Divisibility in Grassmann algebra

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Abstract

Given a $k-$form f and a $l-$form g with $0 \leq l \leq k$, we give necessary and sufficient conditions for the existence of a $(k-l)-$form u verifying

$$f = g \wedge u.$$

1 Introduction

Given a $k-$form f in $\mathbb{R}^n$ (and we denote it by $f \in \Lambda^k(\mathbb{R}^n)$) and $g \in \Lambda^l(\mathbb{R}^n)$ with $0 \leq l \leq k$, we want to find $u \in \Lambda^{k-l}(\mathbb{R}^n)$ such that

$$f = g \wedge u. \tag{1}$$

This is an important issue in several different contexts of differential geometry. The celebrated Cartan lemma (cf., for example, [6] page 75) states that if $g \neq 0$ is a product of $1-$forms, namely

$$g = g_1 \wedge \cdots \wedge g_l \neq 0$$

where $g_1, \cdots, g_l \in \Lambda^1(\mathbb{R}^n)$, then a necessary and sufficient condition for solving (1) is

$$f \wedge g_1 = \cdots = f \wedge g_l = 0.$$

We want here to extend this result to general $l-$forms. Our main result (see Theorem 11 for a more general statement) is the following.

Theorem 1 *Let $0 \leq l \leq k \leq n$ be integers, $f \in \Lambda^k(\mathbb{R}^n)$ and $g \in \Lambda^l(\mathbb{R}^n)$. The three following statements are then equivalent.*

(i) There exists $u \in \Lambda^{k-l}(\mathbb{R}^n)$ verifying

$$f = g \wedge u.$$

(ii) For every $h \in \Lambda^{n-k}(\mathbb{R}^n)$,

$$g \wedge h = 0 \quad \Rightarrow \quad f \wedge h = 0.$$

(iii) For every $0 \leq s \leq n-k$ *and every* $h \in \Lambda^s(\mathbb{R}^n)$,

$$g \wedge h = 0 \quad \Rightarrow \quad f \wedge h = 0.$$

Cartan lemma follows from the above one, once it is noticed that when

$$g = g_1 \wedge \cdots \wedge g_l \neq 0$$

then any $h \in \Lambda^s(\mathbb{R}^n)$ which satisfies $g \wedge h = 0$ is necessarily of the form

$$h = \sum_j g_j \wedge h_j$$

for any $h_j \in \Lambda^{s-1}(\mathbb{R}^n)$.

Closely related to the notion of divisibility is the notion of *prime* or *indecomposable* form, i.e. the $k-$forms that cannot be divided by any $l-$form, $1 \leq l \leq k-1$. We will briefly discuss this matter as well as the question of decomposability into prime forms.

We will also write (cf. Corollary 14) the dual version of (1), where the exterior product is replaced by the interior product, namely we solve

$$u \lrcorner g = f. \tag{2}$$

This last equation has been studied in Bandyopadhyay-Dacorogna-Kneuss [2] in the case where $g \in \Lambda^2(\mathbb{R}^n)$.

In the proof of the above theorem, we are lead to introduce for (2) the notion of *interior annihilator of order s* and for (1) the notion of *exterior annihilator of order s* as well as the concepts of *rank of order s* and *corank of order s* of an exterior form. In the literature only the rank of order 1 seems to be known, see [1], [2], [3], [5], [7].

Finally we will point out that these notions of annihilators, rank and corank seem very well adapted to the pullback equation, since they are invariant by pullback (see Propositions 3 (vii) and 5 (vii)). The pullback equation has been intensively studied in Bandyopadhyay-Dacorogna [1] and in Bandyopadhyay-Dacorogna-Kneuss [2].

Obviously we can replace $\mathbb{R}^n$ by any finite dimensional vector space over a field K of characteristic zero.

2 Exterior and interior annihilators, rank and corank

We write a $k-$form f over $\mathbb{R}^n$, and we denote it by $f \in \Lambda^k(\mathbb{R}^n)$, as

$$f = \sum_{1 \leq i_1 < \cdots < i_k \leq n} f_{i_1 \cdots i_k} e^{i_1} \wedge \cdots \wedge e^{i_k}.$$

We define the *exterior product* of $f \in \Lambda^k(\mathbb{R}^n)$ and $g \in \Lambda^l(\mathbb{R}^n)$ in the usual way

$$f \wedge g \in \Lambda^{k+l}(\mathbb{R}^n).$$

If $0 \leq l \leq k \leq n$, the *interior product* is defined, through the Hodge * operator, by

$$g \lrcorner f = (-1)^{n(k-l)} * [g \wedge (*f)] \in \Lambda^{k-l}(\mathbb{R}^n)$$

while if $l > k$, we set

$$g \lrcorner f = 0.$$

Note that

$$f_{i_1 \cdots i_k} = f \lrcorner \left(e^{i_1} \wedge \cdots \wedge e^{i_k}\right) = * \left[f \wedge \left(* \left(e^{i_1} \wedge \cdots \wedge e^{i_k}\right)\right)\right]. \tag{3}$$

We now define the different annihilating spaces and give some elementary properties.

Definition 2 *Let $0 \leq k \leq n$ and $f \in \Lambda^k(\mathbb{R}^n)$.*

(i) The space of exterior annihilators of f of order s *is the vector space*

$$\mathrm{Anh}_{\wedge}(f, s) = \{h \in \Lambda^s(\mathbb{R}^n) : f \wedge h = 0\}.$$

(ii) The space of interior annihilators of f of order s *is the vector space*

$$\mathrm{Anh}_{\lrcorner}(f, s) = \{h \in \Lambda^s(\mathbb{R}^n) : h \lrcorner f = 0\}.$$

Proposition 3 *Let $0 \leq k \leq n$ and $f, g \in \Lambda^k(\mathbb{R}^n)$.*

(i) The following hold

$$\mathrm{Anh}_{\wedge}(f, n-k) \neq \{0\} \quad \text{and} \quad \mathrm{Anh}_{\lrcorner}(f, k) \neq \{0\}.$$

(ii) The following equivalences hold

$$\begin{aligned} f = 0 \quad &\Leftrightarrow \; \mathrm{Anh}_{\wedge}(f, 0) \neq \{0\} \; \Leftrightarrow \; \mathrm{Anh}_{\wedge}(f, n-k) = \Lambda^{n-k}(\mathbb{R}^n) \\ &\Leftrightarrow \; \mathrm{Anh}_{\lrcorner}(f, 0) \neq \{0\} \; \Leftrightarrow \; \mathrm{Anh}_{\lrcorner}(f, k) = \Lambda^k(\mathbb{R}^n). \end{aligned}$$

(iii) If $0 \leq s \leq t \leq n$, then

$$\mathrm{Anh}_{\wedge}(f, s) \wedge \Lambda^{t-s}(\mathbb{R}^n) \subset \mathrm{Anh}_{\wedge}(f, t)$$

$$\mathrm{Anh}_{\lrcorner}(f, s) \wedge \Lambda^{t-s}(\mathbb{R}^n) \subset \mathrm{Anh}_{\lrcorner}(f, t)$$

(iv) If $0 \leq s \leq n$, then

$$\mathrm{Anh}_{\wedge}(f, s) = \mathrm{Anh}_{\lrcorner}(*f, s).$$

(v) The following inclusion holds

$$\mathrm{Anh}_{\wedge}(f, s) \cup \mathrm{Anh}_{\wedge}(g, s) \subset \mathrm{Anh}_{\wedge}(f \wedge g, s).$$

(vi) If $0 \leq s \leq t \leq n-k$ *and*

$$\operatorname{Anh}_{\wedge}(g,t) \subset \operatorname{Anh}_{\wedge}(f,t)$$

then

$$\operatorname{Anh}_{\wedge}(g,s) \subset \operatorname{Anh}_{\wedge}(f,s).$$

(vii) Let $A \in \mathbb{R}^{n\times n}$ *be an invertible matrix. For* $x \in \mathbb{R}^n$*, let*

$$\varphi(x) = Ax \quad \text{and} \quad \psi(x) = (A^{-1})^t x.$$

Then, for every $1 \leq s \leq n$*,*

$$\varphi^* \left(\operatorname{Anh}_{\wedge}\left(f,s\right)\right) = \operatorname{Anh}_{\wedge}\left(\varphi^*\left(f\right),s\right)$$

$$\psi^* \left(\operatorname{Anh}_{\lrcorner}\left(f,s\right)\right) = \operatorname{Anh}_{\lrcorner}\left(\varphi^*\left(f\right),s\right).$$

Proof *Step 1.* The proofs of (i) to (v) are immediate.

Step 2. We now prove (vi). First, we notice that if $h \in \operatorname{Anh}_{\wedge}(g,s)$, then

$$h \wedge e^{i_1} \wedge \cdots \wedge e^{i_{t-s}} \in \operatorname{Anh}_{\wedge}(g,t), \quad \forall 1 \leq i_1 < \cdots < i_{t-s} \leq n.$$

Thus, by hypothesis,

$$f \wedge h \wedge e^{i_1} \wedge \cdots \wedge e^{i_{t-s}} = 0, \quad \text{for every } 1 \leq i_1 < \cdots < i_{t-s} \leq n,$$

which easily implies the claim, namely

$$f \wedge h = 0.$$

Step 3. The claim (vii) is direct consequence of the two following formulas, see for example (13.35) in [8],

$$\varphi^*(h \wedge g) = \varphi^*(h) \wedge \varphi^*(g) \quad \text{and} \quad \varphi^*(h \lrcorner g) = \psi^*(h) \lrcorner \varphi^*(g).$$

This concludes the proof of the proposition. ■

The next important concept is the notion of rank and corank of a form and it is related to the dimension of the corresponding annihilating spaces.

Definition 4 *Let* $0 \leq k \leq n$ *be integers and* $f \in \Lambda^k(\mathbb{R}^n)$*.*

(i) The rank of order s, $0 \leq s \leq k \leq n$, *of* $f \in \Lambda^k(\mathbb{R}^n)$ *is given by*

$$\operatorname{rank}_s(f) = \binom{n}{s} - \dim(\operatorname{Anh}_{\lrcorner}(f,s)).$$

(ii) The corank of order s, $s+k \leq n$, *of* $f \in \Lambda^k(\mathbb{R}^n)$ *is defined by*

$$\operatorname{corank}_s(f) = \binom{n}{s} - \dim(\operatorname{Anh}_{\wedge}(f,s)).$$

In the literature (see page 1720 in [1], Definition 2.2 in [2], Definition 7.11 in [3], pages 85-88 in [5], page 25 in [7]) the only notion of rank, for an exterior form that is used, is the above rank of order 1. In page 89 of [5] and pages 24-25 of [7], a similar notion to our interior annihilator of order 1 is given. However it is not defined as above and it is not everywhere defined in the same way; but all the definitions are equivalent as it will be seen in Proposition 9. But before that let us gather some elementary properties of rank and corank.

Proposition 5 *Let* $f \in \Lambda^k(\mathbb{R}^n)$.

(i) If $f = 0$, *then, for every* s,

$$\operatorname{corank}_s(f) = \operatorname{rank}_s(f) = 0.$$

(ii) If $f \neq 0$, *then*

$$\operatorname{corank}_{n-k}(f) = \operatorname{rank}_k(f) = 1.$$

(iii) If $0 \leq s + k \leq n$, *then*

$$\operatorname{corank}_s(f) = \operatorname{rank}_s(*f).$$

(iv) If $0 \leq s \leq k$, *then*

$$\operatorname{rank}_s(f) = \operatorname{rank}_{k-s}(f)$$

and if $f \neq 0$, *then*

$$\operatorname{rank}_s(f) \geq \binom{k}{s}.$$

(v) If $0 \leq s + k \leq n$, *then*

$$\operatorname{corank}_s(f) = \operatorname{corank}_{n-(k+s)}(f)$$

and if $f \neq 0$, *then*

$$\operatorname{corank}_s(f) \geq \binom{n-k}{s}.$$

(vi) If $\operatorname{rank}_1(f) = k$, *then*

$$\dim\left(\operatorname{Anh}_{\wedge}(f, 1)\right) = k.$$

(vii) Let $A \in \mathbb{R}^{n\times n}$ *be an invertible matrix. For* $x \in \mathbb{R}^n$, *let*

$$\varphi(x) = Ax.$$

If $0 \leq s \leq k$, *then*

$$\operatorname{rank}_s\left(\varphi^*(f)\right) = \operatorname{rank}_s(f)$$

while if $0 \leq s + k \leq n$, *then*

$$\operatorname{corank}_s\left(\varphi^*(f)\right) = \operatorname{corank}_s(f).$$

Proof The proofs of (i), (ii) and (iii) are elementary.

Step 1. We now discuss the assertion (iv). We here use Notation 8. Let us show that

$$\overline{f}_{\lrcorner,s}=(-1)^{s(k-s)}\left(\overline{f}_{\lrcorner,k-s}\right)^{t}$$

which will prove the assertion using Proposition 9 (i). Indeed, by definition, for any $1\leq i_1<\cdots<i_s\leq n$ and $1\leq j_1<\cdots<j_{k-s}\leq n$, we have

$$\begin{aligned}\left(\overline{f}_{\lrcorner,s}\right)_{i_1\cdots i_s}^{j_1\cdots j_{k-s}}&=f_{i_1\cdots i_s j_1\cdots j_{k-s}}\\&=(-1)^{s(k-s)}f_{j_1\cdots j_{k-s}i_1\cdots i_s}=(-1)^{s(k-s)}\left(\overline{f}_{\lrcorner,k-s}\right)_{j_1\cdots j_{k-s}}^{i_1\cdots i_s}\end{aligned}$$

and thus the claim is proved. The fact that $\operatorname{rank}_s(f)\geq\binom{k}{s}$ is elementary.

Step 2. We then discuss (v). Recalling that $*f\in\Lambda^{n-k}(\mathbb{R}^n)$ and using (iii) and (iv), we have the assertion, since

$$\operatorname{corank}_s(f)=\operatorname{rank}_s(*f)=\operatorname{rank}_{n-k-s}(*f)=\operatorname{corank}_{n-(k+s)}(f).$$

The assertion on the lower bound for the corank follows from (iii) and (iv).

Step 3. We next establish (vi). Since $\operatorname{rank}_1(f)=k$ the $k-$form f is totally decomposable (see [2] and the Remark following Proposition 7), this means that there exist $f_1,\cdots,f_k\in\Lambda^1(\mathbb{R}^n)$ such that

$$f=f_1\wedge\cdots\wedge f_k\,.$$

This clearly shows the assertion, namely

$$\operatorname{Anh}_\wedge(f,1)=\operatorname{span}\{f_1,\cdots,f_k\}\,.$$

Step 4. The final claim (vii) is a direct consequence of Proposition 3 (vii). ■

Before proceeding further, we give some examples.

Example 6 Let $f\in\Lambda^k(\mathbb{R}^n)$ with $f\neq 0$.

(i) We start with the case $k=1$. We have, for $0\leq s+1\leq n$,

$$\operatorname{rank}_1(f)=1\quad\text{and}\quad\operatorname{corank}_s(f)=\binom{n-1}{s}.$$

(ii) We now turn to the case $k=2$. The only invariant that matters is $\operatorname{rank}_1(f)$ and, as will be seen in Proposition 7, it is even. It determines the corank of any order and, according to Proposition 5 (ii),

$$\operatorname{rank}_2(f)=1.$$

(iii) When $k=n$, then

$$\operatorname{rank}_s(f)=\binom{n}{s}$$

while if $k=n-1$, then

$$\operatorname{rank}_s(f)=\binom{n-1}{s}.$$

(iv) Consider the case $k = 3$. We automatically have, according to Proposition 5 (iv),

$$\operatorname{rank}_2(f) = \operatorname{rank}_1(f).$$

However the corank are not uniquely determined by the rank of order 1. Indeed assume that $n = 7$ and

$$f = e^1 \wedge e^2 \wedge e^3 + e^2 \wedge e^4 \wedge e^5 + e^3 \wedge e^6 \wedge e^7$$

$$g = e^1 \wedge e^2 \wedge e^3 + e^1 \wedge e^4 \wedge e^5 + e^1 \wedge e^6 \wedge e^7.$$

Then

$$\operatorname{rank}_1(f) = \operatorname{rank}_1(g) = 7$$

while

$$\operatorname{corank}_1(f) = 7 \quad \text{and} \quad \operatorname{corank}_1(g) = 6.$$

(v) We now give an example of two forms f and g having all their rank and corank equal. It can be proved, however, that they cannot be pulled back one to another. We still are in the case $k = 3$ and this time $n = 6$. We choose

$$f = e^1 \wedge e^2 \wedge e^3 + e^4 \wedge e^5 \wedge e^6$$

$$g = e^1 \wedge e^2 \wedge e^3 + e^1 \wedge e^4 \wedge e^5 + e^2 \wedge e^4 \wedge e^6 + e^3 \wedge e^5 \wedge e^6.$$

Then

$$\operatorname{rank}_1(f) = \operatorname{rank}_1(g) = \operatorname{rank}_2(f) = \operatorname{rank}_2(g) = 6$$

while we naturally have

$$\operatorname{rank}_3(f) = \operatorname{rank}_3(g) = 1.$$

Similarly

$$\operatorname{corank}_1(f) = \operatorname{corank}_1(g) = \operatorname{corank}_2(f) = \operatorname{corank}_2(g) = 6$$

while we automatically have

$$\operatorname{corank}_3(f) = \operatorname{corank}_3(g) = 1.$$

(vi) We finally give an example showing that the rank of order 1 does not determine the rank of higher orders. We let $k = 4$ and

$$f = e^1 \wedge e^2 \wedge e^3 \wedge e^4 + e^5 \wedge e^6 \wedge e^7 \wedge e^8$$

$$g = e^1 \wedge \left[e^2 \wedge e^3 \wedge e^4 + e^5 \wedge e^6 \wedge e^7 + e^2 \wedge e^5 \wedge e^8\right].$$

We have

$$\operatorname{rank}_1(f) = \operatorname{rank}_1(g) = 8$$

$$\operatorname{rank}_2(f) = 12 \quad \text{and} \quad \operatorname{rank}_2(g) = 14.$$

Since the most essential notion is the one of rank of order 1, we gather below some properties of this rank. The next proposition and remark is taken from Bandyopadhyay-Dacorogna-Kneuss [2].

Proposition 7 *Let* $f \in \Lambda^k(\mathbb{R}^n)$, $1 \leq k \leq n$.

(i) If $f \neq 0$ *and* $3 \leq k \leq n$, *then*

$$\operatorname{rank}_1(f) \in \{k, k+2, \cdots, n\}$$

and any of the values in $\{k, k+2, \cdots, n\}$ *can be achieved by the rank of order* 1 *of a* $k-$*form.*

(ii) If $k = 2$, *then the rank of order* 1 *of* f, $f \neq 0$, *is even and any even value less than or equal to* n *can be achieved by the rank of order* 1 *of a* $2-$*form. Moreover* $\operatorname{rank}_1(f) = 2l$ *if and only if*

$$f^l \neq 0 \quad \textit{and} \quad f^{l+1} = 0$$

where $f^l = \underbrace{f \wedge \cdots \wedge f}_{l-times}$.

(iii) If $g \in \Lambda^l(\mathbb{R}^n)$, *then*

$$\operatorname{rank}_1(f \wedge g) \leq \operatorname{rank}_1(f) + \operatorname{rank}_1(g) - \dim\left(\Lambda_f^1 \cap \Lambda_g^1\right)$$

where

$$\Lambda_f^1 = \left\{u \in \Lambda^1(\mathbb{R}^n) : \exists\, h \in \Lambda^{k-1}(\mathbb{R}^n) \quad \textit{with } h \lrcorner f = u\right\}$$

and similarly for Λ_g^1. *Moreover*

$$\operatorname{rank}_1(f \wedge g) = \operatorname{rank}_1(f) + \operatorname{rank}_1(g) \quad \Leftrightarrow \quad \Lambda_f^1 \cap \Lambda_g^1 = \{0\}.$$

Remark (i) For $f \in \Lambda^k(\mathbb{R}^n)$, then the rank of order 1 of f can never be $(k+1)$. In particular when $k = n-1$ and $f \neq 0$, then

$$\operatorname{rank}_1(f) = n - 1.$$

(ii) If $f \in \Lambda^2(\mathbb{R}^n)$ and $g \in \Lambda^1(\mathbb{R}^n)$ with $f \wedge g \neq 0$, then

$$\operatorname{rank}_1(f \wedge g) = \operatorname{rank}_1(f) \pm 1$$

and thus is odd. In particular if $\operatorname{rank}_1(f) = n$ (and therefore n is even), then

$$\operatorname{rank}_1(f \wedge g) = \operatorname{rank}_1(f) - 1 = n - 1.$$

(iii) Let $f \in \Lambda^k(\mathbb{R}^n)$. Then $\operatorname{rank}_1(f) = k$ if and only if

$$f \wedge [g \lrcorner f] = 0, \text{ for every } g \in \Lambda^{k-1}(\mathbb{R}^n)$$

if and only if there exist $f_1, \cdots, f_k \in \Lambda^1(\mathbb{R}^n)$ such that

$$f = f_1 \wedge \cdots \wedge f_k.$$

(iv) From (iii) of the proposition we can infer that if $f \neq 0$, then

$$\operatorname{rank}_1(*f) \geq n - \operatorname{rank}_1(f).$$

When $k = 1$ then the inequality becomes an equality. In general, however, as soon as $2 \leq k \leq n-2$ the inequality can be strict. ■

We now get back to prove that our definition of rank of order 1 is essentially the same as the one adopted by other authors for the definition of the rank of a form. At the same time we relate the notion of rank of a form with the rank of an associated matrix.

Notation 8 *In Proposition 9 and throughout the article we identify a $k-$form to a vector of $\mathbb{R}^{\binom{n}{k}}$ and, to fix the order of the elements of the vector, we adopt the lexicographical order. Let $f \in \Lambda^k(\mathbb{R}^n)$.*

(i) To the linear map

$$g \in \Lambda^s(\mathbb{R}^n) \to g \lrcorner f \in \Lambda^{k-s}(\mathbb{R}^n)$$

we associate a matrix $\overline{f}_{\lrcorner,s} \in \mathbb{R}^{\binom{n}{k-s}\times\binom{n}{s}}$ such that, by abuse of notations,

$$g \lrcorner f = \overline{f}_{\lrcorner,s}\, g \quad \textit{for every } g \in \Lambda^s(\mathbb{R}^n).$$

More explicitly, using the lexicographical order for the columns (index below) and the rows (index above) of the matrix $\overline{f}_{\lrcorner,s}$, we have

$$(\overline{f}_{\lrcorner,s})_{i_1\cdots i_s}^{j_1\cdots j_{k-s}} = f_{i_1\cdots i_s j_1\cdots j_{k-s}}$$

for $1 \leq i_1 < \cdots < i_s \leq n$ and $1 \leq j_1 < \cdots < j_{k-s} \leq n$.

(ii) Similarly, to the linear map

$$g \in \Lambda^s(\mathbb{R}^n) \to g \wedge f \in \Lambda^{k+s}(\mathbb{R}^n)$$

we associate a matrix $\overline{f}_{\wedge,s} \in \mathbb{R}^{\binom{n}{k+s}\times\binom{n}{s}}$ such that, by abuse of notations,

$$f \wedge g = \overline{f}_{\wedge,s}\, g \quad \textit{for every } g \in \Lambda^s(\mathbb{R}^n).$$

As above, the components of the matrix $\overline{f}_{\wedge,s}$ can be written as

$$(\overline{f}_{\wedge,s})_{i_1\cdots i_s}^{j_1\cdots j_{s+k}} = \left(f \wedge e^{i_1} \wedge \cdots \wedge e^{i_s}\right)_{j_1\cdots j_{s+k}}$$

for $1 \leq i_1 < \cdots < i_s \leq n$ and $1 \leq j_1 < \cdots < j_{k+s} \leq n$.

Proposition 9 *Let $f \in \Lambda^k(\mathbb{R}^n)$.*

(i) Let $0 \leq s \leq k \leq n$ and $f \in \Lambda^k(\mathbb{R}^n)$. Then

$$\operatorname{rank}_s(f) = \operatorname{rank}\left(\overline{f}_{\lrcorner,s}\right).$$

(ii) Let $0 \leq s, k \leq s+k \leq n$ *and* $f \in \Lambda^{k}(\mathbb{R}^{n})$. *Then*

$$\operatorname{corank}_{s}(f) = \operatorname{rank}\left(\overline{f}_{\wedge,s}\right).$$

(iii) Let

$$\Lambda_{f}^{1} = \left\{u \in \Lambda^{1}(\mathbb{R}^{n}) : \exists\, g \in \Lambda^{k-1}(\mathbb{R}^{n}) \quad \textit{with } g \lrcorner f = u\right\}$$

then

$$\operatorname{rank}_{1}(f) = \dim \Lambda_{f}^{1}.$$

Proof We start with the proof of (i), the proof of (ii) being similar. Using the definition of $\overline{f}_{\lrcorner,s}$, we see that

$$\ker(\overline{f}_{\lrcorner,s}) = \operatorname{Anh}_{\lrcorner}(f,s).$$

We thus obtain the result, since

$$\operatorname{rank}(\overline{f}_{\lrcorner,s}) = \tbinom{n}{s} - \ker(\overline{f}_{\lrcorner,s}).$$

To show (iii) we use Proposition 5 (iv) and the present proposition (i) to immediately get that

$$\dim \Lambda_{f}^{1} = \operatorname{rank}(\overline{f}_{\lrcorner,k-1}) = \operatorname{rank}(\overline{f}_{\lrcorner,1}) = \operatorname{rank}_{1}(f).$$

This concludes the proof of the proposition. ■

3 The main theorem and some corollaries

We start by recalling Cartan lemma.

Theorem 10 (Cartan lemma) *Let* $1 \leq l < k \leq n$, $f \in \Lambda^{k}(\mathbb{R}^{n})$ *and* $g_{1}, \cdots, g_{l} \in \Lambda^{1}(\mathbb{R}^{n})$ *be such that*

$$g = g_{1} \wedge \cdots \wedge g_{l} \neq 0.$$

Then the four following statements are equivalent.

(i) There exists $u \in \Lambda^{k-l}(\mathbb{R}^{n})$ *satisfying*

$$f = g \wedge u.$$

(ii) The following set of identities hold

$$f \wedge g_{1} = \cdots = f \wedge g_{l} = 0.$$

(iii) For every integer $0 \leq s \leq n-k$

$$f \wedge h = 0 \quad \forall h \in \operatorname{Anh}_{\wedge}(g,s).$$

(iv) The following holds

$$f \wedge h = 0 \quad \forall h \in \operatorname{Anh}_{\wedge}(g,1).$$

Remark (i) Cartan lemma is the equivalence between (i) and (ii). The other two equivalences are our way of rewriting the lemma. The equivalence between (i) and (iii) allows to suppress the hypothesis $g \neq 0$, since then $\mathrm{Anh}_{\wedge}(g, 0) \neq \{0\}$ and the condition

$$f \wedge h = 0 \quad \forall h \in \mathrm{Anh}_{\wedge}(g, 0)$$

is equivalent to $f = 0$.

(ii) The equivalence between (ii), (iii) and (iv) follows from the following observation. When

$$g = g_1 \wedge \cdots \wedge g_l \neq 0$$

then any $h \in \mathrm{Anh}_{\wedge}(g, s)$ is necessarily of the form

$$h = \sum_j g_j \wedge h_j$$

for any $h_j \in \Lambda^{s-1}(\mathbb{R}^n)$. ■

We now have our main result (cf. Theorem 1).

Theorem 11 *Let $0 \leq l \leq k \leq n$ be integers. Let $g \in \Lambda^l(\mathbb{R}^n)$ and $f \in \Lambda^k(\mathbb{R}^n)$. The following assertions are then equivalent.*

(i) There exists $u \in \Lambda^{k-l}(\mathbb{R}^n)$ verifying

$$g \wedge u = f.$$

(ii) The following inclusion holds

$$\mathrm{Anh}_{\wedge}(g, n-k) \subset \mathrm{Anh}_{\wedge}(f, n-k).$$

(iii) For every $0 \leq s \leq n-k$

$$\mathrm{Anh}_{\wedge}(g, s) \subset \mathrm{Anh}_{\wedge}(f, s).$$

(iv) Let $r = \mathrm{rank}(\overline{g}_{\wedge,k-l})$. Looking at f and at the columns of $\overline{g}_{\wedge,k-l}$ as $1-$forms in $\mathbb{R}^{\binom{n}{k}}$, then

$$(\overline{g}_{\wedge,k-l})_{i_1^{(1)}\cdots i_{k-l}^{(1)}} \wedge \cdots \wedge (\overline{g}_{\wedge,k-l})_{i_1^{(r)}\cdots i_{k-l}^{(r)}} \wedge f = 0$$

for every

$$1 \leq i_1^{(1)} < \cdots < i_{k-l}^{(1)} \leq n, \cdots, 1 \leq i_1^{(r)} < \cdots < i_{k-l}^{(r)} \leq n.$$

We now give some corollaries. The first one is immediate.

Corollary 12 *Theorem 11 indeed generalizes Cartan lemma.*

Corollary 13 *A $k-$form f over $\mathbb{R}^n$, is characterized, up to a multiplicative constant, by $\mathrm{Anh}_{\wedge}(f, n-k)$.*

Proof We give two proofs of the corollary. The first one as a consequence of the theorem and the second one in a constructive way.

Proof 1. Clearly f and λf, with $\lambda \neq 0$, verify

$$\mathrm{Anh}_{\wedge}(f, n-k) = \mathrm{Anh}_{\wedge}(\lambda f, n-k).$$

So let us show the converse and let $f, g \in \Lambda^k(\mathbb{R}^n)$ with

$$\mathrm{Anh}_{\wedge}(f, n-k) = \mathrm{Anh}_{\wedge}(g, n-k).$$

Theorem 11 (iii) (with $s = 0$) implies then the existence of $\lambda \in \Lambda^0(\mathbb{R}^n)$ with $g = \lambda f$. Noting that $\lambda \neq 0$ (unless $f = g = 0$), we have the claim.

Proof 2. The sufficient part is as in the first proof. We divide the proof into two steps and assume that

$$\mathrm{Anh}_{\wedge}(f, n-k) = \mathrm{Anh}_{\wedge}(g, n-k)$$

and let us show that $g = \lambda f$.

Step 1. We show that if $f_{i_1 \cdots i_k} = 0$ for some $1 \leq i_1 < \cdots < i_k \leq n$, then

$$g_{i_1 \cdots i_k} = 0.$$

Note that $f_{i_1 \cdots i_k} = 0$ is equivalent, according to (3), to

$$*\left(e^{i_1} \wedge \cdots \wedge e^{i_k}\right) \in \mathrm{Anh}_{\wedge}(f, n-k).$$

Hence, by hypothesis,

$$*\left(e^{i_1} \wedge \cdots \wedge e^{i_k}\right) \in \mathrm{Anh}_{\wedge}(g, n-k)$$

which is equivalent to $g_{i_1 \cdots i_k} = 0$, using (3).

Step 2. If $f = 0$, then $g = 0$ according to Step 1 and the corollary is thus true for any $\lambda \in \mathbb{R}$. We therefore assume that $f \neq 0$ and, without loss of generality, that $f_{1 \cdots k} \neq 0$. Let $1 \leq i_1 < \cdots < i_k \leq n$. We note that, using (3),

$$h = f_{i_1 \cdots i_k}\left[*\left(e^1 \wedge \cdots \wedge e^k\right)\right] - f_{1 \cdots k}\left[*\left(e^{i_1} \wedge \cdots \wedge e^{i_k}\right)\right] \in \mathrm{Anh}_{\wedge}(f, n-k).$$

The hypothesis implies that

$$g \wedge h = 0$$

which is equivalent, in view of (3), to

$$f_{i_1 \cdots i_k} g_{1 \cdots k} - f_{1 \cdots k} g_{i_1 \cdots i_k} = 0$$

and thus

$$g_{i_1 \cdots i_k} = \frac{g_{1 \cdots k}}{f_{1 \cdots k}} f_{i_1 \cdots i_k}\,.$$

Setting

$$\lambda = \frac{g_{1 \cdots k}}{f_{1 \cdots k}}$$

we have the assertion. ■

By duality, we obtain from Theorem 11 the corresponding result for interior equations.

Corollary 14 *Let $0 \le l \le k \le n$ be integers. Let $g \in \Lambda^k(\mathbb{R}^n)$ and $f \in \Lambda^l(\mathbb{R}^n)$. The following statements are then equivalent.*

(i) There exists $u \in \Lambda^{k-l}(\mathbb{R}^n)$ satisfying

$$u \lrcorner g = f.$$

(ii) For every $h \in \Lambda^l(\mathbb{R}^n)$

$$h \lrcorner g = 0 \quad \Rightarrow \quad h \lrcorner f = 0.$$

(iii) For every $0 \le s \le l$ and every $h \in \Lambda^s(\mathbb{R}^n)$

$$h \lrcorner g = 0 \quad \Rightarrow \quad h \lrcorner f = 0.$$

(iv) The following inclusion holds

$$\operatorname{Anh}_{\lrcorner}(g, l) \subset \operatorname{Anh}_{\lrcorner}(f, l).$$

(v) For every $0 \le s \le l$

$$\operatorname{Anh}_{\lrcorner}(g, s) \subset \operatorname{Anh}_{\lrcorner}(f, s).$$

(vi) Let $r = \operatorname{rank}(\overline{g}_{\lrcorner, k-l})$. Seeing f and the columns of $\overline{g}_{\lrcorner, k-l}$ as $1-$forms in $\mathbb{R}^{\binom{n}{l}}$, then

$$(\overline{g}_{\lrcorner, k-l})_{i_1^{(1)} \cdots i_{k-l}^{(1)}} \wedge \cdots \wedge (\overline{g}_{\lrcorner, k-l})_{i_1^{(r)} \cdots i_{k-l}^{(r)}} \wedge f = 0$$

for every

$$1 \le i_1^{(1)} < \cdots < i_{k-l}^{(1)} \le n, \cdots, 1 \le i_1^{(r)} < \cdots < i_{k-l}^{(r)} \le n.$$

Remark The case $k = 2$ has already been discussed in Bandyopadhyay-Dacorogna-Kneuss [2], where the necessary and sufficient condition is expressed differently. It is proved there that if $g \in \Lambda^2(\mathbb{R}^n)$ with $\operatorname{rank}_1(g) = 2s \le n$ and $f \in \Lambda^1(\mathbb{R}^n)$, then there exists $u \in \Lambda^1(\mathbb{R}^n)$ such that

$$f = u \lrcorner g$$

if and only if

$$f \wedge g^s = 0$$

where $g^s = \underbrace{g \wedge \cdots \wedge g}_{s-times}$. ■

Proof of Corollary 14 The equivalences (i) to (v) follow from Theorem 11 and from the following observations

$$f = u \lrcorner g = (-1)^{nl} * (u \wedge (*g)) \quad \Leftrightarrow \quad *f = (-1)^l (u \wedge (*g))$$

$$h \lrcorner g = 0 \quad \Leftrightarrow \quad h \wedge (*g) = 0$$

$$h \lrcorner f = 0 \quad \Leftrightarrow \quad h \wedge (*f) = 0.$$

The equivalence between (i) and (vi) is just Lemma 16 applied to the matrix $\overline{g}_{\lrcorner, k-l}$, since

$$u \lrcorner g = f \quad \Leftrightarrow \quad \overline{g}_{\lrcorner, k-l} u = f.$$

This concludes the proof of the corollary. ■

Our study of divisibility leads in a natural way to the notion of *prime* or *indecomposable* form, i.e. the $k-$forms that cannot be divided by any $l-$form, $1 \leq l \leq k-1$. The main theorem leads immediately to the following result.

Corollary 15 *A $k-$form f is prime (or indecomposable) if and only if for any $1 \leq l \leq k-1$ and any $g \in \Lambda^l(\mathbb{R}^n)$ there exists $h \in \text{Anh}_\wedge(g, n-k)$ such that*

$$f \wedge h \neq 0.$$

Remark (i) If f is prime then

$$\text{Anh}_\wedge(f, 1) = \{0\}.$$

Conversely if $k = 2$ or $k = 3$ and $\text{Anh}_\wedge(f, 1) = \{0\}$, then f is prime.

(ii) In Bandyopadhyay-Dacorogna-Kneuss [2], it is said that a $k-$form f is $l-$decomposable, $1 \leq l \leq k-1$, if and only if there exist $g \in \Lambda^l(\mathbb{R}^n)$ and $h \in \Lambda^{k-l}(\mathbb{R}^n)$ such that

$$f = g \wedge h.$$

With this definition we have

$$\text{corank}_1(f) < n \;\Leftrightarrow\; \text{Anh}_\wedge(f, 1) \neq \{0\} \;\Leftrightarrow\; f \text{ is } 1-\text{decomposable}.$$

Moreover if l is odd, then

$$f \text{ is } l-\text{decomposable} \;\Rightarrow\; \text{corank}_l(f) < \tbinom{n}{l}.$$

The converse is, in general, false. The result is also false, in general, if l is even.

(iii) As observed in [2], the decomposability of a given $k-$form into a product of primes is not unique. One can write, for example,

$$f = \left[e^1 \wedge e^2 + e^3 \wedge e^4\right] \wedge e^3 = e^1 \wedge e^2 \wedge e^3.$$

However the second representation of f is clearly better, in a sense discussed in Bandyopadhyay-Dacorogna-Kneuss (see the remark following Proposition 2.5 in [2]). ■

4 Proof of the main theorem and a special case of the theorem

In the proof of Corollary 14 we have used the following lemma. It will also be used in the proof of Theorem 11.

Lemma 16 *Let $m, n \geq 1$ be integers, $A \in \mathbb{R}^{n \times m}$ a matrix of rank r and $y \in \mathbb{R}^n$. Then there exists $x \in \mathbb{R}^m$ verifying*

$$Ax = y$$

if and only if

$$A_{i_1} \wedge \cdots \wedge A_{i_r} \wedge y = 0 \quad \textit{for every} \quad 1 \leq i_1 < \cdots < i_r \leq m$$

where y is identified to a $1-$form in $\mathbb{R}^n$ and A_k denotes the $k-$th column of A and is identified to a $1-$form in $\mathbb{R}^n$.

Proof *Step 1.* We first prove the necessary part. Assume that there exists $x \in \mathbb{R}^m$ verifying $Ax = y$. Then, writing, $x = (x^1, \cdots, x^m)$, we have

$$y = \sum_{l=1}^{m} A_l x^l.$$

Since the rank of A is r, we get

$$A_{i_1} \wedge \cdots \wedge A_{i_r} \wedge y = \sum_{l=1}^{m} x^l \left(A_{i_1} \wedge \cdots \wedge A_{i_r} \wedge A_l \right) = 0$$

which is our claim.

Step 2. We then turn to the proof of the sufficient part. Since the rank of A is r, we can find $1 \leq i_1 < \cdots < i_r \leq m$ such that

$$A_{i_1} \wedge \cdots \wedge A_{i_r} \neq 0.$$

Since we also have

$$A_{i_1} \wedge \cdots \wedge A_{i_r} \wedge y = 0$$

it follows that y is a linear combination of the A_{i_l}. This means that there exist w^l, $1 \leq l \leq r$ so that

$$y = \sum_{l=1}^{r} w^l A_{i_l}\,.$$

Setting $x = (x^1, \cdots, x^m)$ where

$$x^s = \begin{cases} w^l & \text{if } s = i_l \\ 0 & \text{otherwise} \end{cases}$$

it follows that

$$Ax = y.$$

This concludes the proof of the lemma. ■

We now turn to the proof of the main theorem.

Proof of Theorem 11 *Step 1.* The implications (i) ⇒ (iii) ⇒ (ii) are obvious. The equivalence (i) ⇔ (iv) is just a rewriting of Lemma 16, since

$$u \wedge g = f \quad \Leftrightarrow \quad \overline{g}_{\wedge,k-l} u = f.$$

Step 2. The only nontrivial implication is (ii) ⇒ (iv). Let

$$1 \leq i_1^{(1)} < \cdots < i_{k-l}^{(1)} \leq n, \cdots, 1 \leq i_1^{(r)} < \cdots < i_{k-l}^{(r)} \leq n,$$

recalling that $r = \operatorname{rank}(\overline{g}_{\wedge,k-l})$, and let us prove that

$$(\overline{g}_{\wedge,k-l})_{i_1^{(1)} \cdots i_{k-l}^{(1)}} \wedge \cdots \wedge (\overline{g}_{\wedge,k-l})_{i_1^{(r)} \cdots i_{k-l}^{(r)}} \wedge f = 0 \tag{4}$$

where f and $(\overline{g}_{\wedge,k-l})_{i_1^{(m)} \cdots i_{k-l}^{(m)}}$, $1 \leq m \leq r$ are seen as $1-$forms in $\mathbb{R}^{\binom{n}{k}}$. The equation (4) is equivalent to

$$\det \begin{vmatrix} (\overline{g}_{\wedge,k-l})_{i_1^{(1)} \cdots i_{k-l}^{(1)}}^{j_1^{(1)} \cdots j_k^{(1)}} & \cdots & (\overline{g}_{\wedge,k-l})_{i_1^{(r)} \cdots i_{k-l}^{(r)}}^{j_1^{(1)} \cdots j_k^{(1)}} & (f)^{j_1^{(1)} \cdots j_k^{(1)}} \\ \vdots & & \vdots & \vdots \\ (\overline{g}_{\wedge,k-l})_{i_1^{(1)} \cdots i_{k-l}^{(1)}}^{j_1^{(r+1)} \cdots j_k^{(r+1)}} & \cdots & (\overline{g}_{\wedge,k-l})_{i_1^{(r)} \cdots i_{k-l}^{(r)}}^{j_1^{(r+1)} \cdots j_k^{(r+1)}} & (f)^{j_1^{(r+1)} \cdots j_k^{(r+1)}} \end{vmatrix} = 0 \tag{5}$$

for every $1 \leq j_1^{(1)} < \cdots < j_k^{(1)} \leq n, \cdots, 1 \leq j_1^{(r+1)} < \cdots < j_k^{(r+1)} \leq n$. Expanding the determinant in (5) with respect to the last column and writing, for every $1 \leq m \leq r+1$,

$$c_m = \det \begin{vmatrix} (\overline{g}_{\wedge,k-l})_{i_1^{(1)} \cdots i_{k-l}^{(1)}}^{j_1^{(1)} \cdots j_k^{(1)}} & \cdots & (\overline{g}_{\wedge,k-l})_{i_1^{(r)} \cdots i_{k-l}^{(r)}}^{j_1^{(1)} \cdots j_k^{(1)}} \\ \vdots & & \vdots \\ (\overline{g}_{\wedge,k-l})_{i_1^{(1)} \cdots i_{k-l}^{(1)}}^{j_1^{(m-1)} \cdots j_k^{(m-1)}} & \cdots & (\overline{g}_{\wedge,k-l})_{i_1^{(r)} \cdots i_{k-l}^{(r)}}^{j_1^{(m-1)} \cdots j_k^{(m-1)}} \\ (\overline{g}_{\wedge,k-l})_{i_1^{(1)} \cdots i_{k-l}^{(1)}}^{j_1^{(m+1)} \cdots j_k^{(m+1)}} & \cdots & (\overline{g}_{\wedge,k-l})_{i_1^{(r)} \cdots i_{k-l}^{(r)}}^{j_1^{(m+1)} \cdots j_k^{(m+1)}} \\ \vdots & & \vdots \\ (\overline{g}_{\wedge,k-l})_{i_1^{(1)} \cdots i_{k-l}^{(1)}}^{j_1^{(r+1)} \cdots j_k^{(r+1)}} & \cdots & (\overline{g}_{\wedge,k-l})_{i_1^{(r)} \cdots i_{k-l}^{(r)}}^{j_1^{(r+1)} \cdots j_k^{(r+1)}} \end{vmatrix}$$

we find that (5) is equivalent to

$$\sum_{m=1}^{r+1} (-1)^{m+1} c_m (f)^{j_1^{(m)} \cdots j_k^{(m)}} = 0.$$

The above equation is equivalent (seeing f as a $k-$form and appealing to (3)) to

$$f \wedge \left(\sum_{m=1}^{r+1} (-1)^{m+1} c_m \left[*(e^{j_1^{(m)}} \wedge \cdots \wedge e^{j_k^{(m)}}) \right] \right) = 0.$$

To prove our claim it is sufficient to prove that

$$h = \sum_{m=1}^{r+1} (-1)^{m+1} c_m \left[*(e^{j_1^{(m)}} \wedge \cdots \wedge e^{j_k^{(m)}}) \right] \in \Lambda^{n-k}(\mathbb{R}^n)$$

satisfies

$$g \wedge h = 0$$

which turns out to be equivalent to

$$g \wedge e^{t_1} \wedge \cdots \wedge e^{t_{k-l}} \wedge h = 0 \tag{6}$$

for every $1 \leq t_1 < \cdots < t_{k-l} \leq n$. Since the matrix $\overline{g}_{\wedge,k-l}$ has rank r, we get

$$(\overline{g}_{\wedge,k-l})_{i_1^{(1)} \cdots i_{k-l}^{(1)}} \wedge \cdots \wedge (\overline{g}_{\wedge,k-l})_{i_1^{(r)} \cdots i_{k-l}^{(r)}} \wedge (\overline{g}_{\wedge,k-l})_{t_1 \cdots t_{k-l}} = 0$$

which implies that

$$\det \begin{vmatrix} (\overline{g}_{\wedge,k-l})^{j_1^{(1)} \cdots j_k^{(1)}}_{i_1^{(1)} \cdots i_{k-l}^{(1)}} & \cdots & (\overline{g}_{\wedge,k-l})^{j_1^{(1)} \cdots j_k^{(1)}}_{i_1^{(r)} \cdots i_{k-l}^{(r)}} & (\overline{g}_{\wedge,k-l})^{j_1^{(1)} \cdots j_k^{(1)}}_{t_1 \cdots t_{k-l}} \\ \vdots & & \vdots & \vdots \\ (\overline{g}_{\wedge,k-l})^{j_1^{(r+1)} \cdots j_k^{(r+1)}}_{i_1^{(1)} \cdots i_{k-l}^{(1)}} & \cdots & (\overline{g}_{\wedge,k-l})^{j_1^{(r+1)} \cdots j_k^{(r+1)}}_{i_1^{(r)} \cdots i_{k-l}^{(r)}} & (\overline{g}_{\wedge,k-l})^{j_1^{(r+1)} \cdots j_k^{(r+1)}}_{t_1 \cdots t_{k-l}} \end{vmatrix} = 0.$$

Expanding the above determinant with respect to the last column, we obtain

$$\sum_{m=1}^{r+1} (-1)^{m+1} c_m (\overline{g}_{\wedge,k-l})^{j_1^{(m)} \cdots j_k^{(m)}}_{t_1 \cdots t_{k-l}} = 0.$$

Let us show that this last equation is equivalent to (6), namely

$$g \wedge e^{t_1} \wedge \cdots \wedge e^{t_{k-l}} \wedge h = 0.$$

Noting that from Notation 8 and (3), we have, for every $1 \leq m \leq r+1$,

$$\begin{aligned} (\overline{g}_{\wedge,k-l})^{j_1^{(m)} \cdots j_k^{(m)}}_{t_1 \cdots t_{k-l}} &= \left(g \wedge e^{t_1} \wedge \cdots \wedge e^{t_{k-l}} \right)_{j_1^{(m)} \cdots j_k^{(m)}} \\ &= *(g \wedge e^{t_1} \wedge \cdots \wedge e^{t_{k-l}} \wedge (*(e^{j_1^{(m)}} \wedge \cdots \wedge e^{j_k^{(m)}})). \end{aligned}$$

We therefore obtain that

$$*(g \wedge e^{t_1} \wedge \cdots \wedge e^{t_{k-l}} \wedge h) = \sum_{m=1}^{r+1} (-1)^{m+1} c_m (\overline{g}_{\wedge,k-l})^{j_1^{(m)} \cdots j_k^{(m)}}_{t_1 \cdots t_{k-l}} = 0.$$

This is exactly our claim (6). ■

We conclude our article with the case $k = l + 1$ and a special g.

Theorem 17 *Let $1 \leq p \leq n-1,\ 2 \leq l \leq n-1,$ with $p+l+1, pl \leq n,$ be integers. Let $f \in \Lambda^{l+1}(\mathbb{R}^n)$ and $g \in \Lambda^l(\mathbb{R}^n)$ with*

$$g = [g_1 \wedge \cdots \wedge g_l] + [g_{l+1} \wedge \cdots \wedge g_{2l}] + \cdots + \left[g_{(p-1)l+1} \wedge \cdots \wedge g_{pl}\right]$$

$$g_1 \wedge \cdots \wedge g_{pl} \neq 0$$

The two following statements are then equivalent.

(i) There exists $u \in \Lambda^1(\mathbb{R}^n)$ verifying

$$f = g \wedge u.$$

(ii) For every $i_j \in \{(j-1)\,l+1, \cdots, jl\}$ and $j, s, t \in \{1, \cdots, p\}$ with $s < t$, the following two sets of identities hold

$$f \wedge g_{i_1} \wedge \cdots \wedge g_{i_p} = 0$$

$$f \wedge g_{i_1} \wedge \cdots \wedge g_{i_{s-1}} \wedge g_{i_{s+1}} \wedge \cdots \wedge g_{i_{t-1}} \wedge g_{i_{t+1}} \wedge \cdots \wedge g_{i_p} \wedge G_{st} = 0$$

where

$$G_{st} = \left[g_{(s-1)l+1} \wedge \cdots \wedge g_{sl}\right] + (-1)^{l+1}\left[g_{(t-1)l+1} \wedge \cdots \wedge g_{tl}\right].$$

Remark The case $p = 1$ in the theorem above is Cartan lemma (when $k = l+1$), since the last set of identities is then empty. If $p = 2$ the last set of identities reads as

$$f \wedge g_i \wedge g_j = 0, \quad 1 \leq i \leq l < j \leq 2l$$

$$f \wedge \Big[[g_1 \wedge \cdots \wedge g_l] + (-1)^{l+1}[g_{l+1} \wedge \cdots \wedge g_{2l}]\Big] = 0. \quad \blacksquare$$

Proof *Step 1.* We show that $(i) \Rightarrow (ii)$. For every $1 \leq j \leq p$, we set

$$I_j = \{(j-1)l+1, \cdots, jl\}.$$

Since we trivially have, for every $i_j \in I_j$ and $j, s, t \in \{1, \cdots, p\}$ with $s < t$,

$$g \wedge g_{i_1} \wedge \cdots \wedge g_{i_p} = 0$$

$$g \wedge g_{i_1} \wedge \cdots \wedge g_{i_{s-1}} \wedge g_{i_{s+1}} \wedge \cdots g_{i_{t-1}} \wedge g_{i_{t+1}} \wedge \cdots \wedge g_{i_p} \wedge G_{st} = 0$$

and since

$$f = g \wedge u$$

we immediately get the result.

Step 2. We now prove that $(ii) \Rightarrow (i)$.

Step 2.1. Since

$$g_1 \wedge \cdots \wedge g_{pl} \neq 0,$$

we can assume, without loss of generality, that $g_i = e^i$, $1 \leq i \leq pl$. Under this hypothesis the existence of a u satisfying the equation

$$f = g \wedge u$$

will be implied by the following two sets of identities. The first one is

$$f_{j_1\cdots j_{l+1}} = 0 \tag{7}$$

for every $1 \leq j_1 < \cdots < j_{l+1} \leq n$ and $I_m \not\subset \{j_1, \cdots, j_{l+1}\}$ for every $m \in \{1, \cdots, p\}$. The second one is

$$f_{(s-1)l+1\cdots(sl)\nu} = f_{(t-1)l+1\cdots(tl)\nu} \tag{8}$$

for every $1 \leq s < t \leq p$ and $\nu \in \{1, \cdots, n\} \backslash (I_s \cup I_t)$. The result will follow, if we can show that (7) is implied by the first set of identities in the statement (ii) of the theorem (cf. Step 2.2) and (8) is implied by the second set of identities in the statement (ii) of the theorem (cf. Step 2.3).

Step 2.2. By hypothesis, we have, for every $i_s \in I_s$ and $s \in \{1, \cdots, p\}$,

$$f \wedge e^{i_1} \wedge \cdots \wedge e^{i_p} = 0.$$

We therefore deduce, for every $1 \leq j_1 < \cdots < j_{l+1} \leq n$, that

$$(f \wedge e^{i_1} \wedge \cdots \wedge e^{i_p})_{j_1\cdots j_{l+1} i_1 \cdots i_p} = 0. \tag{9}$$

Let $1 \leq j_1 < \cdots < j_{l+1} \leq n$ with $I_m \not\subset \{j_1, \cdots, j_{l+1}\}$ for every $m \in \{1, \cdots, p\}$. We then choose, for $m \in \{1, \cdots, p\}$,

$$i_m \in I_m \diagdown \{j_1, \cdots, j_{l+1}\}.$$

Applying (9) with these coefficients, we immediately have (7).

Step 2.3. We know that, for every $i_r \in I_r$ and $r, s, t \in \{1, \cdots, p\}$ with $s < t$, the following set of identities hold

$$h = f \wedge e^{i_1} \wedge \cdots \wedge e^{i_{s-1}} \wedge e^{i_{s+1}} \wedge \cdots \wedge e^{i_{t-1}} \wedge e^{i_{t+1}} \wedge \cdots \wedge e^{i_p} \wedge E^{st} = 0$$

where

$$E^{st} = \left[e^{(s-1)l+1} \wedge \cdots \wedge e^{sl}\right] + (-1)^{l+1} \left[e^{(t-1)l+1} \wedge \cdots \wedge e^{tl}\right].$$

Let $1 \leq s < t \leq p$ and $\nu \in \{1, \cdots, n\} \backslash (I_s \cup I_t)$. Choosing, for $r \in \{1, \cdots, p\}$ with $r \neq s, t$, $i_r \in I_r \diagdown \{\nu\}$, we get, in particular,

$$h_{(s-1)l+1\cdots(sl),(t-1)l+1\cdots(tl),\nu,i_1\cdots i_{s-1}i_{s+1}\cdots i_{t-1}i_{t+1}\cdots i_p} = 0.$$

It is easily seen, with the above choice of coefficients, that the last equation is equivalent to (8), namely

$$f_{(s-1)l+1\cdots(sl)\nu} = f_{(t-1)l+1\cdots(tl)\nu}.$$

This concludes the proof of the theorem. ■

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