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REGULARITY AND SELECTING PRINCIPLES FOR IMPLICIT ORDINARY DIFFERENTIAL EQUATIONS

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*This article is dedicated to Paolo Marcellini
on the occasion of his sixtieth birthday*

Abstract. Implicit Ordinary or Partial Differential Equations have been widely studied in recent times, essentially from the existence of solutions point of view. One of the main issues is to select a meaningful solution among the infinitely many ones. The most celebrated principle is the viscosity method. This selection principle is well adapted to convex Hamiltonians, but it is not always applicable to the non-convex setting.

In this work we present an alternative selecting principle that singles out the most regular solutions (which do not always coincide with the viscosity ones). Our method is based on a general regularity theorem for Implicit ODEs. We also provide several examples.

Introduction. Let f be a function from $[a, b] \times \mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. An Implicit Ordinary Differential Equation, as defined by Dacorogna-Marcellini [7], is an equation of the form

$$f(t, u(t), \dot{u}(t)) = 0, \qquad \text{for a.e. } t \text{ in } (a, b), \tag{1}$$

where solutions u are searched in $W_0^{1,\infty}(a, b)$. As it is well known, under suitable compatibility conditions between the data and the function f, this problem admits a priori infinitely many solutions [7].

The purpose of the present work is to determine criteria to select, among these infinitely many solutions, some of them. At the moment there is no general criterion that permits to select a unique solution in all contexts. Depending on the equation some of them are more adapted than the others. Here are some of the most natural criteria.

1) The most widely used criterion is the *viscosity method* introduced in a systematic way by Crandall-Lions [5, 10]. However powerful and general is the method, it essentially applies only when the Hamiltonian f is convex in the last variable.

2) Another natural criterion is the one selecting the *maximal solution*. In many instances, the pointwise supremum of solutions is however not a solution.

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3) We will here emphasize a new criterion selecting the *most regular solution*. Since we are dealing with first order ode's coupled with Dirichlet data, the rule is that there are no C^1 solutions and only Lipschitz ones are to be expected. Therefore it is natural to choose among the many solutions, those that exhibit the smallest number of discontinuities of their derivatives.

In order to make the last principle precise, we first establish a general regularity result. It says that, under only mild and natural assumptions on the boundary data and for a large class of f, the solutions in $W_0^{1,\infty}(a,b)$ to (1) have piecewise continuous derivative (Theorem 1), with a finite number of discontinuities. We therefore define the most regular solutions as those which have derivatives with the least number of discontinuity points. By using this result we show (Theorem 14) that there exist a finite number, that we provide explicitly, of solutions with at most one discontinuity point of the derivative. Furthermore, we show an "essential" uniqueness result (Theorem 17), that is, in the strictly convex case (f strictly convex with respect to $\dot{u}$), there exist two solutions, one positive and one negative, with at most one discontinuity point of the derivative.

We would like to conclude this introduction pointing out that we look at Problem (1) essentially as a differential inclusion. In other words, from our standpoint, Problem (1) is equivalent to find u in $W_0^{1,\infty}(a,b)$ that solves

$$(t, u(t), \dot{u}(t)) \in Z_f, \qquad \text{for a.e. } t \text{ in } (a,b),$$

where Z_f is the set of zeros of f, namely

$$Z_f = \{(t,u,\xi) \in [a,b] \times \mathbb{R} \times \mathbb{R} : f(t,u,\xi) = 0\}.$$

Therefore, no other properties of f, besides the ones related to its set of zeros, will play any role in our work. Nevertheless, we deal throughout the paper with our problem mainly in form (1), since this form is the most commonly studied.

The paper is organized as follows. In Section 1 we present the regularity result for the derivative of the solutions to our problem (1); in Section 2 we address briefly whether the compatibility condition on the boundary data are also necessary for the existence. In Sections 3.1 and 3.2 we recall the viscosity solution method and the maximal solution selecting principle and finally, in Section 3.3, we present our criterion based on choosing the most regular solutions and also the uniqueness result.

1. **Regularity.** In this section we present our regularity results. We prove that for a large class of f, under a compatibility condition in order to guarantee existence, the problem (1) admits a solution $\bar{u}$ in $W_0^{1,\infty}(a,b)$ with piecewise continuous derivative.

We denote by $C_{\text{piec}}([a,b])$ the set of piecewise continuous functions, with a *finite* number of discontinuities, and such that the left and the right limits exist and are finite at every points of $[a,b]$. We also introduce the set

$$C_{k-\text{piec}}([a,b]) := \{u \in C_{\text{piec}}([a,b]) : u \text{ has at most } k \text{ continuity intervals}\},$$

for a given k in $\mathbb{N}$, that is the set of functions in $C_{\text{piec}}([a,b])$ with at most $k-1$ discontinuity points.

We denote also by S_f the set of solutions to (1), i.e.

$$S_f := \{u \in W_0^{1,\infty}(a,b) : f(t,u,\dot{u}) = 0\}.$$

and we say that a function $F(t,u) : [a,b] \times \mathbb{R} \to \mathbb{R}$ is continuous in u and piecewise continuous in t if there exist a finite partition of $[a,b]$ in intervals with extreme points $a^j < b^j$, $j = 1, \cdots, K$, such that F restricted to any $(a^j, b^j) \times \mathbb{R}$ is uniformly continuous.

We start proving the Theorem that has a central role in the present work.

Theorem 1. *Assume that Z_f contains the zeros of the graph of a positive and a negative function, i.e. Z_f contains the zeros of*

$$P_+(t, u, \xi) := \xi - f_+(t, u), \qquad P_-(t, u, \xi) := \xi - f_-(t, u),$$

where $f_+, f_- : [a, b] \times [0, \delta] \to \mathbb{R}$ are continuous in u, for a certain $\delta > 0$, piecewise continuous in t and $f_+ > 0 > f_-$.

Then, (1) admits a $W_0^{1,\infty}(a, b)$ solution $\bar{u}_+ > 0$, with $\dot{\bar{u}}_+$ in $C_{piec}([a, b])$.

Analogously, assume that Z_f contains the zeros of the graph of a positive and a negative function, i.e. Z_f contains the zeros of

$$P_+(t, u, \xi) := \xi - g_+(t, u), \qquad P_-(t, u, \xi) := \xi - g_-(t, u),$$

where $g_+, g_- : [a, b] \times [-\delta, 0] \to \mathbb{R}$ are continuous in u, for a certain $\delta > 0$, piecewise continuous in t and $g_+ > 0 > g_-$.

Then, (1) admits a $W_0^{1,\infty}(a, b)$ solution $\bar{u}_- < 0$, with $\dot{\bar{u}}_-$ in $C_{piec}([a, b])$.

Proof. We can suppose that f_+ and f_- are continuous in t on the entire interval $[a, b]$. The general case follows by proceeding as below for every continuity intervals $[a^j, b^j]$, $j = 1, \cdots, K$. Let $R > 0$ be the maximum between f_+ and $-f_-$ on $[a, b] \times [0, \delta]$. Consider the Cauchy problem

$$\begin{cases} \dot{u}(t) = f_+(t, u(t)), & t > a, \\ u(a) = 0. \end{cases} \tag{2}$$

By the Cauchy-Peano Theorem, this problem admits a local solution $\bar{u}_l$ in $C^1([a, a_1])$, with $a_1 > a$. We also have that any solution to (2) is increasing since $f_+ > 0$. By the continuity of $\bar{u}_l$ and $\bar{u}_l(a) = 0$, we can choose a_1 such that $\bar{u}_l(t)$ is smaller than δ, for any t in $[a, a_1]$. We then consider another Cauchy problem defined by

$$\begin{cases} \dot{u}(t) = f_-(t, u(t)), & t > a_1, \\ u(a_1) = \bar{u}_l(a_1). \end{cases} \tag{3}$$

Again this new problem admits a local solution $\bar{u}_l$ in $C^1([a_1, a_2])$ which is now decreasing since $f_- < 0$. The bound on $|f_-|$ implies that, defining a_2 by

$$a_2 := a_1 + \frac{\bar{u}_l(a_1)}{2R},$$

$\bar{u}_l(t) \geq \bar{u}_l(a_1)/2$, for any t in $[a_1, a_2]$. Indeed, for any t in $[a_1, a_2]$, by definition of R,

$$\bar{u}_l(t) = \bar{u}_l(a_1) + \int_{a_1}^{t} f_-(\tau, \bar{u}_l(\tau))d\tau \geq \bar{u}_l(a_1) - R(t - a_1)$$

Hence, $\bar{u}_l(a_2)$ is greater than $\bar{u}_l(a_1)/2$ and is smaller than $\bar{u}_l(a_1) < \delta$. Denoting

$$f_{(-)^n} := \begin{cases} f_+ & \text{if } n \text{ is even} \\ f_- & \text{if } n \text{ is odd,} \end{cases}$$

by iteration, we can define $\bar{u}_l$ on the entire interval $[a, b]$ where $\bar{u}_l$ is solution to

$$\begin{cases} \dot{u}(t) = f_{(-)^n}(t, u(t)), & t > a_n, \\ u(a_n) = \bar{u}_l(a_n). \end{cases}$$

on $[a_n, a_{n+1}]$, $a_{n+1} := a_n + \min\{\bar{u}_l(a_1)/(2R), b - a_n\}$ and $a_0 := a$. Observe that the iteration process stops after integer part of $2R(b-a)/\bar{u}_l(a_1)$ steps. Furthermore,

$$\bar{u}_l(b) \geq \frac{\bar{u}_l(a_1)}{2} > 0.$$

Analogously, starting from b and proceeding backward in t, we can define a function $\bar{u}_r$ on $[a,b]$ solution to

$$\begin{cases} \dot{u}(t) = f_{(-)^{n+1}}(t, u(t)), & t < b_n, \\ u(b_n) = \bar{u}_r(b_n). \end{cases}$$

on $[b_n, b_{n+1}]$, $b_{n+1} := b_n - \min\{\bar{u}_r(b_1)/(2R), b_n - a\}$ and $b_0 := b$, $\bar{u}_r(b_0) := 0$. The iteration process stops after integer part of $2R(b-a)/\bar{u}_r(b_1)$ steps. Furthermore,

$$\bar{u}_r(a) \geq \frac{\bar{u}_r(b_1)}{2} > 0.$$

Define the function $\bar{u}_+ := \min\{\bar{u}_l, \bar{u}_r\}$. It is continuous on $[a,b]$ with piecewise continuous derivative, $\bar{u}_+(a) = \bar{u}_+(b) = 0$ and it satisfies

$$[\dot{\bar{u}}_+(t) - f_+(t, \bar{u}_+(t))][\dot{\bar{u}}_+(t) - f_-(t, \bar{u}_+(t))] = 0.$$

We therefore conclude that $f(t, \bar{u}_+(t), \dot{\bar{u}}_+(t)) = 0$ for a.e. t in (a,b).

Analogously, we can prove the existence of $\bar{u}_- < 0$ in case of g_+ and g_- defined on $[a,b] \times [-\delta, 0]$. □

We refer to the assumptions of Theorem 1 as the *compatibility condition.*

Theorem 1 is a sharp result (see Section 2). Even if the assumptions are expressed in term of the set of zeros of f and not directly on f, a function f which does not satisfy such assumptions must be built on purpose in an artificial way. However, in what follows we propose an explicit general non-convex class of functions f which satisfy the needed conditions on its set of zeros.

We say that a function $h(\xi) : \mathbb{R} \to \mathbb{R}$ is piecewise strictly monotone increasing (respectively decreasing) if there exists a locally finite partition of intervals of $\mathbb{R}$ such that h is strictly monotone increasing (respectively decreasing) on any of such intervals. Further, a function $q(t,u) : [a,b] \times \mathbb{R} \to \mathbb{R}$ is locally monotone increasing (respectively decreasing) at $u = 0$ if there exists $\delta > 0$ such that

$$q(t,u) \geq q(a,0) \quad (\text{respectively } q(t,u) \leq q(a,0)), \qquad \text{for any } t \text{ in } (a,b),\ u \text{ in } (0,\delta).$$

We say that q is piecewise locally monotone at $u = 0$ if there exits a finite partition of intervals of $[a,b]$ such that q is locally monotone at $u = 0$ on any of such intervals.

Theorem 2. *Let $q(t,u) : [a,b] \times \mathbb{R} \to \mathbb{R}$ be a function continuous with respect to u and piecewise continuous with respect to t and let $h(\xi) : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose that either h is piecewise strictly monotone, or h is piecewise monotone and q is locally piecewise monotone at $u = 0$,*

$$f(t,u,\xi) := q(t,u) + h(\xi),$$

$h(0) < -q(t,0)$, for any t in $[a,b]$, and $h(\xi) \to +\infty$, as $|\xi| \to \infty$.

Then, problem (1) admits two $W_0^{1,\infty}(a,b)$ solutions $\bar{u}_+ > 0 > \bar{u}_-$, with $\dot{\bar{u}}_+$ and $\dot{\bar{u}}_-$ in $C_{piec}([a,b])$.

Proof. Whitout loss of generality, we can suppose that q is continuous with respect to t in the whole interval $[a,b]$. The general case follows by proceeding as below for every continuity intervals $[a^j, b^j]$, $j = 1, \cdots, K$. We split the proof in two cases.

Case 1. Suppose that h is piecewise strictly monotone.

Let $\{I_j\}_{j\in\mathbb{N}}$ be a partition of $(0,+\infty)$ such that h restricted to the interior $\mathrm{int}(I_j)$ of I_j is strictly monotone and the I_j's are maximal. By the strict monotonicity of $h|_{\mathrm{int}(I_j)}$, the inverse functions h_j^{-1} exist and are continuous from $h(\mathrm{int}(I_j))$ to I_j. Furthermore,

$$\bigcup_{j\in\mathbb{N}} h(\mathrm{int}(I_j)) = (\inf_{(0,+\infty)} h, +\infty).$$

By the piecewise strict monotonicity of h, there exists a partition $a_0 := a < a_1 < \cdots < a_{n-1} < b$ of $[a,b]$ and n integer numbers $j_1,\cdots,j_n$ such that

$$\begin{aligned} &h(\mathrm{int}(I_{j_k})) \supset \{-q(t,0) : t\in[a_{k-1},a_k)\}, \text{ for } k=1,\cdots,n-1,\\ &h(\mathrm{int}(I_{j_n})) \supset \{-q(t,0) : t\in[a_{n-1},b]\}.\end{aligned}$$

Therefore, by continuity, there exists also $\delta>0$ such that

$$\begin{aligned} &h(\mathrm{int}(I_{j_k})) \supset \{-q(t,u) : t\in[a_{k-1},a_k), u\in[0,\delta)\}, \text{ for } k=1,\cdots,n-1,\\ &h(\mathrm{int}(I_{j_n})) \supset \{-q(t,u) : t\in[a_{n-1},b], u\in[0,\delta)\}.\end{aligned}$$

Define

$$f_+(t,u) := \begin{cases} h_{j_k}^{-1}(-q(t,u)), & (t,u)\in[a_{k-1},a_k)\times[0,\delta), k=1,\cdots,n-1,\\ h_{j_n}^{-1}(-q(t,u)), & (t,u)\in[a_{n-1},b]\times[0,\delta),\end{cases}$$

or equivalently in a closed form

$$f_+(t,u) := \sum_{k=1}^{n-1} h_{j_k}^{-1}(-q(t,u))\chi_{[a_{k-1},a_k)\times[0,\delta)}(t,u) + h_{j_n}^{-1}(-q(t,u))\chi_{[a_{n-1},b]\times[0,\delta)}(t,u),$$

which is a function from $[a,b]\times[0,\delta)$ with values in $(0,\infty)$.

Similarly, starting from the partition of $(-\infty,0)$ of intervals of strict monotonicity for h, we can define $f_-(t,u) : [a,b]\times[0,\delta)\to(-\infty,0)$ and, considering the left neighbourhood of $u=0$ instead of the right one, we define $g_+(t,u) : [a,b]\times(-\delta,0]\to(0,\infty)$ and $g_-(t,u) : [a,b]\times(-\delta,0]\to(-\infty,0)$ analogously to f_+ and f_-.

By construction, if t, u and ξ are such that

$$[\xi - f_+(t,u)][\xi - f_-(t,u)][\xi - g_+(t,u)][\xi - g_-(t,u)] = 0,$$

then $f(t,u,\xi)=0$. Moreover, one can verify that $f_\pm$, $g_\pm$ are continuous in u and piecewise continuous in t. Therefore, by applying Theorem 1, we conclude that (1) admits two $W_0^{1,\infty}(a,b)$ solutions $\bar u_+ > 0 > \bar u_-$, with $\dot{\bar u}_+$ and $\dot{\bar u}_-$ in $C_{\mathrm{piec}}([a,b])$.

Case 2. Suppose that h is piecewise monotone and q is locally piecewise monotone at $u=0$.

Let $\{I_j\}_{j\in\mathbb{N}}$ be a partition of $(0,+\infty)$ as in *Case 1.* Differently than in the previous case, since h is not a priori strictly monotone, we have

$$\bigcup_{j\in\mathbb{N}} h(\mathrm{int}(I_j)) = (\inf_{(0,+\infty)} h, +\infty)\setminus\{z_i\}_{i\in\mathbb{N}},$$

where $\{z_i\}_{i\in\mathbb{N}}$ is a set of isolated points of $\mathbb{R}$.

By the local piecewise monotonicity at $u=0$ of q and the piecewise monotonicity of h, there exists a partition $a_0 := a < a_1 < \cdots < a_n := b$ of $[a,b]$ and n integer numbers $j_1,\cdots,j_n$ such that $h(\mathrm{int}(I_{j_k})) \supset \{-q(t,0) : t\in(a_{k-1},a_k)\}$, for $k = 1,\cdots,n$. Therefore, by continuity, there exists also $\delta>0$ such that $h(\mathrm{int}(I_{j_k})) \supset \{-q(t,u) : t\in(a_{k-1},a_k), u\in[0,\delta)\}$, for $k=1,\cdots,n$.

Defining f_+ as in *Case 1*, with

$$f_+(a_k,u) := \lim_{t\to a_k^+} h_{j_k}^{-1}(-q(t,u)),$$

for $k = 1, \cdots, n-1$,

$$f_+(b,u) := \lim_{t \to b^-} h_{j_n}^{-1}(-q(t,u)),$$

and analogously f_-, $g_\pm$. By applying Theorem 1, we conclude that (1) admits two $W_0^{1,\infty}(a,b)$ solutions $\bar{u}_+ > 0 > \bar{u}_-$, with $\dot{\bar{u}}_+$ and $\dot{\bar{u}}_-$ in $C_{\text{piec}}([a,b])$. □

We now give several examples to discuss the optimality of Theorem 2. In these examples but the last one there exist solutions in $W_0^{1,\infty}(a,b)$, none of them having piecewise continuous derivative, each one violating a different hypothesis.

Example 3. *Let $h : \mathbb{R} \to \mathbb{R}$ be the continuous function defined by*

$$h(\xi) := \begin{cases} |\xi| - 2, & \xi \in [0, 5/2) \cup [3, \infty), \\ 2(1 - 2^{-2n}) - 1, & \xi \in [3 - 2^{-2n+1}, 3 - 2^{-2n}), n \geq 1, \\ 3|\xi - 3 + 2^{-2n}| + 2(1 - 2^{-2n}) - 1, & \xi \in [3 - 2^{-2n}, 3 - 2^{-2n-1}), n \geq 1, \end{cases}$$

and, for $\xi < 0$, $h(\xi) := h(-\xi)$. Consider, for any t in $[0,1]$ and for any ξ in $\mathbb{R}$,

$$f(t,\xi) := -t - 2 + h(\xi).$$

Then, any solution to (1) in $W_0^{1,\infty}(0,1)$ has infinitely many discontinuities close to the point $t = 1$.

The function h does not satisfy the assumption of piecewise monotonicity.

Example 4. *Let $h : \mathbb{R} \to \mathbb{R}$ be the continuous function defined by*

$$h(\xi) := \begin{cases} |\xi| - 2, & \xi \in [0,2), \\ 0 & \xi \in [2,3), \\ |\xi| - 3, & \xi \in [3, \infty), \end{cases}$$

and, for $\xi < 0$, $h(\xi) := h(-\xi)$. Consider, for any u and ξ in $\mathbb{R}$,

$$f(u,\xi) := u \sin \frac{1}{u} + h(\xi).$$

Then, any solution to (1) in $W_0^{1,\infty}(a,b)$ has infinitely many discontinuities close to the points $t = a$ and $t = b$.

Analogously, if

$$f(t,\xi) := t \sin \frac{1}{t} + h(\xi),$$

then, any solution to (1) in $W_0^{1,\infty}(-1,1)$ has infinitely many discontinuities close to the point $t = 0$.

In both cases the function h is piecewise (but not strictly) monotone but q does not verify the hypothesis of local piecewise monotonicity at $u = 0$.

Example 5. *Let*

$$f(t,u,\xi) := \max\{u^2 - t^4, 0\} + (1 - \xi^2)^2.$$

A solution $\bar{u}$ in $W_0^{1,\infty}(0,1)$ to (1) with derivative in $C_{piec}([0,1])$ must satisfy $|\bar{u}(t)| \leq t^2$ and $\dot{\bar{u}}(t) = 1$ (or $\dot{\bar{u}}(t) = -1$) at every t in a right neighbourhood of 0. Therefore, by the absolute continuity of $\bar{u}$, for any t small enough, we obtain $t^2 \geq |\bar{u}(t)| = |\int_0^t \dot{\bar{u}}| = t$, reaching a contradiction.

Notice that there exists a solution in $W_0^{1,\infty}(0,1)$*. Indeed, one verifies that*

$$\begin{aligned}\bar{u}(t) := & \sum_{n=1}^{\infty}\left[\frac{(a_{n+1}+a_n)^2}{4} - \left|t - \frac{a_{n+1}+a_n}{2}\right|\right]\chi_{(a_{n+1},a_n]}(t)\\ & + \left[1 - \frac{(a_1+3)}{4} - \left|t - \frac{3a_1+1}{4}\right|\right]\chi_{(a_1,(a_1+1)/2]}(t)\\ & + \left[1 - \frac{(a_1+3)}{4} - \left|t - \frac{a_1+3}{4}\right|\right]\chi_{((a_1+1)/2,1]}(t),\end{aligned}$$

where $a_{n+1} := \sqrt{1+4a_n} - (1+a_n)$*,* $a_0 := 1$*, is a function in* $W_0^{1,\infty}(0,1)$ *that solves (1).*

The function f *does not verify the hypothesis* $f(t,0,0) < 0$*, for every* t *in* $[a,b]$*.*

Example 6. *Here is an example where there exists a solution in* $W_0^{1,\infty}(-1,1)$*, but its derivative does not belong to* $C_{piec}([-1,1])$*. Let*

$$f(t,\xi) := \begin{cases} t\xi - 2t^2\sin\dfrac{\pi}{t} + \pi t\cos\dfrac{\pi}{t}, & t \neq 0,\\ 0, & t = 0.\end{cases}$$

A solution $\bar{u}$ *in* $W_0^{1,\infty}(-1,1)$ *to (1) is given by* $\bar{u}(t) := t^2\sin(\pi/t)$*. Its derivative does not belong to* $C_{piec}([-1,1])$*, because* $\lim_{t\to 0}\dot{\bar{u}}(t)$ *is not well defined.*

The function f *does neither verify* $f(t,0) < 0$*, for every* t *in* $(-1,1)$*, nor* $f(t,\xi) \to +\infty$*, as* $|\xi| \to \infty$*, for every* t *in* $(-1,1)$*.*

Example 7. *We give an example of non-existence of solutions in* $W_0^{1,\infty}(0,1)$*. Let*

$$f(u,\xi) := u^2\xi^2 - 1.$$

A solution $\bar{u}$ *in* $W_0^{1,\infty}(0,1)$ *to (1), if it exists, should satisfy* $|\bar{u}(t)\dot{\bar{u}}(t)| = 1$ *and, therefore,* $\bar{u}(t) \neq 0$*, for a.e.* t *in* $(0,1)$*. Since* $\bar{u}(0) = 0$*, by the continuity of* $\bar{u}$*, we obtain that* $\dot{\bar{u}}$ *is unbounded in a neighbourhood of* 0*.*

The function $f(0,\xi)$ *does not tend to* $+\infty$ *as* $|\xi| \to \infty$*.*

2. **On The Existence.** In this section, we would like to address briefly whether the *compatibility condition* on the boundary data stated in Theorem 1, that is sufficient to guarantee the existence of solutions to problem (1), is also necessary.

We discuss the special cases in which f does not depend either on t or on u. We prove that the *compatibility condition* is indeed also a necessary condition for the existence of a positive and a negative solution to (1) in $W_0^{1,\infty}(a,b)$ with $C_{\text{piec}}([a,b])$ derivative. More precisely:

Theorem 8. *Let* $\bar{u}_+ > 0 > \bar{u}_-$ *in* $W_0^{1,\infty}(a,b)$*, with* $\dot{\bar{u}}$ *in* $C_{piec}([a,b])$*, be solutions to (1). Assume that* f *is of the form* $f(t,\xi)$ *or* $f(u,\xi)$*.*

Then, there exist two functions $f_+(t,u), f_-(t,u) : [a,b]\times[-\delta,\delta] \to \mathbb{R}$ *continuous in* u*, for a certain* $\delta > 0$*, piecewise continuous in* t *and* $f_+ > 0 > f_-$ *such that the set* Z_f *of zeros of* f *contains the zeros of*

$$P_+(t,u,\xi) := \xi - f_+(t,u), \qquad P_-(t,u,\xi) := \xi - f_-(t,u).$$

Proof. Case 1. Suppose that $f(t,u,\xi) = f(t,\xi)$ does not depend on u. We can define two piecewise continuous functions f_+ and f_- by

$$f_+(t) := \dot{\bar{u}}_+(t), \quad f_-(t) := \dot{\bar{u}}_-(t),$$

for any t in $[a,b]$. By definition, $\bar{u}_+$ is a solution to $\dot{u}(t)-f_+(t)=0$ and $\bar{u}_-$ is a solution to $\dot{u}(t)-f_-(t)=0$, for almost every t in (a,b). Furthermore, by assumption,

$$f(t,\dot{\bar{u}}_+(t))=0, \quad f(t,\dot{\bar{u}}_-(t))=0.$$

for t in $[a,b]$. We conclude that, if $\xi-f_+(t)=0$ or $\xi-f_-(t)=0$, then $f(t,\xi)=0$, as stated.

Case 2. Suppose that $f(t,u,\xi)=f(u,\xi)$ does not depend on t. From the assumed regularity on $\bar{u}:=\bar{u}_+$, there exists $a_1<b_1$ in (a,b) such that $\bar{u}$ belongs to $C^1([a,a_1]\cup[b_1,b])$ and $\bar{u}$ is respectively increasing on $[a,a_1]$ and decreasing on $[b_1,b]$. We can define two continuous functions f_+ and f_- by

$$f_+(u):=\dot{\bar{u}}(\bar{u}|_{[a,a_1]}^{-1}(u)), \quad f_-(u):=\dot{\bar{u}}(\bar{u}|_{[b_1,b]}^{-1}(u)),$$

for any u in $[0,\min\{\bar{u}(a_1),\bar{u}(b_1)\}]$. By definition, $\bar{u}$ is a solution to $\dot{u}(t)-f_+(u(t))=0$, for t in $[a,a_1]$, $\dot{u}(t)-f_-(u(t))=0$, for t in $[b_1,b]$. That implies, by the changes of variable $t=\bar{u}|_{[a,a_1]}^{-1}(u)$ or $t=\bar{u}|_{[b_1,b]}^{-1}(u)$,

$$f(u,f_+(u))=0, \quad f(u,f_-(u))=0,$$

for any u in $[0,\min\{\bar{u}(a_1),\bar{u}(b_1)\}]$. We infer that, if $\xi-f_+(u)=0$ or $\xi-f_-(u)=0$, then $f(u,\xi)=0$. To conclude, we proceed analogously as above with $\bar{u}:=\bar{u}_-$ in order to extend the definition of f_+ and f_- to the left of 0. $\square$

3. **Selecting Principles.** In this section we present three selecting principles for solutions to (1). Namely, the viscosity method developed by Crandall-Lions, the maximal solutions method and the most regular solutions method.

Before presenting these methods, we would like to recall here a simple example of Georgy [8] showing that these three methods select, in general, different solutions. However if the function f is convex in the variable ξ and increasing in the variable u, then the three methods select the same solution.

Consider the continuous function f given by

$$f(t,\xi):=-t+\begin{cases} -(\xi+1) & \xi \text{ in } (-\infty,-3/2] \\ \xi+2 & \xi \text{ in } (-3/2,-1] \\ |\xi| & \xi \text{ in } (-1,1] \\ -(\xi-2) & \xi \text{ in } (1,3/2] \\ \xi-1 & \xi \text{ in } (3/2,\infty). \end{cases}$$

Let $\beta\in\mathbb{R}$ and $u_0(t)=t\beta/2$. We then look for solutions $u\in u_0+W_0^{1,\infty}(0,2)$ of

$$f(t,\dot{u}(t))=0, \qquad \text{for a.e. } t \text{ in } (0,2). \tag{4}$$

In [8], the following results are proved.

(i) Equation (4) has a solution $u\in u_0+W_0^{1,\infty}(0,2)$ if and only if $\beta\in[-7/2,7/2]$. Moreover, in the interval of existence $\beta\in[-7/2,7/2]$, there always exists a maximal (and a minimal) solution and also a solution with at most two points of discontinuities of the derivative.

(ii) In $\beta\in[-7/2,-2]\cup[2,7/2]$, there exists a solution with at most one point of discontinuity.

(iii) There are no C^1 solutions.

(iv) There exists a viscosity solution if and only if $\beta\in[-3,7/2]$. However, in this interval if, moreover, $\beta\neq-3,-2,7/2$, then any viscosity solution has at least two points of discontinuities of its derivative. Therefore, the viscosity solution is not always the most regular solution. It can also be seen that the viscosity solutions can be different from the maximal one as we add other wells to the function f.

3.1. **The Viscosity Solutions.** In this section, we would like to recall very briefly few facts on the viscosity solutions. For more details, we refer to [1, 5, 10]. The name "viscosity" refers to the so-called vanishing viscosity term $\epsilon\ddot{u}$ (or $\epsilon\Delta u$ in the multidimensional case). Indeed solutions u of (1) are obtained as the limit of the sequence u_ϵ which are solutions of the second order equation

$$\epsilon\ddot{u}_\epsilon(t) + f(t, u_\epsilon(t), \dot{u}_\epsilon(t)) = 0, \qquad t \in (a, b)\,.$$

This method is, by now, more general and does not refer anymore to the above limit procedure and the following definition is now the standard one.

Definition 9. *A continuous function u is a* viscosity solution *to (1) if and only if for all ϕ in $C^1(a,b)$ such that $u - \phi$ has a local maximum at t_0 in (a,b), then*

$$f(t_0, u(t_0), \dot{\phi}(t_0)) \leq 0,$$

and for all ϕ in $C^1(a,b)$ such that $u - \phi$ has a local minimum at t_0 in (a,b), then

$$f(t_0, u(t_0), \dot{\phi}(t_0)) \geq 0.$$

For the existence and uniqueness of the viscosity solutions in $W_0^{1,\infty}(a,b)$ to (1), the basic assumptions are, as already said, the convexity with respect to ξ and the increasing monotonicity with respect to u of $f(t,u,\xi)$ (see [10]). The convexity is essentially a necessary condition not only for the uniqueness of viscosity solutions but also for the existence, as the above example of Georgy shows (for β in $[-7/2, -3)$).

3.2. **The Pointwise Maximal Solution.** In the case where f does not depend explicitly on the variable u, it can easily be proved as in [8] that there exists a maximal solution. The result is however false, in general, if the Hamiltonian depends explicitly on u as will be seen in Example 12.

Theorem 10. *If $Z_f \subset [a,b] \times \mathbb{R}$ is a closed set, bounded with respect to ξ and*

$$S_f = \left\{ u \in W_0^{1,\infty}(a,b) : (t, \dot{u}) \in Z_f \right\} \neq \emptyset,$$

then $\bar{u} \in S_f$ where

$$\bar{u}(t) := \sup_{u \in S_f} u(t), \qquad \text{for any } t \text{ in } [a,b],$$

Remark 11. *In particular, if we assume that $f(t,\xi)$ is continuous, $f(t,0) < 0$ and $f(t,\xi) \to +\infty$, as $|\xi| \to \infty$, for any $t \in [a,b]$, then the set*

$$Z_f = \{(t,\xi) \in [a,b] \times \mathbb{R} : f(t,\xi) = 0\}$$

verifies the assumptions of the theorem.

Proof. Since, by assumption, Z_f is bounded with respect the last variable, there exists $\rho > 0$ such that $Z_f \subset [a,b] \times [-\rho, \rho]$. Hence,

$$\|\dot{u}\|_{L^\infty(a,b)} \leq \rho, \qquad \text{for any } u \text{ in } S_f.$$

Observing that

$$\int_a^b \sup_{u \in S_f} \dot{u} \geq \int_a^b \dot{u} = 0 \geq \int_a^b \inf_{u \in S_f} \dot{u},$$

by continuity it follows that there exists a point $\bar{t}$ in $[a, b]$ such that

$$\int_a^{\bar{t}} \sup_{u \in S_f} \dot{u} + \int_{\bar{t}}^b \inf_{u \in S_f} \dot{u} = 0.$$

(If there exist such two points $\bar{t}_1 < \bar{t}_2$, then $\dot{u} = \sup_{u \in S_f} \dot{u} = \inf_{u \in S_f} \dot{u}$ in $(\bar{t}_1, \bar{t}_2)$.) Set

$$\dot{\bar{u}}(t) := \begin{cases} \sup_{u \in S_f} \dot{u}, & t \in [a, \bar{t}], \\ \inf_{u \in S_f} \dot{u}, & t \in (\bar{t}, b]. \end{cases}$$

By definition, $\bar{u}(t) := \int_a^t \dot{\bar{u}}$ belongs to $W_0^{1,\infty}(a, b)$. Furthermore, for any t in $[a, \bar{t}]$

$$\bar{u}(t) = \int_a^t \sup_{u \in S_f} \dot{u} \geq \int_a^t \dot{u} = u(t),$$

and, as t belongs to $(\bar{t}, b]$,

$$\bar{u}(t) = -\int_t^b \inf_{u \in S_f} \dot{u} \geq -\int_t^b \dot{u} = u(t),$$

for any u in S_f. Hence, $\bar{u} \geq \sup\{u : u \in S_f\}$. We are left to prove that u is a solution. In fact, by the fact that Z_f is closed, we have

$$(t, \dot{\bar{u}}(t)) = \begin{cases} (t, \sup_{u \in S_f} \dot{u}(t)), & t \in [a, \bar{t}], \\ (t, \inf_{u \in S_f} \dot{u}(t)), & t \in (\bar{t}, b], \end{cases}$$

and hence

$$(t, \dot{\bar{u}}(t)) \in \bar{Z}_f = Z_f \, .$$

We conclude that $\bar{u} = \sup_{u \in S_f} u$ is a solution to (4). □

We now emphasize that the result is not true in case Z_f depends on u. In fact, consider the following example.

Example 12. *The pointwise maximal, among all solutions, function might not be a solution in case f depends explicitly on u. If we let*

$$f(u, \xi) := \max\{u^2 - 1, 0\} + (\xi^2 - 1)^2, \ a = 0 \text{ and } b = 4,$$

then

$$\bar{u}(t) = \sup_{u \in S_{Z_f}} u(t) = \begin{cases} t, & t \in [0, 1], \\ 1, & t \in (1, 3], \\ 4 - t, & t \in (3, 4], \end{cases}$$

does not belong to S_f.

In the theorem below, we recall a result contained in [8] giving a necessary and sufficient condition for the existence of a (maximal) solution to (1).

Theorem 13. *Let $f : [a, b] \times \mathbb{R} \to \mathbb{R}$, $f = f(t, \xi)$, be a continuous function. If the set*

$$E(t) := \{\xi \in \mathbb{R} : f(t, \xi) = 0\}$$

is non-empty and bounded for any t in $[a, b]$, then, there exists a maximal solution $\bar{u} \in W_0^{1,\infty}(a, b)$ to (1) if and only if

$$\int_a^b \min\{\xi \in E(t)\} dt \leq 0 \leq \int_a^b \max\{\xi \in E(t)\} dt.$$

3.3. **The Most Regular Solutions.** In this section we present our selecting principle based on choosing the most regular solution. Furthermore, in Theorem 17 we give a uniqueness result under convexity assumption.

To illustrate our purpose, we start with an elementary example, namely the eikonal equation

$$|\dot{u}(t)| = 1, \quad \text{for a.e. } t \text{ in } (0,1). \tag{5}$$

As it is well known, this problem admits infinitely many solutions. For instance, for any integer j, the function

$$\bar{u}_j(t) := \begin{cases} t - i2^{-j}, & t \in [i2^{-j}, (i+2^{-1})2^{-j}],\ i = 0, 1, \cdots, 2^j - 1 \\ (i+1)2^{-j} - t, & t \in [(i+2^{-1})2^{-j}, (i+1)2^{-j}],\ i = 0, 1, \cdots, 2^j - 1 \end{cases}$$

is a solution of (5). Even if we look at the solutions with a fixed number of discontinuity points for the derivative, we obtain infinitely many functions. Indeed, if $k = 3$, the functions

$$\bar{u}_\alpha(t) := t\chi_{[0,\alpha)}(t) + (\alpha - t)\chi_{[\alpha,(\alpha+1)/2)}(t) + (t-1)\chi_{[(\alpha+1)/2,1]}(t)$$

are solutions to (5), for any real $\alpha \in (0, 1/2)$. Analogously, we can find infinitely many solutions for every $k > 3$. Nevertheless, the solutions with exactly one discontinuity point for the derivative are just two, namely

$$\bar{u}_+(t) := 2^{-1} - |t - 2^{-1}|, \qquad \bar{u}_-(t) := -2^{-1} + |t - 2^{-1}|,$$

and there are no C^1 solutions. Incidentally the function $\bar{u}_+$ is the unique viscosity solution of (5).

In the theorem below, under simple assumptions on the set of zeros of f, we can guarantee the existence of a finite number of solutions whose derivative admits at most one discontinuity point. More precisely, we show that if Z_f is the union of n_+ positive and n_- negative graphs of continuous functions

$$f_+^i(t,u), f_-^i(t,u) : [a,b] \times \mathbb{R} \to \mathbb{R}, \quad \text{with} \quad f_-^i \le 0 \le f_+^i\,,$$

then there are solutions to (1) with at most one point of discontinuity of the derivative, and there exist at most $2n_+n_-$ such solutions. If we assume that the zeros do not intersect the plane $\xi = 0$, meaning that

$$f_-^i < 0 < f_+^i\,,$$

we can then prove that the solutions of (1) with minimal number of discontinuity points have exactly one singularity, and there exist at most $2n_+n_-$ solutions of this type.

Note that, we can always suppose, without changing the problem (1), that f_+^i and f_-^i are defined over the entire space $[a,b] \times \mathbb{R}$ (but clearly without supposing that they are continuous on $[a,b] \times \mathbb{R}$ in general) by considering

$$f_+^i(t,u) := \max\{f_+^i(t,u), 0\} \ \text{ and } \ f_-^i(t,u) := \min\{f_-^i(t,u), 0\}.$$

Theorem 14. *Assume that the set Z_f of zeros of f is a finite union of n_+ positive and n_- negative continuous graphs, namely*

$$Z_f = \bigcup_{i=1}^{n_+} Z_{P_+^i} \cup \bigcup_{i=1}^{n_-} Z_{P_-^i}$$

where

$$P_+^i(t,u,\xi) := \xi - f_+^i(t,u), \qquad P_-^i(t,u,\xi) := \xi - f_-^i(t,u),$$

$f_+^i, f_-^i : [a,b]\times\mathbb{R} \to \mathbb{R}$ *are locally Lipschitz with respect to* u*, continuous with respect to* t*,* $f_-^i \le 0 \le f_+^i$ *and one of them has sub-linear growth, i.e.*

$$|f_+^i(t,u)| \quad (\textit{or } |f_-^i(t,u)|) \ \le m|u| + q,$$

for certain $m > 0$*,* $q \in \mathbb{R}$ *and for any* $(t,u) \in [a,b]\times\mathbb{R}$*.*

Then, the solutions to (1) with minimal number of discontinuity points have derivative in $C_{2-piec}([a,b])$*, i.e. the derivative has at most one singularity, and there exist at least* 2 *and at most*

$$2n_+n_-$$

solutions of this type.

Remark 15. *Theorem 14 states the existence of at most* $2n_+n_-$ *solutions. The fact that we might lose some solutions is due to the structure of the zeros of* f*. If, for instance, we assume that the functions* f_+^i *and* f_-^i *do not intersect one with eachother, then, we do not lose any solution and therefore we obtain exactly* $2n_+n_-$ *distinct solutions.*

Proof. We first prove the existence of a solution to (1). For any $i = 1,\cdots,n_+$, consider the Cauchy problem

$$\begin{cases} \dot{u}(t) = f_+^i(t,u(t)), & t > a, \\ u(a) = 0. \end{cases} \tag{6}$$

By the Cauchy-Lipschitz Theorem and the growth assumption on f_+^i this problem admits a global solution $\bar{u}_l^i \in C^1([a,b])$, with $\bar{u}_l^i(a) = 0$. We also have that $\bar{u}_l^i$ is increasing since $f_+ \ge 0$. Analogously, for any $j = 1,\cdots,n_-$, considering the Cauchy problem

$$\begin{cases} \dot{u}(t) = f_-^j(t,u(t)), & t < b, \\ u(b) = 0, \end{cases} \tag{7}$$

we have a C^1 solution $\bar{u}_r^j$ in a right neighbourhood of $t = b$, with $\bar{u}_r^j(b) = 0$ and $\bar{u}_r^j \to +\infty$ at the left extreme of that neighbourhood, and $\bar{u}_r^j$ is decreasing.

Defining

$$\bar{u}^{i,j}(t) := \min\{\bar{u}_l^i(t), \bar{u}_r^j(t)\}, \ \ t \in [a,b],$$

we obtain that $\bar{u}^{i,j}$ belongs to $W_0^{1,\infty}(a,b)$ and $\dot{\bar{u}}^{i,j}$ belongs to the space $C_{2-\text{piec}}([a,b])$. Indeed, by the monotonicity of $\bar{u}_l^i, \bar{u}_r^j$, there exists exactly one point $t_{i,j}$ such that

$$\bar{u}^{i,j}(t) = \bar{u}_l^i(t)\chi_{[a,t_{i,j})}(t) + \bar{u}_r^j(t)\chi_{[t_{i,j},b]}(t).$$

Note that, if $\dot{\bar{u}}_l^i(t_{i,j}) = \dot{\bar{u}}_r^j(t_{i,j})$, then both derivatives $\dot{\bar{u}}_l^i(t_{i,j}), \dot{\bar{u}}_r^j(t_{i,j})$ must be equal to 0 and, hence, $\bar{u}^{i,j}$ has continuous derivative on the whole $[a,b]$. (If we had considered f_-^j sub-linear instead of f_+^i, proceeding as above we would have obtained the same result.)

Now, if we consider problem (6), for every $i = 1,\cdots,n_+$, and problem (7), for every $j = 1,\cdots,n_-$, we define at most $2n_+n_-$ solutions to (1) and no more than $2n_+n_-$, by the local Lipschitzianity of f_+^i and f_-^j. □

If we suppose that f_+^i, f_-^i do not assume the value zero, i.e. $f_-^i < 0 < f_+^i$ on $[a,b]\times\mathbb{R}$, then, from the proof of Theorem 14, it follows that the solutions to (1) with minimal number of discontinuity points have derivative with exactly one singularity. The eikonal equation belongs to this class. We therefore obtain the following corollary.

Corollary 16. *Assume that the set Z_f of zeros of f is finite union of n_+ positive and n_- negative continuous graphs, namely*

$$Z_f = \bigcup_{i=1}^{n_+} Z_{P_+^i} \cup \bigcup_{i=1}^{n_-} Z_{P_-^i}$$

where

$$P_+^i(t,u,\xi) := \xi - f_+^i(t,u), \qquad P_-^i(t,u,\xi) := \xi - f_-^i(t,u),$$

$f_+^i, f_-^i : [a,b]\times\mathbb{R} \to \mathbb{R}$ are locally Lipschitz with respect to u, continuous with respect to t, $f_-^i < 0 < f_+^i$ and one of them has sub-linear growth.

Then, the solutions to (1) with minimal number of discontinuity points have derivative with exactly one singularity, and there exist at least 2 *and at most*

$$2n_+n_-$$

solutions of this type.

Proof. It follows directly from Theorem 14 that, since $\dot{\bar{u}}_r^i < 0 < \dot{\bar{u}}_l^j$ on $[a,b]$ and $\bar{u}^{i,j}(a) = \bar{u}^{i,j}(b) = 0$, the function $\bar{u}^{i,j}$ cannot have continuous derivative on $[a,b]$. □

As consequence of Theorem 14, we obtain that, for $n_+ = n_- = 1$, there exist exactly two solutions, one positive and one negative, to (1) with derivative in $C_{2-\text{piec}}([a,b])$. For instance, the problem of finding u in $W_0^{1,\infty}(a,b)$ solution to

$$|\dot{u}(t)| = f(t,u(t)), \quad \text{for a.e. } t \text{ in } (a,b),$$

with f locally Lipschitz with respect to u and $f \geq 0$, for any $(t,u) \in [a,b]\times\mathbb{R}$, admits exactly a positive and a negative solution in $W_0^{1,\infty}(a,b)$ with derivative with at most one discontinuity.

In the same spirit of the observation above, if we assume the strict convexity of f with respect to ξ, we can prove the essential uniqueness of the most regular solution.

Theorem 17. *Assume that f is a continuously differentiable function, strictly convex with respect to ξ, $f(t,u,0) < 0$ and*

$$f(t,u,\xi) \geq \chi_{[a,b]\times\{(u,\xi)\in\mathbb{R}^2 : |\xi| \geq m|u|+q\}}(t,u,\xi),$$

for certain $m > 0$, q in $\mathbb{R}$ and for any $(t,u) \in [a,b]\times\mathbb{R}$.

Then, the solutions of (1) with minimal number of discontinuity points have derivative in $C_{2-piec}([a,b])$ and there exist exactly 2 *solutions of this type, one positive and one negative.*

Proof. From the Implicit Function Theorem, there exists two functions $f_+(t,u)$ and $f_-(t,u)$ in $C^1([a,b]\times\mathbb{R})$ such that

$$f(t,u,f_\pm(t,u)) = 0.$$

Therefore, the result follows from Corollary 16. □

The eikonal equation (5) satisfies the assumptions of Theorem 17 and, in fact, it admits exactly 2 solutions, one positive and one negative, with one discontinuity point of the derivatives, and it does not admit C^1 solutions.

A particular case of Theorem 14 is provided by a function f which is polynomial in ξ of even degree.

Corollary 18. *Let $f_i : [a,b] \times \mathbb{R} \to \mathbb{R}$, $i = 0, \cdots, 2N$, be continuous functions, locally Lipschitz with respect to u, such that $f_0(t,0) < 0$, for any t in $[a,b]$. Let*

$$f(t,u,\xi) := \sum_{i=0}^{2N} \xi^i f_i(t,u).$$

Then, the solutions of (1) with minimal number of discontinuity points have derivative in $C_{2-piec}([a,b])$ and there exist at least 2 and at most

$$2n_+ n_-$$

solutions of this type, where n_+ is the number of $\xi > 0$ such that $f(t,0,\xi) = 0$ and n_- is the number of $\xi < 0$ such that $f(t,0,\xi) = 0$.

Proof. The proof follows directly from Theorem 14. □

We now discuss the optimality of our Theorem 14 through two simple examples.

Example 19. *If we drop the Lipschitz assumption with respect to u, we might end up with more solutions than the ones stated by Theorem 14. Indeed, consider*

$$f(u,\xi) := (\xi - \sqrt{1-u^2})(\xi + \sqrt{1-u^2}), \quad \text{for a.e. } t \text{ in } (0,2\pi). \tag{8}$$

Equation (1) admits exactly four solutions with continuous derivative, that are $\bar{u}_1(t) := \sin t$, $\bar{u}_2(t) := -\sin t$,

$$\bar{u}_3(t) := \sin(t)\chi_{[0,\pi/2)}(t) + \chi_{[\pi/2,3\pi/2]}(t) - \sin(t)\chi_{(3\pi/2,2\pi]}(t)$$

and $\bar{u}_4(t) := -\bar{u}_3(t)$, while if f_+ and f_- were Lipschitz in u, the solutions would be only two (cf. also Example 21).

Example 20. *The assumption on the zeros of f to be the graph of continuous functions defined for any u in $\mathbb{R}$ is optimal. In fact, consider the modified eikonal equation*

$$\left| u(t) - \frac{3}{8} \right| \chi_{\{|\xi| \le |u|8/3\}}(u(t),\dot{u}(t)) + \frac{3}{8}|\dot{u}(t) - 1|\chi_{\{|\xi| > |u|8/3\}}(u(t),\dot{u}(t)) = 0, \ a.e. \ t \in (0,1).$$

This problem admit infinitely many solutions with derivative in $C_{k-piec}([0,1])$, for any $k \ge 3$, but it does not admit solutions for $k = 1, 2$. Observe that

$$f(u,\xi) = \left| u - \frac{3}{8} \right| \chi_{\{|\xi| \le |u|8/3\}}(u,\xi) + \frac{3}{8}|\xi - 1|\chi_{\{|\xi| > |u|8/3\}}(u,\xi)$$

is convex in ξ (and also in u). The set Z_f is the rectangle with sides parallel to the axis u and ξ of coordinates $\xi = 3/8, -3/8$ and $u = 1, -1$.

Finally we conclude with another example showing that there is non-uniqueness even among smooth solutions.

Example 21. *Consider*

$$\alpha_i \in C^\infty([0,1]), \ i = 1, \cdots, n, \quad \text{with} \quad \int_0^1 \alpha_i(t)\,dt = 0.$$

Let

$$f(t,\xi) = \prod_{i=1}^{n} [\xi - \alpha_i(t)],$$

then

$$u_i(t) = \int_0^t \alpha_i(s)\, ds,\ \ i = 1, \cdots, n,$$

are smooth solutions of

$$f(t, \dot{u}_i(t)) = 0 \quad \textit{with} \quad u_i(0) = u_i(1) = 0.$$

Note that when $n = 2$, *the function* $\xi \to f(t, \xi)$ *is even strictly convex.*

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