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A GLOBAL VERSION OF THE DARBOUX THEOREM WITH OPTIMAL REGULARITY AND DIRICHLET CONDITION

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Abstract. Let $n > 2$ be even; $r \geq 1$ be an integer; $0 < \alpha < 1$; Ω be a bounded, connected, smooth, open set in $\mathbb{R}^n$; and ν be its exterior unit normal. Let $f, g \in C^{r,\alpha}(\overline{\Omega}; \Lambda^2)$ be two symplectic forms (i.e., closed and of rank n) such that $f - g$ is orthogonal to the harmonic fields with vanishing tangential part, $\nu \wedge f, \nu \wedge g \in C^{r+1,\alpha}(\partial\Omega; \Lambda^3)$ and $\nu \wedge f = \nu \wedge g$ on $\partial\Omega$. Moreover assume that $tg + (1-t)f$ has rank n for every $t \in [0,1]$. We will then prove the existence of a $\varphi \in \operatorname{Diff}^{r+1,\alpha}(\overline{\Omega}; \overline{\Omega})$ satisfying

$$\begin{cases} \varphi^*(g) = f & \text{in } \Omega \\ \varphi = \text{id} & \text{on } \partial\Omega. \end{cases}$$

1. Introduction

Let n be an even integer; Ω be a bounded, connected, smooth, open set in $\mathbb{R}^n$; and f and g be two closed 2-forms defined over $\overline{\Omega}$. We wish to find a diffeomorphism $\varphi : \overline{\Omega} \to \overline{\Omega}$ which pulls back g to f and preserves pointwise fixed the boundary of Ω, namely

$$\begin{cases} \varphi^*(g) = f & \text{in } \Omega \\ \varphi = \text{id} & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

To see more analytically the meaning of (1.1), we first write g and f as

$$g = \sum_{1 \leq i < j \leq n} g_{ij}(x)\, dx^i \wedge dx^j \quad \text{and} \quad f = \sum_{1 \leq i < j \leq n} f_{ij}(x)\, dx^i \wedge dx^j.$$

The fact that g and f are closed just means that

$$dg = 0 \;\Leftrightarrow\; \frac{\partial g_{ij}}{\partial x_k} - \frac{\partial g_{ik}}{\partial x_j} + \frac{\partial g_{jk}}{\partial x_i} = 0, \quad 1 \leq i < j < k \leq n$$

$$df = 0 \;\Leftrightarrow\; \frac{\partial f_{ij}}{\partial x_k} - \frac{\partial f_{ik}}{\partial x_j} + \frac{\partial f_{jk}}{\partial x_i} = 0, \quad 1 \leq i < j < k \leq n.$$

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The equation (1.1) reads as

$$\sum_{1\le p<q\le n} g_{pq}\left(\varphi\left(x\right)\right) d\varphi^{p}\wedge d\varphi^{q}=\sum_{1\le i<j\le n} f_{ij}\left(x\right) dx^{i}\wedge dx^{j}\quad \text{in } \Omega,$$

where

$$d\varphi^{p}=\sum_{i=1}^{n}\frac{\partial\varphi^{p}}{\partial x_{i}}dx^{i}.$$

Substituting into the equation, we find that (1.1) is equivalent, for every $1\le i<j\le n$, to

$$\sum_{1\le p<q\le n} g_{pq}\left(\varphi\left(x\right)\right)\left(\frac{\partial\varphi^{p}(x)}{\partial x_{i}}\frac{\partial\varphi^{q}(x)}{\partial x_{j}}-\frac{\partial\varphi^{p}(x)}{\partial x_{j}}\frac{\partial\varphi^{q}(x)}{\partial x_{i}}\right)=f_{ij}\left(x\right),\quad x\in\Omega$$

coupled with the boundary condition $\varphi\left(x\right)=x$ on $\partial\Omega$. Problem (1.1) is therefore a *non-linear* homogeneous-of-degree-2 (in the derivatives) system of $\binom{n}{2}=\frac{n(n-1)}{2}$ first-order partial differential equations subject to a Dirichlet condition on the boundary.

The local version of (1.1) is the celebrated *Darboux theorem* which reads as follows. Let $n=2m$ be an even integer, $x_0\in\mathbb{R}^n$, and ω be the *standard symplectic form,* namely

$$\omega=\sum_{i=1}^{m}dx^{2i-1}\wedge dx^{2i}.$$

Note that since ω has constant components it is necessarily *closed.* Moreover, it has *maximal rank* (the notion of rank will be precisely defined below), namely

$$\operatorname{rank}\left[\omega\right]=n.$$

It is a striking fact that these are the only two invariants of the local problem. Indeed, let f be a closed 2-form defined in a neighborhood of x_0 and with

$$\operatorname{rank}\left[f\left(x_0\right)\right]=n.$$

Then the Darboux theorem asserts that there exist a neighborhood $V\subset\mathbb{R}^n$ of x_0 and a diffeomorphism $\varphi:V\to\varphi\left(V\right)\subset\mathbb{R}^n$ such that $\varphi^{*}\left(\omega\right)=f$ in V and $\varphi\left(x_0\right)=x_0$. Since in the present article we will be interested in deriving optimal regularity, let us first discuss this matter in the local case. The optimal regularity in the local context has been obtained by Bandyopadhyay-Dacorogna in [2] and states that if $r\ge 0$ is an integer, $0<\alpha<1$ and, in addition to the previous hypotheses, $f\in C^{r,\alpha}$ in a neighborhood of x_0, then the diffeomorphism φ is in $C^{r+1,\alpha}$. All the other proofs that can be found in classical books show, at best, that if $f \in C^r$, with $r \geq 1$, then the diffeomorphism φ is only in C^r.

Our main result is Theorem 11, where we prove that Problem (1.1) has, under natural conditions, a solution with optimal regularity. This is a very delicate point and the only existing result in this direction is the one of Bandyopadhyay-Dacorogna [2], where the same result is obtained but under more stringent hypotheses. We should emphasize that our result as well as [2] are not only interesting from the point of view of regularity, but are also the only available global results on a manifold with boundary.

2. Preliminaries

We gather here all the notions that will be needed in the article.

2.1. Notation.

We start by giving some notation.

(i) The set Ω will always denote a bounded, connected, smooth, open set in $\mathbb{R}^n$ and $\nu \in C^\infty(\partial\Omega;\mathbb{R}^n)$ will always be its exterior unit normal.

(ii) Let $r \geq 0$ be an integer and $0 \leq \alpha \leq 1$. The space $C^{r,\alpha}(\overline{\Omega})$ with its norm $\|\cdot\|_{C^{r,\alpha}(\overline{\Omega})}$ and, respectively, $C^{r,\alpha}(\partial\Omega)$ with its norm $\|\cdot\|_{C^{r,\alpha}(\partial\Omega)}$, will denote the space of functions $C^{r,\alpha}$ in $\overline{\Omega}$ and, respectively, on $\partial\Omega$. In order to simplify the notation we will write $\|\cdot\|_{C^{r,\alpha}}$ instead of $\|\cdot\|_{C^{r,\alpha}(\overline{\Omega})}$, in all the proofs (and only there unless mentioned explicitly).

(iii) Let $r \geq 1$ be an integer and $0 \leq \alpha \leq 1$. The set $\mathrm{Diff}^{r,\alpha}(\overline{\Omega};\overline{\Omega})$ is the set of maps φ such that $\varphi, \varphi^{-1} \in C^{r,\alpha}(\overline{\Omega};\overline{\Omega})$.

(iv) Let $0 \leq k \leq n$. A k-form $g : \overline{\Omega} \to \Lambda^k(\mathbb{R}^n)$ will be written as

$$g = \sum_{1 \leq i_1 < \cdots < i_k \leq n} g_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

where $g_{i_1 \cdots i_k} : \overline{\Omega} \to \mathbb{R}$, for every $1 \leq i_1 < \cdots < i_k \leq n$. When $g_{i_1 \cdots i_k} \in C^{r,\alpha}(\overline{\Omega})$ for every $1 \leq i_1 < \cdots < i_k \leq n$, we will write $g \in C^{r,\alpha}(\overline{\Omega};\Lambda^k)$ and similarly for $C^{r,\alpha}(\partial\Omega;\Lambda^k)$. The norm is defined componentwise; for instance,

$$\|g\|^2_{C^{r,\alpha}(\overline{\Omega})} = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \|g_{i_1 \cdots i_k}\|^2_{C^{r,\alpha}(\overline{\Omega})}.$$

Throughout the article we will often identify 1-forms with vector fields and vice versa. In particular, we will see ν as a 1-form over $\partial\Omega$, namely $\nu \in C^\infty(\partial\Omega;\Lambda^1)$.

(v) Let $1 \leq k \leq n-1$, $g, f \in C^0(\overline{\Omega}; \Lambda^k)$, and $u \in C^0(\overline{\Omega}; \Lambda^1)$. The *exterior product of* u *with* g, denoted $u \wedge g$, belongs to $C^0(\overline{\Omega}; \Lambda^{k+1})$ and is defined by

$$u \wedge g = \sum_{1 \leq j_1 < \cdots < j_{k+1} \leq n} \Big(\sum_{\gamma=1}^{k+1} (-1)^{\gamma+1} u_{j_\gamma} g_{j_1 \cdots j_{\gamma-1} j_{\gamma+1} \cdots j_{k+1}} \Big) dx^{j_1} \wedge \cdots \wedge dx^{j_{k+1}}.$$

The *interior product of* u *with* g, denoted $u \lrcorner\, g$, belongs to $C^0(\overline{\Omega}; \Lambda^{k-1})$ and is defined by

$$u \lrcorner\, g = \sum_{1 \leq j_1 < \cdots < j_{k-1} \leq n} \Big(\sum_{\gamma=1}^{k-1} (-1)^{\gamma+1} \sum_{j_{\gamma-1} < i < j_\gamma} g_{j_1 \cdots j_{\gamma-1} i j_\gamma \cdots j_{k-1}}\, u_i \Big)\, dx^{j_1} \wedge \cdots \wedge dx^{j_{k-1}}.$$

The *scalar product of* g *and* f, denoted $\langle g; f \rangle$, belongs to $C^0(\overline{\Omega})$ and is defined by

$$\langle g; f \rangle = \sum_{1 \leq i_1 < \cdots < i_k \leq n} g_{i_1 \cdots i_k} f_{i_1 \cdots i_k}\,.$$

In particular, if $k = 2$, it follows that

$$u \wedge g = \sum_{1 \leq i < j < k \leq n} (u_i g_{jk} - u_j g_{ik} + u_k g_{ij})\, dx^i \wedge dx^j \wedge dx^k,$$

$$u \lrcorner\, g = \sum_{j=1}^{n} \Big(\sum_{i=1}^{n} g_{ij} u_i \Big) dx^j, \qquad \langle g; f \rangle = \sum_{1 \leq i < j \leq n} g_{ij} f_{ij},$$

where we have used the convention $g_{ij} = -g_{ji}$.

(vi) Let $1 \leq k \leq n-1$ and $g \in C^1(\overline{\Omega}; \Lambda^k)$. The *exterior derivative of* g, denoted dg, belongs to $C^0(\overline{\Omega}; \Lambda^{k+1})$ and is defined by

$$dg = \sum_{1 \leq j_1 < \cdots < j_{k+1} \leq n} \Big(\sum_{\gamma=1}^{k+1} (-1)^{\gamma+1} \frac{\partial g_{j_1 \cdots j_{\gamma-1} j_{\gamma+1} \cdots j_{k+1}}}{\partial x_{j_\gamma}} \Big) dx^{j_1} \wedge \cdots \wedge dx^{j_{k+1}}.$$

The k-form g is said to be *closed* if $dg = 0$ in Ω. The *interior derivative of* g, denoted δg, belongs to $C^0(\overline{\Omega}; \Lambda^{k-1})$ and is defined by

$$\delta g = \sum_{1 \leq j_1 < \cdots < j_{k-1} \leq n} \Big(\sum_{\gamma=1}^{k} (-1)^{\gamma+1} \sum_{j_{\gamma-1} < i < j_\gamma} \frac{\partial f_{j_1 \cdots j_{\gamma-1} i j_\gamma \cdots j_{k-1}}}{\partial x_i} \Big)\, dx^{j_1} \wedge \cdots \wedge dx^{j_{k-1}}.$$

In particular, when $k = 2$,

$$dg = \sum_{1 \leq i < j < k \leq n} \Big(\frac{\partial g_{ij}}{\partial x_k} - \frac{\partial g_{ik}}{\partial x_j} + \frac{\partial g_{jk}}{\partial x_i} \Big) dx^i \wedge dx^j \wedge dx^k,$$

$$\delta g = \sum_{j=1}^{n} \Big(\sum_{i=1}^{n} \frac{\partial g_{ij}}{\partial x_i} \Big) dx^j,$$

where we have used the convention $g_{ij} = -g_{ji}$.

(vii) Let $g \in C^0(\overline{\Omega}; \Lambda^2)$. The map $\overline{g} \in C^0(\overline{\Omega}; \mathbb{R}^{n\times n})$ is defined by (where the index below denotes the column and the index above the row) $(\overline{g})_i^j = g_{ij}$, where we have used again the convention $g_{ij} = -g_{ji}$. Hence for every $x \in \overline{\Omega}$ the matrix $\overline{g}(x) \in \mathbb{R}^{n\times n}$ is skew-symmetric, and thus its rank is even. Note that, identifying 1-forms with vector fields,

$$u \lrcorner\, g = \overline{g} u.$$

(viii) Let $g \in C^0(\overline{\Omega}; \Lambda^2)$. The rank of g at $x \in \overline{\Omega}$ is defined by

$$\operatorname{rank}[g(x)] = \operatorname{rank}[\overline{g}(x)].$$

Note that the rank of g is always even. In particular, if $\operatorname{rank}[g] = n$ in $\overline{\Omega}$ (which is equivalent to $\overline{g}(x)$ invertible for every $x \in \overline{\Omega}$) we have, identifying 1-forms with vectors fields,

$$v = u \lrcorner\, g \quad \Leftrightarrow \quad u = (\overline{g})^{-1} v.$$

(ix) Let $\varphi \in C^1(\overline{\Omega}; \overline{\Omega})$ and $g \in C^0(\overline{\Omega}; \Lambda^2)$. The pullback of g by φ, denoted $\varphi^*(g)$, belongs to $C^0(\overline{\Omega}; \Lambda^2)$ and is defined by

$$\varphi^*(g) = \sum_{1\le i<j\le n} g_{ij}(\varphi) d\varphi^i \wedge d\varphi^j,$$

where

$$d\varphi^i = \sum_{\gamma=1}^{n} \frac{\partial \varphi^i}{\partial x_\gamma} dx^\gamma.$$

Explicitly,

$$(\varphi^*(g)) = \sum_{1\le i<j\le n} \Big(\sum_{1\le p<q\le n} g_{pq}(\varphi) \Big(\frac{\partial \varphi^p}{\partial x_i} \frac{\partial \varphi^q}{\partial x_j} - \frac{\partial \varphi^p}{\partial x_j} \frac{\partial \varphi^q}{\partial x_i} \Big) \Big) dx^i \wedge dx^j.$$

(x) Finally, we give the definition of harmonic fields with vanishing tangential or normal part. For properties of such spaces see Section 2.2.

$$\mathcal{H}_T(\Omega; \Lambda^2) = \{\omega \in C^\infty(\overline{\Omega}; \Lambda^2) : d\omega = 0,\ \delta\omega = 0 \text{ in } \Omega,\ \text{and } \nu \wedge \omega = 0 \text{ on } \partial\Omega\}$$

$$\mathcal{H}_N(\Omega; \Lambda^2) = \{\omega \in C^\infty(\overline{\Omega}; \Lambda^2) : d\omega = 0,\ \delta\omega = 0 \text{ in } \Omega,\ \text{and } \nu \lrcorner\, \omega = 0 \text{ on } \partial\Omega\}.$$

2.2. **Hodge decomposition and Poincaré lemma.** We start with the definition of a contractible set.

Definition 1. *An open set* $\Omega \subset \mathbb{R}^n$ *is* contractible *if there exist* $x_0 \in \Omega$ *and* $F \in C^\infty([0,1] \times \Omega; \Omega)$ *such that, for every* $x \in \Omega$,

$$F(0,x) = x_0 \quad \textit{and} \quad F(1,x) = x.$$

We now give some properties of the spaces with vanishing tangential or normal part defined in the previous section (see [10] for a proof).

Proposition 2. *Let* $1 \leq k \leq n-1$, $r \geq 0$ *be integers;* $0 \leq \alpha \leq 1$; Ω *be a bounded, connected, smooth, open set in* $\mathbb{R}^n$*; and* $\nu \in C^\infty(\partial\Omega; \Lambda^1)$ *be its exterior unit normal.*

(i) If $g \in C^1(\overline{\Omega}; \Lambda^k)$ *is such that* $\nu \wedge g = 0$ *on* $\partial\Omega$*, then* $\nu \wedge dg = 0$ *on* $\partial\Omega$.

(ii) If $g \in C^1(\overline{\Omega}; \Lambda^k)$ *is such that* $\nu \lrcorner g = 0$ *on* $\partial\Omega$*, then* $\nu \lrcorner \delta g = 0$ *on* $\partial\Omega$.

(iii) There exists a constant $C = C(r, \Omega)$ *such that for every* $\omega \in \mathcal{H}_T(\Omega; \Lambda^k) \cup \mathcal{H}_N(\Omega; \Lambda^k)$

$$\|\omega\|_{C^{r,\alpha}(\overline{\Omega})} \leq C \|\omega\|_{C^0(\overline{\Omega})}.$$

Moreover, if Ω *is contractible, then* $\mathcal{H}_T(\Omega; \Lambda^k) = \mathcal{H}_N(\Omega; \Lambda^k) = \{0\}$.

We recall the Gauss-Green theorem for k-forms.

Theorem 3 (Gauss-Green theorem)**.** *Let* $1 \leq k \leq n$; Ω *be a bounded, connected, smooth, open set in* $\mathbb{R}^n$; $f \in C^1(\overline{\Omega}; \Lambda^{k-1})$; *and* $g \in C^1(\overline{\Omega}; \Lambda^k)$. *Then*

$$\int_\Omega \langle df; g\rangle + \int_\Omega \langle f; \delta g\rangle = \int_{\partial\Omega} \langle \nu \wedge f; g\rangle = \int_{\partial\Omega} \langle f; \nu \lrcorner g\rangle.$$

We now state the Hodge decomposition theorem (see [10] for a proof).

Theorem 4. *Let* $r \geq 0$ *be an integer;* $0 < \alpha < 1$; Ω *be a bounded, connected, smooth, open set in* $\mathbb{R}^n$; *and* $g \in C^{r,\alpha}(\overline{\Omega}; \Lambda^2)$. *Then the two following decompositions hold true.*

(i) There exist $a \in C^{r+1,\alpha}\left(\overline{\Omega}; \Lambda^1\right)$, $b \in C^{r+1,\alpha}\left(\overline{\Omega}; \Lambda^3\right)$, *and* $h \in \mathcal{H}_T(\Omega; \Lambda^2)$ *such that*

$$\begin{cases} g = da + \delta b + h & \textit{in } \Omega \\ \nu \wedge a = 0 \quad \textit{and} \quad \nu \wedge b = 0 & \textit{on } \partial\Omega. \end{cases}$$

Moreover, there exists a constant $C = C(r, \alpha, \Omega)$ *such that*

$$\|a\|_{C^{r+1,\alpha}} + \|b\|_{C^{r+1,\alpha}} + \|h\|_{C^{r,\alpha}} \leq C \|g\|_{C^{r,\alpha}}.$$

(ii) There exist $a \in C^{r+1,\alpha}(\overline{\Omega};\Lambda^1)$, $b \in C^{r+1,\alpha}(\overline{\Omega};\Lambda^3)$, *and* $h \in \mathcal{H}_N(\Omega;\Lambda^2)$ *such that*

$$\begin{cases} g = da + \delta b + h & \text{in } \Omega \\ \nu \lrcorner a = 0 \quad \text{and} \quad \nu \lrcorner b = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, there exists a constant $C = C(r,\alpha,\Omega)$ *such that*

$$\|a\|_{C^{r+1,\alpha}} + \|b\|_{C^{r+1,\alpha}} + \|h\|_{C^{r,\alpha}} \le C\|g\|_{C^{r,\alpha}}\,.$$

Remark 5. Using Proposition 2 and Theorem 3 we immediately deduce the two following extra properties of Hodge decomposition.

(i) If, in the first decomposition of Theorem 4, g moreover satisfies $dg = 0$ in Ω and $\nu \wedge g = 0$ on $\partial\Omega$, then $\delta\beta = 0$ in Ω.

(ii) If, in the second decomposition of Theorem 4, g moreover satisfies $dg = 0$ in Ω, then $\delta\beta = 0$ in Ω.

We now state a sharp version of the Poincaré lemma with boundary condition as well as an elementary result for 1-forms on the boundary (see [3] for a proof).

Theorem 6. *Let* $r \ge 1$ *and* $n \ge 3$ *be integers;* $0 < \alpha < 1$; *and* Ω *be a bounded, connected, smooth, open set in* $\mathbb{R}^n$.

(i) Let $g \in C^{r,\alpha}\left(\overline{\Omega};\Lambda^2\right)$ *be such that*

$$dg = 0 \text{ in } \Omega, \quad \nu \wedge g = 0 \text{ on } \partial\Omega$$

$$\int_\Omega \langle g;\chi\rangle = 0, \quad \text{for every } \chi \in \mathcal{H}_T\left(\Omega;\Lambda^2\right).$$

Then there exist $\omega \in C^{r+1,\alpha}(\overline{\Omega};\Lambda^1)$ *and a constant* $C = C(r,\alpha,\Omega)$ *such that*

$$\begin{cases} d\omega = g & \text{in } \Omega \\ \omega = 0 & \text{on } \partial\Omega \end{cases}$$

and

$$\|\omega\|_{C^{r+1,\alpha}(\overline{\Omega})} \le C\|g\|_{C^{r,\alpha}(\overline{\Omega})}.$$

Moreover, the correspondence $g \mapsto \omega$ *can be chosen to be linear.*

(ii) Let $f \in C^{r,\alpha}\left(\partial\Omega;\Lambda^1\right)$ *be such that*

$$\nu \wedge f = 0, \quad \text{on } \partial\Omega.$$

Then there exist $b \in C^{r+1,\alpha}\left(\overline{\Omega}\right)$ *and a constant* $C = C(r,\alpha,\Omega)$ *such that*

$$db = f \quad \text{and} \quad b = 0, \quad \text{on } \partial\Omega$$

and

$$\|b\|_{C^{r+1,\alpha}(\overline{\Omega})} \le C\,\|f\|_{C^{r,\alpha}(\partial\Omega)}\,.$$

Finally, we will use in Theorem 23 the following elementary result (cf. Bandyopadhyay [1]).

Proposition 7. *Let $n \geq 2$ be an integer; Ω be a bounded, connected, smooth, open set in $\mathbb{R}^n$ and $F \in C^2\left([0,1] \times \overline{\Omega}; \overline{\Omega}\right)$, $F = F(t,x) = F_t(x)$ be such that $F_t(x) = x$ for every $t \in [0,1]$; and every $x \in \partial\Omega$. Let $g \in C^1\left(\overline{\Omega}; \Lambda^2\right)$ be such that $dg = 0$ in Ω. Then there exists $G \in C^1\left(\overline{\Omega}; \Lambda^1\right)$ such that*

$$\begin{cases} dG = F_1^*(g) - F_0^*(g) & \text{in } \Omega \\ G = 0 & \text{on } \partial\Omega. \end{cases}$$

2.3. **Hölder spaces.** We gather here some properties of Hölder spaces (see [7] for the first five statements and [4] for the last one). In the following result, in order to simplify the notation, we drop the dependence on $\overline{\Omega}$ for the norms.

Theorem 8. *Let Ω be a bounded, connected, smooth, open set in $\mathbb{R}^n$; $r \geq 0$ an integer; and $0 \leq \alpha \leq 1$. We then have the following properties.*

(i) There exists a constant $C = C(r, \Omega)$ such that, for every $f, g \in C^{r,\alpha}(\overline{\Omega})$

$$\|fg\|_{C^{r,\alpha}} \leq C\left(\|f\|_{C^{r,\alpha}} \|g\|_{C^0} + \|f\|_{C^0} \|g\|_{C^{r,\alpha}}\right).$$

(ii) Let $g \in C^{r,\alpha}(\mathbb{R}^n)$ and $f \in C^{r,\alpha}\left(\overline{\Omega}; \mathbb{R}^n\right) \cap C^1\left(\overline{\Omega}; \mathbb{R}^n\right)$. Then

$$\|g \circ f\|_{C^{0,\alpha}} \leq \|g\|_{C^{0,\alpha}} \|f\|_{C^1}^{\alpha} + \|g\|_{C^0}$$

while if $r \geq 1$, there exists a constant $C = C(r, \Omega) > 0$ such that

$$\|g \circ f\|_{C^{r,\alpha}} \leq C\left[\|g\|_{C^{r,\alpha}} \|f\|_{C^1}^{r+\alpha} + \|g\|_{C^1} \|f\|_{C^{r,\alpha}} + \|g\|_{C^0}\right].$$

(iii) Let $A \in C^{r,\alpha}\left(\overline{\Omega}; \mathbb{R}^{n \times n}\right)$ and $c > 0$ be such that

$$\left\|\frac{1}{\det A}\right\|_{C^0}, \ \|A\|_{C^0} \leq c.$$

Then there exists a constant $C = C(c, r, \Omega) > 0$ such that

$$\left\|A^{-1}\right\|_{C^{r,\alpha}} \leq C \|A\|_{C^{r,\alpha}}.$$

(iv) Let $c > 0$, $g \in C^{r+1,\alpha}(\mathbb{R}^n)$, and $u, v \in C^{r,\alpha}\left(\overline{\Omega}; \mathbb{R}^n\right) \cap C^1\left(\overline{\Omega}; \mathbb{R}^n\right)$, with

$$\|u\|_{C^1}, \ \|v\|_{C^1} \leq c.$$

Then there exists $C = C(c, r, \Omega) > 0$ so that, if $r = 0$,

$$\|g \circ u - g \circ v\|_{C^{0,\alpha}} \leq C \|g\|_{C^{1,\alpha}} \|u - v\|_{C^0} + C \|g\|_{C^1} \|u - v\|_{C^{0,\alpha}}$$

while, when $r \geq 1$,

$$\|g \circ u - g \circ v\|_{C^{r,\alpha}} \leq C \|g\|_{C^{r+1,\alpha}} \|u - v\|_{C^0} + C \|g\|_{C^2} \left[\|u\|_{C^{r,\alpha}}\right.$$

$$+ \|v\|_{C^{r,\alpha}}] \|u-v\|_{C^0} + C\|g\|_{C^1} \|u-v\|_{C^{r,\alpha}} \ .$$

(v) Let $s \geq r \geq k \geq 1$ be integers and $0 \leq \alpha, \beta \leq 1$. Then there exists a constant $C = C(s, \Omega) > 0$ such that, for every $f, g \in C^{s+k,\beta}(\overline{\Omega})$

$$\|f\|_{C^{r,\alpha}} \|g\|_{C^{s,\beta}} \leq C\left[\|f\|_{C^{r-k,\alpha}} \|g\|_{C^{s+k,\beta}} + \|g\|_{C^{r-k,\alpha}} \|f\|_{C^{s+k,\beta}}\right].$$

(vi) Let $r \geq 1$ be an integer, $R > 0$, and $C_R = \{f \in C^{r,\alpha}(\overline{\Omega}) : \|f\|_{C^{r,\alpha}} \leq R\}$. Let $\{f_\nu\} \subset C_R$ be a sequence such that $f_\nu \to f$ in $C^0(\overline{\Omega})$ as $\nu \to \infty$. Then $f \in C_R$ and

$$\|f\|_{C^{r,\alpha}} \leq \liminf_{\nu \to \infty} \|f_\nu\|_{C^{r,\alpha}} \ .$$

Finally, we have the following result, which will allow us to regularize functions in an appropriate way (see [7] for a proof).

Theorem 9. *Let Ω be a bounded, connected, smooth, open set in $\mathbb{R}^n$. Let $s \geq r \geq t \geq 0$ be integers and $0 < \alpha, \beta, \gamma < 1$ be such that $t+\gamma \leq r+\alpha \leq s+\beta$. Let $f \in C^{r,\alpha}(\overline{\Omega})$. Then there exist a constant $C = C(s, \Omega) > 0$ and, for every $\epsilon \in (0,1]$, $f_\epsilon \in C^\infty(\overline{\Omega})$ such that*

$$\|f_\epsilon\|_{C^{s,\beta}} \leq \frac{C}{\epsilon^{(s+\beta)-(r+\alpha)}} \|f\|_{C^{r,\alpha}}$$

$$\|f - f_\epsilon\|_{C^{t,\gamma}} \leq C\epsilon^{(r+\alpha)-(t+\gamma)} \|f\|_{C^{r,\alpha}}$$

$$\left\|\frac{d}{d\epsilon} f_\epsilon\right\|_{C^{s,\beta}} \leq \frac{C}{\epsilon^{(s+\beta)-(r+\alpha)+1}} \|f\|_{C^{r,\alpha}}$$

$$\left\|\frac{d}{d\epsilon} f_\epsilon\right\|_{C^{t,\gamma}} \leq C\epsilon^{(r+\alpha)-(t+\gamma)-1} \|f\|_{C^{r,\alpha}} \ .$$

Moreover, defining $F : (0,1] \times \overline{\Omega} \to \mathbb{R}$ by $F(\epsilon, x) = f_\epsilon(x)$, then

$$F \in C^\infty((0,1] \times \overline{\Omega}; \Lambda^k).$$

Remark 10. The construction of f_ϵ is universal, in the sense that the four inequalities remain true replacing $(r, s, t, \alpha, \beta, \gamma)$ by $(r', s', t', \alpha', \beta', \gamma')$ as far as $f \in C^{r',\alpha'}(\overline{\Omega})$ and with a constant $C' = C'(s', \Omega) > 0$.

3. The main theorem

We now state our main theorem. It has been obtained under more restrictive hypotheses by Bandyopadhyay-Dacorogna [2].

Theorem 11. *Let $n > 2$ be even; Ω be a bounded, connected, smooth, open set in $\mathbb{R}^n$; and $\nu \in C^\infty(\partial\Omega; \Lambda^1)$ be its exterior unit normal. Let $r \geq 1$ be an integer and $0 < \alpha < 1$. Let $f, g \in C^{r,\alpha}(\overline{\Omega}; \Lambda^2)$ satisfying $df = dg = 0$ in Ω,*

$$\nu \wedge f, \nu \wedge g \in C^{r+1,\alpha}\left(\partial\Omega; \Lambda^3\right) \quad \text{and} \quad \nu \wedge f = \nu \wedge g \text{ on } \partial\Omega$$

$$\int_\Omega \langle f; \psi\rangle\, dx = \int_\Omega \langle g; \psi\rangle\, dx, \quad \text{for every } \psi \in \mathcal{H}_T(\Omega; \Lambda^2) \tag{3.1}$$

and, for every $t \in [0,1]$,

$$\operatorname{rank}\left[tg + (1-t)\,f\right] = n, \ \text{ in } \overline{\Omega}.$$

Then, there exists $\varphi \in \operatorname{Diff}^{r+1,\alpha}\left(\overline{\Omega}; \overline{\Omega}\right)$ such that

$$\begin{cases} \varphi^*(g) = f & \text{in } \Omega \\ \varphi = \operatorname{id} & \text{on } \partial\Omega. \end{cases}$$

Remark 12. (i) With only minor changes, we can consider, in a similar way, a general homotopy f_t with $f_0 = f$, $f_1 = g$,

$$df_t = 0, \quad \nu \wedge f_t = \nu \wedge f_0 \text{ on } \partial\Omega \quad \text{and} \quad \operatorname{rank}\left[f_t\right] = n \text{ in } \overline{\Omega}$$

$$\int_\Omega \langle f_t; \psi\rangle\, dx = \int_\Omega \langle f_0; \psi\rangle\, dx, \quad \text{for every } \psi \in \mathcal{H}_T(\Omega; \Lambda^2).$$

(ii) If Ω is contractible, then (cf. Proposition 2) $\mathcal{H}_T(\Omega; \Lambda^2) = \{0\}$ and therefore (3.1) is automatically satisfied.

(iii) Note that the extra regularity on f and g holds only on the boundary and only for their tangential parts.

(iv) It is worthwhile to discuss some of the necessary conditions. It can be easily proved that if $\varphi^*(g) = f$, then, for every $x \in \Omega$,

$$\operatorname{rank}\left[f(x)\right] = \operatorname{rank}\left[g(\varphi(x))\right] \quad \text{and} \quad \operatorname{rank}\left[df(x)\right] = \operatorname{rank}\left[dg(\varphi(x))\right].$$

Therefore, if one of the forms is closed, then, necessarily, the other one must be closed. Moreover, since $\varphi = \operatorname{id}$ on $\partial\Omega$, then we necessarily have

$$\nu \wedge f = \nu \wedge g \text{ on } \partial\Omega.$$

4. Proof of the main theorem

Since the proof is rather long we divide it into four subsections. In the first one we construct an appropriate smoothing for 2-forms. The second and the third one are dedicated to proving the theorem with some restrictions on f and g using the flow method (second subsection) and the fixed-point method (third subsection). Finally, in the last one we conclude the proof of the theorem.

4.1. **Smoothing of 2-forms.** The aim of this subsection is to prove the following result.

Theorem 13. *Let* Ω *be a bounded, connected, smooth, open set in* $\mathbb{R}^n$ *and* $\nu \in C^\infty(\partial\Omega;\Lambda^1)$ *be its exterior unit normal. Let* $s \geq r \geq t \geq 0$ *and* $n \geq 3$ *be integers. Let* $0 < \alpha, \beta, \gamma < 1$ *be such that* $t + \gamma \leq r + \alpha \leq s + \beta$. *Let* $g \in C^{r,\alpha}(\overline{\Omega};\Lambda^2)$ *be such that*

$$dg = 0 \text{ in } \Omega \quad \text{and} \quad \nu \wedge g \in C^{s,\beta}\left(\partial\Omega;\Lambda^{k+1}\right).$$

Then for every $\epsilon \in (0,1]$, *there exist* $g_\epsilon \in C^{s,\beta}\left(\overline{\Omega};\Lambda^2\right)$ *and a constant* $C = C(s,\alpha,\beta,\gamma,\Omega) > 0$ *such that*

$$dg_\epsilon = 0 \text{ in } \Omega, \quad \nu \wedge g_\epsilon = \nu \wedge g \text{ on } \partial\Omega$$

$$\int_\Omega \langle g_\epsilon;\psi\rangle = \int_\Omega \langle g;\psi\rangle\,, \quad \text{for every } \psi \in \mathcal{H}_T(\Omega;\Lambda^2)$$

$$\|g_\epsilon - g\|_{C^{t,\gamma}(\overline{\Omega})} \leq C\epsilon^{(r+\alpha)-(t+\gamma)}\|g\|_{C^{r,\alpha}(\overline{\Omega})}$$

$$\|g_\epsilon\|_{C^{s,\beta}(\overline{\Omega})} \leq \frac{C}{\epsilon^{(s+\beta)-(r+\alpha)}}\|g\|_{C^{r,\alpha}(\overline{\Omega})} + C\|\nu \wedge g\|_{C^{s,\beta}(\partial\Omega)}$$

$$\left\|\frac{d}{d\epsilon}g_\epsilon\right\|_{C^{s,\beta}(\overline{\Omega})} \leq \frac{C}{\epsilon^{(s+\beta)-(r+\alpha)+1}}\|g\|_{C^{r,\alpha}(\overline{\Omega})}$$

$$\left\|\frac{d}{d\epsilon}g_\epsilon\right\|_{C^{t,\gamma}(\overline{\Omega})} \leq C\epsilon^{(r+\alpha)-(t+\gamma)-1}\|g\|_{C^{r,\alpha}(\overline{\Omega})}\,.$$

Moreover, defining $\Gamma : (0,1] \times \overline{\Omega} \to \Lambda^2$ *by* $\Gamma(\epsilon,x) = g_\epsilon(x)$, *then*

$$\Gamma \in C^{s,\beta}((0,1]\times\overline{\Omega};\Lambda^2) \quad \text{and} \quad \frac{\partial\Gamma}{\partial\epsilon} \in C^\infty((0,1]\times\overline{\Omega};\Lambda^2).$$

Remark 14. (i) We recall that if Ω is contractible, then (cf. Proposition 2)

$$\mathcal{H}_T(\Omega;\Lambda^2) = \{0\}.$$

(ii) The construction is universal in the sense that the four inequalities remain true replacing $(r,s,t,\alpha,\beta,\gamma)$ by $(r',s',t',\alpha',\beta',\gamma')$ as far as $g \in C^{r',\alpha'}\left(\overline{\Omega};\Lambda^2\right)$ and with a constant $C' = C'(s',\alpha',\beta',\gamma',\Omega) > 0$.

Before starting the proof of the theorem, we need the equivalent of the theorem for functions.

Lemma 15. *Let Ω be a bounded, connected, smooth, open set in $\mathbb{R}^n$. Let $s \geq r \geq t \geq 0$ be integers. Let $0 < \alpha, \beta, \gamma < 1$ be such that $t + \gamma \leq r + \alpha \leq s + \beta$. Let $f \in C^{r,\alpha}\left(\overline{\Omega}\right) \cap C^{s,\beta}\left(\partial\Omega\right)$. Then for every $\epsilon \in (0,1]$, there exist $f_\epsilon \in C^{s,\beta}\left(\overline{\Omega}\right)$ and a constant $C = C\left(s, \alpha, \beta, \gamma, \Omega\right) > 0$ such that*

$$f_\epsilon = f \text{ on } \partial\Omega$$

$$\|f_\epsilon - f\|_{C^{t,\gamma}\left(\overline{\Omega}\right)} \leq C\epsilon^{(r+\alpha)-(t+\gamma)} \|f\|_{C^{r,\alpha}\left(\overline{\Omega}\right)}$$

$$\|f_\epsilon\|_{C^{s,\beta}\left(\overline{\Omega}\right)} \leq \frac{C}{\epsilon^{(s+\beta)-(r+\alpha)}} \|f\|_{C^{r,\alpha}\left(\overline{\Omega}\right)} + C\|f\|_{C^{s,\beta}(\partial\Omega)}$$

$$\left\|\frac{d}{d\epsilon} f_\epsilon\right\|_{C^{s,\beta}\left(\overline{\Omega}\right)} \leq \frac{C}{\epsilon^{(s+\beta)-(r+\alpha)+1}} \|f\|_{C^{r,\alpha}\left(\overline{\Omega}\right)}$$

$$\left\|\frac{d}{d\epsilon} f_\epsilon\right\|_{C^{t,\gamma}\left(\overline{\Omega}\right)} \leq C\epsilon^{(r+\alpha)-(t+\gamma)-1} \|f\|_{C^{r,\alpha}\left(\overline{\Omega}\right)}.$$

Moreover, defining $F : (0,1] \times \overline{\Omega} \to \mathbb{R}$ by $F(\epsilon, x) = f_\epsilon(x)$, then

$$F \in C^{s,\beta}((0,1] \times \overline{\Omega}) \quad \text{and} \quad \frac{\partial F}{\partial \epsilon} \in C^\infty((0,1] \times \overline{\Omega}).$$

Proof. We first find (using Theorem 9) $w_\epsilon \in C^\infty\left(\overline{\Omega}\right)$ and a constant $C_1 = C_1(s, \Omega)$ such that

$$\|w_\epsilon - f\|_{C^{t,\gamma}} \leq C_1\epsilon^{(r+\alpha)-(t+\gamma)} \|f\|_{C^{r,\alpha}}$$

$$\|w_\epsilon\|_{C^{s,\beta}} \leq \frac{C_1}{\epsilon^{(s+\beta)-(r+\alpha)}} \|f\|_{C^{r,\alpha}}$$

$$\left\|\frac{d}{d\epsilon} w_\epsilon\right\|_{C^{s,\beta}} \leq \frac{C_1}{\epsilon^{(s+\beta)-(r+\alpha)+1}} \|f\|_{C^{r,\alpha}}$$

$$\left\|\frac{d}{d\epsilon} w_\epsilon\right\|_{C^{t,\gamma}} \leq C_1\epsilon^{(r+\alpha)-(t+\gamma)-1} \|f\|_{C^{r,\alpha}}.$$

Moreover, defining $W : (0,1] \times \overline{\Omega} \to \mathbb{R}$ by $W(\epsilon, x) = w_\epsilon(x)$, we have $W \in C^\infty((0,1] \times \overline{\Omega})$. We then fix the boundary data as follows. Let $f_\epsilon \in C^\infty(\Omega) \cap C^{s,\beta}\left(\overline{\Omega}\right)$ be the solution of

$$\begin{cases} \Delta f_\epsilon = \Delta w_\epsilon & \text{in } \Omega \\ f_\epsilon = f & \text{on } \partial\Omega \end{cases} \Leftrightarrow \begin{cases} \Delta\left[f_\epsilon - w_\epsilon\right] = 0 & \text{in } \Omega \\ f_\epsilon - w_\epsilon = f - w_\epsilon & \text{on } \partial\Omega. \end{cases}$$

Using Schauder estimates (the classical estimates assume $s \geq 2$; however, they are also valid when $s = 0$ or $s = 1$; see Gilbarg-Hörmander [5], Lieberman [8]–[9], and Widman [12]), there exists $C_2 = C_2(s, \beta, \gamma, \Omega)$ such that

$$\|f_\epsilon\|_{C^{s,\beta}} \leq C_2\left[\, \|w_\epsilon\|_{C^{s,\beta}} + \|f\|_{C^{s,\beta}(\partial\Omega)}\, \right]$$

$$\|f_\epsilon - w_\epsilon\|_{C^{t,\gamma}} \leq C_2 \|f - w_\epsilon\|_{C^{t,\gamma}(\partial\Omega)} \leq C_2 \|f - w_\epsilon\|_{C^{t,\gamma}} .$$

Moreover, defining $F : (0,1] \times \overline{\Omega} \to \mathbb{R}$ by $F(\epsilon, x) = f_\epsilon(x)$, we have $F \in C^{s,\beta}((0,1] \times \overline{\Omega})$. Finally, noticing that $\frac{d}{d\epsilon} f_\epsilon$ satisfies

$$\begin{cases} \Delta \frac{d}{d\epsilon} f_\epsilon = \Delta \frac{d}{d\epsilon} w_\epsilon & \text{in } \Omega \\ \frac{d}{d\epsilon} f_\epsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

there exists $C_3 = C_3(s, \beta, \gamma, \Omega)$ such that

$$\left\| \frac{d}{d\epsilon} f_\epsilon \right\|_{C^{s,\beta}} \leq C_3 \left\| \frac{d}{d\epsilon} w_\epsilon \right\|_{C^{s,\beta}}, \quad \left\| \frac{d}{d\epsilon} f_\epsilon \right\|_{C^{t,\gamma}} \leq C_3 \left\| \frac{d}{d\epsilon} w_\epsilon \right\|_{C^{t,\gamma}}$$

and $\frac{\partial F}{\partial \epsilon} \in C^\infty((0,1] \times \overline{\Omega})$. The combination of the properties of f_ϵ and w_ϵ gives the result. □

We can now go back to the proof of Theorem 13.

Proof. We split the proof into two steps.

Step 1. We first show that we can find $G \in C^{r+1,\alpha}\left(\overline{\Omega}; \Lambda^1\right) \cap C^{s+1,\beta}\left(\partial\Omega; \Lambda^1\right)$, $h \in C^\infty\left(\overline{\Omega}; \Lambda^2\right)$, and a constant $C = C(s, \alpha, \beta, \Omega)$ such that

$$g = dG + h$$

$$\|h\|_{C^{r,\alpha}} + \|G\|_{C^{r+1,\alpha}} \leq C \|g\|_{C^{r,\alpha}}$$

$$\|h\|_{C^{s,\beta}} + \|G\|_{C^{s+1,\beta}(\partial\Omega)} \leq C \left[\|g\|_{C^{0,\alpha}} + \|\nu \wedge g\|_{C^{s,\beta}(\partial\Omega)} \right].$$

We proceed as follows. In the sequel $C_1, C_2, \ldots$ will denote generic constants depending only on s, α, β, and Ω.

(i) We first find (solving the Dirichlet problem component by component) $g^{(1)} \in C^{s,\beta}\left(\overline{\Omega}; \Lambda^2\right)$ satisfying

$$\begin{cases} \Delta g^{(1)} = 0 & \text{in } \Omega \\ g^{(1)} = \nu \lrcorner (\nu \wedge g) & \text{on } \partial\Omega. \end{cases}$$

We also have

$$\left\| g^{(1)} \right\|_{C^{0,\alpha}} \leq C_1 \|g\|_{C^{0,\alpha}}, \quad \left\| g^{(1)} \right\|_{C^{r,\alpha}} \leq C_1 \|g\|_{C^{r,\alpha}}$$

$$\left\| g^{(1)} \right\|_{C^{s,\beta}} \leq C_1 \|\nu \wedge g\|_{C^{s,\beta}(\partial\Omega)} .$$

Since $|\nu| = 1$, we have

$$\nu \wedge g^{(1)} = \nu \wedge g.$$

Observe also that, since $dg = 0$, then, using Proposition 2,

$$\nu \wedge dg^{(1)} = 0$$

and

$$\int_\Omega \left\langle dg^{(1)}; \psi \right\rangle = 0, \text{ for every } \psi \in \mathcal{H}_T\left(\Omega; \Lambda^{k+1}\right).$$

This last identity follows from the fact that, using Theorem 3,

$$\int_\Omega \left\langle dg^{(1)}; \psi \right\rangle = \int_{\partial\Omega} \left\langle \nu \wedge g^{(1)}; \psi \right\rangle = \int_{\partial\Omega} \langle \nu \wedge g; \psi \rangle = \int_\Omega \langle dg; \psi \rangle = 0.$$

(ii) We next solve

$$\begin{cases} dg^{(2)} = -dg^{(1)} & \text{in } \Omega \\ g^{(2)} = 0 & \text{on } \partial\Omega \end{cases}$$

and we have $g^{(2)} \in C^{s,\beta}\left(\overline{\Omega}; \Lambda^k\right)$. This is possible (according to the first statement of Theorem 6) since

$$\nu \wedge dg^{(1)} = 0 \quad \text{and} \quad \int_\Omega \left\langle dg^{(1)}; \psi \right\rangle = 0, \text{ for every } \psi \in \mathcal{H}_T\left(\Omega; \Lambda^{k+1}\right).$$

We also have

$$\left\| g^{(2)} \right\|_{C^{0,\alpha}} \le C_2 \|g\|_{C^{0,\alpha}}, \quad \left\| g^{(2)} \right\|_{C^{r,\alpha}} \le C_2 \|g\|_{C^{r,\alpha}}$$

$$\left\| g^{(2)} \right\|_{C^{s,\beta}} \le C_2 \|\nu \wedge g\|_{C^{s,\beta}(\partial\Omega)}.$$

(iii) We then set $g^{(3)} = g^{(2)} + g^{(1)}$ and observe that $g^{(3)} \in C^{s,\beta}\left(\overline{\Omega}; \Lambda^k\right)$,

$$dg^{(3)} = 0 \quad \text{and} \quad \nu \wedge g^{(3)} = \nu \wedge g^{(1)} = \nu \wedge g.$$

Apply the Hodge decomposition to $g^{(3)}$ and find (in view of Theorem 4 (ii) and Remark 5 (ii))

$$G^{(3)} \in C^{s+1,\beta}\left(\overline{\Omega}; \Lambda^1\right) \quad \text{and} \quad h^{(3)} \in \mathcal{H}_N\left(\Omega; \Lambda^2\right) \subset C^\infty(\overline{\Omega}; \Lambda^2)$$

such that

$$\begin{cases} g^{(3)} = dG^{(3)} + h^{(3)} & \text{in } \Omega \\ \nu \lrcorner G^{(3)} = 0 & \text{on } \partial\Omega. \end{cases}$$

Observe that

$$\left\| h^{(3)} \right\|_{C^{0,\alpha}} \le C_3 \left\| g^{(3)} \right\|_{C^{0,\alpha}} \le C_4 \|g\|_{C^{0,\alpha}}$$

$$\left\| G^{(3)} \right\|_{C^{r+1,\alpha}} \le C_3 \left\| g^{(3)} \right\|_{C^{r,\alpha}} \le C_4 \|g\|_{C^{r,\alpha}}$$

$$\left\| h^{(3)} \right\|_{C^{s,\beta}} + \left\| G^{(3)} \right\|_{C^{s+1,\beta}} \le C_3 \left\| g^{(3)} \right\|_{C^{s,\beta}} \le C_4 \|\nu \wedge g\|_{C^{s,\beta}(\partial\Omega)}.$$

(iv) We now again apply the Hodge decomposition to $g-g^{(3)}$ (cf. Theorem 4 (i) and Remark 5 (i)) to get $G^{(4)} \in C^{r+1,\alpha}\left(\overline{\Omega};\Lambda^1\right)$ and $h^{(4)} \in \mathcal{H}_T\left(\Omega;\Lambda^2\right) \subset C^{\infty}(\overline{\Omega};\Lambda^2)$ such that

$$\begin{cases} g-g^{(3)} = dG^{(4)} + h^{(4)} & \text{in } \Omega \\ \nu \wedge G^{(4)} = 0 & \text{on } \partial\Omega. \end{cases}$$

Furthermore, we find the following estimate:

$$\left\|h^{(4)}\right\|_{C^{r,\alpha}} + \left\|G^{(4)}\right\|_{C^{r+1,\alpha}} \leq C_5 \left\|g-g^{(3)}\right\|_{C^{r,\alpha}} \leq C_6 \left\|g\right\|_{C^{r,\alpha}} .$$

Similarly, as above, we have

$$\left\|G^{(4)}\right\|_{C^{r+1,\alpha}} \leq C_6 \left\|g\right\|_{C^{r,\alpha}} \quad \text{and} \quad \left\|h^{(4)}\right\|_{C^{0,\alpha}} \leq C_6 \left\|g\right\|_{C^{0,\alpha}} .$$

(v) Finally we correct $G^{(4)} \in C^{r+1,\alpha}\left(\overline{\Omega};\Lambda^1\right)$ so as to have

$$G^{(5)} \in C^{r+1,\alpha}\left(\overline{\Omega};\Lambda^1\right)$$

with $G^{(5)} = 0$ on $\partial\Omega$. This easily follows from the second statement of Theorem 6, since we can find $A \in C^{r+2,\alpha}\left(\overline{\Omega}\right)$ such that

$$dA = -G^{(4)} \quad \text{on } \partial\Omega.$$

It then suffices to set $G^{(5)} = G^{(4)} + dA$. We also have

$$\left\|G^{(5)}\right\|_{C^{r+1,\alpha}} \leq C_7 \left\|G^{(4)}\right\|_{C^{r+1,\alpha}} \leq C_8 \left\|g\right\|_{C^{r,\alpha}} .$$

(vi) We conclude that $G = G^{(3)} + G^{(5)}$ and $h = h^{(3)} + h^{(4)}$ have all the desired properties. Indeed, $G \in C^{r+1,\alpha}\left(\overline{\Omega};\Lambda^1\right) \cap C^{s+1,\beta}\left(\partial\Omega;\Lambda^1\right)$, $h \in C^{\infty}\left(\overline{\Omega};\Lambda^2\right)$, and

$$dG = dG^{(3)} + dG^{(5)} = g^{(3)} - h^{(3)} + g - g^{(3)} - h^{(4)} = g - h^{(3)} - h^{(4)} .$$

By construction, we deduce the following estimates:

$$\left\|h\right\|_{C^{r,\alpha}} + \left\|G\right\|_{C^{r+1,\alpha}} \leq C_9 \left\|g\right\|_{C^{r,\alpha}}$$

$$\begin{aligned} \left\|h\right\|_{C^{s,\beta}} + \left\|G\right\|_{C^{s+1,\beta}(\partial\Omega)} &\leq C_{10}\left[\left\|h\right\|_{C^{0,\alpha}} + \left\|\nu \wedge g\right\|_{C^{s,\beta}(\partial\Omega)}\right] \\ &\leq C_{11}\left[\left\|g\right\|_{C^{0,\alpha}} + \left\|\nu \wedge g\right\|_{C^{s,\beta}(\partial\Omega)}\right] \end{aligned}$$

since by Proposition 2 (iii)

$$\left\|h\right\|_{C^{s,\beta}} \leq C_{12} \left\|h\right\|_{C^{0,\alpha}} .$$

Step 2. Applying Lemma 15 on each component of G, we get G_ϵ as in the lemma (in particular $G_\epsilon = G$ on $\partial\Omega$). Setting $g_\epsilon = dG_\epsilon + h$, we have the claim. Before checking the inequalities observe that by construction, using in particular Theorem 2 (i),

$$dg_\epsilon = 0 \text{ in } \Omega, \quad \nu \wedge g_\epsilon = \nu \wedge g \text{ on } \partial\Omega$$

and since $g_\epsilon - g = d\,(G_\epsilon - G)$ with $G_\epsilon - G = 0$ on $\partial\Omega$, we have, using Theorem 3,

$$\int_\Omega \langle g_\epsilon - g; \psi\rangle \, dx = 0, \quad \text{for every } \psi \in \mathcal{H}_T(\Omega; \Lambda^2).$$

In the sequel, $C_1, C_2, \ldots$ will denote generic constants depending only on s, α, β, γ, and Ω. The first inequality follows from

$$\begin{aligned}\|g_\epsilon - g\|_{C^{t,\gamma}} &\le C_1 \|G_\epsilon - G\|_{C^{t+1,\gamma}} \le C_2 \epsilon^{(r+1+\alpha)-(t+1+\gamma)} \|G\|_{C^{r+1,\alpha}} \\ &\le C_3\, \epsilon^{(r+\alpha)-(t+\gamma)} \|g\|_{C^{r,\alpha}} \,.\end{aligned}$$

To obtain the second inequality first note that

$$\begin{aligned}\|g_\epsilon\|_{C^{s,\beta}} &\le C_4 \left[\|h\|_{C^{s,\beta}} + \|G_\epsilon\|_{C^{s+1,\beta}}\right] \\ &\le \frac{C_5}{\epsilon^{(s+1+\beta)-(r+1+\alpha)}} \|G\|_{C^{r+1,\alpha}} + C_5 \left(\|h\|_{C^{s,\beta}} + \|G\|_{C^{s+1,\beta}(\partial\Omega)}\right)\end{aligned}$$

and hence, since $\epsilon \le 1$,

$$\|g_\epsilon\|_{C^{s,\beta}} \le \frac{C_6}{\epsilon^{(s+\beta)-(r+\alpha)}} \|g\|_{C^{r,\alpha}} + C_6 \|\nu \wedge g\|_{C^{s,\beta}(\partial\Omega)} \,.$$

The third one comes from

$$\begin{aligned}\left\|\frac{d}{d\epsilon} g_\epsilon\right\|_{C^{s,\beta}} &\le C_7 \left\|\frac{d}{d\epsilon} G_\epsilon\right\|_{C^{s+1,\beta}} \le \frac{C_8}{\epsilon^{(s+1+\beta)-(r+1+\alpha)+1}} \|G\|_{C^{r+1,\alpha}} \\ &\le \frac{C_9}{\epsilon^{(s+\beta)-(r+\alpha)+1}} \|g\|_{C^{r,\alpha}}\end{aligned}$$

while the last one follows from

$$\begin{aligned}\left\|\frac{d}{d\epsilon} g_\epsilon\right\|_{C^{t,\gamma}} &\le C_{10} \left\|\frac{d}{d\epsilon} G_\epsilon\right\|_{C^{t+1,\gamma}} \le C_{11} \epsilon^{(r+1+\alpha)-(t+\gamma+1)-1} \|G\|_{C^{r+1,\alpha}} \\ &\le C_{12} \epsilon^{(r+\alpha)-(t+\gamma)-1} \|g\|_{C^{r,\alpha}} \,.\end{aligned}$$

Finally, defining $\Gamma : (0,1] \times \overline{\Omega} \to \Lambda^2$ by $\Gamma(\epsilon, x) = g_\epsilon(x)$, we have that $\Gamma \in C^{s,\beta}((0,1] \times \overline{\Omega}; \Lambda^2)$ and $\frac{\partial \Gamma}{\partial \epsilon} \in C^\infty((0,1] \times \overline{\Omega}; \Lambda^2)$, which concludes the proof. □

Finally, we give a elementary result in the same spirit.

Proposition 16. *Let $r \geq 0$ be an integer; $0 \leq \alpha < \beta \leq 1$; $\epsilon_0, C > 0$; and Ω be a bounded, connected, smooth, open set in $\mathbb{R}^n$. Let $f \in C^{r,\alpha}(\overline{\Omega})$ and $f_\epsilon \in C^{r,\beta}(\overline{\Omega})$, $\epsilon \in (0, \epsilon_0]$, be such that, for every $\epsilon \in (0, \epsilon_0]$,*

$$\|f_\epsilon\|_{C^{r,\beta}} \leq \frac{C}{\epsilon^{\beta-\alpha}} \quad \text{and} \quad \frac{\|f_\epsilon - f\|_{C^{r,\alpha}}}{\epsilon^{\beta-\alpha}} \leq C. \tag{4.1}$$

Then $f \in C^{r,(\alpha+\beta)/2}(\overline{\Omega})$ and

$$\|f\|_{C^{r,(\alpha+\beta)/2}} \leq \|f\|_{C^r} + 2C + \frac{2\|f\|_{C^r}}{\epsilon_0^{\beta-\alpha}}\,.$$

Proof. First notice that we trivially have that

$$|\nabla^r f(x) - \nabla^r f(y)| \leq \frac{2\|f\|_{C^r}}{\epsilon_0^{\alpha+\beta}} |x-y|^{(\alpha+\beta)/2},$$

for every $x, y \in \overline{\Omega}$ with $|x - y| \geq \epsilon_0^2$. Let $x, y \in \overline{\Omega}$ with $|x - y| \in (0, \epsilon_0^2)$ and define ϵ_1 by $\epsilon_1^{\beta-\alpha} = |x-y|^{(\beta-\alpha)/2}$. Noticing that $\epsilon_1 \in (0, \epsilon_0)$ and using (4.1), we deduce

$$\begin{aligned}
&|\nabla^r f(x) - \nabla^r f(y)| \\
&\leq |\nabla^r f_{\epsilon_1}(x) - \nabla^r f_{\epsilon_1}(x)| + |\nabla^r f(x) - \nabla^r f_{\epsilon_1}(x) + \nabla^r f_{\epsilon_1}(y) - \nabla^r f(y)| \\
&\leq \frac{C}{\epsilon_1^{\beta-\alpha}} |x-y|^\beta + C\epsilon_1^{\beta-\alpha} |x-y|^\alpha = 2C|x-y|^{(\alpha+\beta)/2},
\end{aligned}$$

which concludes the proof. □

4.2. **The flow method.** We start by recalling a well-known result.

Theorem 17. *Let $r \geq 1$ be an integer; $0 \leq \alpha \leq 1$; $T > 0$; and Ω be a bounded, connected, smooth, open set in $\mathbb{R}^n$. Let $u = u(t, x) = u_t(x)$ with $u \in C^{r,\alpha}((0, T] \times \overline{\Omega}; \mathbb{R}^n)$ be such that $u_t = 0$ on $\partial\Omega$ for every $t \in (0, T]$ and*

$$\int_0^T \|u_t\|_{C^{r,\alpha}(\overline{\Omega})} dt < \infty.$$

Then there exists a unique solution $\varphi \in C^{r,\alpha}([0, T] \times \overline{\Omega}; \overline{\Omega})$, $\varphi = \varphi(t, x) = \varphi_t(x)$, of

$$\begin{cases} \frac{d}{dt}\varphi_t = u_t \circ \varphi_t & 0 < t \leq T \\ \varphi_0 = \mathrm{id}\,. \end{cases}$$

Furthermore, for every $t \in [0, T]$, $\varphi_t \in \mathrm{Diff}^{r,\alpha}(\overline{\Omega}; \overline{\Omega})$ and $\varphi_t = \mathrm{id}$ on $\partial\Omega$.

We now state the flow method introduced by Moser (see [11]).

Theorem 18. *Let $r \geq 1$, $n \geq 2$ be two integers; $0 \leq \alpha \leq 1$; $T > 0$; and Ω be a bounded, connected, smooth, open set in $\mathbb{R}^n$. Let $u \in C^{r,\alpha}([0,T] \times \overline{\Omega}; \mathbb{R}^n)$, $u = u(t,x) = u_t(x)$, and $f \in C^{r,\alpha}([0,T] \times \overline{\Omega}; \Lambda^2)$, $f = f(t,x) = f_t(x)$, be such that, for every $t \in [0,T]$,*

$$u_t = 0 \quad \text{on } \partial\Omega, \quad df_t = 0 \text{ in } \Omega \quad \text{and} \quad d(u_t \lrcorner f_t) = -\frac{d}{dt} f_t \quad \text{in} \quad \Omega.$$

Then, for every $t \in [0,T]$, the solution φ_t of

$$\begin{cases} \frac{d}{dt}\varphi_t = u_t \circ \varphi_t & 0 \leq t \leq T \\ \varphi_0 = \mathrm{id} \end{cases}$$

belongs to $\mathrm{Diff}^{r,\alpha}(\overline{\Omega}; \overline{\Omega})$, satisfies $\varphi_t = \mathrm{id}$ on $\partial\Omega$, and

$$\varphi_t^*(f_t) = \varphi^*(f_0) \quad \text{in} \quad \Omega.$$

We now state and prove a weaker version, from the point of view of regularity, of Theorem 11. It has, however, the advantage of having a simple proof. It has been obtained by Bandyopadhyay-Dacorogna [2].

Theorem 19. *Let $n > 2$ be even; Ω be a bounded, connected, smooth, open set in $\mathbb{R}^n$; and $\nu \in C^\infty(\partial\Omega; \Lambda^1)$ be its exterior unit normal. Let $r \geq 1$ be an integer, $0 < \alpha < 1$, and let $f, g \in C^{r,\alpha}(\overline{\Omega}; \Lambda^2)$ satisfy*

$$df = dg = 0 \text{ in } \Omega, \quad \nu \wedge f = \nu \wedge g \text{ on } \partial\Omega$$

$$\int_\Omega \langle f; \psi \rangle \, dx = \int_\Omega \langle g; \psi \rangle \, dx, \quad \text{for every } \psi \in \mathcal{H}_T(\Omega; \Lambda^2)$$

$$\operatorname{rank}[\, tg + (1-t) f \,] = n, \text{ in } \overline{\Omega} \text{ and for every } t \in [0,1] \,.$$

Then, there exists $\varphi \in \mathrm{Diff}^{r,\alpha}\left(\overline{\Omega}; \overline{\Omega}\right)$ such that

$$\begin{cases} \varphi^*(g) = f & \text{in } \Omega \\ \varphi = \mathrm{id} & \text{on } \partial\Omega. \end{cases}$$

Proof. We solve (using the first statement of Theorem 6)

$$\begin{cases} dw = f - g & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases}$$

and find $w \in C^{r+1,\alpha}(\overline{\Omega}; \Lambda^1)$. Since $\operatorname{rank}[\, tg + (1-t) f \,] = n$, we can find $u \in C^{r,\alpha}([0,1] \times \overline{\Omega}; \mathbb{R}^n)$, $u = u(t,x) = u_t(x)$, so that

$$u_t \lrcorner [tg + (1-t) f] = w$$

which is equivalent to

$$u_t = \left[\, t\overline{g} + (1-t) \overline{f} \,\right]^{-1} w.$$

We then apply Theorem 18 with $f_t = tg + (1-t)f$ to get the result. $\square$

Finally, we prove Theorem 11 for special f and general g with extra regularity and under a smallness assumption.

Proposition 20. *Let $n > 2$ be even; Ω be a bounded, connected, smooth, open set in $\mathbb{R}^n$; and $\nu \in C^\infty(\partial\Omega; \Lambda^1)$ be its exterior unit normal. Let $r \geq 1$ be an integer, $0 < \alpha < \beta < 1$ and $g \in C^{r,\beta}(\overline{\Omega}; \Lambda^2)$ with*

$$\nu \wedge g \in C^{r+1,\alpha}(\partial\Omega; \Lambda^3), \quad dg = 0 \quad \text{and} \quad \operatorname{rank}[g] = n \quad \text{in } \overline{\Omega}.$$

Then, for every ϵ small, there exist

$$g_\epsilon \in C^{r+1,\alpha}(\overline{\Omega}; \Lambda^2) \quad \text{and} \quad \varphi_\epsilon \in \operatorname{Diff}^{r+1,\alpha}(\overline{\Omega}; \overline{\Omega})$$

such that

$$\begin{cases} \varphi_\epsilon^*(g_\epsilon) = g & \text{in } \Omega \\ \varphi_\epsilon = \mathrm{id} & \text{on } \partial\Omega \end{cases}$$

$$dg_\epsilon = 0 \text{ in } \Omega, \quad \nu \wedge g_\epsilon = \nu \wedge g \text{ on } \partial\Omega$$

$$\int_\Omega \langle g_\epsilon; \psi \rangle = \int_\Omega \langle g; \psi \rangle, \ \forall \psi \in \mathcal{H}_T(\Omega; \Lambda^2)$$

$$\lim_{\epsilon \to 0} \|g_\epsilon - g\|_{C^{r,\alpha}(\overline{\Omega})} = 0.$$

Proof. Step 1 (definition of g_ϵ). Apply Theorem 13 and get, for every $\epsilon \in (0,1]$, $g_\epsilon \in C^{r+1,\alpha}(\overline{\Omega}; \Lambda^2)$ and a constant $C_1 = C_1(r, \alpha, \beta, \Omega)$ such that

$$dg_\epsilon = 0 \quad \text{in } \Omega, \quad \nu \wedge g_\epsilon = \nu \wedge g \quad \text{on } \partial\Omega \tag{4.2}$$

$$\int_\Omega \langle g_\epsilon; \psi \rangle = \int_\Omega \langle g; \psi \rangle, \quad \text{for every } \psi \in \mathcal{H}_T(\Omega; \Lambda^2) \tag{4.3}$$

$$\|g_\epsilon\|_{C^{r+1,\alpha}} \leq \frac{C_1}{\epsilon^{1+\alpha-\beta}} \|g\|_{C^{r,\beta}} + C_1 \|\nu \wedge g\|_{C^{r+1,\alpha}(\partial\Omega)} \tag{4.4}$$

$$\|g_\epsilon - g\|_{C^{r,\alpha}} \leq C_1 \epsilon^{\beta-\alpha} \|g\|_{C^{r,\beta}} \tag{4.5}$$

$$\left\| \frac{d}{d\epsilon} g_\epsilon \right\|_{C^{0,\alpha}} \leq C_1 \|g\|_{C^{1,\alpha}} \quad \text{and} \quad \left\| \frac{d}{d\epsilon} g_\epsilon \right\|_{C^{r,\alpha}} \leq \frac{C_1}{\epsilon^{1+\alpha-\beta}} \|g\|_{C^{r,\beta}}. \tag{4.6}$$

Moreover, defining $G : (0,1] \times \overline{\Omega} \to \Lambda^2$ by $G(\epsilon, x) = g_\epsilon(x)$, we have

$$G \in C^{r+1,\alpha}((0;1] \times \overline{\Omega}; \Lambda^2) \quad \text{and} \quad \frac{\partial}{\partial \epsilon} G \in C^\infty((0;1] \times \overline{\Omega}; \Lambda^2). \tag{4.7}$$

Since $\operatorname{rank}[g] = n$ in $\overline{\Omega}$ (which is equivalent to $\det(\overline{g})(x) \neq 0$ for every $x \in \overline{\Omega}$) and since (4.5) holds there exists $\overline{\epsilon} \leq 1$ such that, for every $\epsilon \in (0, \overline{\epsilon}]$,

$$\|g_\epsilon\|_{C^0} \leq 2\|g\|_{C^0}, \quad \text{and} \quad \|1/\det(\overline{g}_\epsilon)\|_{C^0} \leq 2\|1/\det(\overline{g})\|_{C^0}. \tag{4.8}$$

Hence combining (4.8) with Theorem 8 (iii), we deduce that, for every $\epsilon \in (0, \overline{\epsilon}]$

$$\|(\overline{g}_\epsilon)^{-1}\|_{C^{r+1,\alpha}} \leq C_2 \|g_\epsilon\|_{C^{r+1,\alpha}}, \tag{4.9}$$

where $C_2 = C_2(r, \Omega, \|g\|_{C^0}, \|1/\det(\overline{g})\|_{C^0})$.

Step 2. In this step we will show that, for every $\epsilon \in (0, \overline{\epsilon}]$, there exist $u_\epsilon \in C^{r+1,\alpha}(\overline{\Omega}; \Lambda^1)$ and a constant $C_3 = C_3(r, \alpha, \beta, \Omega, \|g\|_{C^{1,\alpha}}, \|1/\det(\overline{g})\|_{C^0})$ such that

$$d(u_\epsilon \lrcorner\, g_\epsilon) = -\frac{d}{d\epsilon} g_\epsilon \quad \text{in } \Omega \tag{4.10}$$

$$\|u_\epsilon\|_{C^{r+1,\alpha}} \leq \frac{C_3}{\epsilon^{1+\alpha-\beta}} \|g\|_{C^{r,\beta}} + C_3 \|\nu \wedge g\|_{C^{r+1,\alpha}(\partial\Omega)}\,. \tag{4.11}$$

Moreover, defining $u : (0, \overline{\epsilon}] \times \overline{\Omega} \to \Lambda^1$ by $u(\epsilon, x) = u_\epsilon(x)$, we will show that $u \in C^{r+1,\alpha}((0, \overline{\epsilon}] \times \overline{\Omega}; \Lambda^1)$.

Step 2.1. Since (4.2), (4.3), and (4.7) hold, using the first statement of Theorem 6, we can find, for every $\epsilon \in (0, \overline{\epsilon}]$, $w_\epsilon \in C^\infty(\overline{\Omega}; \Lambda^1)$ and a constant $C_4 = C_4(r, \alpha, \Omega)$ such that

$$dw_\epsilon = -\frac{d}{d\epsilon} g_\epsilon \quad \text{in } \Omega, \quad w_\epsilon = 0 \quad \text{on } \partial\Omega$$

$$\|w_\epsilon\|_{C^{1,\alpha}} \leq C_4 \left\| \frac{d}{d\epsilon} g_\epsilon \right\|_{C^{0,\alpha}} \quad \text{and} \quad \|w_\epsilon\|_{C^{r+1,\alpha}} \leq C_4 \left\| \frac{d}{d\epsilon} g_\epsilon \right\|_{C^{r,\alpha}}. \tag{4.12}$$

Moreover, defining $w : (0, \overline{\epsilon}] \times \overline{\Omega} \to \Lambda^1$ by $w(\epsilon, x) = w_\epsilon(x)$, we have $w \in C^\infty((0, \overline{\epsilon}] \times \overline{\Omega}; \Lambda^1)$.

Step 2.2. Since by (4.8) we have, for every $\epsilon \in (0, \overline{\epsilon}]$, $\operatorname{rank}[g_\epsilon] = n$ in $\overline{\Omega}$, there exists a unique $u_\epsilon : \overline{\Omega} \to \Lambda^1$ satisfying

$$u_\epsilon \lrcorner\, g_\epsilon = w_\epsilon\,.$$

Note that $u_\epsilon \in C^{r+1,\alpha}(\overline{\Omega}; \Lambda^1)$ and that $u_\epsilon = 0$ on $\partial\Omega$. Moreover, defining $u : (0, \overline{\epsilon}] \times \overline{\Omega} \to \Lambda^1$ by $u(\epsilon, x) = u_\epsilon(x)$, we have $u \in C^{r+1,\alpha}((0, \overline{\epsilon}] \times \overline{\Omega}; \Lambda^1)$.

Step 2.3. To show Step 2, it only remains to prove (4.11). Using Theorem 8 (i), (4.4), (4.6), (4.8), (4.9), and (4.12), it follows, recalling that $\epsilon \leq 1$ and that $r \geq 1$,

$$\begin{aligned}
&\|u_\epsilon\|_{C^{r+1,\alpha}} = \|(\overline{g_\epsilon})^{-1} w_\epsilon\|_{C^{r+1,\alpha}} \\
&\leq C_5 \|(\overline{g_\epsilon})^{-1}\|_{C^{r+1,\alpha}} \|w_\epsilon\|_{C^0} + C_5 \|(\overline{g_\epsilon})^{-1}\|_{C^0} \|w_\epsilon\|_{C^{r+1,\alpha}} \\
&\leq C_6 \|g_\epsilon\|_{C^{r+1,\alpha}} \|w_\epsilon\|_{C^{1,\alpha}} + C_6 \|w_\epsilon\|_{C^{r+1,\alpha}} \\
&\leq C_7 \|g_\epsilon\|_{C^{r+1,\alpha}} \left\| \frac{d}{d\epsilon} g_\epsilon \right\|_{C^{0,\alpha}} + C_7 \left\| \frac{d}{d\epsilon} g_\epsilon \right\|_{C^{r,\alpha}}
\end{aligned}$$

$$\leq C_8\Big(\frac{1}{\epsilon^{1+\alpha-\beta}}\|g\|_{C^{r,\beta}}+\|\nu\wedge g\|_{C^{r+1,\alpha}(\partial\Omega)}\Big)\|g\|_{C^{1,\alpha}}+\frac{C_8}{\epsilon^{1+\alpha-\beta}}\|g\|_{C^{r,\beta}}$$
$$\leq \frac{C_9}{\epsilon^{1+\alpha-\beta}}\|g\|_{C^{r,\beta}}+C_9\|\nu\wedge g\|_{C^{r+1,\alpha}(\partial\Omega)},$$

where $C_i=C_i(r,\alpha,\beta,\Omega,\|g\|_{C^{1,\alpha}},\|1/\det(\overline{g})\|_{C^0})$. This shows the assertion.

Step 3. We can now conclude the proof.

Step 3.1 Since $u\in C^{r+1,\alpha}((0,\overline{\epsilon}]\times\overline{\Omega}:\mathbb{R}^n)$, $u_\epsilon=0$ on $\partial\Omega$, and since, by (4.11),

$$\int_0^{\overline{\epsilon}}\|u_\epsilon\|_{C^{r+1,\alpha}}d\epsilon<\infty$$

we deduce, using Theorem 17, that the solution $\varphi:[0,\overline{\epsilon}]\times\overline{\Omega}\to\overline{\Omega}$, $\varphi(\epsilon,x)=\varphi_\epsilon(x)$, of

$$\begin{cases}\frac{d}{d\epsilon}\varphi_\epsilon=u_\epsilon(\varphi_\epsilon) & 0<\epsilon\leq\overline{\epsilon}\\ \varphi_0=\mathrm{id}\end{cases}$$

satisfies

$$\varphi\in C^{r+1,\alpha}([0,\overline{\epsilon}]\times\overline{\Omega};\overline{\Omega}) \tag{4.13}$$

and that, for every $\epsilon\in[0,\overline{\epsilon}]$, $\varphi_\epsilon\in\mathrm{Diff}^{r+1,\alpha}(\overline{\Omega};\overline{\Omega})$ and $\varphi_\epsilon=\mathrm{id}$ on $\partial\Omega$.

Step 3.2. Since (4.10) holds, we deduce, using Theorem 18, that for every $0<\epsilon_1\leq\epsilon_2\leq\overline{\epsilon}$,

$$\varphi^*_{\epsilon_2}(g_{\epsilon_2})=\varphi^*_{\epsilon_1}(g_{\epsilon_1})\quad\text{in }\Omega.$$

Since, using (4.5) and (4.13),

$$\lim_{\epsilon\to0}\|\varphi_\epsilon-\varphi_0\|_{C^1}=\lim_{\epsilon\to0}\|g_\epsilon-g\|_{C^0}=0,$$

we immediately infer that, for every $\epsilon\in(0,\overline{\epsilon}]$,

$$\varphi^*_\epsilon(g_\epsilon)=\varphi^*_0(g)=g\,.$$

This concludes the proof of the proposition. □

4.3. **The fixed-point method.** We start by giving an abstract result that is particularly useful when dealing with non-linear problems, once good estimates are known for the linearized problem. We give it under a general form, because we will need it this way in Theorem 23. Our theorem will lean on the following hypotheses.

(H_{XY}) Let $X_1\supset X_2$ be Banach spaces and $Y_1\supset Y_2$ be normed spaces such that the following property holds: If

$$u_\nu\xrightarrow{X_1}u\quad\text{and}\quad\|u_\nu\|_{X_2}\leq r,$$

then $u \in X_2$ and

$$\|u\|_{X_2} \leq r.$$

(H_L) Let $L : X_2 \to Y_2$ be such that there exists $L^{-1} : Y_2 \to X_2$ a linear right inverse of L (namely $L \circ L^{-1} = \text{id}$ in Y_2) and such that there exist $k_1, k_2 > 0$ satisfying

$$\|L^{-1} f\|_{X_i} \leq k_i \|f\|_{Y_i} \quad \text{for every } f \in Y_2 \text{ and } i = 1, 2.$$

(H_Q) Let $\rho > 0$ and

$$Q : B_\rho = \{u \in X_2 : \|u\|_{X_1} \leq \rho\} \to Y_2$$

be such that $Q(0) = 0$ and for every $u, v \in B_\rho$, the following two inequalities hold:

$$\|Q(u) - Q(v)\|_{Y_1} \leq c_1(\|u\|_{X_1}, \|v\|_{X_1})\|u - v\|_{X_1} \tag{4.14}$$

$$\|Q(v)\|_{Y_2} \leq c_2\left(\|v\|_{X_1}, \|v\|_{X_2}\right), \tag{4.15}$$

where $c_1, c_2 : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ are separately increasing.

Theorem 21 (Fixed-point theorem)**.** *Let X_1, X_2, Y_1, Y_2, L, and Q satisfy (H_{XY}), (H_L), and (H_Q). Then, for every $f \in Y_2$ satisfying*

$$2k_1\|f\|_{Y_1} \leq \rho \quad \text{and} \quad 2k_1 c_1(2k_1\|f\|_{Y_1}, 2k_1\|f\|_{Y_1}) \leq 1 \tag{4.16}$$

$$c_2(2k_1\|f\|_{Y_1}, 2k_2\|f\|_{Y_2}) \leq \|f\|_{Y_2} \tag{4.17}$$

there exists $u \in B_\rho \subset X_2$ such that

$$Lu = Q(u) + f \quad \text{and} \quad \|u\|_{X_i} \leq 2k_i\|f\|_{Y_i} \;\; i = 1, 2. \tag{4.18}$$

Proof. We set $N(u) = Q(u) + f$. We next define

$$B = \{u \in X_2 : \|u\|_{X_i} \leq 2k_i\|f\|_{Y_i}, \; i = 1, 2\}.$$

We endow B with the $\|\cdot\|_{X_1}$ norm; the property (H_{XY}) ensures that B is closed. We now want to show that $L^{-1}N : B \to B$ is a contraction mapping (cf. Claims 1 and 2 below). Applying the Banach fixed-point theorem we will have indeed found a solution satisfying (4.18) recalling that $L \circ L^{-1} = \text{id}$.

Claim 1. Let us first show that $L^{-1}N$ is a contraction on B. To show this, let $u, v \in B$ and use (4.14) and (4.16) to get that

$$\begin{aligned}\|L^{-1}N(u) - L^{-1}N(v)\|_{X_1} &\leq k_1\|N(u) - N(v)\|_{Y_1} = k_1\|Q(u) - Q(v)\|_{Y_1} \\ &\leq k_1 c_1(\|u\|_{X_1}, \|v\|_{X_1})\|u - v\|_{X_1} \\ &\leq k_1 c_1(2k_1\|f\|_{Y_1}, 2k_1\|f\|_{Y_1})\|u - v\|_{X_1} \leq \frac{1}{2}\|u - v\|_{X_1}.\end{aligned}$$

Claim 2. We next show $L^{-1}N : B \to B$ is well-defined. First, note that

$$\|L^{-1}N(0)\|_{X_1} \le k_1\|N(0)\|_{Y_1} = k_1\|f\|_{Y_1} \,.$$

Therefore, using Claim 1, we obtain

$$\begin{aligned}\|L^{-1}N(u)\|_{X_1} &\le \|L^{-1}N(u) - L^{-1}N(0)\|_{X_1} + \|L^{-1}N(0)\|_{X_1} \\ &\le \frac{1}{2}\|u\|_{X_1} + k_1\|f\|_{Y_1} \le 2k_1\|f\|_{Y_1}.\end{aligned}$$

It remains to show that

$$\|L^{-1}N(u)\|_{X_2} \le 2k_2\|f\|_{Y_2} \,.$$

Using (4.15), we have

$$\begin{aligned}\|L^{-1}N(u)\|_{X_2} &\le k_2\|N(u)\|_{Y_2} \le k_2\|Q(u)\|_{Y_2} + k_2\|f\|_{Y_2} \\ &\le k_2c_2\left(\|u\|_{X_1}, \|u\|_{X_2}\right) + k_2\|f\|_{Y_2} \\ &\le k_2\left[c_2(2k_1\|f\|_{Y_1}, 2k_2\|f\|_{Y_2}) + \|f\|_{Y_2}\right]\end{aligned}$$

and hence, appealing to (4.17),

$$\|L^{-1}N(u)\|_{X_2} \le k_2\left[\|f\|_{Y_2} + \|f\|_{Y_2}\right] = 2k_2\|f\|_{Y_2} \,.$$

This concludes the proof of Claim 2 and thus of the theorem. □

The following estimate will play a crucial role in the proof of Theorem 23.

Lemma 22. *Let $n \ge 2$ and Ω be a bounded, connected, smooth, open set in $\mathbb{R}^n$. Let $r \ge 1$ and $n \ge 2$ be integers, $c > 0$, and $0 \le \gamma \le \alpha \le 1$. Let $g \in C^{r+1,\alpha}(\overline{\Omega}; \Lambda^2)$ with $dg = 0$ in Ω and $u \in C^{r+1,\alpha}(\overline{\Omega}; \mathbb{R}^n)$ with $(\mathrm{id} + u)(\overline{\Omega}) \subset \overline{\Omega}$. Set*

$$Q(u) = g - (\mathrm{id} + u)^*\,(g) + d\,[u \lrcorner\, g] \,.$$

Then there exists a constant $C = C\,(c, r, \Omega)$ such that the following estimates hold:

$$\|Q(u) - Q(v)\|_{C^{0,\gamma}} \le C\|g\|_{C^{2,\gamma}}(\|u\|_{C^{1,\gamma}} + \|v\|_{C^{1,\gamma}})\|u - v\|_{C^{1,\gamma}}$$

$$\|Q(u)\|_{C^{r,\alpha}} \le C\,\|g\|_{C^{r+1,\alpha}}\,\|u\|_{C^1} + C\,\|g\|_{C^1}\,\|u\|_{C^{r+1,\alpha}}\,\|u\|_{C^1} \,.$$

for every $u, v \in C^{r+1,\alpha}(\overline{\Omega}; \mathbb{R}^n)$ with

$$(\mathrm{id} + u)(\overline{\Omega}), (\mathrm{id} + v)(\overline{\Omega}) \subset \overline{\Omega} \quad \textit{and} \quad \|u\|_{C^{1,\gamma}}, \|v\|_{C^{1,\gamma}} \le c.$$

Proof. We divide the proof into four steps. In the proof we will constantly use Theorem 8 (i).

Step 1. We start with some notation. The form g will be written as

$$g=\sum_{1\leq i<j\leq n} g_{ij}dx^i\wedge dx^j.$$

We first need to write $(\operatorname{id}+u)^*g$ in a different way. For this we observe that we have, for $1\leq i<j\leq n$,

$$(dx^i+du^i)\wedge(dx^j+du^j)=dx^i\wedge dx^j+\left[du^i\wedge dx^j+dx^i\wedge du^j\right]+du^i\wedge du^j.$$

We can therefore write

$$\begin{aligned}\left[(\operatorname{id}+u)^*(g)\right](x)=g\,(x+u)+&\sum_{1\leq i<j\leq n} g_{ij}(x+u)\left[du^i\wedge dx^j+dx^i\wedge du^j\right]\\&+\sum_{1\leq i<j\leq n} g_{ij}(x+u)du^i\wedge du^j.\end{aligned}$$

We will also use, for $1\leq i<j\leq n$,

$$d\left[u\lrcorner\left(dx^i\wedge dx^j\right)\right]=du^i\wedge dx^j+dx^i\wedge du^j.$$

Step 2. We have, since $dg=0$ in Ω, that

$$d\left[u\lrcorner g\right]=\sum_{1\leq i<j\leq n} g_{ij}\left[du^i\wedge dx^j+dx^i\wedge du^j\right]+\sum_{1\leq i<j\leq n}\langle\operatorname{grad} g_{ij};u\rangle\, dx^i\wedge dx^j.$$

In order to get the right estimates, we rewrite $Q(u)$, defined by

$$Q(u)=g-(\operatorname{id}+u)^*(g)+d\left[u\lrcorner g\right],$$

as

$$Q(u)=Q_1(u)-Q_2(u)-Q_3(u)$$

where

$$\begin{aligned}Q_1(u)&=\sum_{1\leq i<j\leq n}\left[g_{ij}-g_{ij}(\operatorname{id}+u)\right]\left[du^i\wedge dx^j+dx^i\wedge du^j\right]\\Q_2(u)&=\sum_{1\leq i<j\leq n}\left[g_{ij}(\operatorname{id}+u)-g_{ij}-\langle\operatorname{grad} g_{ij};u\rangle\right]dx^i\wedge dx^j\\Q_3(u)&=\sum_{1\leq i<j\leq n} g_{ij}(\operatorname{id}+u)du^i\wedge du^j.\end{aligned}$$

Step 3. We now establish the first estimate for each of the Q_p, $p=1,2,3$. So let $u,v\in C^{r+1,\alpha}(\overline{\Omega};\mathbb{R}^n)$ with

$$(\operatorname{id}+u)(\overline{\Omega}),(\operatorname{id}+v)(\overline{\Omega})\subset\overline{\Omega}\quad\text{and}\quad\|u\|_{C^{1,\gamma}},\|v\|_{C^{1,\gamma}}\leq c.$$

In the sequel C_i will denote constants that depend only on c and Ω. Since in all cases we will make the estimates component by component, we immediately drop the sum signs. Before starting we recall (cf. Theorem 8 (ii) and (iv)) that there exists a constant $C_1 = C_1(c, \Omega)$ such that, for every $f \in C^{r,\gamma}(\overline{\Omega})$ and every $w_1, w_2 \in C^1(\overline{\Omega}; \overline{\Omega})$ with $\|w_1\|_{C^1}, \|w_2\|_{C^1} \leq c$,

$$\|f \circ w_1\|_{C^{0,\gamma}} \leq C_1 \|f\|_{C^{0,\gamma}}$$

$$\|f \circ w_1 - f \circ w_2\|_{C^{0,\gamma}} \leq C_1 \|f\|_{C^{1,\gamma}} \|w_1 - w_2\|_{C^{0,\gamma}}$$

$$\|f \circ w_1 - f \circ w_2\|_{C^0} \leq C_1 \|f\|_{C^1} \|w_1 - w_2\|_{C^0} \ .$$

Estimate for Q_1. We have

$$\begin{aligned} &\|Q_1(u) - Q_1(v)\|_{C^{0,\gamma}} \\ &= \| \left[g_{ij}(\mathrm{id}) - g_{ij}(\mathrm{id}+u)\right] \left[du^i \wedge dx^j\right] - \left[g_{ij}(\mathrm{id}) - g_{ij}(\mathrm{id}+v)\right] \left[dv^i \wedge dx^j\right] \|_{C^{0,\gamma}} \\ &\leq \|\left[g_{ij}(\mathrm{id}+v) - g_{ij}(\mathrm{id}+u)\right] \left[dv^i \wedge dx^j\right] \|_{C^{0,\gamma}} \\ &+ \|\left[g_{ij}(\mathrm{id}+u) - g_{ij}(\mathrm{id})\right] \left[\left[dv^i - du^i\right] \wedge dx^j\right] \|_{C^{0,\gamma}} \ . \end{aligned}$$

We therefore get

$$\begin{aligned} \|Q_1(u) - Q_1(v)\|_{C^{0,\gamma}} &\leq C_2 \|[g_{ij}(\mathrm{id}+v) - g_{ij}(\mathrm{id}+u)]\|_{C^0} \|v\|_{C^{1,\gamma}} \\ &+ C_2 \|[g_{ij}(\mathrm{id}+v) - g_{ij}(\mathrm{id}+u)]\|_{C^{0,\gamma}} \|v\|_{C^1} \\ &+ C_2 \|[g_{ij}(\mathrm{id}+u) - g_{ij}(\mathrm{id})]\|_{C^0} \|u - v\|_{C^{1,\gamma}} \\ &+ C_2 \|[g_{ij}(\mathrm{id}+u) - g_{ij}(\mathrm{id})]\|_{C^{0,\gamma}} \|u - v\|_{C^1} \ . \end{aligned}$$

Hence we get

$$\begin{aligned} &\|Q_1(u) - Q_1(v)\|_{C^{0,\gamma}} \\ &\leq C_3 \|g\|_{C^1} \|v - u\|_{C^0} \|v\|_{C^{1,\gamma}} + C_3 \|g\|_{C^{1,\gamma}} \|v - u\|_{C^{0,\gamma}} \|v\|_{C^1} \\ &+ C_3 \|g\|_{C^1} \|u\|_{C^0} \|u - v\|_{C^{1,\gamma}} + C_3 \|g\|_{C^{1,\gamma}} \|u\|_{C^{0,\gamma}} \|u - v\|_{C^1} \ . \end{aligned}$$

We thus have

$$\|Q_1(u) - Q_1(v)\|_{C^{0,\gamma}} \leq C \|g\|_{C^{1,\gamma}} (\|u\|_{C^{1,\gamma}} + \|v\|_{C^{1,\gamma}}) \|u - v\|_{C^{1,\gamma}} \ .$$

Estimate for Q_2. For Q_2 we proceed in the following way. We first observe that

$$\begin{aligned} Q_2(u) &= \int_0^1 \frac{d}{dt} \left[\left(g_{ij}(x + tu) - t \langle \operatorname{grad} g_{ij}(x); u \rangle \right) dx^i \wedge dx^j \right] dt \\ &= \int_0^1 \left[\langle \operatorname{grad} g_{ij}(x + tu) - \operatorname{grad} g_{ij}(x); u \rangle dx^i \wedge dx^j \right] dt. \end{aligned}$$

We therefore obtain

$$\|Q_2(u)-Q_2(v)\|_{C^{0,\gamma}} \leq \int_0^1 \|\langle \operatorname{grad} g_{ij}(\mathrm{id}+tu)-\operatorname{grad} g_{ij}(\mathrm{id});u\rangle$$
$$-\langle \operatorname{grad} g_{ij}(\mathrm{id}+tv)-\operatorname{grad} g_{ij}(\mathrm{id});v\rangle\|_{C^{0,\gamma}}\,dt$$
$$\leq \int_0^1 \{\|\langle \operatorname{grad} g_{ij}(\mathrm{id}+tu)-\operatorname{grad} g_{ij}(\mathrm{id}+tv);u\rangle\|_{C^{0,\gamma}}$$
$$+\|\langle \operatorname{grad} g_{ij}(\mathrm{id}+tv)-\operatorname{grad} g_{ij}(\mathrm{id});u-v\rangle\|_{C^{0,\gamma}}\}dt$$

and hence

$$\|Q_2(u)-Q_2(v)\|_{C^{0,\gamma}}$$
$$\leq C_2\int_0^1 \{\|\operatorname{grad} g_{ij}(\mathrm{id}+tu)-\operatorname{grad} g_{ij}(\mathrm{id}+tv)\|_{C^{0,\gamma}}\|u\|_{C^0}$$
$$+\|\operatorname{grad} g_{ij}(\mathrm{id}+tu)-\operatorname{grad} g_{ij}(\mathrm{id}+tv)\|_{C^0}\|u\|_{C^{0,\gamma}}$$
$$+\|\operatorname{grad} g_{ij}(\mathrm{id}+tv)-\operatorname{grad} g_{ij}(\mathrm{id})\|_{C^{0,\gamma}}\|u-v\|_{C^0}$$
$$+\|\operatorname{grad} g_{ij}(\mathrm{id}+tv)-\operatorname{grad} g_{ij}(\mathrm{id})\|_{C^0}\|u-v\|_{C^{0,\gamma}}\}dt.$$

This leads to

$$\|Q_2(u)-Q_2(v)\|_{C^{0,\gamma}}$$
$$\leq C_3\|g\|_{C^{2,\gamma}}\|u-v\|_{C^{0,\gamma}}\|u\|_{C^0}+C_3\|g\|_{C^2}\|u-v\|_{C^0}\|u\|_{C^{0,\gamma}}$$
$$+C_3\|g\|_{C^{2,\gamma}}\|v\|_{C^{0,\gamma}}\|u-v\|_{C^0}+C_3\|g\|_{C^2}\|v\|_{C^0}\|u-v\|_{C^{0,\gamma}}\,.$$

We therefore have the estimate

$$\|Q_2(u)-Q_2(v)\|_{C^{0,\gamma}} \leq C\|g\|_{C^{2,\gamma}}(\|u\|_{C^{0,\gamma}}+\|v\|_{C^{0,\gamma}})\|u-v\|_{C^{0,\gamma}}\,.$$

Estimate for Q_3. It remains to prove the estimate for Q_3, namely

$$\|Q_3(u)-Q_3(v)\|_{C^{0,\gamma}} = \|g_{ij}(\mathrm{id}+v)dv^i\wedge dv^j-g_{ij}(\mathrm{id}+u)du^i\wedge du^j\|_{C^{0,\gamma}}$$
$$\leq \|g_{ij}(\mathrm{id}+v)(dv^i\wedge dv^j-du^i\wedge du^j)\|_{C^{0,\gamma}}$$
$$+\|(g_{ij}(\mathrm{id}+u)-g_{ij}(\mathrm{id}+v))du^i\wedge du^j\|_{C^{0,\gamma}},$$

which leads to (recalling that $\|u\|_{C^{1,\gamma}}\,,\|v\|_{C^{1,\gamma}}\leq c$)

$$\|Q_3(u)-Q_3(v)\|_{C^{0,\gamma}} \leq C_3\|g\|_{C^{0,\gamma}}(\|u\|_{C^{1,\gamma}}+\|v\|_{C^{1,\gamma}})\|u-v\|_{C^{1,\gamma}}$$
$$+C_3\|g\|_{C^{1,\gamma}}\|u-v\|_{C^{0,\gamma}}\|u\|_{C^{1,\gamma}}$$

and thus

$$\|Q_3(u)-Q_3(v)\|_{C^{0,\gamma}} \leq C\|g\|_{C^{1,\gamma}}(\|u\|_{C^{1,\gamma}}+\|v\|_{C^{1,\gamma}})\|u-v\|_{C^{1,\gamma}},$$

proving the estimate for Q_3.

Step 4. We next establish the second estimate for each of the Q_p , $p = 1, 2, 3$. So let $u \in C^{r+1,\alpha}(\overline{\Omega}; \mathbb{R}^n)$ with

$$(\mathrm{id} + u)(\overline{\Omega}), (\mathrm{id} + v)(\overline{\Omega}) \subset \overline{\Omega} \quad \text{and} \quad \|u\|_{C^{1,\gamma}}, \|v\|_{C^{1,\gamma}} \le c.$$

As before, C_i will denote constants that depend only on c, r, and Ω. Since in all cases we will make the estimates component by component, we drop the sum signs. We recall (cf. Theorem 8 (ii)) that there exists a constant $C_1 = C_1(c, r, \Omega)$ such that, for every $f \in C^{r,\alpha}(\overline{\Omega})$ and every $w \in C^{r,\alpha}(\overline{\Omega}; \overline{\Omega})$ with $\|w\|_{C^1} \le c$,

$$\|f \circ w\|_{C^{r,\alpha}} \le C_1 \|f\|_{C^{r,\alpha}} + C_1 \|f\|_{C^1} \|w\|_{C^{r,\alpha}} .$$

We also have, for every $u \in C^{r+1,\alpha}(\overline{\Omega}; \mathbb{R}^n)$ with $(\mathrm{id} + u)(\overline{\Omega}) \subset \overline{\Omega}$ and $\|u\|_{C^1} \le c$,

$$\|g \circ (\mathrm{id} + u) - g \circ \mathrm{id}\|_{C^{r,\alpha}} \le C_1 \|g\|_{C^{r+1,\alpha}} \|u\|_{C^1} + C_1 \|g\|_{C^1} \|u\|_{C^{r+1,\alpha}} .$$

Indeed, from Theorem 8 (iv) we have

$$\begin{aligned}\|g \circ (\mathrm{id} + u) - g \circ \mathrm{id}\|_{C^{r,\alpha}} &\le C_2 \|g\|_{C^{r+1,\alpha}} \|u\|_{C^0} \\ &\quad + C_2 \|g\|_{C^2} \left[1 + \|u\|_{C^{r,\alpha}}\right] \|u\|_{C^0} + C_2 \|g\|_{C^1} \|u\|_{C^{r,\alpha}}\end{aligned}$$

and from Theorem 8 (v) we get

$$\|g\|_{C^2} \|u\|_{C^{r,\alpha}} \le C_3 \left[\|g\|_{C^{r+1,\alpha}} \|u\|_{C^1} + \|g\|_{C^1} \|u\|_{C^{r+1,\alpha}}\right] . \tag{4.19}$$

Combining the two estimates, we have our claim.

Estimate for Q_1. We have

$$\begin{aligned}\|Q_1(u)\|_{C^{r,\alpha}} &= \| \left[g_{ij}(\mathrm{id}) - g_{ij}(\mathrm{id} + u)\right] \left[du^i \wedge dx^j\right] \|_{C^{r,\alpha}} \\ &\le C_2 \|[g_{ij}(\mathrm{id} + u) - g_{ij}(\mathrm{id})]\|_{C^0} \|u\|_{C^{r+1,\alpha}} \\ &+ C_2 \|[g_{ij}(\mathrm{id} + u) - g_{ij}(\mathrm{id})]\|_{C^{r,\alpha}} \|u\|_{C^1} .\end{aligned}$$

We therefore get (bearing in mind that $\|u\|_{C^{1,\gamma}} \le c$)

$$\begin{aligned}\|Q_1(u)\|_{C^{r,\alpha}} &\le C_3 \|g\|_{C^1} \|u\|_{C^0} \|u\|_{C^{r+1,\alpha}} \\ &\quad + C_3 \|g\|_{C^{r+1,\alpha}} \|u\|_{C^1}^2 + C_3 \|g\|_{C^1} \|u\|_{C^{r+1,\alpha}} \|u\|_{C^1}\end{aligned}$$

and thus

$$\|Q_1(u)\|_{C^{r,\alpha}} \le C \|g\|_{C^{r+1,\alpha}} \|u\|_{C^1} + C \|g\|_{C^1} \|u\|_{C^{r+1,\alpha}} \|u\|_{C^1} .$$

Estimate for Q_2. As before we have that

$$Q_2(u) = \int_0^1 \frac{d}{dt} \left[\left(g_{ij}(x + tu) - t \langle \operatorname{grad} g_{ij}(x); u \rangle\right) dx^i \wedge dx^j \right] dt$$

$$= \int_0^1 \left[\langle \operatorname{grad} g_{ij}(x+tu) - \operatorname{grad} g_{ij}(x); u\rangle dx^i \wedge dx^j\right] dt.$$

We therefore obtain

$$\|Q_2(u)\|_{C^{r,\alpha}} \leq C_2 \int_0^1 \{\|\operatorname{grad} g_{ij}(\operatorname{id}+tu) - \operatorname{grad} g_{ij}(\operatorname{id})\|_{C^{r,\alpha}}\|u\|_{C^0}$$
$$+ \|\operatorname{grad} g_{ij}(\operatorname{id}+tu) - \operatorname{grad} g_{ij}(\operatorname{id})\|_{C^0}\|u\|_{C^{r,\alpha}}\}dt$$

and hence

$$\|Q_2(u)\|_{C^{r,\alpha}} \leq C_2 \int_0^1 \{[\|\operatorname{grad} g_{ij}(\operatorname{id}+tu)\|_{C^{r,\alpha}} + \|\operatorname{grad} g_{ij}\|_{C^{r,\alpha}}]\, \|u\|_{C^0}$$
$$+\|\operatorname{grad} g_{ij}(\operatorname{id}+tu) - \operatorname{grad} g_{ij}(\operatorname{id})\|_{C^0}\|u\|_{C^{r,\alpha}}\}dt.$$

This leads to

$$\|Q_2(u)\|_{C^{r,\alpha}} \leq C_3\,[\|g\|_{C^{r+1,\alpha}} + \|g\|_{C^2}\|u\|_{C^{r,\alpha}}]\,\|u\|_{C^0} + C_3\|g\|_{C^2}\|u\|_{C^0}\|u\|_{C^{r,\alpha}}\,.$$

From (4.19) we get

$$\|Q_2(u)\|_{C^{r,\alpha}} \leq C\,\|g\|_{C^{r+1,\alpha}}\,\|u\|_{C^1} + C_3\|g\|_{C^1}\|u\|_{C^{r+1,\alpha}}\|u\|_{C^1}\,.$$

Estimate for Q_3. We immediately have

$$\|Q_3(u)\|_{C^{r,\alpha}} = \|g_{ij}(\operatorname{id}+u)du^i \wedge du^j\|_{C^{r,\alpha}}$$
$$\leq C_2\|g(\operatorname{id}+u)\|_{C^{r,\alpha}}\|du^i \wedge du^j\|_{C^0} + C_2\|g\|_{C^0}\|du^i \wedge du^j\|_{C^{r,\alpha}}\,.$$

Since $\|u\|_{C^{1,\gamma}} \leq c$, we get

$$\|Q_3(u)\|_{C^{r,\alpha}} \leq C_3\,[\|g\|_{C^{r,\alpha}} + \|g\|_{C^1}\|u\|_{C^{r,\alpha}}]\,\|u\|_{C^1}^2$$
$$+ C_3\|g\|_{C^0}\|u\|_{C^1}\|u\|_{C^{r+1,\alpha}}$$

and thus (recall that $\|u\|_{C^{1,\gamma}} \leq c$)

$$\|Q_3(u)\|_{C^{r,\alpha}} \leq C\|g\|_{C^{r,\alpha}}\|u\|_{C^1} + C\|g\|_{C^1}\|u\|_{C^{r+1,\alpha}}\|u\|_{C^1}\,.$$

The combination of the three estimates gives the proof of the lemma. □

With the help of the two above result we now prove Theorem 11 with some restrictions on f and g.

Theorem 23. *Let $n > 2$ be even; Ω be a bounded, connected, smooth, open set in $\mathbb{R}^n$; and $\nu \in C^\infty(\partial\Omega; \Lambda^1)$ be its exterior unit normal. Let $r \geq 1$ be an integer and $0 < \gamma \leq \alpha < 1$. Let $g \in C^{r+1,\alpha}(\overline{\Omega}; \Lambda^2)$ and $f \in C^{r,\alpha}(\overline{\Omega}; \Lambda^2)$ be such that*

$$df = dg = 0 \text{ in } \Omega, \quad \nu \wedge f = \nu \wedge g \text{ on } \partial\Omega$$

$$\int_\Omega \langle f;\psi\rangle\,dx = \int_\Omega \langle g;\psi\rangle\,dx \quad \text{for every } \psi\in\mathcal{H}_T(\Omega;\Lambda^2)$$

$$\operatorname{rank}[g] = n \text{ in } \overline{\Omega}.$$

Let $c>0$ be such that $\|g\|_{C^1}$, $\left\|\frac{1}{det(\overline{g})}\right\|_{C^0}\le c$ and define

$$\theta(g) = \frac{1}{\|g\|^2_{C^{1,\gamma}}}\min\left\{\|g\|_{C^{1,\gamma}}\,,\ \frac{1}{\|g\|_{C^{2,\gamma}}}\,,\ \frac{1}{\|g\|_{C^{r+1,\alpha}}}\right\}.$$

There exists $C=C(c,r,\alpha,\gamma,\Omega)>0$ such that if

$$\|f-g\|_{C^{0,\gamma}}\le C\theta(g) \quad \text{and} \quad \|f-g\|_{C^{0,\gamma}}\le C\frac{\|f-g\|_{C^{r,\alpha}}}{\|g\|_{C^{1,\gamma}}\|g\|_{C^{r+1,\alpha}}}, \tag{4.20}$$

then there exists $\varphi\in\operatorname{Diff}^{r+1,\alpha}\left(\overline{\Omega};\overline{\Omega}\right)$ satisfying

$$\varphi^*(g) = f \text{ in } \Omega \quad \text{and} \quad \varphi = \operatorname{id} \text{ on } \partial\Omega. \tag{4.21}$$

Furthermore, there exists $\widetilde{C}=\widetilde{C}(c,r,\alpha,\gamma,\Omega)>0$ such that

$$\|\varphi-\operatorname{id}\|_{C^{r+1,\alpha}}\le\widetilde{C}\|g\|_{C^{r+1,\alpha}}\|f-g\|_{C^{r,\alpha}}\,.$$

Remark 24. Note that we necessarily have $\nu\wedge f\in C^{r+1,\alpha}(\partial\Omega;\Lambda^3)$ since $g\in C^{r+1,\alpha}(\overline{\Omega};\Lambda^2)$ and $\nu\wedge f=\nu\wedge g$ on $\partial\Omega$.

Proof. The theorem will follow from Theorem 21. We divide the proof into five steps; the first four ones to verify the hypotheses of the theorem and the last one to conclude.

Step 1. We define the spaces as follows:

$$X_1 = C^{1,\gamma}(\overline{\Omega};\mathbb{R}^n) \quad \text{and} \quad Y_1 = C^{0,\gamma}(\overline{\Omega};\Lambda^2)$$

$$X_2 = \{a\in C^{r+1,\alpha}(\overline{\Omega};\mathbb{R}^n) : a = 0 \text{ on } \partial\Omega\}$$

$$Y_2 = \left\{b\in C^{r,\alpha}(\overline{\Omega};\Lambda^2) : \left[\begin{array}{c} db=0 \text{ in } \Omega,\quad \nu\wedge b = 0 \text{ on } \partial\Omega, \\ \int_\Omega\langle b;\psi\rangle\,dx = 0,\ \forall\,\psi\in\mathcal{H}_T(\Omega;\Lambda^2)\end{array}\right.\right\}.$$

It is easily seen that they satisfy Hypothesis (H_{XY}) of Theorem 21 (see Theorem 8 (vi)).

Step 2. Define $L: X_2\to Y_2$ by $La = d[a\lrcorner g] = b$. We will show that there exists $L^{-1}: Y_2\to X_2$ a linear right inverse of L and a constant $K_1 = K_1(c,r,\alpha,\gamma,\Omega)$ such that, defining $k_1 = K_1\|g\|_{C^{1,\gamma}}$ and $k_2 = K_1\|g\|_{C^{r+1,\alpha}}$ we get

$$\|L^{-1}b\|_{X_i}\le k_i\|b\|_{Y_i} \ \text{ for every } b\in Y_2 \text{ and } i=1,2.$$

Once this is shown, (H_L) of Theorem 21 will be satisfied.

Step 2.1. Indeed we first solve, using the first statement of Theorem 6, the equation

$$\begin{cases} dw = b & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases}$$

and find $w \in C^{r+1,\alpha}(\overline{\Omega};\Lambda^1)$ and $C_1 = C_1(r,\alpha,\gamma,\Omega) > 0$ such that

$$\|w\|_{C^{r+1,\alpha}} \le C_1 \|b\|_{C^{r,\alpha}} \quad \text{and} \quad \|w\|_{C^{1,\gamma}} \le C_1 \|b\|_{C^{0,\gamma}} .$$

Moreover, the correspondence $b \mapsto w$ can be chosen to be linear.

Step 2.2. Since $\operatorname{rank}[g] = n$, we can find a unique $a \in C^{r+1,\alpha}(\overline{\Omega};\mathbb{R}^n)$ so that $a \lrcorner g = w$, which is equivalent to $a = [\,\overline{g}\,]^{-1} w$. Define $L^{-1} : Y_2 \to X_2$ by $L^{-1}(b) = a$. First note that L^{-1} is linear and that

$$L \circ L^{-1} = \mathrm{id} \quad \text{on } Y_2 .$$

Moreover, using Theorem 8 (i) and (iii) and Step 2.1, we can find constants $C_i = C_i(c,r,\alpha,\gamma,\Omega)$, $i = 2,3,4$, such that

$$\begin{aligned} \|a\|_{C^{r+1,\alpha}} &\le C_2 \left\|(\overline{g})^{-1}\right\|_{C^{r+1,\alpha}} \|w\|_{C^0} + C_2 \left\|(\overline{g})^{-1}\right\|_{C^0} \|w\|_{C^{r+1,\alpha}} \\ &\le C_3 \|g\|_{C^{r+1,\alpha}} \|b\|_{C^{0,\gamma}} + C_3 \|g\|_{C^0} \|b\|_{C^{r,\alpha}} \le C_4 \|g\|_{C^{r+1,\alpha}} \|b\|_{C^{r,\alpha}} \end{aligned}$$

and similarly

$$\|a\|_{C^{1,\gamma}} \le C_4 \|g\|_{C^{1,\gamma}} \|b\|_{C^{0,\gamma}} .$$

So that the claim of Step 2 is valid.

Step 3. We define $Q(u) = g - (\mathrm{id} + u)^*(g) + d[u \lrcorner g]$. We will verify that Property (H_Q) of Theorem 21 holds with $\rho \le 1/(2n)$.

Step 3.1. According to Lemma 22, there exists a constant $K_2 = K_2(r,\Omega)$ such that the following estimates hold:

$$\|Q(u) - Q(v)\|_{C^{0,\gamma}} \le K_2 \|g\|_{C^{2,\gamma}} (\|u\|_{C^{1,\gamma}} + \|v\|_{C^{1,\gamma}}) \|u - v\|_{C^{1,\gamma}}$$

$$\|Q(u)\|_{C^{r,\alpha}} \le K_2 \|g\|_{C^{r+1,\alpha}} \|u\|_{C^1} + K_2 \|g\|_{C^1} \|u\|_{C^{r+1,\alpha}} \|u\|_{C^1}$$

for every $u, v \in C^{r+1,\alpha}(\overline{\Omega};\mathbb{R}^n)$ with

$$(\mathrm{id} + u)(\overline{\Omega}), (\mathrm{id} + v)(\overline{\Omega}) \subset \overline{\Omega} \quad \text{and} \quad \|u\|_{C^{1,\gamma}}, \|v\|_{C^{1,\gamma}} \le 1/(2n) .$$

We finally let

$$c_1(t_1,t_2) = K_2 \|g\|_{C^{2,\gamma}} (t_1 + t_2)$$

$$c_2(t_1,t_2) = K_2 \|g\|_{C^{r+1,\alpha}} t_1 + K_2 \|g\|_{C^1} t_1 t_2 .$$

Note that if $F(t,x) = x + tu(x)$ and $\|u\|_{C^1} \le 1/(2n)$, then for every $t \in [0,1]$,

$$\det(\nabla_x F(t,\cdot)) = \det(I + t\,\nabla u) > 0 \quad \text{in } \overline{\Omega}.$$

Therefore, if $u = 0$ on $\partial\Omega$, then we get that

$$F(t,x) \in \overline{\Omega} \quad \text{for every } (t,x) \in [0,1] \times \overline{\Omega}.$$

Step 3.2. Let us check that $Q : \{u \in X_2 : \|u\|_{X_1} \le 1/(2n)\} \to Y_2$. We have to prove that

$$Q(0) = 0, \quad dQ(u) = 0 \text{ in } \Omega, \quad \nu \wedge Q(u) = 0 \text{ on } \partial\Omega$$

$$\int_\Omega \langle Q(u); \psi \rangle \, dx = 0, \ \forall \psi \in \mathcal{H}_T(\Omega; \Lambda^2).$$

(i) The fact that $Q(0) = 0$ is evident.

(ii) The second condition follows immediately since $dg = 0$ and

$$dQ(u) = dg - (\mathrm{id} + u)^* (dg) + dd\,[u \lrcorner g].$$

(iii) The third one is true since $u = 0$ on $\partial\Omega$. Indeed, clearly (using the notation Q_i used in the proof of Lemma 22)

$$Q_1(u) = Q_2(u) = 0 \text{ on } \partial\Omega.$$

Since $u = 0$ on $\partial\Omega$, each of du^i and du^j is parallel to the normal ν. Thus, $du^i \wedge du^j = 0$ on $\partial\Omega$ for every $i < j$, which implies that

$$Q_3(u) = 0 \text{ on } \partial\Omega.$$

Thus, we have in fact proved that $Q(u) = 0$ on $\partial\Omega$.

(iv) Choosing $F(t,x) = x + tu(x)$ in Proposition 7, we find that there exists Φ such that

$$\begin{cases} d\Phi = g - (\mathrm{id} + u)^* (g) & \text{in } \Omega \\ \Phi = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $\Psi = \Phi + u \lrcorner g$ satisfies

$$\begin{cases} d\Psi = Q(u) & \text{in } \Omega \\ \Psi = 0 & \text{on } \partial\Omega \end{cases}$$

we have the claim (by Theorem 3), namely

$$\int_\Omega \langle Q(u); \psi \rangle \, dx = 0, \ \forall \psi \in \mathcal{H}_T(\Omega; \Lambda^2).$$

Step 4. With the definition of L and Q in hand, we now rewrite (4.21) as follows. Setting $\varphi = \mathrm{id} + u$, the equation $\varphi^*(g) = f$ becomes

$$\begin{aligned} Lu = d\,[u \lrcorner g] &= f - (\mathrm{id} + u)^* g + d\,[u \lrcorner g] \\ &= f - g + [g - (\mathrm{id} + u)^* g + d\,[u \lrcorner g]] = f - g + Q(u). \end{aligned}$$

In order to apply Theorem 21, it remains to see how the hypotheses

$$
\begin{gathered}
2k_1\|f-g\|_{C^{0,\gamma}} \leq 1/(2n) \\
2k_1c_1(2k_1\|f-g\|_{C^{0,\gamma}}, 2k_1\|f-g\|_{C^{0,\gamma}}) \leq 1 \\
c_2(2k_1\|f-g\|_{C^{0,\gamma}}, 2k_2\|f-g\|_{C^{r,\alpha}}) \leq \|f-g\|_{C^{r,\alpha}}
\end{gathered}
\tag{4.22}
$$

translate in our context.

(i) The first one leads to

$$\|f-g\|_{C^{0,\gamma}} \leq \frac{1}{4nK_1\|g\|_{C^{1,\gamma}}}\,.$$

(ii) The second one gives

$$\|f-g\|_{C^{0,\gamma}} \leq \frac{1}{8K_1^2K_2\|g\|_{C^{1,\gamma}}^2\|g\|_{C^{2,\gamma}}}\,.$$

(iii) The third condition reads as

$$
\begin{aligned}
&K_2\|g\|_{C^{r+1,\alpha}}\,(2K_1\|g\|_{C^{1,\gamma}}\|f-g\|_{C^{0,\gamma}}) \\
&+K_2\,\|g\|_{C^1}\,(2K_1\|g\|_{C^{1,\gamma}}\|f-g\|_{C^{0,\gamma}})\,(2K_1\|g\|_{C^{r+1,\alpha}}\|f-g\|_{C^{r,\alpha}}) \\
&\leq \|f-g\|_{C^{r,\alpha}}\,.
\end{aligned}
$$

Note that the above condition is satisfied if

$$2K_1K_2\|g\|_{C^{r+1,\alpha}}\|g\|_{C^{1,\gamma}}\|f-g\|_{C^{0,\gamma}} \leq \frac{1}{2}\|f-g\|_{C^{r,\alpha}}$$

and

$$4K_1^2K_2\,\|g\|_{C^1}\,\|g\|_{C^{1,\gamma}}\,\|g\|_{C^{r+1,\alpha}}\,\|f-g\|_{C^{0,\gamma}}\|f-g\|_{C^{r,\alpha}} \leq \frac{1}{2}\|f-g\|_{C^{r,\alpha}}\,.$$

The first one leads to

$$\|f-g\|_{C^{0,\gamma}} \leq \frac{\|f-g\|_{C^{r,\alpha}}}{4K_1K_2\|g\|_{C^{r+1,\alpha}}\|g\|_{C^{1,\gamma}}}$$

while the second one is satisfied if

$$\|f-g\|_{C^{0,\gamma}} \leq \frac{1}{8K_1^2K_2\,\|g\|_{C^{r+1,\alpha}}\,\|g\|_{C^{1,\gamma}}^2}\,.$$

Combining the four conditions we have just obtained, we find that, letting

$$\theta\,(g) = \frac{1}{\|g\|_{C^{1,\gamma}}^2}\min\left\{\|g\|_{C^{1,\gamma}}\,,\,\frac{1}{\|g\|_{C^{2,\gamma}}}\,,\,\frac{1}{\|g\|_{C^{r+1,\alpha}}}\right\},$$

there exists $C = C(c, r, \alpha, \gamma, \Omega) > 0$ such that the inequalities (4.22) are satisfied if

$$\|f - g\|_{C^{0,\gamma}} \leq C\theta(g) \quad \text{and} \quad \|f - g\|_{C^{0,\gamma}} \leq C \frac{\|f - g\|_{C^{r,\alpha}}}{\|g\|_{C^{1,\gamma}} \|g\|_{C^{r+1,\alpha}}}.$$

Step 5. The hypotheses of Theorem 21 having been satisfied, we conclude that there exists $u \in C^{r+1,\alpha}(\overline{\Omega}; \mathbb{R}^n)$, with $\|u\|_{C^{1,\gamma}} \leq 1/(2n)$, satisfying

$$Lu = d[u \lrcorner g] = f - g + Q(u) = f - (\mathrm{id} + u)^*(g) + d[u \lrcorner g].$$

Letting $\varphi = \mathrm{id} + u$, we therefore have found that

$$\varphi^*(g) = f \text{ in } \Omega.$$

Since $u = 0$ on $\partial\Omega$, we have that $\varphi = \mathrm{id}$ on $\partial\Omega$. Since $\|u\|_{C^1} \leq 1/(2n)$, we deduce that

$$\det(\nabla\varphi) > 0 \quad \text{in } \overline{\Omega}$$

and therefore we find that $\varphi \in \mathrm{Diff}^{r+1,\alpha}(\overline{\Omega}; \overline{\Omega})$. This concludes the proof of the theorem. □

4.4. **Conclusion.** We can now go back to the proof of Theorem 11 using an iteration scheme involving appropriate regularization.

Proof. We split the proof into three steps.

Step 1 (approximation of g and f). Choose $\gamma \in (0, \alpha)$ and $\delta > 0$ with $2\delta < \alpha - \gamma$. We next regularize g and f with the help of Theorem 13 and construct for every $\epsilon \in (0, 1]$, $g_\epsilon, f_\epsilon \in C^{r+1,\alpha}(\overline{\Omega}; \Lambda^2)$ such that

$$dg_\epsilon = df_\epsilon = 0, \quad \nu \wedge g_\epsilon = \nu \wedge g = \nu \wedge f = \nu \wedge f_\epsilon \text{ on } \partial\Omega$$

$$\int_\Omega \langle g_\epsilon; \psi \rangle = \int_\Omega \langle g; \psi \rangle = \int_\Omega \langle f; \psi \rangle = \int_\Omega \langle f_\epsilon; \psi \rangle, \ \forall \psi \in \mathcal{H}_T(\Omega; \Lambda^2)$$

$$\|g_\epsilon - g\|_{C^{0,\gamma}} \leq C\epsilon^{r+\alpha-\gamma} \|g\|_{C^{r,\alpha}}$$

$$\|g_\epsilon - g\|_{C^{1,\gamma}} \leq C\epsilon^{r-1+\alpha-\gamma} \|g\|_{C^{r,\alpha}}$$

$$\|g_\epsilon\|_{C^{r+1,\alpha}} \leq \frac{C}{\epsilon} \|g\|_{C^{r,\alpha}} + C \|\nu \wedge g\|_{C^{r+1,\alpha}(\partial\Omega)}$$

$$\|g_\epsilon\|_{C^{r,\alpha+2\delta}} \leq \frac{C}{\epsilon^{2\delta}} \|g\|_{C^{r,\alpha}} + C \|\nu \wedge g\|_{C^{r,\alpha+2\delta}(\partial\Omega)}$$

$$\|g_\epsilon\|_{C^{2,\gamma}} \leq \frac{C}{\epsilon} \|g\|_{C^{1,\gamma}} + C \|\nu \wedge g\|_{C^{2,\gamma}(\partial\Omega)},$$

where $C = C(r, \alpha, \gamma, \delta, \Omega) > 0$ and similarly for f and f_ϵ. Note that, using the first inequality above, for every ϵ small enough, we have that

$$\mathrm{rank}[tg_\epsilon + (1 - t) f_\epsilon] = n, \quad \text{in } \overline{\Omega} \text{ and for every } t \in [0, 1].$$

Step 2. In this step we show that, for every ϵ small enough, there exist $\varphi_{1,\epsilon}, \varphi_{3,\epsilon} \in \operatorname{Diff}^{r+1,\alpha}\left(\overline{\Omega};\overline{\Omega}\right)$ such that

$$\begin{cases} \varphi_{1,\epsilon}^{*}\left(g_{\epsilon}\right)=g & \text{in } \Omega \\ \varphi_{1,\epsilon}=\mathrm{id} & \text{on } \partial\Omega \end{cases} \quad \text{and} \quad \begin{cases} \varphi_{3,\epsilon}^{*}\left(f_{\epsilon}\right)=f & \text{in } \Omega \\ \varphi_{3,\epsilon}=\mathrm{id} & \text{on } \partial\Omega. \end{cases}$$

For this we will use a combination of Proposition 20 and Theorem 23. We only show the assertion for g, the one with f being proved exactly in the same way.

Step 2.1. We start with some preliminary estimates. Using the second inequality in Step 1 we deduce that, for every ϵ small enough, recalling that $r \geq 1$ and $\gamma < \alpha$,

$$\frac{1}{2}\|g_{\epsilon}\|_{C^{1,\gamma}} \leq \|g\|_{C^{1,\gamma}} \leq 2\|g_{\epsilon}\|_{C^{1,\gamma}} \quad \text{and} \quad \left\|\frac{1}{\det(\overline{g}_{\epsilon})}\right\|_{C^{0}} \leq 2\left\|\frac{1}{\det(\overline{g})}\right\|_{C^{0}}.$$

In what follows ϵ will always be assumed small enough. Combining the left-hand side of the previous inequality with the third and fifth inequalities in Step 1, we deduce that there exists $D_1 > 0$ a constant independent of ϵ such that, defining

$$\theta\left(g_{\epsilon}\right)=\frac{1}{\|g_{\epsilon}\|_{C^{1,\gamma}}^{2}} \min\left\{\|g_{\epsilon}\|_{C^{1,\gamma}}, \frac{1}{\|g_{\epsilon}\|_{C^{2,\gamma}}}, \frac{1}{\|g_{\epsilon}\|_{C^{r+1,\alpha}}}\right\},$$

we have $\theta\left(g_{\epsilon}\right) \geq D_1\epsilon$. Hence, since $\|g_{\epsilon}-g\|_{C^{0,\gamma}} \leq C\epsilon^{r+\alpha-\gamma}$, $r \geq 1$ and $\gamma < \alpha$, we immediately deduce

$$\lim_{\epsilon \to 0} \frac{\|g_{\epsilon}-g\|_{C^{0,\gamma}}}{\theta\left(g_{\epsilon}\right)}=0.$$

Note also that there exists $D_2 > 0$, a constant independent of ϵ, such that

$$\frac{\|g_{\epsilon}-g\|_{C^{r,\alpha}}}{\|g_{\epsilon}\|_{C^{1,\gamma}}\|g_{\epsilon}\|_{C^{r+1,\alpha}}} \geq D_2\epsilon\|g_{\epsilon}-g\|_{C^{r,\alpha}}. \tag{4.23}$$

Step 2.2. We show the assertion by considering two cases. In the first one we use Proposition 20 to obtain the assertion while in the second one we use Theorem 23.

(i) Suppose that

$$\lim_{\epsilon \to 0} \frac{\|g_{\epsilon}-g\|_{C^{r,\alpha}}}{\epsilon^{2\delta}}=0.$$

Combining the above equation with the fact that, by the fourth inequality in Step 1 (where $D_3 > 0$ is independent of ϵ)

$$\|g_{\epsilon}\|_{C^{r,\alpha+2\delta}} \leq \frac{D_3}{\epsilon^{2\delta}},$$

we immediately deduce from Proposition 16 that $g \in C^{r,\alpha+\delta}(\overline{\Omega};\Lambda^2)$. The assertion then follows directly from Proposition 20 once it is noticed, using Remark 14 (ii), that the g_ϵ constructed in Proposition 20 are the same as the ones defined in Step 1.

(ii) Suppose that

$$\liminf_{\epsilon\to 0}\frac{\|g_\epsilon - g\|_{C^{r,\alpha}}}{\epsilon^{2\delta}} > 0.$$

Therefore, there exists $D_4 > 0$ a constant independent of ϵ, such that

$$\|g_\epsilon - g\|_{C^{r,\alpha}} \geq D_4\epsilon^{2\delta}.$$

Using the previous inequality, (4.23), and the fact that

$$\|g_\epsilon - g\|_{C^{0,\gamma}} \leq C\epsilon^{r+\alpha-\gamma},$$

we obtain, recalling that $r \geq 1$ and that $2\delta < \alpha - \gamma$,

$$\lim_{\epsilon\to 0}\frac{\|g_\epsilon - g\|_{C^{0,\gamma}}}{\left\{\frac{\|g_\epsilon - g\|_{C^{r,\alpha}}}{\|g_\epsilon\|_{C^{1,\gamma}}\|g_\epsilon\|_{C^{r+1,\alpha}}}\right\}} = 0.$$

Hence, recalling that

$$\lim_{\epsilon\to 0}\frac{\|g_\epsilon - g\|_{C^{0,\gamma}}}{\theta(g_\epsilon)} = 0 \quad \text{and} \quad \|g_\epsilon\|_{C^1}, \left\|\frac{1}{\det(\overline{g}_\epsilon)}\right\|_{C^0} \leq D_5\,,$$

where D_5 is independent of ϵ, we have the claim using Theorem 23.

Step 3. Since

$$\begin{cases} dg_\epsilon = df_\epsilon = 0 & \text{in } \Omega \\ \nu \wedge g_\epsilon = \nu \wedge f_\epsilon & \text{on } \partial\Omega \\ \int_\Omega \langle g_\epsilon;\psi\rangle = \int_\Omega \langle f_\epsilon;\psi\rangle & \text{for every } \psi \in \mathcal{H}_T(\Omega;\Lambda^2) \\ \operatorname{rank}\left[tg_\epsilon + (1-t)\,f_\epsilon\right] = n & \text{in } \overline{\Omega} \text{ and for every } t\in[0,1], \end{cases}$$

we can apply Theorem 19 to find $\varphi_{2,\epsilon} \in \operatorname{Diff}^{r+1,\alpha}\left(\overline{\Omega};\overline{\Omega}\right)$ such that

$$\begin{cases} \varphi_{2,\epsilon}^{*}(g_\epsilon) = f_\epsilon & \text{in } \Omega \\ \varphi_{2,\epsilon} = \text{id} & \text{on } \partial\Omega. \end{cases}$$

The claimed solution is then given by

$$\varphi = \varphi_{1,\epsilon}^{-1}\circ\varphi_{2,\epsilon}\circ\varphi_{3,\epsilon}\,.$$

This achieves the proof of the theorem. □

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