The pullback equation for degenerate forms

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Abstract

We discuss the existence of a diffeomorphism $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\varphi^* (g) = f$$

where $f, g : \mathbb{R}^n \to \Lambda^k$ are closed differential forms and $2 \leq k \leq n-1$.

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1 Introduction

In this article we discuss the existence of a diffeomorphism $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\varphi^* (g) = f \tag{1}$$

where $f, g : \mathbb{R}^n \to \Lambda^k (\mathbb{R}^n)$, $f(x), g(x) \neq 0$ for every $x \in \mathbb{R}^n$, are closed differential forms (i.e. $df = dg = 0$), $2 \leq k \leq n$

$$g = \sum_{1 \leq i_1 < \cdots < i_k \leq n} g_{i_1 \cdots i_k} (x) \, dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

and similarly for f. The meaning of (1) is that

$$\sum_{1 \leq i_1 < \cdots < i_k \leq n} g_{i_1 \cdots i_k} (\varphi (x)) \, d\varphi^{i_1} \wedge \cdots \wedge d\varphi^{i_k} = \sum_{1 \leq i_1 < \cdots < i_k \leq n} f_{i_1 \cdots i_k} (x) \, dx^{i_1} \wedge \cdots \wedge dx^{i_k} .$$

We will concentrate mainly on local existence, though in some special cases we will obtain global results. A condition which is somewhat necessary (cf. Theorem 9) to solve this problem is that the rank of the two forms should be the same (for a precise definition see below), namely

$$\operatorname{rank} g = \operatorname{rank} f.$$

It will turn out that, when $k=2$, $k=n-1$ or $k=n$, if the rank of the two forms is equal and constant then, at least locally, there exists a diffeomorphism φ satisfying (1).

This problem has been intensively studied in two main cases, that we call *non-degenerate*.

Case 1: $k=2$, $n=2m$ and $\operatorname{rank} g = \operatorname{rank} f = n$, this is the celebrated Darboux theorem (cf., for example, Abraham-Marsden-Ratiu [2], McDuff-Salamon [13] or Taylor [22]) which states that if g is the standard symplectic form, namely

$$g = \omega_m = \sum_{i=1}^{m} dx^{2i-1} \wedge dx^{2i}$$

and $f : \mathbb{R}^n \to \Lambda^2(\mathbb{R}^n)$ with $df = 0$ and $\operatorname{rank} g = \operatorname{rank} f = n$, then there exists a diffeomorphism φ defined locally such that

$$\varphi^*(g) = f.$$

This result was significantly improved in Bandyopadhyay-Dacorogna [3] in two main directions:

- optimal regularity in Hölder spaces was obtained (see Theorem 18 in [3], which is restated in Theorem 10 below),

- more strikingly a global result with Dirichlet boundary data was established (cf. Theorems 13 and 15 in [3]).

Case 2: $k=n$ and $\operatorname{rank} g = \operatorname{rank} f = n$ (which is equivalent to $f(x), g(x) \neq 0$ for every $x \in \mathbb{R}^n$). This is the fundamental result of Moser in [15], later also considered by Banyaga [4], Cupini-Dacorogna-Kneuss [7], Dacorogna [8], Reimann [17], Rivière-Ye [18], Tartar [21], Ye [23] and Zehnder [24]. The optimal regularity in Hölder spaces has been obtained in Dacorogna-Moser [10].

We here want to concentrate our attention on *degenerate* cases which are of two types.

Case 3: $k=2$ and $\operatorname{rank} g = \operatorname{rank} f = 2l$ with $2 \leq 2l < n$. This case is a degenerate version of Darboux theorem and has also been intensively studied (see, for example, Abraham-Marsden [1]), the result is that if

$$g = \omega_l = \sum_{i=1}^{l} dx^{2i-1} \wedge dx^{2i}$$

and $f : \mathbb{R}^n \to \Lambda^2(\mathbb{R}^n)$ with $df = 0$ and $\operatorname{rank} g = \operatorname{rank} f = 2l$, then there exists a diffeomorphism φ defined locally such that

$$\varphi^*(g) = f.$$

The usual proof uses Frobenius theorem reducing the dimension to $\mathbb{R}^{2l}$ and then appeals to the non-degenerate case. We here give two proofs. The first one (cf. Theorem 11) inspired by the classical one but using the theorem of

Bandyopadhyay-Dacorogna (cf. Theorem 10) instead of the standard Darboux theorem; it allows us to improve the known regularity. We also give a totally different proof (cf. Theorem 20) using the flow method. This last proof seems to be more appropriate if one wants to obtain global results.

Case 4: $3 \leq k \leq n-1$. In this context we necessarily have (recall that $f(x), g(x) \neq 0$ for every $x \in \mathbb{R}^n$)

$$\operatorname{rank} g = \operatorname{rank} f \in \{k, k+2, \cdots, n\}$$

and thus when $k = n-1$ we only have $\operatorname{rank} g = \operatorname{rank} f = n-1$. The case of $k-$forms, with $3 \leq k \leq n-1$, has been much less studied than the other ones and it is considerably harder. In particular, except when $k = n-1$, the conservation of the rank is not the only necessary condition. The notion of decomposability of a $k-$form enters into play (see below). We are able to handle (cf. Theorem 21) completely the case $\operatorname{rank} g = \operatorname{rank} f = k$ (and so completely the case $k = n-1$) and obtain a non-trivial regularity result, though not optimal. We can also deal with several other cases (cf. Theorems 24, 26, 28) but the general case is still far from complete. For previous work on this subject, we refer, notably for the case $k = n-1$, to Bandyopadhyay-Dacorogna in [3].

We should also point out that when $k = n$, the problem of degeneracy also occurs in a different context, for example when $g > 0$ and f may become 0 or even is allowed to change sign. Of course then one cannot expect to have a diffeomorphism. This has been studied in Cupini-Dacorogna-Kneuss [7].

2 Some algebraic preliminaries

2.1 Notations

We denote a $k-$form $g : \mathbb{R}^n \to \Lambda^k(\mathbb{R}^n)$ by

$$g = \sum_{1 \leq i_1 < \cdots < i_k \leq n} g_{i_1 \cdots i_k}(x)\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Sometimes it will be more convenient to assign meaning to $g_{i_1 \cdots i_k}$ for any $k-$index and we will let

$$g_{i_1 \cdots i_k} = (\operatorname{sgn} \sigma)\, g_{i_{\sigma(1)} \cdots i_{\sigma(k)}}$$

where σ is a permutation of $\{1, \cdots, k\}$.

Throughout the paper, to avoid burdening the notations, we will identify, when necessary, $k-$forms with vectors in $\mathbb{R}^{\binom{n}{k}}$. We moreover adopt the following standard notations.

1) Let $0 \leq k \leq n$, we recall that the *Hodge* $*-$*operator* associates to $f \in \Lambda^k(\mathbb{R}^n)$ the $(n-k)-$form

$$*f \in \Lambda^{n-k}(\mathbb{R}^n).$$

2) For $0 \leq l, k \leq n$, $f \in \Lambda^l(\mathbb{R}^n)$ and $g \in \Lambda^k(\mathbb{R}^n)$, the *exterior product* is denoted by

$$f \wedge g \in \Lambda^{k+l}(\mathbb{R}^n).$$

3) For $f, g \in \Lambda^k(\mathbb{R}^n)$ we write *inner product* as

$$\langle f; g \rangle = \sum_{1 \leq i_1 < \cdots < i_k \leq n} f_{i_1 \cdots i_k} g_{i_1 \cdots i_k} \in \mathbb{R}.$$

4) For $0 \leq l \leq k \leq n$, $f \in \Lambda^l(\mathbb{R}^n)$ and $g \in \Lambda^k(\mathbb{R}^n)$, the *interior product* will then be defined by

$$f \lrcorner g = (-1)^{n(k-l)} * [f \wedge (*g)] \in \Lambda^{k-l}(\mathbb{R}^n).$$

In particular if $f \in \Lambda^1(\mathbb{R}^n)$ and $g \in \Lambda^k(\mathbb{R}^n)$, we find

$$\begin{aligned} f \lrcorner g &= \sum_{1 \leq i_1 < \cdots < i_k \leq n} g_{i_1 \cdots i_k} \sum_{r=1}^{k} (-1)^{r-1} f_{i_r} dx^{i_1} \wedge \cdots \wedge dx^{i_{r-1}} \wedge dx^{i_{r+1}} \wedge \cdots \wedge dx^{i_k} \\ &= (-1)^{k-1} \sum_{1 \leq i_1 < \cdots < i_{k-1} \leq n} \sum_{i_k=1}^{n} g_{i_1 \cdots i_k} f_{i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}} \end{aligned} \tag{2}$$

which turns out when $k = 2$ to give

$$f \lrcorner g = -\sum_{i=1}^{n} \sum_{j=1}^{n} g_{ij} f_j dx^i.$$

If $l > k$, we let

$$f \lrcorner g = 0.$$

Note that if $l = k$, then

$$f \lrcorner g = \langle f; g \rangle.$$

5) The *exterior derivative* is denoted by

$$d : C^1\left(\mathbb{R}^n; \Lambda^k\right) \to C^0\left(\mathbb{R}^n; \Lambda^{k+1}\right).$$

6) Let $\varphi : \mathbb{R}^n \to \mathbb{R}^N$

$$x = \left(x^1, \cdots, x^N\right) = \varphi(y) = \varphi\left(y^1, \cdots, y^n\right) = \left(\varphi^1(y), \cdots, \varphi^N(y)\right)$$

and $g : \mathbb{R}^N \to \Lambda^k\left(\mathbb{R}^N\right)$, $0 \leq k \leq \min\{n, N\}$,

$$g = \sum_{1 \leq i_1 < \cdots < i_k \leq N} g_{i_1 \cdots i_k}(x) \, dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

We define the *pullback* of g by φ, denoted by $f = \varphi^*(g) : \mathbb{R}^n \to \Lambda^k(\mathbb{R}^n)$, through

$$\sum_{1 \leq j_1 < \cdots < j_k \leq n} f_{j_1 \cdots j_k}(y) \, dy^{j_1} \wedge \cdots \wedge dy^{j_k} = \sum_{1 \leq i_1 < \cdots < i_k \leq N} g_{i_1 \cdots i_k}(\varphi(y)) \, d\varphi^{i_1} \wedge \cdots \wedge d\varphi^{i_k}.$$

In particular if $N = n$ and $k = 2$, then the above equation is, in terms of components, a system of $\binom{n}{2}$ first order pdes, namely

$$\sum_{1 \le p < q \le n} g_{pq} \left(\varphi \left(x \right) \right) \left(\frac{\partial \varphi^p}{\partial x^i} \frac{\partial \varphi^q}{\partial x^j} - \frac{\partial \varphi^p}{\partial x^j} \frac{\partial \varphi^q}{\partial x^i} \right) = f_{ij} \, , \text{ for every } 1 \le i < j \le n.$$

This explains the reason for which we use, throughout the article, the notations of differential geometry rather than those with local coordinates, which are more familiar to analysts but are unfortunately much heavier.

We have as a direct consequence of the definitions the following.

Proposition 1 *Let $f \in \Lambda^k \left(\mathbb{R}^n \right)$, $g \in \Lambda^l \left(\mathbb{R}^n \right)$ and $h \in \Lambda^1 \left(\mathbb{R}^n \right)$, then*

$$h \lrcorner \left(f \wedge g \right) = \left(h \lrcorner f \right) \wedge g + \left(-1 \right)^{kl} \left(h \lrcorner g \right) \wedge f$$

and, in particular,

$$h \lrcorner \left(h \wedge g \right) = \left(h \lrcorner h \right) \wedge g + \left(-1 \right)^{k} \left(h \lrcorner g \right) \wedge h.$$

If, moreover, l is even and m is an integer, then

$$h \lrcorner g^{m+1} = \left(m+1 \right) \left(h \lrcorner g \right) \wedge g^m$$

where $g^m = \underbrace{g \wedge \cdots \wedge g}_{m-times}$.

Finally, if $l = k-1$, then

$$\langle f ; g \wedge h \rangle = \left(-1 \right)^{k-1} \langle h \lrcorner f ; g \rangle \, .$$

2.2 The rank of a form

The notion of the rank of a $2-$form is standard, although it is usually presented in a different way. However the corresponding one for a $k-$form, $3 \le k \le n$, is less standard but can be found, for example, in [3], [11] or [20]. To define the rank it is more convenient to represent $u \to u \lrcorner g$ as a matrix operating on a vector.

Definition 2 *Let $g \in \Lambda^k \left(\mathbb{R}^n \right)$, $1 \le k \le n$.*

(i) *The* alternate representation *of g is the matrix $\overline{g} \in \mathbb{R}^{\binom{n}{k-1} \times n}$ defined through (cf. (2))*

$$u \lrcorner g = \overline{g} \, u, \text{ for every } u \in \Lambda^1 \left(\mathbb{R}^n \right)$$

where, as already said and by abuse of notations, in the left hand side the $\left(k-1 \right) -$form $u \lrcorner g$ is identified with a vector in $\mathbb{R}^{\binom{n}{k-1}}$ while in the right hand side of the definition the $1-$form u is identified with a vector in $\mathbb{R}^n$. More explicitly, using the lexicographical order for the columns (index below) and the rows (index above) of the matrix $\overline{g}$, we have

$$\left(\overline{g} \right)_i^{j_1 \cdots j_{k-1}} = g_{i j_1 \cdots j_{k-1}}$$

for $1 \leq i \leq n$ *and* $1 \leq j_1 < \cdots < j_{k-1} \leq n$.

(ii) *The* rank *of the* $k-$*form* g *is then, by definition, the rank of the matrix* $\overline{g}$, *so we can write*

$$\operatorname{rank} g = \operatorname{rank} \overline{g}.$$

When $k = 2$, the matrix $\overline{g} \in \mathbb{R}^{n \times n}$ and it is given by

$$\overline{g} = (g_{ij}) \in \mathbb{R}^{n \times n} \quad \text{with} \quad g_{ij} = -g_{ji}\,.$$

We immediately have the following elementary result.

Proposition 3 *Let* $g \in \Lambda^k (\mathbb{R}^n)\,,\ 1 \leq k \leq n.$

(i) *Define the vector space*

$$\Lambda_g^1 = \left\{ a \in \Lambda^1 (\mathbb{R}^n) : \exists\, b \in \Lambda^{k-1} (\mathbb{R}^n) \quad \textit{with } b \lrcorner g = a \right\},$$

then

$$\operatorname{rank} g = \dim \Lambda_g^1\,.$$

(ii) *If* $g \neq 0$ *and* $3 \leq k \leq n$, *then*

$$\operatorname{rank} g \in \{k, k+2, \cdots, n\}$$

and any of the values in $\{k, k+2, \cdots, n\}$ *can be achieved by the rank of a* $k-$*form.*

(iii) *If* $k = 2$, *then the rank of* g, $g \neq 0$, *is even and any even value less than or equal to* n *can be achieved by the rank of a* $2-$*form. Moreover* $\operatorname{rank} g = 2l$ *if and only if*

$$g^l \neq 0 \quad \textit{and} \quad g^{l+1} = 0$$

where $g^l = \underbrace{g \wedge \cdots \wedge g}_{l-times}$.

(iv) *If* $f \in \Lambda^l (\mathbb{R}^n)\,,$ *then*

$$\operatorname{rank} (f \wedge g) \leq \operatorname{rank} f + \operatorname{rank} g - \dim \left(\Lambda_f^1 \cap \Lambda_g^1 \right).$$

Moreover,

$$\operatorname{rank} (f \wedge g) = \operatorname{rank} f + \operatorname{rank} g \quad \Leftrightarrow \quad \Lambda_f^1 \cap \Lambda_g^1 = \{0\}\,.$$

(v) *If* $A \in GL_n (\mathbb{R})\,,$ *then*

$$\operatorname{rank} (\varphi^* (g)) = \operatorname{rank} g$$

where $\varphi (x) = Ax$.

Remark **(i)** If $g \in \Lambda^1(\mathbb{R}^n)$ and $g \neq 0$, then

$$\operatorname{rank} g = 1.$$

If $g \in \Lambda^n(\mathbb{R}^n)$ with $g \neq 0$, then

$$\operatorname{rank} g = n.$$

(ii) For $g \in \Lambda^k(\mathbb{R}^n)$, then the rank of g can never be $(k+1)$. In particular when $k = n-1$ and $g \neq 0$, then

$$\operatorname{rank} g = n-1.$$

(iii) When $k = 2$ and $\operatorname{rank} g = 2l = n$, then

$$\det \overline{g} = \frac{1}{l!}\left(g^l\right)^2 > 0.$$

(iv) If $g \in \Lambda^2(\mathbb{R}^n)$ and $f \in \Lambda^1(\mathbb{R}^n)$ with $f \wedge g \neq 0$, then

$$\operatorname{rank}(f \wedge g) = \operatorname{rank} g \pm 1$$

and thus is odd. In particular if $\operatorname{rank} g = n$ (and therefore n is even), then

$$\operatorname{rank}(f \wedge g) = \operatorname{rank} g - 1 = n - 1.$$

(v) Let $g \in \Lambda^k(\mathbb{R}^n)$. Then $\operatorname{rank} g = k$ if and only if (cf. Proposition 5 below)

$$g \wedge [f \lrcorner g] = 0$$

for every $f \in \Lambda^{k-1}(\mathbb{R}^n)$.

(vi) From (iv) of the proposition we can infer that if $g \neq 0$, then

$$\operatorname{rank}(*g) \geq n - \operatorname{rank} g.$$

When $k = 1$ then the inequality becomes an equality. In general, however, as soon as $2 \leq k \leq n-2$ the inequality can be strict.

2.3 Decomposability of a form

The other important algebraic concept is the notion of decomposability.

Definition 4 *Let $g \in \Lambda^k(\mathbb{R}^n)$, $1 \leq k \leq n$.*

(i) *Let $1 \leq l \leq k - l$. We say that g is* $l-$decomposable *if there exist $a \in \Lambda^l(\mathbb{R}^n)$ and $b \in \Lambda^{k-l}(\mathbb{R}^n)$ such that*

$$g = a \wedge b.$$

We say that g is indecomposable *if it is not $l-$decomposable for any l.*

(ii) *We say that g is* totally decomposable *if there exist $a_1, \cdots, a_k \in \Lambda^1(\mathbb{R}^n)$ such that*

$$g = a_1 \wedge \cdots \wedge a_k\,.$$

We now gather some properties whose proofs are elementary; the last result is known as Cartan lemma (see, for example, [16]).

Proposition 5 *Let $1 \leq k \leq n$ and $g \in \Lambda^k(\mathbb{R}^n)$ with $g \neq 0$.*

(i) *The form g is $1-$decomposable meaning that there exist $a \in \Lambda^1(\mathbb{R}^n)$ and $b \in \Lambda^{k-1}(\mathbb{R}^n)$ such that*

$$g = a \wedge b$$

if and only if there exists $a \in \Lambda^1(\mathbb{R}^n)$, $a \neq 0$, such that

$$g \wedge a = 0$$

if and only if there exists $f \in \Lambda^{k-1}(\mathbb{R}^n)$ such that $f \lrcorner g \neq 0$ and

$$g \wedge [f \lrcorner g] = 0.$$

(ii) *The form g is totally decomposable if and only if* $\operatorname{rank} g = k$ *if and only if*

$$g \wedge [f \lrcorner g] = 0$$

for every $f \in \Lambda^{k-1}(\mathbb{R}^n)$.

(iii) *If k is odd and if* $\operatorname{rank} g = k + 2$, *then g is $1-$decomposable.*

(iv) *Let $1 \leq l \leq k \leq n$ and $a_1, \cdots, a_l \in \Lambda^1(\mathbb{R}^n)$ be such that*

$$a_1 \wedge \cdots \wedge a_l \neq 0.$$

Then there exists $f \in \Lambda^{k-l}(\mathbb{R}^n)$ verifying

$$g = f \wedge a_1 \wedge \cdots \wedge a_l$$

if and only if

$$g \wedge a_1 = \cdots = g \wedge a_l = 0.$$

Remark (i) In the literature the second definition is standard; it goes back to Cartan and such a form is, sometimes, also called *pure* or only *decomposable*. The first definition seems to be new. Note that we have restricted $1 \leq l \leq k - l$, since a $k-$form is l decomposable if and only if it is $(k - l)$ decomposable.

(ii) It is important to stress that, in general, the rank of a $k-$form and its decomposability are two distinct notions. The only exceptions are the cases $k = 2, n - 1, n$, where the rank determines completely the decomposability.

- If $k = 2$, we have only two possibilities. Either $\operatorname{rank} g = 2$ and then (according to (ii) of Proposition 5) the form g is totally decomposable, or $\operatorname{rank} g \geq 4$ and it is indecomposable.

- If $k = n - 1, n$ then we are automatically in the framework of (ii) of Proposition 5 and therefore all forms are totally decomposable.

(iii) We should point out that a form is *not uniquely* decomposable into indecomposable forms. Indeed consider

$$g = \left[dx^1 \wedge dx^2 + dx^3 \wedge dx^4\right] \wedge dx^3 = dx^1 \wedge dx^2 \wedge dx^3$$

and observe that it is a product of one indecomposable $2-$form of rank 4 and one (indecomposable) $1-$form and, at the same time, a product of three (indecomposable) $1-$forms. However only the second one is an optimal decomposition of g, in the sense that

$$g = g_{k_1} \wedge \cdots \wedge g_{k_s}$$

with $k_1 + \cdots + k_s = k$, $g_{k_i} \in \Lambda^{k_i}(\mathbb{R}^n)$ indecomposable and

$$\operatorname{rank} g = \sum_{i=1}^{s} \operatorname{rank} g_{k_i} \, .$$

An optimal decomposition of the above type does not always exist as the following example shows. Let $g \in \Lambda^4\left(\mathbb{R}^6\right)$ with $\operatorname{rank} g = 6$ and given by

$$g = a \wedge b = \left[dx^1 \wedge dx^2 + dx^3 \wedge dx^4\right] \wedge \left[dx^1 \wedge dx^2 + dx^5 \wedge dx^6\right].$$

Note that $a, b \in \Lambda^2\left(\mathbb{R}^6\right)$ are indecomposable and $\operatorname{rank} a = \operatorname{rank} b = 4$. It can be seen that g cannot be optimally decomposed in the above sense.

(iv) The statement (iii) in the proposition is, in general, false when k is even. Indeed the form $g \in \Lambda^4(\mathbb{R}^n)$ given by

$$g = dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 + dx^1 \wedge dx^2 \wedge dx^5 \wedge dx^6 + dx^3 \wedge dx^4 \wedge dx^5 \wedge dx^6$$

is not $1-$decomposable (although it is $2-$decomposable) while $\operatorname{rank} g = k + 2 = 6$.

(v) When $k = 3$, $g \neq 0$ and $\operatorname{rank} g$ is even then g is indecomposable. This easily follows from the fact that if g is $1-$decomposable, there exists $a \in \Lambda^1(\mathbb{R}^n)$ and $b \in \Lambda^2(\mathbb{R}^n)$ so that $g = a \wedge b$ and therefore $\operatorname{rank} g$ is odd.

2.4 Solutions to some algebraic equations

We here provide an example, that will be used later, on how to solve equations involving the interior or the exterior product.

Proposition 6 *Let $g \in \Lambda^2(\mathbb{R}^n)$ with $\operatorname{rank} g = 2l \leq n$ and $a \in \Lambda^1(\mathbb{R}^n)$. There exists $u \in \Lambda^1(\mathbb{R}^n)$ such that*

$$a = u \lrcorner g$$

if and only if

$$a \wedge g^l = 0.$$

Remark (i) If $n = 2l$, then $a \wedge g^l = 0$ is always satisfied (since then $a \wedge g^l \in \Lambda^{n+1}(\mathbb{R}^n)$) and therefore there always exists $u \in \Lambda^1(\mathbb{R}^n)$ such that $a = u \lrcorner g$.

(ii) More generally if l is even, $g \in \Lambda^k(\mathbb{R}^n)$ with $g^{l+1} = 0$ $(lk \leq n)$ and there exists $u \in \Lambda^1(\mathbb{R}^n)$ such that $a = u \lrcorner g$, then necessarily (cf. Proposition 1)

$$a \wedge g^l = 0.$$

However the converse is then, in general, not true.

Before starting our proof we will need the following lemma, whose proof is elementary (see [12]).

Lemma 7 *Let $2 \leq 2l \leq n$ be integers and $g \in \Lambda^2(\mathbb{R}^n)$. Then, for every $1 \leq m \leq 2l$,*

$$\frac{1}{l}\left(g^l\right)_{i_1 \cdots i_{2l}} = \sum_{j=1}^{m-1} (-1)^{j+m+1} g_{i_j i_m} \left(g^{l-1}\right)_{i_1 \cdots \widehat{i_j} \cdots \widehat{i_m} \cdots i_{2l}} + \sum_{j=m+1}^{2l} (-1)^{j+m+1} g_{i_m i_j} \left(g^{l-1}\right)_{i_1 \cdots \widehat{i_m} \cdots \widehat{i_j} \cdots i_{2l}}$$

for every set of $(2l)-$indices $1 \leq i_1, \cdots, i_{2l} \leq n$.

Notation 8 *When in an index we write $\hat{\imath}$, this means that i is omitted. For example*

$$h_{1 \cdots \widehat{4} \cdots k} = h_{1235 \cdots k}\,.$$

We will adopt this notation throughout the present article.

We can now turn our attention to the proof of Proposition 6.

Proof *Step 1.* We start with the necessary part. Since $\operatorname{rank} g = 2l$, we have $g^l \neq 0$ and $g^{l+1} = 0$. We hence deduce, using Proposition 1, that

$$0 = u \lrcorner g^{l+1} = (l+1)(u \lrcorner g) \wedge g^l = (l+1)\, a \wedge g^l,$$

which implies that $a \wedge g^l = 0$.

Step 2. We now prove the sufficiency part. Without loss of generality, we can assume that $(g^l)_{1 \cdots (2l)} \neq 0$. Since $a \wedge g^l = 0$ we have, for every $2l+1 \leq k \leq n$,

$$a^k (g^l)_{1 \cdots (2l)} = \sum_{i=1}^{2l} (-1)^i a^i (g^l)_{1 \cdots \hat{\imath} \cdots (2l) k}\,. \tag{3}$$

Note that (3) is trivially valid for $1 \leq k \leq 2l$. For every $1 \leq k \leq n$, it follows, from Lemma 7 (with $m = 2l$, $i_j = j$ if $j < i$, $i_j = j+1$ if $i \leq j \leq 2l-1$ and $i_{2l} = k$), that

$$\frac{1}{l}(g^l)_{1 \cdots \hat{\imath} \cdots (2l) k} = \sum_{j=1}^{i-1} (-1)^{j+1} g_{jk} (g^{l-1})_{1 \cdots \hat{\jmath} \cdots \hat{\imath} \cdots (2l)} + \sum_{j=i+1}^{2l} (-1)^j g_{jk} (g^{l-1})_{1 \cdots \hat{\imath} \cdots \hat{\jmath} \cdots (2l)}\,.$$

We thus deduce from (3) that, for every $1 \leq k \leq n$,

$$a^k (g^l)_{1 \cdots (2l)} = l \sum_{i=1}^{2l} (-1)^i a^i \left[\begin{array}{c} \sum_{j=1}^{i-1} (-1)^{j+1} g_{jk} (g^{l-1})_{1 \cdots \hat{\jmath} \cdots \hat{\imath} \cdots (2l)} \\ + \sum_{j=i+1}^{2l} (-1)^j g_{jk} (g^{l-1})_{1 \cdots \hat{\imath} \cdots \hat{\jmath} \cdots (2l)} \end{array} \right]$$

and thus

$$a^k(g^l)_{1\cdots(2l)} = -l\sum_{j=1}^{2l}(-1)^{j+1}g_{jk}\left[\begin{array}{c}\sum_{i=1}^{j-1}(-1)^i a^i(g^{l-1})_{1\cdots\widehat{i}\cdots\widehat{j}\cdots(2l)}\\ -\sum_{i=j+1}^{2l}(-1)^i a^i(g^{l-1})_{1\cdots\widehat{j}\cdots\widehat{i}\cdots(2l)}\end{array}\right].$$

We now define $u \in \Lambda^1(\mathbb{R}^n)$ by

$$u^j = 0 \quad \text{if } 2l+1 \le j \le n$$

while, if $1 \le j \le 2l$,

$$u^j = \frac{(-1)^{j+1}\,l}{(g^l)_{1\cdots(2l)}}\left[\sum_{i=1}^{j-1}(-1)^i a^i(g^{l-1})_{1\cdots\widehat{i}\cdots\widehat{j}\cdots(2l)} - \sum_{i=j+1}^{2l}(-1)^i a^i(g^{l-1})_{1\cdots\widehat{j}\cdots\widehat{i}\cdots(2l)}\right]$$

It follows from the definition of u that, for every $1 \le k \le n$,

$$(u \lrcorner g)^k = -\sum_{j=1}^{2l} g_{jk}u^j = a^k\,.$$

In other words, we have $u \lrcorner g = a$. This is the desired claim. ■

2.5 Necessary conditions for the pullback equation

In the following theorem we gather elementary necessary conditions (statements (ii), (iii) and (iv) below are straightforward and were already mentioned in [3]).

Theorem 9 *Let $\Omega \subset \mathbb{R}^n$ be a smooth domain, $1 \le k \le n$, $f, g \in C^1(\overline{\Omega};\Lambda^k)$ and $\varphi \in \operatorname{Diff}^1\left(\overline{\Omega};\mathbb{R}^n\right)$ be such that*

$$\varphi^*(g) = f \text{ in } \Omega.$$

Then the following conditions hold.

(i) *The forms $f(x)$ and $g(\varphi(x))$ satisfy, for every $x \in \Omega$,*

$$\operatorname{rank} g(\varphi(x)) = \operatorname{rank} f(x)\,.$$

(ii) *If $dg = 0$ in Ω, then*

$$df = 0.$$

(iii) *If Ω is bounded, $\varphi\left(\overline{\Omega}\right) = \overline{\Omega}$ and $n = mk$ with m an integer, then*

$$\int_\Omega f^m = \int_\Omega g^m$$

where $f^m = \underbrace{f\wedge\cdots\wedge f}_{m-times}$.

(iv) *If $\varphi(x) = x$ for $x \in \partial\Omega$, then*

$$f \wedge \nu = g \wedge \nu \text{ on } \partial\Omega$$

where ν is the exterior unit normal to Ω.

Remark Note that pullback preserves not only the rank but also the decomposability in the sense that if

$$g = a \wedge b, \quad a \in \Lambda^l(\mathbb{R}^n), \quad b \in \Lambda^{k-l}(\mathbb{R}^n)$$

and $\varphi^*(g) = f$, then $c = \varphi^*(a) \in \Lambda^l(\mathbb{R}^n)$, $d = \varphi^*(b) \in \Lambda^{k-l}(\mathbb{R}^n)$ are such that

$$f = c \wedge d.$$

3 New proofs of Darboux theorem in the degenerate case

3.1 The main result

We start by recalling the classical Darboux theorem. The version that we quote below gives the optimal regularity result, which is a particularly delicate point, and has been established by Bandyopadhyay-Dacorogna (see Theorem 18 in [3]).

Theorem 10 (Darboux theorem with optimal regularity) *Let $r \geq 0$ and $n = 2m \geq 4$ be integers. Let $0 < \alpha < 1$ and $x_0 \in \mathbb{R}^n$. Let ω_m be the standard symplectic form*

$$\omega_m = \sum_{i=1}^{m} dx^{2i-1} \wedge dx^{2i}.$$

Let ω be a $2-$form. The two following statements are then equivalent.

(i) *The $2-$form ω is closed, is in $C^{r,\alpha}$ in a neighbourhood of x_0 and verifies*

$$\operatorname{rank} \omega(x_0) = n.$$

(ii) *There exist a neighbourhood V of x_0 and $\varphi \in \operatorname{Diff}^{r+1,\alpha}(V;\mathbb{R}^n)$ such that*

$$\varphi^*(\omega_m) = \omega \text{ in } V \quad \text{and} \quad \varphi(x_0) = x_0\,.$$

Remark (i) We refer to [3] for global results including Dirichlet data.

(ii) When $r = 0$, the hypothesis $d\omega = 0$ is to be understood in the sense of distributions.

(iii) The theorem is still valid when $n = 2$, but it is then the result of Dacorogna-Moser [10] (cf. Theorem 14.6 in [9]).

(iv) We will use throughout the following notations. Let $\Omega \subset \mathbb{R}^n$ be open, r a non-negative integer and $0 < \alpha \leq 1$.

- We denote by $C^{r,\alpha}\left(\overline{\Omega}\right)$ the usual set of Hölder functions and by $C^{r,\alpha}\left(\overline{\Omega};\Lambda^k\right)$ the set of $k-$forms

$$g = \sum_{1 \leq i_1 < \cdots < i_k \leq n} g_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

so that $g_{i_1 \cdots i_k} \in C^{r,\alpha}\left(\overline{\Omega}\right)$.

- The sets $\text{Diff}^r(\overline{\Omega};\mathbb{R}^n)$ and $\text{Diff}^{r,\alpha}(\overline{\Omega};\mathbb{R}^n)$ denote the sets of diffeomorphisms φ so that $\varphi \in C^r\left(\overline{\Omega};\mathbb{R}^n\right)$ and $\varphi^{-1} \in C^r\left(\varphi\left(\overline{\Omega}\right);\mathbb{R}^n\right)$, $C^{r,\alpha}$ respectively.

- When $\alpha = 0$, sometimes, by abuse of notations, we write indistinctly $C^{r,0}$ or C^r.

The main result of the present section is the following version of Darboux theorem for degenerate forms.

Theorem 11 *Let $n \geq 3$, $r, l \geq 1$ be integers and $0 < \alpha < 1$. Let $x_0 \in \mathbb{R}^n$ and ω_l be the standard symplectic form of rank $2l < n$, namely*

$$\omega_l = \sum_{i=1}^{l} dx^{2i-1} \wedge dx^{2i}.$$

Let ω be a $C^{r,\alpha}$ closed $2-$form such that

$$\operatorname{rank} \omega = 2l \text{ in a neighbourhood of } x_0\,.$$

Then there exist a neighbourhood V of x_0 and $\varphi \in \text{Diff}^{r,\alpha}(V;\mathbb{R}^n)$ such that

$$\varphi^*\left(\omega_l\right) = \omega \text{ in } V \quad \text{and} \quad \varphi\left(x_0\right) = x_0\,.$$

Remark (i) The theorem is standard in the C^∞ case. In all proofs that we have seen the regularity that is established is, at best, that if $\omega \in C^r$ then $\varphi \in C^{r-1}$. However our result asserts that ω and φ have the same regularity in Hölder spaces. This is, of course, better but still not optimal as in the non-degenerate case of Theorem 10.

(ii) The standard proof uses Frobenius theorem to reduce the dimension so that the forms have maximal rank and then apply the classical Darboux theorem. We will follow this path but using the more sophisticated Theorem 10. We will also provide a completely different proof in Section 3.4; it will use an argument based on the flow method. Still a different proof can be found in [3] when $n = 2l + 1$.

Proof *Step 1.* Without loss of generality, we can assume $x_0 = 0$. We first find, appealing to Theorem 31, two neighbourhoods V, Ω of 0 and $\psi \in \text{Diff}^{r,\alpha}(V;\Omega)$ with $\psi(0) = 0$ and

$$\psi^*(\omega)\left(x^1,\cdots,x^n\right) = \widetilde{\omega}\left(x^1,\cdots,x^{2l}\right) = \sum_{1\leq i<j\leq 2l} \widetilde{\omega}_{ij}\left(x^1,\cdots,x^{2l}\right) dx^i \wedge dx^j.$$

Therefore $\psi^*(\omega) = \widetilde{\omega} \in C^{r-1,\alpha}$ in a neighbourhood of 0 in $\mathbb{R}^{2l}$ and $\operatorname{rank}\widetilde{\omega} = 2l$ in a neighbourhood of 0.

Step 2. We then apply Theorem 10 to $\widetilde{\omega}$ and find a neighbourhood $U \subset \mathbb{R}^{2l}$ of 0 and $\chi \in \text{Diff}^{r,\alpha}(U;\mathbb{R}^{2l})$, with $\chi(0) = 0$, such that

$$\chi^*\left(\omega_l\right) = \widetilde{\omega} \text{ in } U.$$

Setting

$$\widetilde{\chi}(y) = \widetilde{\chi}\left(y^1,\cdots,y^{2l},y^{2l+1},\cdots,y^n\right) = \left(\chi\left(y^1,\cdots,y^{2l}\right),y^{2l+1},\cdots,y^n\right)$$

and $\varphi = \widetilde{\chi} \circ \psi^{-1}$, we have the claim. ■

3.2 Poincaré lemma with constraint

We start with a generalization of Poincaré lemma that will play the central role in our second proof (Theorem 20) of Darboux theorem in the degenerate case.

Theorem 12 *Let $2 \leq 2l \leq n$ be integers and $x_0 \in \mathbb{R}^n$. Let f, g be two C^∞ closed $2-$forms such that, in a neighbourhood of x_0,*

$$f \wedge g^l = 0 \quad \textit{and} \quad \operatorname{rank} g = 2l.$$

Then there exist a neighbourhood V of x_0 and $\omega \in C^\infty(V; \Lambda^1)$ such that

$$d\omega = f, \quad \omega(x_0) = 0 \quad \textit{and} \quad \omega \wedge g^l = 0.$$

Remark We can easily replace C^∞ by C^r, but a refined version of our construction finds ω only in C^{r-1} (see [12]).

Proof The theorem is easily proved if we can invoke Theorem 11. However, since we want to use the present theorem to establish Theorem 11, we have to find an independent proof.

Step 1. Without loss of generality, we can assume that $x_0 = 0$. Appealing to the classical Poincaré lemma, we can find a neighbourhood V of 0 and $u \in C^\infty(V; \Lambda^1)$ with $u(0) = 0$ and $du = f$ in V. We then set

$$\omega = u - dv.$$

Our result will follow if we can find $v \in C^\infty(V)$ verifying

$$\begin{cases} dv \wedge g^l = u \wedge g^l & \text{in } V \\ dv(0) = 0. \end{cases} \tag{4}$$

Step 2. Since $g^l \neq 0$, we can find a matrix $A \in GL_n(\mathbb{R})$, so that if $\varphi(x) = Ax$, then

$$(\varphi^*(g^l))_{1\cdots(2l)}(0) > 0.$$

The problem (4) is then equivalent to finding $w \in C^\infty(V)$ such that

$$\begin{cases} dw \wedge \varphi^*(g^l) = \varphi^*(u) \wedge \varphi^*(g^l) & \text{in } V \\ dw(0) = 0. \end{cases}$$

Indeed it is enough to set $v = (\varphi^{-1})^*(w)$ to have a solution of (4). So from now on we will assume, without loss of generality, that

$$(g^l)_{1\cdots(2l)}(0) > 0.$$

Step 3. We then solve (4) by induction on n, l being fixed. In the case $n = 2l$ nothing is to be proved, just choose $v = 0$. So we assume that the result has been proven for $n = 2l + j$, $j \geq 0$, and let us prove it for $n = 2l + j + 1$. We therefore assume that we can find a neighbourhood $\widetilde{V} \subset \mathbb{R}^{n-1}$ of $0 \in \mathbb{R}^{n-1}$ and $h \in C^\infty(\widetilde{V})$ with

$$\begin{cases} dh \wedge \widetilde{g}^l = \widetilde{u} \wedge \widetilde{g}^l \\ dh(0) = 0 \end{cases}$$

whenever $\widetilde{g} \in C^\infty(\widetilde{V}; \Lambda^2(\mathbb{R}^{n-1}))$ and $\widetilde{u} \in C^\infty(\widetilde{V}; \Lambda^1(\mathbb{R}^{n-1}))$ verify

$$d\widetilde{u} \wedge \widetilde{g}^l = 0, \quad d\widetilde{g} = 0, \quad \widetilde{g}^{l+1} = 0 \quad \text{and} \quad (\widetilde{g}^l)_{1\cdots 2l}(0) \neq 0$$

and let us prove that it holds for n. To establish this result we proceed in five substeps.

Step 3.1. It follows from Proposition 19 that there exist V and W neighbourhoods of $0 \in \mathbb{R}^n$ and $\varphi \in \mathrm{Diff}^\infty(W; V)$ with $\varphi(0) = 0$ and such that

$$\varphi^*(g) = \sum_{i=1}^{l} dx^{2i-1} \wedge dx^{2i} + \sum_{\substack{1 \leq i < j \leq n \\ 2l < j}} r_{ij} dx^i \wedge dx^j$$

for some $r_{ij} \in C^\infty(W)$. Therefore solving (4) is equivalent to finding a solution v of

$$\begin{cases} dv \wedge (\varphi^*(g))^l = \varphi^*(u) \wedge (\varphi^*(g))^l \\ dv(0) = 0. \end{cases}$$

So from now on we will assume, upon substitution of W, $\varphi^*(g)$ and $\varphi^*(u)$ by V, g and u, that

$$g = \sum_{i=1}^{l} dx^{2i-1} \wedge dx^{2i} + \sum_{\substack{1 \leq i < j \leq n \\ 2l < j}} r_{ij} dx^i \wedge dx^j \tag{5}$$

and we therefore have to find $v \in C^\infty(V)$ satisfying (4) only for g as in (5). Note that for such a g we have

$$(g^l)_{1\cdots(2l)} = l\,! \neq 0.$$

Step 3.2. We then solve, by the method of characteristics, the Cauchy problem for the first order partial differential equation

$$\begin{cases} (dv \wedge g^l)_{1\cdots(2l)n} = (u \wedge g^l)_{1\cdots(2l)n} \\ v(x^1, \cdots, x^{n-1}, 0) = h(x^1, \cdots, x^{n-1}) \end{cases} \tag{6}$$

where $h \in C^r(\widetilde{V})$ is a solution, which exists by hypothesis of induction, of

$$\begin{cases} dh \wedge i_n^*(g)^l = i_n^*(u) \wedge i_n^*(g)^l \\ dh(0) = 0. \end{cases} \tag{7}$$

and

$$i_n(x^1, \cdots, x^{n-1}) = (x^1, \cdots, x^{n-1}, 0)$$

Observe that

$$i_n^*(v) = h \quad \Leftrightarrow \quad v(x^1, \cdots, x^{n-1}, 0) = h(x^1, \cdots, x^{n-1}).$$

Note that we can apply the hypothesis of induction since

$$dg = 0, \quad du \wedge g^l = 0, \quad g^{l+1} = 0 \quad \text{and} \quad (g^l)_{1\cdots(2l)}(0) \neq 0$$

imply

$$d(i_n^*(g)) = 0,\ d(i_n^*(u)) \wedge i_n^*(g)^l = 0,\ i_n^*(g)^{l+1} = 0 \text{ and } (i_n^*(g))_{1\cdots(2l)}(0) \neq 0.$$

Step 3.3. It now remains to prove that the solution v of (6) is indeed a solution of (4). We therefore have to show that

$$dv \wedge g^l = u \wedge g^l \tag{8}$$

and

$$dv(0) = 0. \tag{9}$$

Using (6), (7) and the fact that $u(0) = 0$, we easily see that (9) holds. We here only establish (8). Lemma 15 implies that to show (8) we only need to prove

$$\left(dv \wedge g^l\right)_{1\cdots(2l)k} = (u \wedge g^l)_{1\cdots(2l)k}\,, \quad 2l+1 \leq k \leq n. \tag{10}$$

Step 3.4. We now prove (10). Define, for every $2l+1 \leq k \leq n$,

$$L_k v = (dv \wedge g^l)_{1\cdots(2l)k} \quad \text{and} \quad w_k = L_k v - (u \wedge g^l)_{1\cdots(2l)k}\,.$$

Since we already have from (6) that $w_n = 0$, our claim (10) reduces to proving that

$$w_k = 0 \quad \text{for every } 2l+1 \leq k \leq n-1. \tag{11}$$

Since

$$0 = f \wedge g^l = du \wedge g^l$$

and (5) holds, we can apply Lemma 16 and Lemma 17 to obtain

$$\begin{aligned} L_n w_k &= L_n L_k v - L_n \left((u \wedge g^l)_{1\cdots(2l)k}\right) \\ &= L_k L_n v - L_k \left((u \wedge g^l)_{1\cdots(2l)n}\right) = L_k w_n = 0. \end{aligned}$$

Assume, cf. Step 3.5, that we can prove that

$$w_k(x^1, \cdots, x^{n-1}, 0) = 0 \tag{12}$$

we will then have, by uniqueness of the solutions of the Cauchy problem, that the only solution of

$$\begin{cases} L_n w_k = 0 \\ \ \ w_k = 0 \qquad \text{on } x^n = 0. \end{cases}$$

is $w_k = 0$. This is exactly our claim (11).

Step 3.5. Finally we show (12), which is equivalent to proving that $i_n^*(w_k) = 0$. We have that, recalling that $i_n^*(v) = h$,

$$\begin{aligned} i_n^*(w_k) &= i_n^*\left((dv \wedge g^l - u \wedge g^l)_{1\cdots(2l)k}\right) = \left(i_n^*(dv \wedge g^l - u \wedge g^l)\right)_{1\cdots(2l)k} \\ &= \left(d(i_n^*(v)) \wedge i_n^*(g^l)) - i_n^*(u) \wedge i_n^*(g^l)\right)_{1\cdots(2l)k} \end{aligned}$$

and thus, appealing to (7),

$$i_n^*(w_k) = \left(dh \wedge i_n^*(g^l) - i_n^*(u) \wedge i_n^*(g^l)\right)_{1\cdots(2l)k} = 0.$$

This concludes the proof of the theorem. ■

With substantially the same proof we can get a global result (see Kneuss [12]).

Theorem 13 *Let $2 \leq 2l < n$ and $g, f \in C^\infty(\mathbb{R}^n; \Lambda^2)$ be closed. Assume that g is of the form*

$$g = \sum_{i=1}^{l} dx^{2i-1} \wedge dx^{2i} + \sum_{\substack{1 \leq i < j \leq n \\ 2l < j}} g_{ij}\, dx^i \wedge dx^j$$

for some $g_{ij} \in C^\infty(\mathbb{R}^n)$ and, for every $x \in \mathbb{R}^n$,

$$g^{l+1}(x) = 0 \quad \text{and} \quad |g(x)| \leq a\,|x| + b$$

where $a, b > 0$ are constants. Then there exists $w \in C^\infty(\mathbb{R}^n; \Lambda^1)$ so that the following equations are satisfied on the whole of $\mathbb{R}^n$

$$dw = f \quad \text{and} \quad w \wedge g^l = 0.$$

3.3 Some technical algebraic lemmas

In this subsection we gather all algebraic lemmas that we have used in the proof of Theorem 12. In the sequel we will adopt Notation 8.

Lemma 14 *Let $2 \leq 2l < n$ be integers and $g \in \Lambda^2(\mathbb{R}^n)$ with $\operatorname{rank} g = 2l$ and of the form*

$$g = \sum_{i=1}^{l} dx^{2i-1} \wedge dx^{2i} + \sum_{\substack{1 \leq i < j \leq n \\ 2l < j}} g_{ij}\, dx^i \wedge dx^j.$$

Then, for every $1 \leq i, j \leq 2l < k \leq n$, the following holds

$$\begin{aligned} &(g^l)_{1\cdots\widehat{i}\cdots(2l)n}\,(g^l)_{1\cdots\widehat{j}\cdots(2l)k} - (g^l)_{1\cdots\widehat{i}\cdots(2l)k}\,(g^l)_{1\cdots\widehat{j}\cdots(2l)n} \\ &= \begin{cases} 0 & \text{if } i = j \\ l!\,(g^l)_{1\cdots\widehat{i}\cdots\widehat{j}\cdots(2l)kn} & \text{if } i < j \\ -l!\,(g^l)_{1\cdots\widehat{j}\cdots\widehat{i}\cdots(2l)kn} & \text{if } i > j \end{cases} \end{aligned}$$

Remark When $l = 1$ then $g^2 = 0$ and the conclusion of the lemma is immediate and reads as

$$g_{2n} g_{1k} - g_{2k} g_{1n} = g_{kn} = g_{12} g_{kn} .$$

Proof We begin by noting that, when $i = j$, the proof is trivial. We prove the result for $i < j$, which in turn implies the case $i > j$. Since $\operatorname{rank} g = 2l$, we have

$$g^{l+1} = 0. \tag{13}$$

Moreover, the special structure of g gives

$$g_{12} = g_{34} = \cdots = g_{(2l-1)(2l)} = 1 \tag{14}$$

$$g_{ij} = 0, \text{ for every } 1 \le i < j \le 2l \text{ with } \{i, j\} \ne \{2s-1, 2s\}, 1 \le s \le l. \tag{15}$$

We define $\epsilon : \mathbb{N} \to \{-1, 1\}$ as

$$\epsilon(i) = \begin{cases} -1 & \text{if } i \text{ is even} \\ 1 & \text{if } i \text{ is odd} \end{cases}$$

Note that $\{i + \epsilon(i), i\} = \{2s - 1, 2s\}$ for some $s \in \mathbb{N}$.

Step 1. We first show that, for every $1 \le i \le 2l < k \le n$, we have

$$(g^l)_{1 \cdots \widehat{i} \cdots (2l)k} = l!\, g_{(i+\epsilon(i))k} . \tag{16}$$

We assume that i is even (the case i odd, can be handled exactly in same way), in which case $i + \epsilon(i) = i - 1$. It follows, from Lemma 7, that

$$\begin{aligned} (g^l)_{1 \cdots \widehat{i} \cdots (2l)k} = l & \sum_{j=1}^{i+\epsilon(i)-1} (-1)^{j+i+\epsilon(i)+1} g_{j(i+\epsilon(i))} \left(g^{l-1}\right)_{1 \cdots \widehat{j} \cdots \widehat{i+\epsilon(i)} \widehat{i} \cdots (2l)k} \\ + \; l & \sum_{j=i+1}^{2l} (-1)^{j+i+\epsilon(i)} g_{(i+\epsilon(i))j} \left(g^{l-1}\right)_{1 \cdots \widehat{i+\epsilon(i)} \widehat{i} \cdots \widehat{j} \cdots (2l)k} \\ + \; l & \, g_{(i+\epsilon(i))k} \left(g^{l-1}\right)_{1 \cdots \widehat{i+\epsilon(i)} \widehat{i} \cdots \cdots (2l)} . \end{aligned} \tag{17}$$

Using (15), it follows that $g_{j(i+\epsilon(i))} = 0$, for every $1 \le j \le i + \epsilon(i) - 1$ and $g_{(i+\epsilon(i))j} = 0$, for every $i + 1 \le j \le 2l$. We now appeal to (14) and (15) to deduce that

$$g^{l-1}_{1 \cdots \widehat{i+\epsilon(i)} \widehat{i} \cdots \cdots (2l)} = (l-1)! . \tag{18}$$

Finally, we combine (17) and (18) to arrive at (16).

Step 2. Thanks to Step 1, to prove the lemma, it is enough to show that

$$l!\, g_{(i+\epsilon(i))n}\, g_{(j+\epsilon(j))k} - l!\, g_{(i+\epsilon(i))k}\, g_{(j+\epsilon(j))n} = (g^l)_{1 \cdots \widehat{i} \cdots \widehat{j} \cdots (2l)kn} . \tag{19}$$

Recall that we discuss only the case $i < j$. We will consider two cases to establish (19).

Case 1: $\{i,j\} \neq \{2s-1, 2s\}$, for every $1 \leq s \leq l$. We only deal with the case i even (the case i odd is handled similarly). This implies that $i + \epsilon(i) = i - 1$. From Lemma 7 it follows that

$$\begin{aligned}(g^l)_{1\cdots\widehat{i}\cdots\widehat{j}\cdots(2l)kn} = l &\sum_{t=1}^{i+\epsilon(i)-1} (-1)^{t+i+\epsilon(i)+1} g_{t(i+\epsilon(i))}\, (g^{l-1})_{1\cdots\widehat{t}\cdots\widehat{i+\epsilon(i)}\widehat{i}\cdots\widehat{j}\cdots(2l)kn} \\ &+ l \sum_{t=i+1}^{j-1} (-1)^{t+i+\epsilon(i)} g_{(i+\epsilon(i))t}\, (g^{l-1})_{1\cdots\widehat{i+\epsilon(i)}\widehat{i}\cdots\widehat{t}\cdots\widehat{j}\cdots(2l)kn} \\ &+ l \sum_{t=j+1}^{2l} (-1)^{t+i+\epsilon(i)+1} g_{(i+\epsilon(i))t}\, (g^{l-1})_{1\cdots\widehat{i+\epsilon(i)}\widehat{i}\cdots\widehat{j}\cdots\widehat{t}\cdots(2l)kn} \\ &- l\, g_{(i+\epsilon(i))k}\, (g^{l-1})_{1\cdots\widehat{i+\epsilon(i)}\widehat{i}\cdots\widehat{j}\cdots(2l)n} \\ &+ l\, g_{(i+\epsilon(i))n}\, (g^{l-1})_{1\cdots\widehat{i+\epsilon(i)}\widehat{i}\cdots\widehat{j}\cdots(2l)k}\,.\end{aligned}$$

Proceeding exactly as in Step 1, we can show that

$$(g^{l-1})_{1\cdots\widehat{i+\epsilon(i)}\widehat{i}\cdots\widehat{j}\cdots(2l)k} = (l-1)!\, g_{(j+\epsilon(j))k}\,. \tag{20}$$

Using (15) and (20), it follows that

$$(g^l)_{1\cdots\widehat{i}\cdots\widehat{j}\cdots(2l)kn} = -l!\, g_{(i+\epsilon(i))k}\, g_{(j+\epsilon(j))n} + l!\, g_{(i+\epsilon(i))n}\, g_{(j+\epsilon(j))k}$$

which is nothing else than (19).

Case 2: $\{i,j\} = \{2s-1, 2s\}$, for some $1 \leq s \leq l$. Since $(g^{l+1})_{1\cdots(2l)kn} = 0$, we get, using once more Lemma 7, that

$$\begin{aligned}0 = (g^{l+1})_{1\cdots(2l)kn} = (l+1) &\sum_{t=1}^{2s-2} (-1)^{t+2s} g_{t(2s-1)}\, (g^l)_{1\cdots\widehat{t}\cdots\widehat{2s-1}\cdots(2l)kn} \\ &+ (l+1) g_{(2s-1)(2s)}\, (g^l)_{1\cdots\widehat{2s-1}\widehat{2s}\cdots(2l)kn} \\ &+ (l+1) \sum_{t=2s+1}^{2l} (-1)^{t+2l} g_{(2s-1)t} (g^l)_{1\cdots\widehat{2s-1}\cdots\widehat{t}\cdots(2l)kn} \\ &- (l+1) g_{(2s-1)k}\, (g^l)_{1\cdots\widehat{2s-1}\cdots(2l)n} \\ &+ (l+1) g_{(2s-1)n}\, (g^l)_{1\cdots\widehat{2s-1}\cdots(2l)k}\,.\end{aligned}$$

We next invoke Step 1, (14) and (15), to deduce that

$$0 = (g^l)_{1\cdots\widehat{2s-1}\widehat{2s}\cdots(2l)kn} - l!\, g_{(2s-1)k}\, g_{(2s)n} + l!\, g_{(2s-1)n}\, g_{(2s)k}\,. \tag{21}$$

Note that we then have $i = 2s-1$, $j = 2s$, $i + \epsilon(i) = 2s$ and $j + \epsilon(j) = 2s-1$. Hence (21) implies that

$$l!\, g_{(i+\epsilon(i))n}\, g_{(j+\epsilon(j))k} - l!\, g_{(i+\epsilon(i))k}\, g_{(j+\epsilon(j))n} = (g^l)_{1\cdots\widehat{i}\cdots\widehat{j}\cdots(2l)kn}$$

which is exactly (19). This finishes the proof. ■

The next lemma has been used in Step 3.3 of the proof of Theorem 12.

Lemma 15 *Let $2 \leq 2l \leq n$ be integers, $g \in \Lambda^2(\mathbb{R}^n)$ and $\omega \in \Lambda^1(\mathbb{R}^n)$ with*

$$g^{l+1} = 0, \quad (g^l)_{12\cdots\cdots(2l-1)(2l)} \neq 0$$

$$(\omega \wedge g^l)_{1\cdots(2l)k} = 0, \quad \text{for every } 2l+1 \leq k \leq n. \tag{22}$$

Then

$$\omega \wedge g^l = 0.$$

Proof *Step 1.* Without loss of generality, we can assume that

$$(g^l)_{12\cdots\cdots(2l-1)(2l)} = 1.$$

It follows from (22) that, for every $2l+1 \leq k \leq n$,

$$\omega_1 (g^l)_{2\cdots(2l)k} - \omega_2 (g^l)_{13\cdots(2l)k} + \cdots - \omega_{2l} (g^l)_{1\cdots(2l-1)k} + \omega_k (g^l)_{1\cdots(2l)} = 0$$

and therefore

$$\omega_k = \sum_{s=1}^{2l} (-1)^s \omega_s \, (g^l)_{1\cdots\widehat{s}\cdots(2l)k} \, . \tag{23}$$

Note that (23) is trivially valid for $1 \leq k \leq 2l$.

Step 2. We have to prove that

$$(\omega \wedge g^l)_{i_1\cdots i_{2l+1}} = 0, \text{ for every } 1 \leq i_1 < \cdots < i_{2l+1} \leq n.$$

Using (23), it follows that

$$\begin{aligned}
(\omega \wedge g^l)_{i_1\cdots i_{2l+1}} &= \sum_{j=1}^{2l+1} (-1)^{j+1} \omega_{i_j} \, (g^l)_{i_1\cdots\widehat{i_j}\cdots i_{2l+1}} \\
&= \sum_{j=1}^{2l+1} (-1)^{j+1} \left[\sum_{s=1}^{2l} (-1)^s \omega_s (g^l)_{1\cdots\widehat{s}\cdots(2l)i_j} \right] (g^l)_{i_1\cdots\widehat{i_j}\cdots i_{2l+1}} \\
&= \sum_{s=1}^{2l} (-1)^s \omega_s \left[\sum_{j=1}^{2l+1} (-1)^{j+1} (g^l)_{1\cdots\widehat{s}\cdots(2l)i_j} \, (g^l)_{i_1\cdots\widehat{i_j}\cdots i_{2l+1}} \right].
\end{aligned}$$

Step 3. Let

$$A_s = \sum_{j=1}^{2l+1} (-1)^{j+1} (g^l)_{1\cdots\widehat{s}\cdots(2l)i_j} \, (g^l)_{i_1\cdots\widehat{i_j}\cdots i_{2l+1}} \, .$$

If we show that $A_s = 0$, for every $1 \leq s \leq 2l$, the lemma will be proved. Using Lemma 7, we have

$$\begin{aligned}
(g^l)_{1\cdots\widehat{s}\cdots(2l)i_j} &= l \sum_{r=1}^{s-1} (-1)^{r+1} g_{r i_j} \, (g^{l-1})_{1\cdots\widehat{r}\cdots\widehat{s}\cdots(2l)} \\
&\quad + l \sum_{r=s+1}^{2l} (-1)^r g_{r i_j} \, (g^{l-1})_{1\cdots\widehat{s}\cdots\widehat{r}\cdots(2l)}.
\end{aligned}$$

We therefore find

$$A_s = l\sum_{j=1}^{2l+1}(-1)^{j+1}(g^l)_{i_1\cdots\widehat{i_j}\cdots i_{2l+1}}\left[\begin{array}{c}\sum_{r=1}^{s-1}(-1)^{r+1}g_{ri_j}(g^{l-1})_{1\cdots\widehat{r}\cdots\widehat{s}\cdots(2l)}\\ +\sum_{r=s+1}^{2l}(-1)^{r}g_{ri_j}(g^{l-1})_{1\cdots\widehat{s}\cdots\widehat{r}\cdots(2l)}\end{array}\right]$$

and hence

$$\begin{aligned}A_s &= l\sum_{r=1}^{s-1}(-1)^{r+1}(g^{l-1})_{1\cdots\widehat{r}\cdots\widehat{s}\cdots(2l)}\left[\sum\nolimits_{j=1}^{2l+1}(-1)^{j+1}g_{ri_j}(g^l)_{i_1\cdots\widehat{i_j}\cdots i_{2l+1}}\right]\\ &\quad+l\sum_{r=s+1}^{2l}(-1)^{r}(g^{l-1})_{1\cdots\widehat{s}\cdots\widehat{r}\cdots(2l)}\left[\sum\nolimits_{j=1}^{2l+1}(-1)^{j+1}g_{ri_j}(g^l)_{i_1\cdots\widehat{i_j}\cdots i_{2l+1}}\right].\end{aligned}$$

This leads (according to Lemma 7) to

$$\begin{aligned}A_s &= \frac{l}{l+1}\sum_{r=1}^{s-1}(-1)^{r+1}(g^{l-1})_{1\cdots\widehat{r}\cdots\widehat{s}\cdots(2l)}(g^{l+1})_{ri_1\cdots i_{2l+1}}\\ &\quad+\frac{l}{l+1}\sum_{r=s+1}^{2l}(-1)^{r}(g^{l-1})_{1\cdots\widehat{s}\cdots\widehat{r}\cdots(2l)}(g^{l+1})_{ri_1\cdots i_{2l+1}}\\ &= 0\end{aligned}$$

where we have used the fact that $g^{l+1}=0$. This finishes the proof. ■

The following two lemmas have been used in Step 3.4 of Theorem 12.

Lemma 16 *Let $2\le 2l<n$ be integers, $\Omega\subset\mathbb{R}^n$ an open set. Let $g\in C^\infty(\Omega;\Lambda^2)$ be closed with* $\operatorname{rank} g=2l$ *in Ω and of the form*

$$g(x)=\sum_{i=1}^{l}dx^{2i-1}\wedge dx^{2i}+\sum_{\substack{1\le i<j\le n\\ 2l<j}}g_{ij}(x)\,dx^i\wedge dx^j$$

where $g_{ij}\in C^\infty(\Omega)$. Then, for every $2l+1\le k\le n$,

$$L_nL_k=L_kL_n \tag{24}$$

where $L_k:C^\infty(\Omega)\to C^\infty(\Omega)$, $2l+1\le k\le n$, is defined by

$$L_k(z)=(dz\wedge g^l)_{1\cdots(2l)k}\,.$$

Proof We begin by the noting that the structure of g implies that

$$(g^l)_{1\cdots(2l)}=l!\quad\text{in }\Omega. \tag{25}$$

For $z\in C^\infty(\Omega)$, we have

$$L_k(z)=(dz\wedge g^l)_{1\cdots(2l)k}=z_{x^k}\,l!+\sum_{i=1}^{2l}(-1)^{i+1}z_{x^i}(g^l)_{1\cdots\widehat{i}\cdots(2l)k}$$

where we have denoted partial differentiation of z by x^k with z_{x^k}. We therefore find

$$\begin{aligned} L_n L_k(z) &= L_n\left(z_{x^k}\, l! + \sum_{i=1}^{2l}(-1)^{i+1} z_{x^i}(g^l)_{1\cdots\widehat{i}\cdots(2l)k}\right) \\ &= \left(z_{x^k}\, l! + \sum_{i=1}^{2l}(-1)^{i+1} z_{x^i}(g^l)_{1\cdots\widehat{i}\cdots(2l)k}\right)_{x^n} l! \\ &+ \sum_{j=1}^{2l}(-1)^{j+1}\left(z_{x^k}\, l! + \sum_{i=1}^{2l}(-1)^{i+1} z_{x^i}(g^l)_{1\cdots\widehat{i}\cdots(2l)k}\right)_{x^j}(g^l)_{1\cdots\widehat{j}\cdots(2l)n}\,. \end{aligned}$$

Setting

$$A_1 = l!^2 z_{x^k x^n}\,, \quad A_2 = l!\sum_{i=1}^{2l}(-1)^{i+1} z_{x^i x^n}(g^l)_{1\cdots\widehat{i}\cdots(2l)k}$$

$$A_3 = l!\sum_{i=1}^{2l}(-1)^{i+1} z_{x^i}\left((g^l)_{1\cdots\widehat{i}\cdots(2l)k}\right)_{x^n}$$

$$A_4 = l!\sum_{j=1}^{2l}(-1)^{j+1} z_{x^k x^j}(g^l)_{1\cdots\widehat{j}\cdots(2l)n}$$

$$A_5 = \sum_{i,j=1}^{2l}(-1)^{i+1}(-1)^{j+1} z_{x^i x^j}(g^l)_{1\cdots\widehat{i}\cdots(2l)k}(g^l)_{1\cdots\widehat{j}\cdots(2l)n}$$

$$A_6 = \sum_{i,j=1}^{2l}(-1)^{i+1}(-1)^{j+1} z_{x^i}\left((g^l)_{1\cdots\widehat{i}\cdots(2l)k}\right)_{x^j}(g^l)_{1\cdots\widehat{j}\cdots(2l)n}$$

we find that

$$L_n L_k(z) = A_1 + A_2 + A_3 + A_4 + A_5 + A_6\,.$$

Note that A_1, $A_2 + A_4$ and A_5 are symmetric in k and n. Therefore, for proving that $L_k L_n(z) = L_n L_k(z)$, it is enough to show that $A_3 + A_6$ is symmetric in k and n, which is equivalent to

$$\begin{aligned} &\sum_{i=1}^{2l}(-1)^{i+1} z_{x^i}\left[\begin{array}{c} l!\left((g^l)_{1\cdots\widehat{i}\cdots(2l)k}\right)_{x^n} \\ +\sum_{j=1}^{2l}(-1)^{j+1}\left((g^l)_{1\cdots\widehat{i}\cdots(2l)k}\right)_{x^j}(g^l)_{1\cdots\widehat{j}\cdots(2l)n} \end{array}\right] \\ &= \sum_{i=1}^{2l}(-1)^{i+1} z_{x^i}\left[\begin{array}{c} l!\left((g^l)_{1\cdots\widehat{i}\cdots(2l)n}\right)_{x^k} \\ +\sum_{j=1}^{2l}(-1)^{j+1}\left((g^l)_{1\cdots\widehat{i}\cdots(2l)n}\right)_{x^j}(g^l)_{1\cdots\widehat{j}\cdots(2l)k} \end{array}\right]. \end{aligned} \tag{26}$$

To prove this, note first that, for every $2l+1 \le k \le n$

$$\sum_{j=1}^{2l}(-1)^{j+1}\left((g^l)_{1\cdots\widehat{j}\cdots(2l)k}\right)_{x^j} = 0 \tag{27}$$

since $(dg^l)_{1\cdots(2l)k} = 0$, g being closed, and $\left((g^l)_{1\cdots(2l)}\right)_{x^k} = 0$ according to (25). Hence, it follows from (27) that (26) is equivalent to

$$\sum_{i=1}^{2l} (-1)^{i+1} z_{x^i} C_i = 0$$

where

$$\begin{aligned} C_i = {} & l!\, \left((g^l)_{1\cdots\widehat{i}\cdots(2l)k}\right)_{x^n} - l!\, \left((g^l)_{1\cdots\widehat{i}\cdots(2l)n}\right)_{x^k} \\ & + \sum_{j=1}^{2l} (-1)^{j+1} \left((g^l)_{1\cdots\widehat{i}\cdots(2l)k} (g^l)_{1\cdots\widehat{j}\cdots(2l)n}\right)_{x^j} \\ & - \sum_{j=1}^{2l} (-1)^{j+1} \left((g^l)_{1\cdots\widehat{i}\cdots(2l)n} (g^l)_{1\cdots\widehat{j}\cdots(2l)k}\right)_{x^j} . \end{aligned}$$

To finish the proof, it is enough to prove that $C_i = 0$, for every $1 \leq i \leq 2l$. Indeed, using Lemma 14, we deduce that

$$\begin{aligned} C_i = {} & l!\, \left((g^l)_{1\cdots\widehat{i}\cdots(2l)k}\right)_{x^n} - l!\, \left((g^l)_{1\cdots\widehat{i}\cdots(2l)n}\right)_{x^k} \\ & + l! \sum_{j=1}^{i-1} (-1)^{j+1} \left((g^l)_{1\cdots\widehat{j}\cdots\widehat{i}\cdots(2l)kn}\right)_{x^j} + l! \sum_{j=i+1}^{2l} (-1)^{j} \left((g^l)_{1\cdots\widehat{i}\cdots\widehat{j}\cdots(2l)kn}\right)_{x^j} \\ = {} & l!\, (dg^l)_{1\cdots\widehat{i}\cdots(2l)kn} = 0 \end{aligned}$$

since g is closed. This finishes the proof of the lemma. ■

Lemma 17 *Let $2 \leq 2l < n$ be integers, $\Omega \subset \mathbb{R}^n$ an open set. Let $g \in C^\infty(\Omega; \Lambda^2)$ be closed with* $\operatorname{rank} g = 2l$ *in Ω and of the form*

$$g(x) = \sum_{i=1}^{l} dx^{2i-1} \wedge dx^{2i} + \sum_{\substack{1 \leq i < j \leq n \\ 2l < j}} g_{ij}(x)\, dx^i \wedge dx^j$$

where $g_{ij} \in C^\infty(\Omega)$. Let $u \in C^\infty(\Omega)$ be such that

$$du \wedge g^l = 0 \quad \text{in } \Omega.$$

Then, for every integer $2l + 1 \leq k \leq n$, the following holds

$$L_n((u \wedge g^l)_{1\cdots(2l)k}) = L_k((u \wedge g^l)_{1\cdots(2l)n})$$

where $L_k : C^\infty(\Omega) \to C^\infty(\Omega)$, $2l + 1 \leq k \leq n$, is given by

$$L_k(z) = (dz \wedge g^l)_{1\cdots(2l)k} .$$

Proof We divide the proof of the lemma into three steps.

Step 1. We have, since $(g^l)_{1\cdots(2l)} = l!$,

$$\begin{aligned} L_k((u \wedge g^l)_{1\cdots(2l)n}) &= (d[(u \wedge g^l)_{1\cdots(2l)n}] \wedge g^l)_{1\cdots(2l)k} \\ &= l! \left((u \wedge g^l)_{1\cdots(2l)n}\right)_{x^k} \\ &\quad + \sum_{i=1}^{2l} (-1)^{i+1} \left((u \wedge g^l)_{1\cdots(2l)n}\right)_{x^i} (g^l)_{1\cdots\widehat{i}\cdots(2l)k} . \end{aligned}$$

Since $(dg^l)_{1\cdots(2l)k} = 0$, g being closed, and $((g^l)_{1\cdots(2l)})_{x^k} = 0$, it follows that

$$\sum_{i=1}^{2l} (-1)^{i+1} \left((g^l)_{1\cdots\widehat{i}\cdots(2l)k}\right)_{x^i} = 0$$

and therefore

$$\begin{aligned} L_k((u \wedge g^l)_{1\cdots(2l)n}) &= l! \left((u \wedge g^l)_{1\cdots(2l)n}\right)_{x^k} \\ &\quad + \sum_{i=1}^{2l} (-1)^{i+1} \left((u \wedge g^l)_{1\cdots(2l)n} (g^l)_{1\cdots\widehat{i}\cdots(2l)k}\right)_{x^i} . \end{aligned}$$

Similarly, we have

$$\begin{aligned} L_n((u \wedge g^l)_{1\cdots(2l)k}) &= l! \left((u \wedge g^l)_{1\cdots(2l)k}\right)_{x^n} \\ &\quad + \sum_{i=1}^{2l} (-1)^{i+1} \left((u \wedge g^l)_{1\cdots(2l)k} (g^l)_{1\cdots\widehat{i}\cdots(2l)n}\right)_{x^i} . \end{aligned}$$

We then set, for $1 \leq i \leq 2l$,

$$A_i = (u \wedge g^l)_{1\cdots(2l)n} (g^l)_{1\cdots\widehat{i}\cdots(2l)k} - (u \wedge g^l)_{1\cdots(2l)k} (g^l)_{1\cdots\widehat{i}\cdots(2l)n} .$$

In order to prove the lemma, we therefore have to show the following

$$l! \left((u \wedge g^l)_{1\cdots(2l)n}\right)_{x^k} - l! \left((u \wedge g^l)_{1\cdots(2l)k}\right)_{x^n} + \sum_{i=1}^{2l} (-1)^{i+1} \left(A_i\right)_{x^i} = 0. \quad (28)$$

Step 2. In this step, we prove that, for every $1 \leq i \leq 2l$,

$$A_i = l! \, (u \wedge g^l)_{1\cdots\widehat{i}\cdots(2l)kn} . \quad (29)$$

To show this, we note that

$$\begin{aligned} A_i = &\left(\sum_{j=1}^{2l} (-1)^{j+1} u^j (g^l)_{1\cdots\widehat{j}\cdots(2l)n} + l! \, u^n\right) (g^l)_{1\cdots\widehat{i}\cdots(2l)k} \\ &- \left(\sum_{j=1}^{2l} (-1)^{j+1} u^j (g^l)_{1\cdots\widehat{j}\cdots(2l)k} + l! \, u^k\right) (g^l)_{1\cdots\widehat{i}\cdots(2l)n} . \end{aligned}$$

We therefore get

$$A_i = l!\, u^n (g^l)_{1\cdots\widehat{i}\cdots(2l)k} - l!\, u^k (g^l)_{1\cdots\widehat{i}\cdots(2l)n}$$
$$+ \sum_{j=1}^{2l} (-1)^{j+1} u^j \left[(g^l)_{1\cdots\widehat{j}\cdots(2l)n} (g^l)_{1\cdots\widehat{i}\cdots(2l)k} - (g^l)_{1\cdots\widehat{j}\cdots(2l)k} (g^l)_{1\cdots\widehat{i}\cdots(2l)n} \right].$$

Invoking Lemma 14 at this point, it follows that

$$A_i = l!\, u^n (g^l)_{1\cdots\widehat{i}\cdots(2l)k} - l!\, u^k (g^l)_{1\cdots\widehat{i}\cdots(2l)n}$$
$$+ \sum_{j=1}^{i-1} (-1)^{j+1} l!\, u^j (g^l)_{1\cdots\widehat{j}\cdots\widehat{i}\cdots(2l)kn} + \sum_{j=i+1}^{2l} (-1)^j l!\, u^j (g^l)_{1\cdots\widehat{i}\cdots\widehat{j}\cdots(2l)kn}$$
$$= l!\, (u \wedge g^l)_{1\cdots\widehat{i}\cdots(2l)kn}\,.$$

Step 3. We finally use (29) in (28) to deduce that

$$l!\, \left((u \wedge g^l)_{1\cdots(2l)n} \right)_{x^k} - l!\, \left((u \wedge g^l)_{1\cdots(2l)k} \right)_{x^n} +$$
$$l! \sum_{i=1}^{2l} (-1)^{i+1} \left((u \wedge g^l)_{1\cdots\widehat{i}\cdots(2l)kn} \right)_{x^i} =$$
$$l!\, \left(d(u \wedge g^l) \right)_{1\cdots(2l)kn} = 0,$$

since $dg = 0$ and $du \wedge g^l = 0$. The proof is finished. ■

The following lemma is standard, cf. [12], [19].

Lemma 18 *Let $2 \leq 2l \leq n$ and $g \in \Lambda^2(\mathbb{R}^n)$ with $g^l \neq 0$ and $g^{l+1} = 0$. Then there exists $A \in GL_n(\mathbb{R})$, letting $\varphi(x) = Ax$, such that*

$$\varphi^*(g) = \omega_l = \sum_{i=1}^{l} dx^{2i-1} \wedge dx^{2i}.$$

Moreover, if $n = 2l$, then, in addition to the above conclusion, φ can be chosen so that $\varphi^1(x) = x^1$.

The final proposition has been used in Step 3.1 of Theorem 12.

Proposition 19 *Let $2 \leq 2l < n$ be integers and $x_0 \in \mathbb{R}^n$. Let g be a C^∞ closed $2-$form with*

$$g^l \neq 0 \quad \text{and} \quad g^{l+1} = 0 \quad \text{in a neighbourhood of } x_0\,. \tag{30}$$

Then there exist U and V two neighbourhoods of x_0, $r_{ij} \in C^\infty(U)$ and $\varphi \in \operatorname{Diff}^\infty(U;V)$ such that $\varphi(x_0) = x_0$ and

$$\varphi^*(g) = \sum_{i=1}^{l} dx^{2i-1} \wedge dx^{2i} + \sum_{\substack{1 \leq i < j \leq n \\ 2l < j}} r_{ij}\, dx^i \wedge dx^j.$$

Proof *Step 1.* Without loss of generality, we can assume that $x_0 = 0$. In addition, using Lemma 18, we can take (as in Step 2 of Theorem 12)

$$g(0) = \omega_l = \sum_{i=1}^{l} dx^{2i-1} \wedge dx^{2i}.$$

We next introduce some notations. We let

$$x = (y, z) = \left(x^1, \cdots, x^{2l}, x^{2l+1}, \cdots, x^n\right) \in \mathbb{R}^{2l} \times \mathbb{R}^{n-2l}$$

and we define, for every $z \in \mathbb{R}^{n-2l}$, the map $i_z : \mathbb{R}^{2l} \to \mathbb{R}^n$ through

$$i_z\,(y) = (y, z) = x.$$

Step 2. We define, for every $x = (y, z) \in \mathbb{R}^{2l} \times \mathbb{R}^{n-2l}$ with $|x|$ small, the 2−form

$$g_{z,t} : \mathbb{R}^{2l} \to \Lambda^2\left(\mathbb{R}^{2l}\right) \quad \text{by} \quad g_{z,t}\,(y) = i_z^*\left[tg + (1-t)\omega_l\right](y)\,.$$

Note that

$$g_{z,0}\,(y) = i_z^*\left[\omega_l\right] = \omega_l \quad \text{and} \quad g_{z,1}\,(y) = \sum_{1 \le i < j \le 2l} g_{ij}\,(x)\,dx^i \wedge dx^j.$$

Our assumption in Step 1 leads to $g_{z,t}\,(0) = \omega_l$ and therefore, in a sufficiently small cube C centred at $0 \in \mathbb{R}^{2l} \times \mathbb{R}^{n-2l}$, we can ensure that

$$\left(g_{z,t}\right)^l(y) > 0 \quad \text{for every } \ (y,z) \in C \text{ and } t \in [0,1]. \tag{31}$$

Furthermore, $g_{z,t}$ has the property that

$$d_y g_{z,t} = 0 \quad \text{in } C, \quad \text{for every } t \in [0,1], \tag{32}$$

where d_y is understood as the exterior differential operator involving only the variable $y = \left(x^1, \cdots, x^{2l}\right)$, namely $d_y g_{z,t} = 0$ is equivalent to

$$\frac{\partial\left(g_{z,t}\right)_{ij}}{\partial x^k} - \frac{\partial\left(g_{z,t}\right)_{ik}}{\partial x^j} + \frac{\partial\left(g_{z,t}\right)_{jk}}{\partial x^i} = 0 \quad \text{for every } i,j,k = 1, \cdots, 2l.$$

Step 3. Using (31), (32) and Poincaré lemma, we find a C^∞ vector field $u_{z,t} : \mathbb{R}^{2l} \to \mathbb{R}^{2l}$ such that

$$d_y\left(u_{z,t} \lrcorner\, g_{z,t}\right) = -\frac{d}{dt} g_{z,t} = \omega_l - i_z^*\left[g\right] \quad \text{in } C, \quad \text{for every } t \in [0,1].$$

We now consider the initial value problem, for every $x = (y,z) \in C$,

$$\frac{d}{dt}\varphi_{z,t} = u_{z,t}(\varphi_{z,t}) \quad \text{and} \quad \varphi_{z,0}(y) = y.$$

Using Moser flow method, we deduce that, up to restricting the set C,

$$\varphi_{z,1}^{*}\left(g_{z,1}\right)=g_{z,0}=\omega_{l}$$

which means that

$$\sum_{1\leq i<j\leq 2l}\left(g_{z,1}\right)_{ij}\left(\varphi_{z,1}\left(y\right)\right)d_{y}\varphi_{z,1}^{i}\wedge d_{y}\varphi_{z,1}^{j}=\omega_{l} \tag{33}$$

where for $u:\mathbb{R}^{n}\to\mathbb{R}$ we set

$$d_{y}u=\sum_{i=1}^{2l}\frac{\partial u}{\partial x^{i}}dx^{i}\quad\text{and}\quad d_{z}u=\sum_{i=2l+1}^{n}\frac{\partial u}{\partial x^{i}}dx^{i}.$$

We finally let, for $x=(y,z)\in C$,

$$\varphi\left(x\right)=\left(\varphi_{z,1}\left(y\right),z\right)$$

and we claim that this is the diffeomorphism we are looking for. Indeed first observe that

$$\sum_{1\leq i<j\leq 2l}\left(g_{z,1}\right)_{ij}\left(\varphi_{z,1}\left(y\right)\right)d_{y}\varphi_{z,1}^{i}\wedge d_{y}\varphi_{z,1}^{j}=\sum_{1\leq i<j\leq 2l}g_{ij}\left(\varphi\left(y\right)\right)d_{y}\varphi^{i}\wedge d_{y}\varphi^{j}. \tag{34}$$

We moreover have

$$\begin{aligned}\varphi^{*}\left(g\right)&=\sum_{1\leq i<j\leq n}g_{ij}\left(\varphi\left(y\right)\right)d\varphi^{i}\wedge d\varphi^{j}\\&=\sum_{1\leq i<j\leq 2l}g_{ij}\left(\varphi\left(y\right)\right)d\varphi^{i}\wedge d\varphi^{j}+\sum_{\substack{1\leq i<j\leq n\\2l<j}}g_{ij}\left(\varphi\left(y\right)\right)d\varphi^{i}\wedge d\varphi^{j}\\&=\sum_{1\leq i<j\leq 2l}g_{ij}\left(\varphi\left(y\right)\right)\left(d_{y}\varphi^{i}+d_{z}\varphi^{i}\right)\wedge\left(d_{y}\varphi^{j}+d_{z}\varphi^{j}\right)+\sum_{\substack{1\leq i<j\leq n\\2l<j}}g_{ij}\left(\varphi\left(y\right)\right)d\varphi^{i}\wedge dx^{j}\end{aligned}$$

and thus

$$\begin{aligned}\varphi^{*}\left(g\right)=&\sum_{1\leq i<j\leq 2l}g_{ij}\left(\varphi\left(y\right)\right)d_{y}\varphi^{i}\wedge d_{y}\varphi^{j}\\&+\sum_{1\leq i<j\leq 2l}g_{ij}\left(\varphi\left(y\right)\right)d_{y}\varphi^{i}\wedge d_{z}\varphi^{j}+\sum_{1\leq i<j\leq 2l}g_{ij}\left(\varphi\left(y\right)\right)d_{z}\varphi^{i}\wedge d_{y}\varphi^{j}\\&+\sum_{1\leq i<j\leq 2l}g_{ij}\left(\varphi\left(y\right)\right)d_{z}\varphi^{i}\wedge d_{z}\varphi^{j}+\sum_{\substack{1\leq i<j\leq n\\2l<j}}g_{ij}\left(\varphi\left(y\right)\right)d\varphi^{i}\wedge dx^{j}.\end{aligned} \tag{35}$$

Appealing to (33), (34) and (35), we get

$$\varphi^{*}\left(g\right)=\omega_{l}+\sum_{\substack{1\leq i<j\leq n\\2l<j}}r_{ij}\,dx^{i}\wedge dx^{j}$$

for appropriate r_{ij} . This finishes the proof. ∎

3.4 An alternative proof of Darboux theorem in the degenerate case

The second proof that we provide in this section (with the help of the previous sections) may seem to be much longer than the first one given in Theorem 11. However this is misleading because to really compare the length of the two proofs, one should take into account the full proof of Frobenius theorem and the one of the non-degenerate Darboux theorem (Theorem 10). It also seems that the present proof is more appropriate if one wants to look for global results.

Theorem 20 *Let $n \geq 3$ and $x_0 \in \mathbb{R}^n$. Let ω_l be the standard symplectic form of rank $2l < n$, namely*

$$\omega_l = \sum_{i=1}^{l} dx^{2i-1} \wedge dx^{2i}.$$

Let ω be a C^∞ closed $2-$form such that

$$\operatorname{rank} \omega = 2l \textit{ in a neighbourhood of } x_0 \, .$$

Then there exist a neighbourhood V of x_0 and $\varphi \in \operatorname{Diff}^\infty(V; \mathbb{R}^n)$ such that

$$\varphi^* (\omega_l) = \omega \textit{ in } V \quad \textit{and} \quad \varphi (x_0) = x_0 \, .$$

Proof *Step 1*. It is easy to see (cf. [12]) that we can construct a homotopy h with the following properties. There exist a neighbourhood V_1 of x_0 and $h \in C^\infty(V_1 \times [0,1]; \Lambda^2)$, $h(x,t) = h_t(x)$, such that, for every $(x,t) \in V_1 \times [0,1]$,

$$dh_t = 0, \quad h_t^l \neq 0 \quad \text{and} \quad h_t^{l+1} = 0 \tag{36}$$

while

$$h_0 = \omega(x_0) \quad \text{and} \quad h_1 = \omega.$$

Step 2. Since (36) holds and

$$h_t^l \wedge \frac{\partial h_t}{\partial t} = \frac{1}{l+1} \frac{\partial h_t^{l+1}}{\partial t} = 0$$

we can apply Theorem 12. We therefore can find a neighbourhood $V_2 \subset V_1$ of x_0 and $w \in C^\infty(V_2 \times [0,1]; \mathbb{R}^n)$, $w(x,t) = w_t(x)$, satisfying, for every $(x,t) \in V_2 \times [0,1]$,

$$dw_t = -\frac{\partial h_t}{\partial t}, \quad w_t \wedge h_t^l = 0 \quad \text{and} \quad w_t(x_0) = 0.$$

We then apply Proposition 6 to find $u \in C^\infty(V_2 \times [0,1]; \mathbb{R}^n)$, $u(x,t) = u_t(x)$, with

$$u_t \lrcorner h_t = w_t \quad \text{and} \quad u_t(x_0) = 0.$$

Step 3. We next find the flow, associated to the vector field u_t,

$$\begin{cases} \frac{d}{dt}\varphi_t(x) = u_t(\varphi_t(x)) & t \in [0,1] \\ \varphi_0 = \operatorname{id}. \end{cases}$$

Classical results (cf. for example Theorem 10 in [3]) show the existence of a neighbourhood $V_3 \subset V_2$ of x_0 such that

$$\varphi_1^*(h_1) = h_0 \text{ in } V_3 \quad \text{and} \quad \varphi_1(x_0) = x_0\,.$$

Step 4. Since h_0 is constant, we can use Lemma 18 to find a linear diffeomorphism ψ so that

$$\psi^*(h_0) = \omega_l = \sum_{i=1}^{l} dx^{2i-1} \wedge dx^{2i}.$$

Letting $\varphi = \varphi_1 \circ \psi$ we have the claim. ■

4 Simultaneous resolutions and the case of $k-$forms when $3 \leq k \leq n-1$

4.1 A general theorem for forms of rank k

Our first result concerns $k-$forms of minimal non-zero rank.

Theorem 21 *Let $2 \leq k \leq n$, $r \geq 1$ be integers, $0 < \alpha < 1$ and $x_0 \in \mathbb{R}^n$. Let f, g be two $C^{r,\alpha}$ closed $k-$forms satisfying*

$$\operatorname{rank} f = \operatorname{rank} g = k \text{ in a neighbourhood of } x_0\,.$$

Then there exists a diffeomorphism φ of class $C^{r,\alpha}$ such that, in a neighbourhood of x_0,

$$\varphi^*(g) = f \quad \text{and} \quad \varphi(x_0) = x_0\,.$$

Moreover if $k = n$, the diffeomorphism φ can be chosen of class $C^{r+1,\alpha}$.

Remark The theorem solves completely the case $k = n-1$. Moreover in this case the condition on the rank is equivalent to

$$f(x_0) \neq 0 \quad \text{and} \quad g(x_0) \neq 0.$$

Proof The statement with $k = n$ is the result of Dacorogna-Moser [10] (cf. Theorem 14.6 in [9]) and will be used below after reduction of dimension. Note that, it is enough to find a $\varphi \in \operatorname{Diff}^{r,\alpha}(\Omega; \mathbb{R}^n)$ such that

$$\varphi^*(dx^1 \wedge \cdots dx^k) = f \quad \text{in } \Omega,$$

where Ω is a neighbourhood of x_0 and we take, without loss of generality, $x_0 = 0$. Using Theorem 31, we find two neighbourhood V, Ω of 0 in $\mathbb{R}^n$ and $\psi \in \operatorname{Diff}^{r,\alpha}(V; \Omega)$ satisfying $\psi(0) = 0$ and

$$\psi^*(f)(x^1, \cdots, x^n) = a(x^1, \cdots, x^k)\, dx^1 \wedge \cdots \wedge dx^k,$$

where $a \in C^{r-1,\alpha}(W)$ and W is a sufficiently small open ball in $\mathbb{R}^k$ containing 0. Note that $a \neq 0$ in W, since $\operatorname{rank} f = k$. We now apply the result of Dacorogna-Moser [10] to find $\chi \in \operatorname{Diff}^{r,\alpha}(W;W)$ such that

$$\chi^*(c\,dx^1 \wedge \cdots \wedge dx^k) = a\,dx^1 \wedge \cdots \wedge dx^k \text{ in } W,$$

where

$$c = \frac{1}{\operatorname{meas} W} \int_W a.$$

At this point, we can perform a linear change of variables which allows us to assume that $c = 1$. Next, we construct a map $\widetilde{\chi} \in C^{r,\alpha}(W \times \mathbb{R}^{n-k};\mathbb{R}^n)$ by

$$\widetilde{\chi}(y) = (\chi(y^1,\cdots,y^k), y^{k+1},\cdots,y^n), \text{ for every } y = (y^1,\cdots,y^n) \in W \times \mathbb{R}^{n-k}.$$

Finally, we define $\varphi : \Omega \to \mathbb{R}^n$ as

$$\varphi = \widetilde{\chi} \circ \psi^{-1}.$$

Hence we have found $\varphi \in \operatorname{Diff}^{r,\alpha}(\Omega;\mathbb{R}^n)$ satisfying

$$\varphi^*(dx^1 \wedge \cdots dx^k) = f \quad \text{in } \Omega$$

as wished. ■

We have as an immediate corollary the following.

Corollary 22 *Let $2 \leq k \leq n$, $r \geq 1$ be integers, $0 < \alpha < 1$ and $x_0 \in \mathbb{R}^n$. Let f be a $C^{r,\alpha}$ closed $k-$form such that*

$$\operatorname{rank} f = k \text{ in a neighbourhood of } x_0\,.$$

Then there exists a diffeomorphism φ of class $C^{r,\alpha}$ such that, in a neighbourhood of x_0,

$$f = \nabla\varphi^1 \wedge \cdots \wedge \nabla\varphi^k \quad \text{and} \quad \varphi(x_0) = x_0\,.$$

Proof It is enough to choose

$$g = dx^1 \wedge \cdots \wedge dx^k$$

in the theorem. ■

The corollary reads in a more analytical way when $k = n-1$ (cf. also [5]), since the exterior derivative of an $(n-1)-$form is then essentially the classical divergence operator.

Corollary 23 *Let $r \geq 1$ be an integer and $0 < \alpha < 1$ and $x_0 \in \mathbb{R}^n$. Let f be a $C^{r,\alpha}$ vector field satisfying*

$$f(x_0) \neq 0 \quad \text{and} \quad \operatorname{div} f = 0 \text{ in a neighbourhood of } x_0\,.$$

Then there exists a diffeomorphism φ of class $C^{r,\alpha}$ such that, in a neighbourhood of x_0,

$$f = *\left(\nabla\varphi^1 \wedge \cdots \wedge \nabla\varphi^{n-1}\right) \quad \text{and} \quad \varphi(x_0) = x_0\,.$$

4.2 Simultaneous resolutions and applications

We start with a simultaneous resolution of closed $1-$forms. We will also obtain a global result with Dirichlet data. We follow here the idea explained in [3].

Theorem 24 *Let $r \geq 0$, $1 \leq m \leq n$ be integers and $x_0 \in \mathbb{R}^n$. Let $b_1, \cdots, b_m$, $a_1, \cdots, a_m$ be C^r closed $1-$forms satisfying, in a neighbourhood of x_0 ,*

$$b_1 \wedge \cdots \wedge b_m \neq 0 \quad \text{and} \quad a_1 \wedge \cdots \wedge a_m \neq 0.$$

Then there exists a C^{r+1} diffeomorphism such that $\varphi(x_0) = x_0$ and, in a neighbourhood of x_0 ,

$$\varphi^*(b_i) = a_i\,, \text{ for every } i = 1, \cdots, m.$$

Remark (i) When $r = 0$, the fact that the forms are closed has to be understood in the sense of distributions.

(ii) It is interesting to compare the above theorem and Theorem 21. In view of Proposition 5, we know that any $m-$form g with $\operatorname{rank} g = m$, is a product of $1-$forms $b_1, \cdots, b_m$ so that

$$g = b_1 \wedge \cdots \wedge b_m\,;$$

however we do not know, a priori, that $b_1, \cdots, b_m$ are closed if g is closed (and even that $b_1, \cdots, b_m \in C^r$ if $g \in C^r$); we know it only a posteriori from the conclusion of Theorem 21, but we have lost one degree of regularity, namely $b_1, \cdots, b_m \in C^{r-1,\alpha}$. Therefore if we assume that $b_1, \cdots, b_m$ are closed, then the above theorem is better from the point of view of regularity than Theorem 21.

(iii) When $m = n$ and $g \in C^0$ it is, in general, impossible (according to [6] and [14]) to find closed $1-$forms $b_1, \cdots, b_n \in C^0$ so that

$$g = b_1 \wedge \cdots \wedge b_n\,;$$

although, in view of Theorem 21, we can do so if $g \in C^{0,\alpha}$, finding even that $b_1, \cdots, b_n \in C^{0,\alpha}$.

Proof We assume that $m = n$, otherwise we choose $1 \leq k_1 < \cdots < k_{n-m} \leq n$ and $1 \leq l_1 < \cdots < l_{n-m} \leq n$ such that

$$b_1 \wedge \cdots \wedge b_m \wedge dx^{k_1} \wedge \cdots \wedge dx^{k_{n-m}} \neq 0,$$

$$a_1 \wedge \cdots \wedge a_m \wedge dx^{l_1} \wedge \cdots \wedge dx^{l_{n-m}} \neq 0$$

and we prove the theorem for

$$\left\{b_1, \cdots, b_m, dx^{k_1}, \cdots, dx^{k_{n-m}}\right\} \quad \text{and} \quad \left\{a_1, \cdots, a_m, dx^{l_1}, \cdots, dx^{l_{n-m}}\right\}.$$

As usual, we may assume that

$$a_i = dx^i, \quad i = 1, \cdots, n.$$

Since $(b_1 \wedge \cdots \wedge b_n)(x_0) \neq 0$, there exists $j_1 \in \{1, \cdots, n\}$ satisfying $b_1^{j_1}(x_0) \neq 0$ which, in turn, implies that

$$b_1^{j_1}(x) \neq 0, \text{ for every } x \in U_1,$$

where U_1 is some ball around x_0. We now use Proposition 19 of [3] to find $\varphi_1 \in \operatorname{Diff}^{r+1}(U_1; \mathbb{R}^n)$ satisfying $\varphi_1(x_0) = x_0$ and

$$\varphi_1^*(b_1) = dx^{j_1}, \text{ in } U_1$$

as well as

$$\varphi_1^p(x) = x^p, \text{ for every } p \in \{1, \cdots, n\},\, p \neq j_1.$$

Having found φ_1, we note that $\varphi_1^*(b_2)^{j_2}(x_0) \neq 0$, for some $j_2 \in \{1, \cdots, n\}$ with $j_2 \neq j_1$. This follows from the fact that

$$\varphi_1^*(b_1 \wedge \cdots \wedge b_n)(x_0) = dx^{j_1} \wedge \varphi_1^*(b_2)(x_0) \cdots \wedge \varphi_1^*(b_n)(x_0) \neq 0.$$

We hence deduce that there exists a ball $U_2 \subset U_1$ centred at x_0 such that $\varphi_1^*(b_2)^{j_2}(x) \neq 0$, for every $x \in U_2$. Invoking Proposition 19 of [3] again, we find $\varphi_2 \in \operatorname{Diff}^{r+1}(U_2; \mathbb{R}^n)$ such that $\varphi_2(x_0) = x_0$,

$$\varphi_2^*(\varphi_1^*(b_1)) = dx^{j_2}, \text{ in } U_2,$$

and

$$\varphi_2^p(x) = x^p, \text{ for every } p \in \{1, \cdots, n\},\, p \neq j_2.$$

Continuing this process for n steps, we find a finite sequence $U_n \subset \cdots \subset U_1$ of balls centred at x_0 and $\varphi_q \in \operatorname{Diff}^{r+1}(U_q; \mathbb{R}^n)$, for every $q = 1, \cdots, n$, such that $\varphi_q(x_0) = x_0$ and

$$\varphi_q^*\left(\varphi_{q-1}^* \circ \cdots \circ \varphi_1^*(b_q)\right) = dx^{j_q}, \text{ for every } q = 1, \cdots, n,$$

where

$$j_q \notin \{j_1, \cdots, j_{q-1}\}, \text{ for every } q = 2, \cdots, n.$$

Furthermore, we have, for every $q = 1, \cdots, n$,

$$\varphi_q^p(x) = x^p, \text{ for every } p \in \{1, \cdots, n\},\, p \neq j_q.$$

Finally, we set $V = U_n$ and we define $\varphi \in \operatorname{Diff}^{r+1}(V; \mathbb{R}^n)$ by

$$\varphi = \varphi_1 \circ \cdots \circ \varphi_n.$$

By construction we have

$$\varphi^*(b_i) = dx^{j_i}, \text{ for every } i = 1, \cdots, n.$$

Finally, composing φ with a linear map leads us to the map we have been looking for. This proves the theorem. ■

It is interesting to see that the above theorem can also be global.

Theorem 25 *Let $\Omega \subset \mathbb{R}^n$ be a bounded, smooth and simply connected domain. Let $r \geq 0$ and $1 \leq m \leq n$ be integers. Let $b_1, \cdots, b_m \in C^r\left(\overline{\Omega}; \Lambda^1\right)$ be closed forms. For all $i = 1, \cdots, m-1$ and $j, k = 1, \cdots, m$, define $B_{jk}^{(i)} \in C^r(\overline{\Omega})$ by*

$$B_{jk}^{(1)} = b_j^k \quad \text{and} \quad B_{jk}^{(i+1)} = B_{ii}^{(i)} B_{jk}^{(i)} - B_{ji}^{(i)} B_{ik}^{(i)} .$$

Assume that, for every $i = 1, \cdots, m$,

$$B_{ii}^{(i)} > 0 \text{ in } \overline{\Omega} \quad \text{and} \quad b_i \wedge \nu = dx^i \wedge \nu \text{ on } \partial\Omega$$

where ν is the exterior unit normal to Ω. Then there exists $\varphi \in \operatorname{Diff}^{r+1}\left(\overline{\Omega}; \overline{\Omega}\right)$ satisfying

$$\begin{aligned} \varphi^*(b_i) &= dx^i \text{ in } \Omega, \text{ for every } i = 1, \cdots, m \\ \varphi^*\left(dx^i\right) &= dx^i, \text{ for every } i = m+1, \cdots, n. \\ \varphi(x) &= x \text{ on } \partial\Omega. \end{aligned}$$

Proof We prove the theorem only for $m = 2$ and we write $b_1 = b$ and $b_2 = a$. The general case is proved similarly (cf. [12] for details). Note that we have

$$B_{11}^{(1)} = b^1 > 0 \quad \text{and} \quad B_{22}^{(2)} = (b \wedge a)_{12} > 0.$$

Since $b^1 > 0$, using Proposition 19 of [3], we find a diffeomorphism

$$\psi(x) = \left(\psi^1(x), x^2, \cdots, x^n\right)$$

such that

$$\psi^*(b) = dx^1 \text{ in } \Omega \quad \text{and} \quad \psi(x) = x \text{ on } \partial\Omega.$$

Moreover, it follows, since $\psi^*(b) = dx^1$, that, if $j \neq 1$,

$$b^1(\psi) \frac{\partial \psi^1}{\partial x^j} + b^j(\psi) = 0$$

and thus

$$\frac{\partial \psi^1}{\partial x^j} = -\frac{b^j(\psi)}{b^1(\psi)} .$$

We therefore deduce that

$$\begin{aligned} \psi^*(a) &= \sum_{j=1}^n a^j(\psi)\, d\psi^j = a^1(\psi)\, d\psi^1 + \sum_{j=2}^n a^j(\psi)\, dx^j \\ &= a^1(\psi) \sum_{j=1}^n \frac{\partial \psi^1}{\partial x^j} dx^j + \sum_{j=2}^n a^j(\psi)\, dx^j \\ &= a^1(\psi) \frac{\partial \psi^1}{\partial x^1} dx^1 + \sum_{j=2}^n \left(a^j(\psi) + a^1(\psi) \frac{\partial \psi^1}{\partial x^j} \right) dx^j \\ &= a^1(\psi) \frac{\partial \psi^1}{\partial x^1} dx^1 + \sum_{j=2}^n \left(a^j(\psi) - a^1(\psi) \frac{b^j(\psi)}{b^1(\psi)} \right) dx^j. \end{aligned}$$

Since $(b \wedge a)_{12} > 0$, invoking Proposition 19 of [3] again, we find a diffeomorphism χ, where
$$\chi(x) = \left(x^1, \chi^2(x), x^3, \cdots, x^n\right),$$
such that
$$\chi^*(\psi^*(a)) = dx^2 \text{ in } \Omega \quad \text{and} \quad \chi(x) = x \text{ on } \partial\Omega.$$
Setting
$$\varphi = \psi \circ \chi$$
we observe that, in Ω,
$$\varphi^*(b) = dx^1 \quad \text{and} \quad \varphi^*(a) = dx^2$$
while $\varphi(x) = x$ on $\partial\Omega$. This finishes the proof. ■

We next generalize Theorem 24 by mixing $1-$forms and $2-$forms.

Theorem 26 *Let $m, l \geq 0$ be integers and $x_0 \in \mathbb{R}^n$. Let $b_1, \cdots, b_m, a_1, \cdots, a_m$ be C^∞ closed $1-$forms satisfying, in a neighbourhood of x_0,*
$$b_1 \wedge \cdots \wedge b_m \neq 0 \quad \textit{and} \quad a_1 \wedge \cdots \wedge a_m \neq 0.$$
Let $g_1, \cdots, g_l, f_1, \cdots, f_l$ be C^∞ closed $2-$forms verifying, in a neighbourhood of x_0,
$$\operatorname{rank} g_i = \operatorname{rank} f_i = 2s_i\,,\ i = 1, \cdots, l$$
$$\begin{aligned} \operatorname{rank}[g_1 \wedge \cdots \wedge g_l \wedge b_1 \wedge \cdots \wedge b_m] &= \operatorname{rank}[f_1 \wedge \cdots \wedge f_l \wedge a_1 \wedge \cdots \wedge a_m] \\ &= 2(s_1 + \cdots + s_l) + m \leq n. \end{aligned}$$
Then there exists a C^∞ diffeomorphism φ such that $\varphi(x_0) = x_0$ and, in a neighbourhood of x_0,
$$\varphi^*(g_i) = f_i \quad \textit{for every } i = 1, \cdots, l$$
$$\varphi^*(b_i) = a_i \quad \textit{for every } i = 1, \cdots, m.$$

Remark (i) When $m = 0$, respectively $l = 0$, the theorem is to be understood as a statement only on $2-$forms, respectively only on $1-$forms (in this last case see Theorem 24).

(ii) When $g_i, f_i \in C^{r,\alpha}$ and $b_j, a_j \in C^{r,\alpha}$, we have $\varphi \in \operatorname{Diff}^{r-l+1,\alpha}$.

(iii) Of course the theorem applies to $k-$forms, $k = 2l + m$, of the type
$$G = g_1 \wedge \cdots \wedge g_l \wedge b_1 \wedge \cdots \wedge b_m \quad \text{and} \quad F = f_1 \wedge \cdots \wedge f_l \wedge a_1 \wedge \cdots \wedge a_m\,.$$
We therefore obtain that there exists a diffeomorphism φ such that
$$\varphi^*(G) = F$$
generalizing the result obtained in [3].

It is interesting to contrast the algebraic result of Proposition 5 (iii) with the analytical result of the above theorem, where it is essential to require that the $1-$forms and the $2-$forms be closed.

Example 27 *Although every constant* $3-$*form of* $\operatorname{rank} = 5$ *is a linear pullback (according to Proposition 5 (iii)) of*

$$G = g \wedge b$$

where

$$g = dx^1 \wedge dx^2 + dx^3 \wedge dx^4 \quad \textit{and} \quad b = dx^5,$$

there does exist a (non-constant) closed form $F \in \Lambda^3(\mathbb{R}^n)$ *with* $\operatorname{rank} F = 5$ *which cannot be pulled back to* G*. One such example is*

$$F = f \wedge a$$

where

$$f = \frac{1}{(x^3)^4 + 1} dx^1 \wedge dx^5 + dx^3 \wedge dx^4 \quad \textit{and} \quad a = ((x^3)^2 + 1)dx^1 + ((x^3)^4 + 1)dx^2.$$

The result follows from the following observations.

1) Any $1-$*divisor* c *of* F *must be of the form* $c = \lambda a$ *where* λ *takes value in* $\mathbb{R}$*. Indeed if this is not the case we have that the* $1-$*form* c *is linearly independent of* a*. We therefore have*

$$F \wedge a = F \wedge c = 0 \quad \textit{and} \quad c \wedge a \neq 0.$$

Appealing to Proposition 5 (iv), we deduce that $\operatorname{rank} F = 3,$ *a contradiction.*

2) Note that λa *is not closed in any open set unless* λ *is identically zero.*

3) Therefore, if there exists a local diffeomorphism φ *satisfying*

$$F = \varphi^*(G) = \varphi^*(g) \wedge \varphi^*(b),$$

it follows from our aforementioned observations that

$$\varphi^*(b) = \varphi^*(dx^5) = \lambda a.$$

But this leads to contradiction because the form on the left hand side is closed and non-zero whereas the form on the right hand side is not closed.

We now turn our attention to the proof of Theorem 26.

Proof It is enough to prove the theorem when

$$f_j = \sum_{i=(s_1+\cdots+s_{j-1})+1}^{s_1+\cdots+s_j} dx^{2i-1} \wedge dx^{2i}, \quad j = 1, \cdots, l$$

and when

$$a_i = dx^{2(s_1+\cdots+s_l)+i} \quad \text{for every } i = 1, \cdots, m.$$

We establish the result by induction on l. When $l = 0$, we are in the situation of Theorem 24 which has already been proved. Let us suppose that the theorem is true for $l = k$. It remains to prove the result when $l = k + 1$. We find, using Theorem 11, a neighbourhood V_1 of x_0 and $\varphi_1 \in \text{Diff}^{\infty}(V_1; \mathbb{R}^n)$ such that $\varphi_1(x_0) = x_0$ and

$$\varphi_1^*(g_1) = f_1 \text{ in } V_1 .$$

Since

$$\begin{aligned}
&\text{rank}\,(\varphi_1^*(g_1) \wedge \cdots \wedge \varphi_1^*(g_l) \wedge \varphi_1^*(b_1) \wedge \cdots \wedge \varphi_1^*(b_m)) \\
&= \text{rank}\,(\varphi_1^*(g_1 \wedge \cdots \wedge g_l \wedge b_1 \wedge \cdots \wedge b_m)) \\
&= \text{rank}(g_1 \wedge \cdots \wedge g_l \wedge b_1 \wedge \cdots \wedge b_m) = 2(s_1 + \cdots + s_l) + m
\end{aligned}$$

it follows from Proposition 3 (iv) that

$$\begin{aligned}
&\text{rank}\left(dx^1 \wedge \cdots \wedge dx^{2s_1} \wedge \varphi_1^*(g_2) \wedge \cdots \wedge \varphi_1^*(g_l) \wedge \varphi_1^*(b_1) \wedge \cdots \wedge \varphi_1^*(b_m)\right) \\
&= 2(s_1 + \cdots + s_l) + m.
\end{aligned}$$

Hence, using induction hypothesis, there exists $V_2 \subset V_1$ of x_0 and $\varphi_2 \in \text{Diff}^{\infty}(V_2; \mathbb{R}^n)$ such that $\varphi_2(x_0) = x_0$ and, for every $i = 2, \cdots, l$, $j = 1, \cdots, 2s_1$ and $k = 1, \cdots, m$,

$$\varphi_2^*(\varphi_1^*(g_i)) = f_i\,, \quad \varphi_2^*(dx^j) = dx^j \quad \text{and} \quad \varphi_2^*(b_k) = dx^{2(s_1+\cdots+s_l)+k},$$

Note, in particular, that $\varphi_2^*(\varphi_1^*(g_1)) = \varphi_2^*(f_1) = f_1$. Setting, choosing if necessary a smaller V_2 ,

$$\varphi = \varphi_1 \circ \varphi_2$$

we have $\varphi \in \text{Diff}^{\infty}(V_2; \mathbb{R}^n)$ with the claimed properties. ■

In the same spirit we have the following result, a particular case of which was proved in [3].

Theorem 28 *Let $n = 2m$ be even and $x_0 \in \mathbb{R}^n$. Let a, b be C^{∞} closed $1-$forms with*

$$a\,(x_0) \neq 0 \quad \textit{and} \quad b\,(x_0) \neq 0.$$

Let f, g be C^{∞} closed $2-$forms such that

$$\text{rank}\, f\,(x_0) = \text{rank}\, g\,(x_0) = n = 2m.$$

Then there exists a C^{∞} diffeomorphism φ such that, in a neighbourhood of x_0 ,

$$\varphi^*\,(b) = a \quad \textit{and} \quad \varphi^*\,(g) = f.$$

Proof It is enough to prove the theorem when

$$f = \omega_m = \sum_{i=1}^{m} dx^{2i-1} \wedge dx^{2i} \quad \text{and} \quad a = dx^1.$$

Using Theorem 24, it follows that there exists a C^{∞} diffeomorphism φ defined in a small ball U centred at x_0 such that

$$\varphi_1^*(b) = dx^1 \text{ in } U \quad \text{and} \quad \varphi_1(x_0) = x_0\,.$$

We now define a homotopy g_t through

$$g_t(x) = \varphi_1^*(g)(tx + (1-t)x_0) \quad \text{for every } x \in U \text{ and } t \in [0,1].$$

Note that, g_t is closed and non-degenerate (choosing U smaller if necessary), for every $t \in [0,1]$. We now use the example in Subsection 6.3 in [3] to find φ_2 such that

$$\varphi_2^*(g_1) = g_0 \quad \text{and} \quad \varphi_2^*(dx^1) = dx^1.$$

Finally, we find, according to Lemma 18, $A \in GL_n(\mathbb{R})$ satisfying

$$\varphi_3^*(g_0) = \omega_m \quad \text{and} \quad \varphi_3^*(dx^1) = dx^1$$

where $\varphi_3(x) = Ax$. It is now easy to check that the map

$$\varphi = \varphi_1 \circ \varphi_2 \circ \varphi_3$$

is the one that we are looking for. This finishes the proof. ■

5 Appendix: reduction of dimension

We turn our attention to a very useful result, which is well known in the case of $2-$forms. But it can be extended in a straightforward way to the case of $k-$forms; it seems however that this extension has never been noticed. We will provide two proofs of the theorem. The first one is based on Frobenius theorem, while the second one is much more elementary and self contained. Both versions lead to the same result when $k = n-1$, while the first one is better from the point of view of regularity when $2 \leq k \leq n-2$.

We begin by recalling few notions and results related to the theory of differential forms. For details, see [2], [22].

Notation 29 (Lie derivative and involutive family) *Let $U \subset \mathbb{R}^n$ be open, let $a, b \in C^1(U;\mathbb{R}^n)$ and let $\omega \in C^1(U;\Lambda^k)$.*

1. *$\mathcal{L}_a\omega$ stands for the* Lie derivative *of ω with respect to a. Recall that, for a vector field b, $\mathcal{L}_a b$ is also known as the* Lie bracket *of a with b and is denoted by $[a,b]$.*
2. Cartan formula *states that*
$$\mathcal{L}_a\omega = a \lrcorner d\omega + d(a \lrcorner \omega) \tag{37}$$
and moreover
$$[a,b] \lrcorner \omega = \mathcal{L}_a(b \lrcorner \omega) - b \lrcorner (\mathcal{L}_a\omega). \tag{38}$$
3. *For $a_1, \cdots, a_m \in C^1(U;\mathbb{R}^n)$, we say that $\{a_1, \cdots, a_m\}$ is an* involutive family *if, for every $1 \leq i,j \leq m$, there exist $c_{ij}^p \in C^0(U)$, $1 \leq p \leq m$, satisfying*
$$[a_i, a_j](x) = \sum_{p=1}^{m} c_{ij}^p(x) a_m(x), \text{ for every } x \in U.$$

We now recall Frobenius theorem.

Theorem 30 (Frobenius theorem) *Let $\Omega \subset \mathbb{R}^n$ be open, $r \geq 1$ and $1 \leq m < n$ be integers. Let $0 \leq \alpha \leq 1$ and $x_0 \in \Omega$. Let $a_1, \cdots, a_m \in C^{r,\alpha}(\Omega; \mathbb{R}^n)$ be an involutive family satisfying, for every $x \in \Omega$,*

$$\{a_1(x), \cdots, a_m(x)\} \quad \text{are linearly independent.}$$

Then, there exist two neighbourhoods $U, V \subset \Omega$ of x_0 and $\varphi \in \operatorname{Diff}^{r,\alpha}(V; U)$ such that

$$\varphi(x_0) = x_0$$

and, for every $x \in V$ and $1 \leq i \leq m$,

$$\frac{\partial \varphi}{\partial x^i}(x) \in \operatorname{span}\{(a_1 \circ \varphi)(x), \cdots, (a_m \circ \varphi)(x)\}.$$

The main result on dimension reduction is the following.

Theorem 31 (Reduction of dimension) *Let $r \geq 1$, $1 \leq k \leq l \leq n-1$ be integers, $0 \leq \alpha \leq 1$ and $x_0 \in \mathbb{R}^n$. Let g be a $C^{r,\alpha}$ closed $k-$form verifying*

$$\operatorname{rank} g = l \quad \text{in a neighbourhood of } x_0.$$

Then, there exist two neighbourhoods U, V of x_0 and $\varphi \in \operatorname{Diff}^{r,\alpha}(V; U)$ with $\varphi(x_0) = x_0$ and such that, for every $x = (x^1, \cdots, x^n) \in V$,

$$\begin{aligned}\varphi^*(g)\left(x^1, \cdots, x^n\right) &= f\left(x^1, \cdots, x^l\right) \\ &= \sum_{1 \leq i_1 < \cdots < i_k \leq l} f_{i_1 \cdots i_k}\left(x^1, \cdots, x^l\right) dx^{i_1} \wedge \cdots \wedge dx^{i_k}.\end{aligned}$$

Thus $f = \varphi^*(g)$ can be seen as a $k-$form with maximal rank (i.e. $\operatorname{rank} f = l$) on $\mathbb{R}^l$.

Before starting with the two proofs of the theorem, we need the following simple lemma, the proof of which is straightforward.

Lemma 32 *Let $U, V \subset \mathbb{R}^n$ be open, $g \in C^0(U; \Lambda^k)$ and $a \in C^0(U; \mathbb{R}^n)$ such that*

$$a(x) \lrcorner\, g(x) = 0 \quad \text{for every } x \in U.$$

Let $\varphi \in \operatorname{Diff}^1(V; U)$ satisfy

$$\frac{\partial \varphi}{\partial x^n} = a \circ \varphi \quad \text{in } V.$$

Then, for every $x \in V$,

$$(\varphi^* g)_{i_1 \cdots i_{k-1} n}(x) = 0 \quad \text{for every } 1 \leq i_1 < \cdots < i_{k-1} \leq n-1.$$

We now turn our attention to the first proof of our theorem.

First proof of Theorem 31 We divide the proof into four steps.

Step 1. Since $\operatorname{rank} g = l \leq n-1$, it is easy to find a neighbourhood U_1 of x_0 and $a_i \in C^{r,\alpha}(U_1; \mathbb{R}^n)$ for every $l+1 \leq i \leq n$ such that, for every $x \in U_1$,

$$\{a_{l+1}(x), \cdots, a_n(x)\} \text{ are linearly independent} \tag{39}$$

and

$$\operatorname{span}\{a_{l+1}(x), \cdots, a_n(x)\} = \ker \overline{g}(x); \tag{40}$$

so, in particular, for every $l+1 \leq i \leq n$ and every $x \in U_1$,

$$a_i(x) \lrcorner\, g(x) = 0.$$

Step 2. We now show that the family $\{a_{l+1}, \cdots, a_n\}$ of vector fields is involutive, i.e. for every $l+1 \leq i, j \leq m$, there exist $c_{ij}^p \in C^0(U_1)$, $l+1 \leq p \leq m$, satisfying

$$[a_i, a_j](x) = \sum_{p=l+1}^{n} c_{ij}^p(x) a_p(x) \quad \text{for every } x \in U_1 .$$

It follows, from the fact that g is closed, (37), (38) and (40), that

$$\begin{aligned} [a_i, a_j] \lrcorner\, g &= \mathcal{L}_{a_i}(a_j \lrcorner\, g) - a_j \lrcorner\, (\mathcal{L}_{a_i} g) = -a_j \lrcorner\, (\mathcal{L}_{a_i} g) \\ &= -a_j \lrcorner\, (a_i \lrcorner\, dg + d(a_i \lrcorner\, g)) = 0 \quad \text{in } U_1 . \end{aligned}$$

Hence, we have $[a_i, a_j](x) \in \ker \overline{g}(x)$, for every $x \in U_1$, from where, using (39) and (40), the existence of unique coefficients $c_{ij}^p(x)$, for every $x \in U_1$, follows. It is easy to check that $c_{ij}^p \in C^0(U_1)$.

Step 3. Therefore, using Theorem 30, we find two neighbourhoods U, V of x_0, $U, V \subset U_1$ and $\varphi \in \operatorname{Diff}^{r,\alpha}(V; U)$ such that $\varphi(x_0) = x_0$ and

$$\frac{\partial \varphi}{\partial x^i}(x) \in \ker \overline{g}(\varphi(x)) \quad \text{for every } x \in V \text{ and } l+1 \leq i \leq n. \tag{41}$$

It remains to show that satisfy

$$\varphi^*(g)_{i_1 \cdots i_k} = 0 \quad \text{in } V,$$

for every $1 \leq i_1 < \cdots < i_k \leq n$ with $i_k \geq l+1$. This follows at once from Lemma 32 and (41).

Step 4. Finally, since $dg = 0$, we have $d\varphi^*(g) = 0$. Hence, on writing

$$\varphi^*(g) = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant l} r_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

we obtain, for every $s \geq l+1$,

$$\frac{\partial r_{i_1 \cdots i_k}}{\partial x^s} = (d\varphi^*(g))_{i_1 \cdots i_k s} = 0.$$

We thus have the result. ■

We finish this article with the second proof of the theorem. As already said it is much more elementary but it gives a less sharp regularity; indeed we will only be able to establish that

$$\varphi \in \operatorname{Diff}^{r+l-n+1,\alpha}(V;U).$$

Note that when $k = n-1$ (and thus $l = n-1$) both proofs give the same result.

Second proof of Theorem 31 Without loss of generality, we assume that $x_0 = 0$. In the following proof, U, V stand for generic neighbourhoods of 0.

Step 1. Since $\operatorname{rank} g = l \leq n-1$, there exists $a \in C^{r,\alpha}(U;\mathbb{R}^n)$ satisfying

$$a(x) \neq 0 \quad \text{and} \quad a(x) \lrcorner\, g(x) = 0 \quad \text{for every } x \in U. \tag{42}$$

Step 2. We first find $\psi = \psi(x,t) \in \mathbb{R}^n$, for $|x|$ and $|t|$ small, such that

$$\frac{\partial}{\partial t}\psi(x,t) = a(\psi(x,t)) \quad \text{and} \quad \psi(x,0) = x.$$

Since $a(0) \neq 0$, there exist $b^1, \cdots, b^{n-1} \in \mathbb{R}^n$ so that

$$\left\{b^1, \cdots, b^{n-1}, a(0)\right\} \text{ are linearly independent.}$$

Let $B \in \mathbb{R}^{n\times(n-1)}$ be the matrix whose ith column is b^i. Finally define

$$\varphi_n(x) = \varphi_n(x^1, \cdots, x^n) = \psi\left(B(x^1, \cdots, x^{n-1}), x^n\right)$$

and observe that

$$\varphi_n(0) = 0, \quad \det \nabla\varphi_n(0) \neq 0 \quad \text{and} \quad \frac{\partial \varphi_n}{\partial x^n} = a \circ \varphi_n \quad \text{in } V.$$

We therefore have found a small neighbourhood V of 0 and $\varphi_n \in \operatorname{Diff}^{r,\alpha}(V;U)$ such that

$$\varphi_n(0) = 0 \quad \text{and} \quad \frac{\partial \varphi_n}{\partial x^n} = a \circ \varphi_n \quad \text{in } V. \tag{43}$$

Step 3. From (42), (43) and Lemma 32, it follows that, for every $1 \leq i_1 < \cdots < i_{k-1} \leq n-1$,

$$\varphi_n^*(g)_{i_1 \cdots i_{k-1} n} = 0 \quad \text{in } V,$$

and therefore

$$\varphi_n^*(g) \in C^{r-1,\alpha}(V;\Lambda^k(\mathbb{R}^{n-1})).$$

If $l = n-1$, the proof is finished. Henceforth, we will assume that $1 \leq l < n-1$. Since $\operatorname{rank}(\varphi_n^*(g)) = l$, repeating the argument aforementioned, we find $\varphi_{n-1} \in \operatorname{Diff}^{r-1,\alpha}(V,U)$ satisfying, for every $1 \leq i_1 < \cdots < i_{k-1} \leq n-2$,

$$\varphi_{n-1}^*(\varphi_n^*(g))_{i_1 \cdots i_{k-1}(n-1)} = 0 \quad \text{in } V.$$

After repeating the same argument $n-l$ times, we set

$$\varphi = \varphi_n \circ \cdots \circ \varphi_l \, .$$

It is clear that $\varphi \in \operatorname{Diff}^{r+l-n+1,\alpha}(V;U)$ and

$$\varphi^*(g) \in C^{r+l-n,\alpha}(V;\Lambda^k(\mathbb{R}^l)).$$

Step 4. Furthermore, since $dg = 0$, it follows, exactly as in Step 4 of the first proof of Theorem 31, that the coefficients of $\varphi^*(g)$ do not depend on $x^{l+1}, \cdots, x^n$. This finishes the proof. ■

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