j, s \le n$, by

$$E_{ij}^s = \begin{cases} 0 & \text{if } i = j \\ 0 & \text{if } s \neq i \text{ or } s \neq j \\ \nu_j & \text{if } s = i \text{ and } i \neq j \\ -\nu_i & \text{if } s = j \text{ and } i \neq j. \end{cases}$$

For $f \in C^1(\partial\Omega)$ we denote its derivative in the direction of E_{ij} by $\partial_{ij}[f]$, that is

$$\partial_{ij}[f](x) = \begin{cases} \left[\frac{\partial}{\partial t} f(c_{ij}(t))\right]_{t=0} & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases}$$

where $c_{ij}(t)$ is any curve lying in $\partial\Omega$, which satisfies $c_{ij}(0) = x$ and $\dot{c}_{ij}(0) = E_{ij}$. It turns out, that if f has been extended to $\mathbb{R}^n$, then

$$\partial_{ij}[f] = \frac{\partial f}{\partial x_i}\nu_j - \frac{\partial f}{\partial x_j}\nu_i = (df \wedge \nu)_{ij}$$

and thus this expression is independent of the choice of the extension. With this notation we can prove the following.

Proposition 6 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^2 boundary and unit outward normal ν. Then for every $\alpha \in \Lambda^k$ the maps $\alpha \to \nu \wedge L^\nu(\alpha)$ and $\alpha \to \nu \lrcorner K^\nu(\alpha)$ depend only on the values of ν on $\partial\Omega$.*

Proof *Step 1.* First extend ν as a C^1 vector field to $\mathbb{R}^n$ and let

$$\begin{cases} \frac{d}{dt}\phi_t = \nu(\phi_t) & \text{if } t \geq 0 \\ \phi_0(x) = x & \text{for every } x \in \Omega. \end{cases}$$

We therefore get

$$\frac{\partial}{\partial t}\Big|_{t=0} d\phi_t^{i_1}(x) \wedge \ldots \wedge d\phi_t^{i_k}(x) = \sum_{\gamma=1}^{k} dx^{i_1} \wedge \ldots \wedge dx^{i_{\gamma-1}} \wedge d\nu_{i_\gamma} \wedge dx^{i_{\gamma+1}} \wedge \ldots \wedge dx^{i_k}. \tag{19}$$

Setting this into the definition of L^ν yields

$$L^\nu(\alpha) = \sum_{i_1<\cdots<i_k} \alpha_{i_1\cdots i_k}(x) \sum_{\gamma=1}^{k}\sum_{s=1}^{n} \frac{\partial \nu_{i_\gamma}}{\partial x_s} dx^{i_1} \wedge \ldots \wedge dx^{i_{\gamma-1}} \wedge dx^s \wedge dx^{i_{\gamma+1}} \wedge \ldots \wedge dx^{i_k}. \tag{20}$$

Step 2. We claim that for $\alpha = \sum_I \alpha_I dx^I$

$$\nu \wedge L^\nu(\alpha) = \sum_I \sum_{\gamma=1}^{k} \sum_{1\leq r<s\leq n} (-1)^\gamma \alpha_I \partial_{rs}[\nu_{i_\gamma}] dx^r \wedge dx^s \wedge dx^{I\setminus\{i_\gamma\}}. \tag{21}$$

Once we have shown this, it follows that both $\nu \wedge L^\nu(\alpha)$ and $\nu \lrcorner K^\nu(\alpha)$ (see (9)) only depend on the values of ν on $\partial\Omega$. Using (20) we obtain

$$\begin{aligned} &\nu \wedge L^\nu(\alpha) \\ &= \sum_I \alpha_I \sum_{\gamma=1}^{k}\sum_{r,s=1}^{n} \frac{\partial \nu_{i_\gamma}}{\partial x_s}\nu_r dx^r \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_{\gamma-1}} \wedge dx^s \wedge dx^{i_{\gamma+1}} \wedge \ldots \wedge dx^{i_k} \\ &= \sum_I \alpha_I \sum_{\gamma=1}^{k}\sum_{r,s=1}^{n} (-1)^{\gamma-1} \frac{\partial \nu_{i_\gamma}}{\partial x_s}\nu_r\, dx^r \wedge dx^s \wedge dx^{i_1} \wedge \ldots \wedge \widehat{dx}^{i_\gamma} \wedge \cdots \wedge dx^{i_k}. \end{aligned}$$

We now split the sum over the r, s as

$$\sum_{r,s=1}^{n} = \sum_{r<s} + \sum_{r>s}.$$

In the second sum of these two we interchange the roles of r and s. Recalling that $dx^r \wedge dx^s = -dx^s \wedge dx^r$, we have indeed established (21). ■

Proposition 7 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^2 boundary. Let α be a $0-$form. Then*

$$K^\nu(\alpha) = [(n-1)\kappa]\,\alpha$$

where κ is the mean curvature of the hypersurface $\partial\Omega$.

Proof Recall that the outward unit normal ν has been extended on a neighborhood of $\partial\Omega$ so that $|\nu| = 1$. Let α be a $0-$form. Due to the definition of K^ν, see (8), we obtain

$$\begin{aligned} K^\nu(\alpha) &= * \left[\alpha \, \frac{\partial}{\partial t}\Big|_{t=0} d\phi_t^1 \wedge \cdots \wedge d\phi_t^n\right] \\ &= * \left[\alpha \textstyle\sum_{j=1}^n dx^1 \wedge \cdots \wedge dx^{j-1} \wedge d\nu_j \wedge dx^{j+1} \wedge \cdots \wedge dx^n\right]. \end{aligned}$$

Only the derivative with respect to j, $\frac{\partial \nu_j}{\partial x_j} dx^j$ in $d\nu_j$ has a contribution. Hence

$$K^\nu(\alpha) = *\alpha \sum_{j=1}^n \frac{\partial \nu_j}{\partial x_j}\, dV = \alpha \operatorname{div} \nu.$$

Since the divergence of ν is equal (see for instance Krantz-Parks [7]) to $(n-1)\kappa$, if $|\nu| = 1$ near $\partial\Omega$, the proposition follows. ■

Using Proposition 7, we can rewrite Theorem 3 in the case of $1-$forms with vanishing tangential component as follows.

Corollary 8 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^2 boundary and unit outward normal ν. Every α, $\beta \in C^1(\overline{\Omega}; \Lambda^1)$ with $\nu \wedge \alpha = \nu \wedge \beta = 0$ satisfy*

$$\int_\Omega (\langle d\alpha; d\beta\rangle + \langle \delta\alpha; \delta\beta\rangle - \langle \nabla\alpha; \nabla\beta\rangle) = \int_{\partial\Omega} (n-1)\,\kappa\,\langle \alpha; \beta\rangle.$$

We present another possibility to express the main theorem for $1-$forms (for a proof see Csató [2]).

Proposition 9 *Let $\alpha = \sum \alpha_i dx^i$ and $\beta = \sum \beta_i dx^i$ be $1-$forms. Then, for every $x \in \partial\Omega$, the following identity*

$$\langle \nu \wedge L^\nu(\alpha); \nu \wedge \beta\rangle + \langle \nu \lrcorner K^\nu(\alpha); \nu \lrcorner \beta\rangle = -\sum_{i,j=1}^n \alpha_i \beta_j \sum_{r=1}^n \mathbb{I}_x(E_{ir}, E_{jr})$$

is valid, where $\mathbb{I}_x$ is the second fundamental form of the hypersurface $\partial\Omega$ at x.

4 An application to Gaffney inequality and a generalization

Gaffney inequality is essentially based on the fact that the first boundary integral in Theorem 3 drops, whenever one of the conditions $\nu \wedge \omega = 0$ or $\nu \lrcorner \omega = 0$ is satisfied.

We first need to define the Sobolev spaces of forms with vanishing tangential, respectively normal component on the boundary, namely

$$\begin{aligned} W_T^{1,2}(\Omega;\Lambda^k) &= \{\omega \in W^{1,2}(\Omega;\Lambda^k), \quad \nu \wedge \omega = 0 \quad \text{on } \partial\Omega\} \\ W_N^{1,2}(\Omega;\Lambda^k) &= \{\omega \in W^{1,2}(\Omega;\Lambda^k), \quad \nu \lrcorner \omega = 0 \quad \text{on } \partial\Omega\}. \end{aligned}$$

The spaces $C_T^r(\overline{\Omega};\Lambda^k)$ and $C_N^r(\overline{\Omega};\Lambda^k)$ are defined in an analogous way. The norms in these spaces are defined componentwise. The following density argument holds true, see for instance Csató [2], Iwaniec-Scott-Stroffolini [6] or Morrey [9].

Theorem 10 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^2 boundary. Then $C_T^1(\overline{\Omega};\Lambda^k)$ is dense in $W_T^{1,2}(\Omega;\Lambda^k)$ and $C_N^1(\overline{\Omega};\Lambda^k)$ is dense in $W_N^{1,2}(\Omega;\Lambda^k)$.*

To obtain Gaffney inequality from the main Theorem 3 we will need the following elementary result.

Proposition 11 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^2 boundary. Then there exists $C = C(\Omega) > 0$ such that, for any $0 < \epsilon < 1$,*

$$\int_{\partial\Omega} u^2 \leq \epsilon \int_\Omega |\nabla u|^2 + \frac{C}{\epsilon}\int_\Omega |u|^2$$

for every $u \in W^{1,2}(\Omega)$.

Proof Due to the density of $C^1\left(\overline{\Omega}\right)$in $W^{1,2}(\Omega)$ and the continuous imbedding $W^{1,2}(\Omega) \hookrightarrow L^2(\partial\Omega)$ it suffices to show the inequality for all $u \in C^1(\overline{\Omega})$. As $\partial\Omega$ is of class C^2, we can extend the unit outward normal vector ν to a $C^1(\overline{\Omega})$ function. Hence $|\nu|$ and the divergence $|\operatorname{div}\nu|$ are bounded in Ω by some $C = C(\Omega) > 0$. Using integration by parts yields to

$$\begin{aligned} \int_{\partial\Omega} u^2 &= \int_{\partial\Omega} u^2 \sum_{i=1}^n \nu_i^2 = \int_{\partial\Omega} \langle u^2\,\nu;\nu\rangle \\ &= \int_\Omega \operatorname{div}(u^2\,\nu) = \int_\Omega u^2 \operatorname{div}\nu + \int_\Omega \langle \nabla u^2;\nu\rangle \\ &\leq \int_\Omega |\operatorname{div}\nu|\,u^2 + \int_\Omega |\nu|\,|\nabla u^2| \leq C\int_\Omega u^2 + C\int_\Omega 2\,|u|\,|\nabla u|. \end{aligned}$$

Since

$$2\,C\,|u|\,|\nabla u| \leq \epsilon\,|\nabla u|^2 + \frac{C^2}{\epsilon}\,u^2$$

we have the desired result. ■

Theorem 12 (Gaffney Inequality) *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^2 boundary. Then there exists a constant $C = C(\Omega) > 0$ such that*

$$\|\omega\|_{W^{1,2}}^2 \leq C \left(\|d\omega\|_{L^2}^2 + \|\delta\omega\|_{L^2}^2 + \|\omega\|_{L^2}^2 \right)$$

for every $\omega \in W_T^{1,2}(\Omega; \Lambda^k) \cup W_N^{1,2}(\Omega; \Lambda^k)$.

Proof Appealing to Theorem 3 and the properties of L^ν and K^ν there exists continuous functions $f_{IJ} \in C(\partial\Omega)$, depending only on the geometry of $\partial\Omega$ and on k, such that

$$\int_\Omega \left(|d\omega|^2 + |\delta\omega|^2 \right) = \int_\Omega |\nabla\omega|^2 + \int_{\partial\Omega} \sum_{I,J} f_{IJ}\omega_I\omega_J$$

for every $\omega \in C_T^1(\overline{\Omega}; \Lambda^k) \cup C_N^1(\overline{\Omega}; \Lambda^k)$. In particular, since $\partial\Omega$ is compact, there exists a constant $C = C(n, k, \Omega)$ such that

$$\int_\Omega \left(|d\omega|^2 + |\delta\omega|^2 \right) \geq \int_\Omega |\nabla\omega|^2 - C \int_{\partial\Omega} |\omega|^2. \tag{22}$$

Combining this with Theorem 10 and Proposition 11, we have Gaffney inequality. ■

We just saw that the proof of Gaffney inequality is essentially based on the fact that the first boundary integral in Theorem 3 drops, whenever the tangential or normal component of ω vanishes. In that case no derivatives of ω occur in the boundary integral and one obtains the estimate (22). We now discuss the possibility of extending Theorem 12 to more general conditions than those of vanishing tangential or normal components. We give in Theorem 14 two ways of generalizing Gaffney inequality. But before proceeding further we need the following algebraic lemma.

Lemma 13 *(i) Let $2k \leq n$, k odd, ω a $k-$form, ν a $1-$form and λ a $(n-2k)-$form such that*

$$*[\nu \wedge \omega] = \lambda \wedge (\nu \lrcorner \omega).$$

Then

$$\langle \nu \wedge d(\nu \lrcorner \omega); \nu \wedge \omega \rangle + \langle \nu \lrcorner \delta(\nu \wedge \omega); \nu \lrcorner \omega \rangle = -\langle \nu \wedge d\lambda \wedge (\nu \lrcorner \omega); *(\nu \lrcorner \omega) \rangle.$$

(ii) Let $2k \geq n$, $(n-k)$ be odd, ω a $k-$form, ν a $1-$form and λ a $(2k-n)-$form such that

$$\nu \lrcorner \omega = \lambda \wedge *(\nu \wedge \omega).$$

Then

$$\langle \nu \wedge d(\nu \lrcorner \omega); \nu \wedge \omega \rangle + \langle \nu \lrcorner \delta(\nu \wedge \omega); \nu \lrcorner \omega \rangle = \langle \nu \wedge d\lambda \wedge *(\nu \wedge \omega); \nu \wedge \omega \rangle.$$

Proof *Step 1.* Let $\omega \in \Lambda^k(\mathbb{R}^n)$, $\nu \in \Lambda^1(\mathbb{R}^n)$ and $dV = dx^1 \wedge \cdots \wedge dx^n$. It follows, from the definitions of the interior product and the interior derivative as well as (3), that

$$\begin{aligned}&\langle \nu \wedge d(\nu \lrcorner \omega); \nu \wedge \omega \rangle dV + \langle \nu \lrcorner \delta(\nu \wedge \omega); \nu \lrcorner \omega \rangle dV \\ &\qquad = \nu \wedge \left[d(\nu \lrcorner \omega) \wedge (*(\nu \wedge \omega)) + (-1)^k (\nu \lrcorner \omega) \wedge d(*(\nu \wedge \omega)) \right].\end{aligned} \tag{23}$$

Step 2. We first prove (i). We set the equality $*[\nu \wedge \omega] = \lambda \wedge (\nu \lrcorner \omega)$ into the right-hand side of (23), which yields

$$\begin{aligned}&\nu \wedge [d(\nu \lrcorner \omega) \wedge \lambda \wedge (\nu \lrcorner \omega) - (\nu \lrcorner \omega) \wedge d(\lambda \wedge (\nu \lrcorner \omega))] \\ &\qquad = -\nu \wedge (\nu \lrcorner \omega) \wedge d\lambda \wedge (\nu \lrcorner \omega) + \nu \wedge A\end{aligned}$$

where

$$A = d(\nu \lrcorner \omega) \wedge \lambda \wedge (\nu \lrcorner \omega) + (-1)^{n-k} (\nu \lrcorner \omega) \wedge \lambda \wedge d(\nu \lrcorner \omega).$$

Using again that k is odd one can easily verify that A vanishes. It therefore follows from (23) and the above two identities that

$$\begin{aligned}&\langle \nu \wedge d(\nu \lrcorner \omega); \nu \wedge \omega \rangle dV + \langle \nu \lrcorner \delta(\nu \wedge \omega); \nu \lrcorner \omega \rangle dV \\ &= -\nu \wedge (\nu \lrcorner \omega) \wedge d\lambda \wedge (\nu \lrcorner \omega) = -\nu \wedge d\lambda \wedge (\nu \lrcorner \omega) \wedge **(\nu \lrcorner \omega) \\ &= -\langle \nu \wedge d\lambda \wedge (\nu \lrcorner \omega); *(\nu \lrcorner \omega) \rangle dV.\end{aligned}$$

Step 3. The proof of (ii) is analogous to that of (i) by setting the equality $\nu \lrcorner \omega = \lambda \wedge *(\nu \wedge \omega)$ into the right-hand side of (23). ■

Theorem 14 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^2 boundary, ν the unit outward normal to $\partial\Omega$ and $0 \le k \le n$.*

(i) Let $2k \le n$ with k odd. Let $\lambda \in C^1(\partial\Omega; \Lambda^{n-2k})$. Then there exists a constant $C = C(n, k, \partial\Omega, \lambda)$ such that

$$\|\nabla \omega\|^2_{W^{1,2}} \le C \left(\|d\omega\|^2_{L^2} + \|\delta\omega\|^2_{L^2} + \|\omega\|^2_{L^2} \right)$$

for every $\omega \in C^1(\overline{\Omega}; \Lambda^k)$ satisfying

$$*(\nu \wedge \omega) = \lambda \wedge (\nu \lrcorner \omega) \quad \text{on } \partial\Omega.$$

(ii) Let $2k \ge n$ with $(k-n)$ odd. Let $\lambda \in C^1(\partial\Omega; \Lambda^{2k-n})$. Then there exists a constant $C = C(n, k, \partial\Omega, \lambda)$ such that

$$\|\nabla \omega\|^2_{W^{1,2}} \le C \left(\|d\omega\|^2_{L^2} + \|\delta\omega\|^2_{L^2} + \|\omega\|^2_{L^2} \right)$$

for every $\omega \in C^1(\overline{\Omega}; \Lambda^k)$ verifying

$$\nu \lrcorner \omega = \lambda \wedge *(\nu \wedge \omega) \quad \text{on } \partial\Omega.$$

Proof We prove (i). The proof of (ii) is completely analogous. Due to Theorem 3 and Lemma 13 (i)

$$\int_\Omega |d\omega|^2 + \int_\Omega |\delta\omega|^2 - \int_\Omega |\nabla\omega|^2$$
$$= \int_{\partial\Omega} \left(\langle \nu \wedge d\lambda \wedge (\nu \lrcorner \omega); *(\nu \lrcorner \omega) \rangle + \langle L^\nu (\nu \wedge \omega); \nu \wedge \omega \rangle + \langle K^\nu (\nu \lrcorner \omega); \nu \lrcorner \omega \rangle \right).$$

The regularity assumption $\lambda \in C^1(\partial\Omega; \Lambda^{n-2k})$ implies that $\nu \wedge d\lambda$, which is well defined by (12), is a continuous function on $\partial\Omega$. One can now proceed exactly as in the proof of Gaffney inequality. ■

We give the following example to Part (i) of Theorem 14.

Example 15 Let $k = 1$ and $n \geq 3$. Hence $n - 2k = n - 2$. It will be more convenient to calculate with $*\lambda$ than with λ, so we suppose that $*\lambda \in C^1(\partial\Omega; \Lambda^{n-2})$. In that case the condition $*(\nu \wedge \omega) = (*\lambda) \wedge (\nu \lrcorner \omega)$ can be written as

$$\nu \wedge \omega = (\nu \lrcorner \omega)\lambda \quad \text{on } \partial\Omega$$

which consists of the $\binom{n}{2}$ equations

$$\nu_i\, \omega_j - \nu_j\, \omega_i = \lambda_{ij} \sum_{l=1}^{n} \nu_l\, \omega_l \quad \text{for } 1 \leq i < j \leq n.$$

To make the example even simpler assume that

$$H = \partial\Omega \cap \{x \in \mathbb{R}^n : x_n = 0\}$$

contains a relatively open set. Furthermore suppose that, for every $x \in H$,

$$\lambda_{ij}(x) = 0 \quad \text{if } j \neq n \text{ and } 1 \leq i < j.$$

At every $x \in H$, we have

$$\nu \wedge \omega = 0 \quad \Leftrightarrow \quad \omega_1 = \cdots = \omega_{n-1} = 0$$

$$\nu \lrcorner \omega = 0 \quad \Leftrightarrow \quad \omega_n = 0$$

whereas

$$\nu \wedge \omega = (\nu \lrcorner \omega)\lambda \quad \Leftrightarrow \quad \omega_i + \lambda_{in}\omega_n = 0, \text{ for every } i = 1, \cdots, n-1.$$

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