On the equation $\det \nabla u = f$ with no sign hypothesis

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January 19, 2009

Abstract

We prove existence of $u \in C^k\left(\overline{\Omega};\mathbb{R}^n\right)$ satisfying

$$\begin{cases} \det \nabla u\,(x) = f\,(x) & x \in \Omega \\ u\,(x) = x & x \in \partial\Omega \end{cases}$$

where $k \geq 1$ is an integer, Ω is a bounded smooth domain and $f \in C^k\left(\overline{\Omega}\right)$ satisfies

$$\int_\Omega f\,(x)\,dx = \operatorname{meas}\Omega$$

with no sign hypothesis on f.

1 Introduction

In this article, we discuss the existence of $u : \overline{\Omega} \subset \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\begin{cases} \det \nabla u\,(x) = f\,(x) & x \in \Omega \\ u\,(x) = x & x \in \partial\Omega \end{cases} \tag{1}$$

where Ω is a bounded smooth domain. Clearly the divergence theorem implies that a necessary condition for solving (1) is

$$\int_\Omega f\,(x)\,dx = \operatorname{meas}\Omega. \tag{2}$$

When $f > 0$, this problem has generated a considerable amount of work since the seminal article of Moser [11], notably by Banyaga [1], Dacorogna [3], Reimann [12], Tartar [15], Zehnder [17]. The next important step appeared in Dacorogna-Moser [6], where the regularity problem was handled, in particular it was shown that if $f \in C^{r,\alpha}\left(\overline{\Omega}\right)$, then a mapping u can be found in $C^{r+1,\alpha}\left(\overline{\Omega};\overline{\Omega}\right)$. Posterior contributions can also be found in Burago-Kleiner [2], Mc Mullen [9], Rivière-Ye [13] and Ye [16]. It should be emphasized that, when $f > 0$, the solution is necessarily a diffeomorphism.

The aim of this article is to remove the hypothesis $f > 0$ and to consider any f satisfying (2), with no restriction on its sign. Of course the solution will then not be a diffeomorphism; although if $f \geq 0$, and under further restrictions, it can be a homeomorphism. Our main result is the following (cf. Theorem 2 for a more general statement).

Theorem 1 *Let $k \geq 1$ be an integer, $\Omega \subset \mathbb{R}^n$ be the unit ball and $f \in C^k\left(\overline{\Omega}\right)$ with*

$$\int_\Omega f(x)\,dx = \operatorname{meas}\Omega.$$

Then there exists $u \in C^k\left(\overline{\Omega};\mathbb{R}^n\right)$ verifying

$$\begin{cases} \det \nabla u(x) = f(x) & x \in \Omega \\ u(x) = x & x \in \partial\Omega. \end{cases}$$

Our proof cannot use the flow method introduced by Moser and does not use either the fixed point method developed in [6]. It is more constructive. Some extensions of this theorem, in particular to more general domains Ω, are considered below (cf. Propositions 11 and 12). We also point out that our method does not produce, as the one in [6] did when $f > 0$, a gain in regularity.

We should also emphasize that when f is negative in some part, then it might be that $u\left(\overline{\Omega}\right) \not\subset \overline{\Omega}$. This indeed happens if $f < 0$ in some part of $\partial\Omega$ (cf. Proposition 4).

We would now like to conclude with a qualitative remark. If $g > 0$ and

$$\int_\Omega f(x)\,dx = \int_\Omega g(x)\,dx,$$

then the theorem is still valid (cf. Theorem 2) and there exists a solution of

$$\begin{cases} g(u(x)) \det \nabla u(x) = f(x) & x \in \Omega \\ u(x) = x & x \in \partial\Omega \end{cases} \tag{3}$$

with no restriction on the sign of f. However if g vanishes in at least one point, and even if $f \equiv 1$, then the problem becomes, in general, unsolvable. More precisely, if $f \equiv 1$ (or more generally if $f > 0$), then the following assertions are true (see Proposition 8).

(i) If g has at least one zero, then there is no C^1 solution of (3).

(ii) If $g \geq 0$ and has only a countable number of zeroes, then there exists a continuous (but not C^1) weak solution of (3).

2 Notations

We gather here the main notations that will be used throughout the article. We let $\Omega, O \subset \mathbb{R}^n$ be bounded open sets.

- Balls in $\mathbb{R}^n$ are denoted by

$$B_\epsilon(x) := \{y \in \mathbb{R}^n : |y - x| < \epsilon\}$$

and when $x = 0$ we just write B_ϵ instead of $B_\epsilon(0)$.

- For $g \in C^0(\mathbb{R}^n)$, $\Phi \in C^1(\overline{\Omega}; \mathbb{R}^n)$ and $x \in \overline{\Omega}$, we let, as in differential geometry,

$$\Phi^*(g)(x) := g(\Phi(x)) \det \nabla\Phi(x).$$

- The set diffeomorphisms of class (k, α), $k \geq 1$ an integer and $\alpha \in [0, 1]$, is denoted by

$$\operatorname{Diff}^{k,\alpha}(\overline{\Omega}; \overline{O}) := \left\{\Phi : \Phi \in C^{k,\alpha}(\overline{\Omega}; \overline{O}) \text{ and } \Phi^{-1} \in C^{k,\alpha}(\overline{O}; \overline{\Omega})\right\}.$$

If $\alpha = 0$, we simply write $\operatorname{Diff}^k(\overline{\Omega}; \overline{O})$.

- For homeomorphisms, we let

$$\operatorname{Hom}(\overline{\Omega}; \overline{O}) := \left\{\Phi : \Phi \in C^0(\overline{\Omega}; \overline{O}) \text{ and } \Phi^{-1} \in C^0(\overline{O}; \overline{\Omega})\right\}.$$

- For $A \subset \mathbb{R}^n$, the characteristic function of A is defined as

$$1_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

- In many instances, we will write, for $g \in C^k(\mathbb{R}^n)$ and $f \in C^k(\overline{\Omega})$, $\operatorname{supp}(g - f) \subset \Omega$ meaning that the support of $[g|_{\overline{\Omega}} - f]$ is contained in Ω.

3 Main result

The main result of our paper (also valid in the framework of Hölder spaces $C^{k,\alpha}$) is the following one.

Theorem 2 *Let $k \geq 1$ be an integer and $\Omega \subset \mathbb{R}^n$ be an open set, such that $\overline{\Omega}$ is C^{k+1}-diffeomorphic to $\overline{B_1}$. Let also $g \in C^k(\mathbb{R}^n)$ and $f \in C^k(\overline{\Omega})$ be such that*

$$\inf_{x \in \mathbb{R}^n} g(x) > 0 \quad \text{and} \quad \int_\Omega f = \int_\Omega g.$$

Then there exists $\Phi \in C^k(\overline{\Omega}; \mathbb{R}^n)$ such that

$$\begin{cases} \Phi^*(g) = f & \text{in } \Omega \\ \Phi = \operatorname{id} & \text{on } \partial\Omega. \end{cases}$$

Moreover Φ has the extra following three properties.

(i) If $\operatorname{supp}(g - f) \subset \Omega$, *then Φ can be defined so that* $\operatorname{supp}(\Phi - \operatorname{id}) \subset \Omega$.

(ii) If $f \geq 0$, then Φ can be chosen so that $\Phi \in C^k(\overline{\Omega}; \overline{\Omega})$.

(iii) If $f \geq 0$ and $f^{-1}(0) \cap \Omega$ is countable, then Φ can be defined so that $\Phi \in \operatorname{Hom}(\overline{\Omega}; \overline{\Omega})$.

Remark 3 *(i) By "$\overline{\Omega}$ is C^{k+1}-diffeomorphic to $\overline{B_1}$" we mean that there exists* $\Phi_1 \in \operatorname{Diff}^{k+1}(\mathbb{R}^n; \mathbb{R}^n)$ *such that*

$$\Phi_1(\overline{B_1}) = \overline{\Omega}$$

and

$$\inf_{x \in \mathbb{R}^n} \det \nabla \Phi_1(x) > 0.$$

In particular $\overline{B_1}$ is C^{k+1}-diffeomorphic to $\overline{B_1}$.

(ii) Throughout the article we will assume $n \geq 2$. When $n = 1$, the result is trivial and the solution is unique.

(iii) If f is negative in some part of $\partial\Omega$, then any Φ must go out of $\overline{\Omega}$ (cf. Proposition 4). However if, for example, $f > 0$ on $\partial\Omega$, then Φ can be chosen so that $\Phi\left(\overline{\Omega}\right) = \overline{\Omega}$. For details we refer to Kneuss [8]. Note that when $n = 1$, the condition $f > 0$ on $\partial\Omega$, is not sufficient to guarantee that $\Phi\left(\overline{\Omega}\right) \subset \overline{\Omega}$.

The proof is rather long and relies on the results of Sections 5, 6 and 7. However in order to motivate all the technical lemmas of these sections, we now give the proof of the theorem, based on these intermediate results.

Proof. We split the proof into seven steps. In the course of the proof, we use several times (32), namely

$$(\Phi \circ \Psi)^* = \Psi^* \circ \Phi^*.$$

Step 1. Since $\overline{\Omega}$ is C^{k+1}-diffeomorphic to $\overline{B_1}$, there exists $\Phi_1 \in \operatorname{Diff}^{k+1}(\mathbb{R}^n; \mathbb{R}^n)$ with $\Phi_1(\overline{B_1}) = \overline{\Omega}$ and

$$\inf_{x \in \mathbb{R}^n} \det \nabla \Phi_1(x) > 0.$$

Step 2 (positive radial integration). Applying Lemma 26 to $\Phi_1^*(f) \in C^k(\overline{B_1})$, we find that there exists $\Phi_2 \in \operatorname{Diff}^\infty(\overline{B_1}; \overline{B_1})$ satisfying

$$(\Phi_1 \circ \Phi_2)^*(f)(0) > 0 \quad \text{and} \quad \operatorname{supp}(\Phi_2 - \operatorname{id}) \subset B_1$$

with

$$\int_0^r s^{n-1}(\Phi_1 \circ \Phi_2)^*(f)(s\frac{x}{|x|})ds > 0, \quad \text{for every } x \neq 0 \text{ and } r \in (0, 1]. \qquad (4)$$

Notice that

$$\int_{B_1} (\Phi_1 \circ \Phi_2)^*(f) = \int_{B_1} \Phi_1^*(f) = \int_\Omega f.$$

Step 3 (radial solution). Applying Lemma 17 to $\Phi_1^*(g)$ and $(\Phi_1 \circ \Phi_2)^*(f)$, we infer that there exists $\Phi_3 \in C^k(\overline{B_1};\mathbb{R}^n)$ such that

$$\begin{cases} (\Phi_1 \circ \Phi_3)^*(g) = (\Phi_1 \circ \Phi_2)^*(f) & \text{in } B_1 \\ \Phi_3 = \mathrm{id} & \text{on } \partial B_1 . \end{cases}$$

This is possible, since $\inf_{x\in\mathbb{R}^n} \Phi_1^*(g)(x) > 0$, $(\Phi_1 \circ \Phi_2)^*(f)(0) > 0$,

$$\int_{B_1} \Phi_1^*(g) = \int_\Omega g = \int_\Omega f = \int_{B_1} \Phi_1^*(f) = \int_{B_1} (\Phi_1 \circ \Phi_2)^*(f)$$

and (4) holds.

Step 4 (conclusion). By the previous steps, we have that

$$\Phi := \Phi_1 \circ \Phi_3 \circ \Phi_2^{-1} \circ \Phi_1^{-1} \in C^k(\overline{\Omega};\mathbb{R}^n)$$

satisfies

$$\begin{cases} \Phi^*(g) = f & \text{in } \Omega \\ \Phi = \mathrm{id} & \text{on } \partial\Omega \end{cases}$$

since

$$\begin{aligned} \Phi^*(g) &= \left[(\Phi_1 \circ \Phi_2)^{-1}\right]^* \circ [\Phi_1 \circ \Phi_3]^* (g) \\ &= \left[(\Phi_1 \circ \Phi_2)^{-1}\right]^* \circ [\Phi_1 \circ \Phi_2]^* (f) = f. \end{aligned}$$

Step 5. We now discuss (i). If $\mathrm{supp}(g - f) \subset \Omega$, then

$$\mathrm{supp}(\Phi_1^*(g) - (\Phi_1 \circ \Phi_2)^*(f)) \subset B_1 .$$

Therefore, by Lemma 17 (i), we can define Φ_3 such that

$$\mathrm{supp}(\Phi_3 - \mathrm{id}) \subset B_1 .$$

Finally, we get

$$\mathrm{supp}(\Phi - \mathrm{id}) \subset \Omega.$$

Thus Statement (i) is established.

Step 6. We now consider Statement (ii). Since $f \geq 0$, we have $(\Phi_1 \circ \Phi_2)^*(f) \geq 0$ and then by Lemma 17 (ii), we can choose $\Phi_3 \in C^k(\overline{B_1};\overline{B_1})$. Eventually we get $\Phi \in C^k(\overline{\Omega};\overline{\Omega})$.

Step 7. As far as (iii) is concerned, we have from Lemma 17 (iii) that $\Phi_3 \in \mathrm{Hom}(\overline{B_1};\overline{B_1})$. Since $\Phi_1 \in \mathrm{Diff}^{k+1}(\overline{B_1};\overline{\Omega})$ and $\Phi_2 \in \mathrm{Diff}^\infty(\overline{B_1};\overline{B_1})$, we have the claim. ■

4 Remarks, extensions and related results

In this section $\Omega \subset \mathbb{R}^n$ is a bounded connected open set.

We start by showing that if $f < 0$ in some parts of $\partial\Omega$, then any solution of

$$\begin{cases} \Phi^*(g) = f & \text{in } \Omega \\ \Phi = \text{id} & \text{on } \partial\Omega \end{cases} \tag{5}$$

must go out of $\overline{\Omega}$, more precisely $\Phi\left(\overline{\Omega}\right) \not\subset \overline{\Omega}$.

Proposition 4 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set of class C^1 and $\Phi \in C^1(\overline{\Omega};\mathbb{R}^n)$ with $\Phi = \text{id}$ on $\partial\Omega$. If there exists $\overline{x} \in \partial\Omega$ such that $\det\nabla\Phi(\overline{x}) < 0$ then*

$$\Phi(\overline{\Omega}) \not\subset \overline{\Omega}. \tag{6}$$

Proof. *Step 1 (simplification).* By hypothesis there exists $\Psi \in \text{Diff}^1(\overline{B_1};\Psi(\overline{B_1}))$ with $\Psi(0) = \overline{x}$ and

(i) $\Psi(\overline{B_1} \cap \{x_n = 0\}) \subset \partial\Omega$

(ii) $\Psi(\overline{B_1} \cap \{x_n > 0\}) \subset \Omega$

(iii) $\Psi(\overline{B_1} \cap \{x_n < 0\}) \subset (\overline{\Omega})^c$.

Therefore using that $\Phi(\overline{x}) = \overline{x}$, we can choose $\epsilon > 0$ small enough so that $\widetilde{\Phi} : \overline{B_\epsilon} \cap \{x_n \geq 0\} \to \mathbb{R}^n$,

$$\widetilde{\Phi}(x) := \Psi^{-1}(\Phi(\Psi(x)))$$

is well defined. We observe that $\widetilde{\Phi}$ satisfies

$$\widetilde{\Phi} = \text{id} \quad \text{on } \overline{B_\epsilon} \cap \{x_n = 0\} \quad \text{and} \quad \det\nabla\widetilde{\Phi}(0) = \det\nabla\Phi(\overline{x}) < 0. \tag{7}$$

To prove (6) it is enough to show that

$$\widetilde{\Phi}(\overline{B_{\epsilon'}} \cap \{x_n > 0\}) \subset \{x_n < 0\}, \tag{8}$$

for a certain $0 < \epsilon' \leq \epsilon$.

Step 2. We now show (8). Using (7), we immediately obtain

$$\frac{\partial\widetilde{\Phi}_n}{\partial x_n}(0) = \det\nabla\widetilde{\Phi}(0) = \det\nabla\Phi(x) < 0$$

and therefore by continuity, there exists $0 < \epsilon' \leq \epsilon$ such that

$$\frac{\partial\widetilde{\Phi}_n}{\partial x_n} < 0 \quad \text{in } B_{\epsilon'}\,. \tag{9}$$

Combining (9) and the fact that $\widetilde{\Phi}_n(0) = 0$ (by (7)) we get (8). ■

We next prove, under suitable assumptions, that a classical solution of (5) is necessarily a weak solution (see Definition 5 and Lemma 7). We then prove that g and f do not play the same role in (5) (see Proposition 8).

Definition 5 *Let $g, f \in C^0(\overline{\Omega})$. We say that $\Phi \in \operatorname{Hom}(\overline{\Omega}; \overline{\Omega})$ is a* weak solution *of* (5) *if*

$$\begin{cases} \int_{\Phi(E)} g = \int_E f & \text{for every open } E \subset \Omega \\ \Phi = \operatorname{id} & \text{on } \partial\Omega. \end{cases} \tag{10}$$

Remark 6 *If $\Phi \notin \operatorname{Hom}(\overline{\Omega}; \overline{\Omega})$, then the right notion of weak solution of (5) is with the first equation in* (10) *replaced (see [7] page 106) by*

$$\int_E f(x)\,dx = \int_{\mathbb{R}^n} g(y) \deg(\Phi, E, y) dy$$

where deg *stands for the topological degree (see Appendix).*

Lemma 7 *Suppose that $g, f \in C^0(\overline{\Omega})$ and $\Phi \in C^1(\overline{\Omega}; \overline{\Omega}) \cap \operatorname{Hom}(\overline{\Omega}; \overline{\Omega})$. Then Φ is a classical solution if and only if Φ is a weak solution.*

Proof. It will be seen in Proposition 31 that, if $\Phi \in C^1(\overline{\Omega}; \overline{\Omega}) \cap \operatorname{Hom}(\overline{\Omega}; \overline{\Omega})$ and $\Phi = \operatorname{id}$ on $\partial\Omega$, then $\det \nabla\Phi(x) \geq 0$ and

$$\operatorname{int}\left((\det \nabla\Phi)^{-1}(0) \right) = \emptyset. \tag{11}$$

(i) Suppose that Φ is a classical solution of (5) and let $E \subset \Omega$ be an open set. Consider

$$E_+ := E \cap \{x \in \Omega : \det \nabla\Phi(x) > 0\}$$
$$E_0 := E \cap \{x \in \Omega : \det \nabla\Phi(x) = 0\}.$$

Since $g(\Phi(x)) \det \nabla\Phi(x) = f(x)$, we have $f \equiv 0$ in E_0. By Sard theorem (see (72))

$$\operatorname{meas}(\Phi(E_0)) = 0.$$

Thus, by the change of variables formula and Φ being one to one, we obtain

$$\int_{\Phi(E)} g = \int_{\Phi(E_+ \cup E_0)} g = \int_{\Phi(E_+)} g + \int_{\Phi(E_0)} g$$
$$= \int_{\Phi(E_+)} g = \int_{E_+} f = \int_E f.$$

Hence Φ is a weak solution of (5).

(ii) Assume now that Φ is a weak solution of (5). Let $x \in \Omega$ be such that $\det \nabla\Phi(x) > 0$. Then $\Phi \in \operatorname{Diff}^1(\overline{B_r(x)}; \Phi\left(\overline{B_r(x)}\right))$ for some suitable small r. By the assumptions and the change of variables formula, for every $0 < \rho < r$, we have

$$\int_{B_\rho(x)} g(\Phi(y)) \det \nabla\Phi(y) dy = \int_{\Phi(B_\rho(x))} g(z)\,dz = \int_{B_\rho(x)} f(z)\,dz.$$

Letting $\rho \to 0$ we obtain

$$g\left(\Phi\left(x\right)\right) \det \nabla\Phi(x) = f\left(x\right).$$

By continuity, we conclude that the above equality holds true for every

$$x \in \text{closure}\left\{y \in \Omega : \det \nabla\Phi(y) > 0\right\} = \overline{\Omega}$$

in view of (11). ■

We now show that in our problem (5), the functions g and f do not play the same role.

Proposition 8 *The following three statements hold true.*

(i) If $g \in C^0(\mathbb{R}^n)$, $f \in C^0(\overline{\Omega})$, $f > 0$ and $g^{-1}(0) \cap \overline{\Omega} \neq \emptyset$, then there exists no solution $\Phi \in C^1(\overline{\Omega};\mathbb{R}^n)$ to (5).

(ii) Let $f, g \in C^0(\overline{\Omega})$ satisfy

$$f > 0, \quad g \geq 0 \quad \text{and} \quad \int_\Omega f = \int_\Omega g.$$

If there exists a weak solution of (5)*, then*

$$\text{int}(g^{-1}(0) \cap \Omega) = \emptyset. \tag{12}$$

(iii) Let $\overline{\Omega}$ be C^2-diffeomorphic to $\overline{B_1}$ and $f, g \in C^1(\overline{\Omega})$ be such that

$$f > 0, \quad g \geq 0, \quad g^{-1}(0) \cap \Omega \text{ is countable} \quad \text{and} \quad \int_\Omega f = \int_\Omega g.$$

Then there exists a weak solution of (5)*.*

Proof. *(i)* We proceed by contradiction. Assume that $\Phi \in C^1(\overline{\Omega};\mathbb{R}^n)$ is a solution of (5). Since $\Phi = \text{id}$ on $\partial\Omega$, then (see (74))

$$\Phi(\overline{\Omega}) \supset \overline{\Omega}.$$

Thus, there exists $z \in \overline{\Omega}$ such that $\Phi(z) \in \overline{\Omega}$ and $g(\Phi(z)) = 0$, which is the desired contradiction, since

$$g(\Phi(z)) \det \nabla\Phi(z) = f\left(z\right) > 0.$$

(ii) Let $\Phi \in \text{Hom}(\overline{\Omega};\overline{\Omega})$ satisfy (10) with $f > 0$. If (12) is not true, then there exists $B_\epsilon(z)$ such that

$$B_\epsilon(z) \subset g^{-1}(0) \cap \Omega.$$

Let $E = \Phi^{-1}\left(B_\epsilon(z)\right) \subset \Omega$ which is open (and non-empty) by continuity of Φ. From (10) we get that

$$0 < \int_E f = \int_{\Phi(E)} g = 0,$$

which is absurd.

(iii) From Theorem 2 iii) we find that there exists $\Psi \in C^1(\overline{\Omega};\overline{\Omega}) \cap \mathrm{Hom}(\overline{\Omega};\overline{\Omega})$, such that

$$\Psi^*(f) = g \quad \text{and} \quad \Psi = \text{id} \ \text{ on } \partial\Omega\,.$$

For every open set $E \subset \Omega$ we have, from Lemma 7,

$$\int_{\Psi(E)} f = \int_E g.$$

Then, $\Phi := \Psi^{-1}$ satisfies (10). ■

In the following proposition, we state a necessary condition (see (13)) for the existence of a one to one solution of (5). Moreover, we show that not all solutions of (5), verifying (13), are one to one. Notice that Lemma 28 shows that if $\Phi \in C^0(\overline{\Omega};\mathbb{R}^n)$ is one to one and $\Phi = \text{id}$ on $\partial\Omega$, then $\Phi \in \mathrm{Hom}(\overline{\Omega};\overline{\Omega})$.

Proposition 9 *Let*

$$g \in C^0(\mathbb{R}^n), \quad f \in C^0(\overline{\Omega}), \quad \inf_{x\in\mathbb{R}^n} g(x) > 0 \quad \textit{and} \quad \int_\Omega f = \int_\Omega g.$$

Then the following claims hold true.

(i) If $\Phi \in C^1(\overline{\Omega};\mathbb{R}^n)$ is a one to one solution of (5), then

$$f \geq 0 \quad \textit{and} \quad \mathrm{int}(f^{-1}(0)) = \emptyset. \tag{13}$$

(ii) If f satisfies (13), then not all solutions $\Phi \in C^1(\overline{\Omega};\mathbb{R}^n)$ of (5) are one to one.

Proof. *(i)* By Lemma 28, we have that $\Phi \in \mathrm{Hom}(\overline{\Omega};\overline{\Omega})$. Applying Proposition 31, we have the claim.

(ii) We provide a counterexample in two dimensions. Let $f \in C^1(\overline{B_1})$ be such that $f \geq 0$,

$$f^{-1}(0) = \{(t,0) \,:\, t \in [1/2, 3/4]\}, \quad f \equiv 1 \text{ on a neighborhood of } 0$$

and, for $x \neq 0$,

$$\int_0^1 s\, f(s\frac{x}{|x|})ds = \frac{1}{2}\,.$$

Define next $\alpha : \overline{B_1} \setminus \{0\} \to [0,1]$, through

$$\frac{\alpha(x)^2}{2} = \int_0^{|x|} s\, f(s\frac{x}{|x|})ds.$$

As in the proof of Lemma 17, the function

$$\Phi(x) := \alpha(x)\frac{x}{|x|}$$

is in $C^1(\overline{B_1};\overline{B_1})$ with

$$\Phi^*(1) = f \quad \text{and} \quad \Phi = \text{id} \ \text{on } \partial B_1 \,.$$

Since $\Phi(1/2, 0) = \Phi(3/4, 0)$, then Φ is not one to one. ■

The next proposition can be proved with the same techniques as the one developed here and we refer to [8] for details.

Proposition 10 *Let $k \geq 1$ be an integer, $g \in C^k(\mathbb{R}^n)$ and $f \in C^k(\overline{B_1})$ satisfy*

$$\inf_{x \in \mathbb{R}^n} g(x) > 0 \quad \textit{and} \quad \int_{B_1} g = \int_{B_1} f.$$

Then there exist $\gamma = \gamma(n, k, g, f)$ and $\epsilon = \epsilon(n, k, g, f)$ such that for every $h_1, h_2 \in C^k(\overline{B_1})$ satisfying

$$\int_{B_1} h_i = \int_{B_1} g \quad \textit{and} \quad \|h_i - f\|_{C^k} \leq \epsilon, \quad i = 1, 2,$$

there exist $\Phi_{h_i} \in C^k(\overline{B_1}; \mathbb{R}^n)$, $i = 1, 2$, with

$$\Phi^*_{h_i}(g) = h_i \quad \textit{and} \quad \Phi_{h_i} = \text{id} \ \textit{on } \partial B_1$$

and

$$\|\Phi_{h_1} - \Phi_{h_2}\|_{C^k} \leq \gamma \|h_1 - h_2\|_{C^k} \,.$$

We conclude this section with two extensions of Theorem 2 (cf. [8]) to more general domains Ω. For example domains with a finite number of holes or general domains but with only a finite number of connected components where f is not positive.

Proposition 11 *Let $k \geq 1$ be an integer. Let Ω be an open set such that $\overline{\Omega}$ is C^{k+1}-diffeomorphic to*

$$\overline{B_1} \setminus \bigcup_{i=1}^{N} B_{\delta_i}(x_i)$$

with $\overline{B_{\delta_i}(x_i)}$ pairwise disjoint and contained in B_1, and denote by Φ_1 such a diffeomorphism. If $g \in C^k(\mathbb{R}^n)$ and $f \in C^k(\overline{\Omega})$ satisfy

$$\inf_{x \in \mathbb{R}^n} g(x) > 0, \quad f > 0 \quad \textit{in } \Phi_1^{-1}\left(\textstyle\bigcup_{i=1}^N \partial B_{\delta_i}(x_i)\right)$$

and

$$\int_\Omega f = \int_\Omega g,$$

then there exists $\Phi \in C^k(\overline{\Omega}; \mathbb{R}^n)$ verifying (5).

The following three properties also hold.

(i) If $\text{supp}(g - f) \subset \Omega$, then Φ can be defined so that $\text{supp}(\Phi - \text{id}) \subset \Omega$.

(ii) If $f \geq 0$ or if $f > 0$ on $\partial\Omega$, then Φ can be chosen so that $\Phi \in C^k(\overline{\Omega}; \overline{\Omega})$.

(iii) If $f \geq 0$ and $f^{-1}(0) \cap \Omega$ is countable, then Φ can be defined so that $\Phi \in \text{Hom}(\overline{\Omega}; \overline{\Omega})$.

Proposition 12 *Let $k \geq 1$ be an integer. Let Ω be an open set of class C^k and suppose that $f, g \in C^k(\overline{\Omega})$ satisfy*

$$g > 0 \text{ in } \overline{\Omega}, \quad f > 0 \text{ on } \partial\Omega \quad \text{and} \quad \int_\Omega f = \int_\Omega g.$$

Suppose that $W_1, \cdots, W_m$ are open sets such that

$$\begin{cases} \overline{W_i} \subset \Omega \text{ and } \overline{W_i} \text{ is } C^{k+1} \text{ diffeomorphic to } \overline{B_1} & 1 \leq i \leq m \\ \overline{W_i} \cap \overline{W_j} = \emptyset & 1 \leq i \neq j \leq m \\ f^{-1}((-\infty, 0]) \subset \bigcup_{i=1}^m W_i\,. \end{cases}$$

Then, there exists $\Phi \in C^k(\overline{\Omega}; \overline{\Omega})$ solution of (5).

Moreover, if $\operatorname{supp}(g-f) \subset \Omega$, *then Φ can be defined so that* $\operatorname{supp}(\Phi - \operatorname{id}) \subset \Omega$.

5 Preliminary results

We now recall a result of [6].

Theorem 13 (Dacorogna-Moser theorem) *Let $k \geq 1$ be an integer, Ω be a bounded connected open set of class C^k and let $f, g \in C^k(\overline{\Omega})$ be such that*

$$f \cdot g > 0 \text{ in } \overline{\Omega} \quad \text{and} \quad \int_\Omega f = \int_\Omega g.$$

Then there exists $\Phi \in \operatorname{Diff}^k(\overline{\Omega}; \overline{\Omega})$ such that

$$\begin{cases} \Phi^*(g) = f & \text{in } \Omega \\ \Phi = \operatorname{id} & \text{on } \partial\Omega. \end{cases}$$

Furthermore, if $\operatorname{supp}(g-f) \subset \Omega$, *then Φ can be chosen so that* $\operatorname{supp}(\Phi - \operatorname{id}) \subset \Omega$.

We have as an immediate corollary the following.

Corollary 14 *Let $k \geq 1$ be an integer, $f, g \in C^k(\overline{\Omega})$ and let $V \subset \Omega$ be a connected open set such that*

$$f \cdot g > 0 \text{ in } V, \quad \int_V f = \int_V g \quad \text{and} \quad \operatorname{supp}(f-g) \subset V.$$

Then there exists $\Phi \in \operatorname{Diff}^k(\overline{V}, \overline{V})$ such that

$$\Phi^*(g) = f \text{ in } V \quad \text{and} \quad \operatorname{supp}(\Phi - \operatorname{id}) \subset V.$$

Proof. We surely can find an open set W of class C^k such that

$$\overline{W} \subset V \quad \text{and} \quad \operatorname{supp}(f-g) \subset W.$$

Using Theorem 13, we have the claim. ■

Proposition 15 *Let $k \geq 1$ be an interger and $R > 1$. Let also $f, g \in C^k(\overline{B_R})$ be such that $f, g > 0$ in $\overline{B_R}$ and*

$$\int_{B_1} f = \int_{B_1} g \quad , \quad \int_{B_R} f = \int_{B_R} g.$$

There exists $\Phi \in \mathrm{Diff}^k(\overline{B_R}; \overline{B_R})$ such that

$$\begin{cases} \Phi^*(g) = f & \text{in } B_R \\ \Phi = \mathrm{id} & \text{on } \partial B_1 \cup \partial B_R. \end{cases} \tag{14}$$

Proof. We decompose the proof into two steps.

Step 1. Since $f - g \in C^k(\overline{B_R})$, then, for example, $f - g \in C^{k-1,1/2}(\overline{B_R})$; therefore, using Lemma 16, there exists $u \in C^{k,1/2}(\overline{B_R}; \mathbb{R}^n)$ (in particular in $C^k(\overline{B_R}; \mathbb{R}^n)$) such that

$$\begin{cases} \mathrm{div}(u) = f - g & \text{in } B_R \\ u = 0 & \text{on } \partial B_1 \cup \partial B_R. \end{cases}$$

Step 2. Let $v \in C^k([0,1] \times \overline{B_R}; \mathbb{R}^n)$, $v(t,x) = v_t(x)$, be defined by

$$v_t(x) := \frac{u(x)}{tg(x) + (1-t)f(x)}\,.$$

We then define $\Psi_t(x) : [0,1] \times \overline{B_R} \to \mathbb{R}^n$ as the solution of

$$\begin{cases} \frac{d}{dt}[\Psi_t(x)] = v_t(\Psi_t(x)) & t > 0 \\ \Psi_0(x) = x. \end{cases}$$

Using classical results about ODE, recalling that $v_t \equiv 0$ on $\partial B_1 \cup \partial B_R$, we have, for every $t \in [0,1]$, that

$$\Psi_t \in \mathrm{Diff}^k(\overline{B_R}; \overline{B_R}) \quad \text{and} \quad \Psi_t = \mathrm{id} \text{ on } \partial B_1 \cup \partial B_R\,.$$

Finally, it can be easily shown, see e.g. [5] p. 540, that $\Phi := \Psi_1$ verifies (14). ■

In Proposition 15 we used the following lemma.

Lemma 16 *Let $k \geq 0$ be an integer, $\alpha \in (0,1)$ and $R > 1$. Let also $f \in C^{k,\alpha}(\overline{B_R})$ be such that*

$$\int_{B_1} f = \int_{B_R} f = 0.$$

There exists $u \in C^{k+1,\alpha}(\overline{B_R}; \mathbb{R}^n)$ such that

$$\begin{cases} \mathrm{div}(u) = f & \text{in } B_R \\ u = 0 & \text{on } \partial B_1 \cup \partial B_R. \end{cases} \tag{15}$$

Proof. We split the proof into four steps.

Step 1. Using a classical result about the divergence, see e.g. [5] p. 531, there exist $w_1 \in C^{k+1,\alpha}(\overline{B_1};\mathbb{R}^n)$ and $v \in C^{k+1,\alpha}(\overline{B_R};\mathbb{R}^n)$ such that

$$\begin{cases} \operatorname{div}(w_1) = f & \text{in } B_1 \\ w_1 = 0 & \text{on } \partial B_1 \end{cases} \tag{16}$$

and

$$\begin{cases} \operatorname{div}(v) = f & \text{in } B_R \\ v = 0 & \text{on } \partial B_R. \end{cases} \tag{17}$$

Step 2. Let $w_2 \in C^{k+1,\alpha}(\overline{B_1};\mathbb{R}^n)$ be defined by $w_2 := w_1 - v$. Using (16) and (17), we obtain

$$\begin{cases} \operatorname{div}(w_2) = 0 & \text{in } B_1 \\ w_2 = -v & \text{on } \partial B_1. \end{cases} \tag{18}$$

Since $\operatorname{div}(w_2) = 0$, there exists, by Poincaré lemma (see e.g. [4]),

$$H = (H_{ij})_{1 \le i < j \le n} \in \mathbb{R}^{n(n-1)/2}$$

with $H_{ij} \in C^{k+2,\alpha}(\overline{B_1})$ and

$$w_2 = \operatorname{rot}^* H$$

where

$$\operatorname{rot}^* H = ((\operatorname{rot}^* H)_1, \cdots, (\operatorname{rot}^* H)_n)$$

and

$$(\operatorname{rot}^* H)_i = \sum_{j=1}^{i-1} \frac{\partial H_{ji}}{\partial x_j} - \sum_{j=i+1}^{n} \frac{\partial H_{ij}}{\partial x_j}.$$

Step 3. For all $1 \le i < j \le n$ let $\widetilde{H}_{ij} \in C^{k+2,\alpha}(\overline{B_R})$ be such that

$$\widetilde{H}_{ij} = H_{ij} \quad \text{in } \overline{B_1}.$$

Let also $\phi \in C^\infty(\mathbb{R}^n)$ be such that

$$\begin{cases} \phi \equiv 1 & \text{in } B_{(1+R)/2} \\ \phi \equiv 0 & \text{in } (B_{(1+2R)/3})^c. \end{cases}$$

Finally let $w \in C^{k+1,\alpha}(\overline{B_R};\mathbb{R}^n)$ be defined by $w := \operatorname{rot}^*(\phi \widetilde{H})$.

Step 4. Let us show that $u \in C^{k+1,\alpha}(\overline{B_R};\mathbb{R}^n)$ defined by $u := v + w$ verifies (15). Using (17), we have

$$\operatorname{div}(u) = \operatorname{div}(v) + \operatorname{div}(w) = f + 0 = f \quad \text{in } B_R\,.$$

Using the definition of ϕ we have $w = 0$ on ∂B_R and therefore, using (17),

$$u = v + w = 0 \text{ on } \partial B_R\,.$$

Using again the definition of ϕ we obtain $w = \mathrm{rot}^*(\widetilde{H}) = \mathrm{rot}^*(H) = w_2$ in $\overline{B_1}$. Combining this with (16) and (18) we have

$$u = v + w = v + w_2 = w_1 = 0 \text{ on } \partial B_1 ,$$

which concludes the proof of the lemma. ■

In Step 3 of the proof of our main theorem (Theorem 2), we used the following lemma.

Lemma 17 (Radial solution) *Let $k \geq 1$ be an integer, $g \in C^k(\mathbb{R}^n)$ and $f \in C^k(\overline{B_1})$ be such that $\inf_{x \in \mathbb{R}^n} g(x) > 0,\ f(0) > 0,$*

$$\int_{B_1} g = \int_{B_1} f$$

and, for every $x \neq 0$ and $r \in (0, 1]$,

$$\int_0^r s^{n-1} f(s\frac{x}{|x|})ds > 0. \tag{19}$$

Then there exists $\Phi \in C^k(\overline{B_1}; \mathbb{R}^n)$ verifying

$$\begin{cases} \Phi^*(g) = f & \text{in } B_1 \\ \Phi = \mathrm{id} & \text{on } \partial B_1 . \end{cases}$$

The three following statements are also valid.

(i) If $\mathrm{supp}(g - f) \subset B_1$ then Φ can be chosen so that

$$\mathrm{supp}(\Phi - \mathrm{id}) \subset B_1 .$$

(ii) If for every $x \neq 0$ and $r \in [0, 1]$,

$$\int_r^1 s^{n-1} f(s\frac{x}{|x|})ds \geq 0 \tag{20}$$

then Φ can be assumed in $C^k(\overline{B_1}; \overline{B_1})$. In particular (20) is always verified if $f \geq 0$.

(iii) If

$$f \geq 0 \quad \text{and} \quad f^{-1}(0) \cap B_1 \text{ is countable,} \tag{21}$$

then Φ can be assumed in $\mathrm{Hom}(\overline{B_1}; \overline{B_1})$.

Remark 18 *Notice that the assumption $f^{-1}(0) \cap B_1$ countable can be weakened as*

$$f^{-1}(0) \cap \left[0, \frac{x}{|x|}\right]$$

does not contain intervals for every $x \neq 0$.

Proof. *Step 1 (definition of an auxiliary function).* Since $f(0) > 0$ and (19) holds, we can find $0 < \epsilon < 1/6$ such that

$$f > 0 \text{ in } B_{2\epsilon} \quad \text{and} \quad \min_{x \neq 0} \int_{2\epsilon}^{1} s^{n-1} f(s\frac{x}{|x|}) ds > 0. \tag{22}$$

We define $\eta \in C^{\infty}([0,1];[0,1])$ as

$$\eta(s) = \begin{cases} 1 & \text{if } 0 \leq s \leq \epsilon \\ 0 & \text{if } 2\epsilon \leq s \leq 1. \end{cases}$$

If $\operatorname{supp}(g - f) \subset B_1$ (in particular $f > 0$ on ∂B_1) we modify the definition of ϵ and η as follows. We assume that

$$\eta(s) = \begin{cases} 1 & \text{if } 0 \leq s \leq \epsilon \text{ or } 1 - \epsilon \leq s \leq 1 \\ 0 & \text{if } 2\epsilon \leq s \leq 1 - 2\epsilon \end{cases}$$

where $0 < \epsilon < 1/6$ is such that

$$f > 0 \text{ in } B_{2\epsilon} \cup \left(\overline{B_1} \setminus B_{1-2\epsilon}\right) \quad \text{and} \quad \min_{x \neq 0} \int_{2\epsilon}^{1-2\epsilon} s^{n-1} f(s\frac{x}{|x|}) ds > 0. \tag{23}$$

Define next $\overline{f} : \overline{B_1} \setminus \{0\} \to \mathbb{R}$ as

$$\overline{f}(x) = \overline{f}(\frac{x}{|x|}) := \frac{\int_0^1 s^{n-1}(1 - \eta(s)) f(s\frac{x}{|x|}) ds}{\int_0^1 s^{n-1}(1 - \eta(s)) ds}.$$

It is easy to see that $\overline{f} \in C^k(\overline{B_1} \setminus \{0\})$ and, by (22) or (23), $\overline{f} > 0$. We now define

$$h(x) := \eta(|x|) f(x) + (1 - \eta(|x|)) \overline{f}(x).$$

Observe that h satisfies

$$\begin{cases} h > 0 \text{ on } \overline{B_1}, \quad h = f \text{ in } B_\epsilon \\ \int_0^1 s^{n-1} h(s\frac{x}{|x|}) ds = \int_0^1 s^{n-1} f(s\frac{x}{|x|}) ds, \text{ for every } x \neq 0 \end{cases} \tag{24}$$

and is in $C^k(\overline{B_1})$. Furthermore, if $\operatorname{supp}(g - f) \subset B_1$ then $h = f$ in a neighborhood of ∂B_1.

Step 2. Define

$$h_0 := \min_{x \in \overline{B_1}} h(x), \quad g_0 := \inf_{x \in \mathbb{R}^n} g(x) > 0,$$

$$m := \min\{g_0, h_0\}/2 \quad \text{and} \quad A := \max_{x \neq 0} \max_{r \in (0,1]} \int_0^r s^{n-1} f(s\frac{x}{|x|}) ds < \infty.$$

Define $R > 1$ large enough in order to have

$$\frac{mR^n}{n} > A.$$

We now construct a function $\widetilde{g} \in C^k(\overline{B_R})$ such that $\widetilde{g} \geq m$ in $\overline{B_R}$,

$$\widetilde{g} = h \quad \text{in } \overline{B_1},$$

$$\int_{B_1} g = \int_{B_1} \widetilde{g} \quad \text{and} \quad \int_{B_R} g = \int_{B_R} \widetilde{g}. \tag{25}$$

Using (24), we first observe $\int_{B_1} h = \int_{B_1} g$ and so the first identity in (25) is automatically verified. Let $\overline{h} \in C^k(\overline{B_R})$ be an extension of h such that $\overline{h} > m$ in $\overline{B_R}$. For all $\epsilon > 0$ let $\rho_\epsilon \in C^\infty(\mathbb{R}^n)$ be such that $0 \leq \rho_\epsilon \leq 1$ and

$$\rho_\epsilon \equiv \begin{cases} 1 & \text{in } \overline{B_1}; \\ 0 & \text{in } (B_{1+\epsilon})^c. \end{cases}$$

For all $\epsilon > 0$ small enough, it is clear that there exists a unique $D(\epsilon) \in \mathbb{R}$ such that the function

$$\widetilde{g}_\epsilon := \rho_\epsilon \overline{h} + (1 - \rho_\epsilon) D(\epsilon) \in C^k(\overline{B_R})$$

verifies

$$\int_{B_R} \widetilde{g}_\epsilon = \int_{B_R} g.$$

It is easy to see that we can choose ϵ_1 small enough in order to have

$$D(\epsilon_1) > m.$$

The function $\widetilde{g} := \widetilde{g}_{\epsilon_1}$ has all the required properties.

Since $g, \widetilde{g} > 0$, $g, \widetilde{g} \in C^k(\overline{B_R})$ and (25) holds, there exists, using Proposition 15, $\Phi_1 \in \operatorname{Diff}^k(\overline{B_R}; \overline{B_R})$ such that

$$\begin{cases} \Phi_1^*(g) = \widetilde{g} & \text{in } B_R \\ \Phi_1 = \text{id} & \text{on } \partial B_1 \cup \partial B_R. \end{cases} \tag{26}$$

Since $\widetilde{g} \geq m$ in $\overline{B_R}$, we have, by definition of R, that

$$\int_0^R s^{n-1} \widetilde{g}(s \frac{x}{|x|}) ds > A. \tag{27}$$

Step 3 (radial solution). Let $\alpha : \overline{B_1} \setminus \{0\} \to \mathbb{R}$ be such that

$$\int_0^{\alpha(x)} s^{n-1} \widetilde{g}(s \frac{x}{|x|}) ds = \int_0^{|x|} s^{n-1} f(s \frac{x}{|x|}) ds. \tag{28}$$

Since $\widetilde{g} > 0$, by (19), (27) and the definition of A, α is well defined and satisfies $\alpha \in [0, R]$. Moreover using again (19), $\alpha(x) > 0$ if $x \in \overline{B_1} \setminus \{0\}$. Using (24) (and the fact that $\widetilde{g} = f$ in a neighborhood of ∂B_1 if $\operatorname{supp}(g - f) \subset B_1$), we get

(i) $\alpha(x) = |x|$ in B_ϵ,

(ii) $\alpha(x) = 1$ on ∂B_1 (and $\alpha(x) = |x|$ in a neighborhood of ∂B_1 if $\operatorname{supp}(g - f) \subset B_1$)),

(iii) if (20) holds then $\alpha \in [0,1]$,

(iv) if (21) holds, then

$$\alpha(x) \neq \alpha(rx), \quad \text{for every } x \in \overline{B_1} \setminus \{0\} \text{ and } r \in [0,1). \tag{29}$$

Thus, by the implicit function theorem, we have that the function $\alpha \in C^k(\overline{B_1} \setminus \{0\})$, since $\widetilde{g} > 0$ and $\alpha(x) > 0$ if $x \in \overline{B_1} \setminus \{0\}$. Moreover, since $\alpha(x) = |x|$ in B_ϵ, in fact the function $x \to \alpha(x)/|x|$ is $C^k(\overline{B_1})$. Let us show that

$$\Phi_2(x) := \frac{\alpha(x)}{|x|} x,$$

is in $C^k(\overline{B_1}; \mathbb{R}^n)$ and verifies

$$\begin{cases} \Phi_2^*(\widetilde{g}) = f & \text{in } B_1 \\ \Phi_2 = \text{id} & \text{on } \partial B_1 \,. \end{cases}$$

In fact, by the properties of α, it easily follows that $\Phi_2 \in C^k(\overline{B_1}; \overline{B_R})$ (and $\Phi_2 \in C^k(\overline{B_1}; \overline{B_1})$ if (20) holds). We also see that $\Phi_2 = \text{id}$ on ∂B_1 (and also on a neighborhood of ∂B_1 if $\operatorname{supp}(g - f) \subset B_1$). Appealing to Lemma 19, we obtain

$$\det \nabla \Phi_2(x) = \frac{\alpha^{n-1}(x)}{|x|^n} \sum_{i=1}^n \frac{\partial \alpha(x)}{\partial x_i} x_i \,. \tag{30}$$

Computing the derivative of (28) with respect to x_i, we get

$$\begin{aligned} &\alpha^{n-1}(x) \widetilde{g}(\Phi_2(x)) \frac{\partial \alpha(x)}{\partial x_i} + \sum_{j=1}^n \int_0^{\alpha(x)} s^n \frac{\partial \widetilde{g}}{\partial x_j}(s \frac{x}{|x|}) \left(\frac{|x|\delta_{ij} - \frac{x_i x_j}{|x|}}{|x|^2} \right) ds \\ &= |x|^{n-1} f(x) \frac{x_i}{|x|} + \sum_{j=1}^n \int_0^{|x|} s^n \frac{\partial f}{\partial x_j}(s \frac{x}{|x|}) \left(\frac{|x|\delta_{ij} - \frac{x_i x_j}{|x|}}{|x|^2} \right) ds, \end{aligned}$$

where $\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ otherwise. Multiplying by x_i the above equality, adding up the terms with respect to i and using

$$\sum_{i=1}^n x_i \left(\frac{|x|\delta_{ij} - \frac{x_i x_j}{|x|}}{|x|^2} \right) = 0, \quad 1 \le j \le n,$$

we obtain

$$\alpha^{n-1}(x) \widetilde{g}(\Phi_2(x)) \sum_{i=1}^n x_i \frac{\partial \alpha}{\partial x_i}(x) = |x|^n f(x).$$

This equality, together with (30), implies that $\Phi_2^*(\widetilde{g}) = f$.

Step 4 (conclusion). Defining $\Phi \in C^k(\overline{B_1}; \mathbb{R}^n)$ by

$$\Phi = \Phi_1 \circ \Phi_2 \,,$$

it is obvious to see that

$$\begin{cases} \Phi^*(g) = f & \text{in } B_1 \\ \Phi = \text{id} & \text{on } \partial B_1 \, . \end{cases}$$

Indeed

$$\Phi^*(g) = (\Phi_1 \circ \Phi_2)^*(g) = \Phi_2^*(\Phi_1^*(g)) = \Phi_2^*(\widetilde{g}) = f.$$

Step 5. It remains to prove the statement (iii). We claim that Φ is one to one. From (29), we already know that it is one to one on $\overline{B_1} \setminus \{0\}$. By (28) and the assumption (21), we obtain

$$0 = \Phi\,(0) \neq \Phi\,(x)\,, \quad \text{for every } x \in \overline{B_1} \setminus \{0\}.$$

Hence Φ is one to one. Moreover, by (74) in the Appendix, Φ is onto and thus $\Phi \in \text{Hom}(\overline{B_1}; \overline{B_1})$. ■

In Step 3 of the previous lemma, we used the following elementary result.

Lemma 19 *Let $\lambda \in C^1(\overline{B_1})$ and $\Phi \in C^1(\overline{B_1}; \mathbb{R}^n)$, $\Phi(x) := \lambda(x)x$. Then*

$$\det \nabla\Phi(x) = \lambda^n(x) + \lambda^{n-1}(x) \sum_{i=1}^{n} x_i \frac{\partial \lambda}{\partial x_i}(x).$$

In particular, if $\lambda(x) = \alpha(x)/|x|$, for some α, then

$$\det \nabla\Phi(x) = \frac{\alpha^{n-1}(x)}{|x|^n} \sum_{i=1}^{n} x_i \frac{\partial \alpha}{\partial x_i}(x).$$

Proof. Since $\nabla\Phi = \lambda\,\text{Id} + \nabla\lambda \otimes x$ and $\nabla\lambda \otimes x$ is a rank-one matrix, the first equality holds true. The second one easily follows. ■

6 Uniform concentration of mass

We start with an elementary lemma.

Lemma 20 *Let $c \in C^0([0,1]; B_1)$. Then for every $\epsilon > 0$ such that $c([0,1]) + B_\epsilon \subset B_1$, there exists $\Phi_\epsilon \in \text{Diff}^\infty(\overline{B_1}; \overline{B_1})$ satisfying*

$$\Phi_\epsilon(c(0)) = c(1) \quad \textit{and} \quad \text{supp}(\Phi_\epsilon - \text{id}) \subset c([0,1]) + B_\epsilon \, .$$

Proof. For every $\epsilon > 0$ such that $c([0,1]) + B_\epsilon \subset B_1$ define $\eta_\epsilon \in C_0^\infty(\mathbb{R}^n; [0,1])$ such that

$$\eta_\epsilon = \begin{cases} 1 & \text{in } B_{\epsilon/4} \\ 0 & \text{in } \left(B_{\epsilon/2}\right)^c . \end{cases}$$

Set, for $a \in \mathbb{R}^n$,

$$\eta_{a,\epsilon}(x) := \eta_\epsilon(x - a).$$

We then have

$$\delta\|\nabla\eta_{a,\epsilon}\|_{C^0} = \delta\|\nabla\eta_\epsilon\|_{C^0} \leq 1/2, \tag{31}$$

for a suitable $\delta = \delta(\epsilon) > 0$. Let $x_i \in B_1$, $1 \leq i \leq N$, with $x_1 = c(0)$, $x_N = c(1)$, be such that

$$\begin{cases} x_i \in c([0,1]) & 1 \leq i \leq N \\ |x_{i+1} - x_i| < \delta & 1 \leq i \leq N-1 \end{cases}$$

and define

$$\Phi_i(x) := x + \eta_{x_i,\epsilon}(x)(x_{i+1} - x_i), \quad 1 \leq i \leq N-1.$$

Since (31) holds and $\operatorname{supp}(\Phi_i - \mathrm{id}) \subset c([0,1]) + B_\epsilon \subset B_1$, we have $\det \nabla\Phi_i > 0$ and $\Phi_i = \mathrm{id}$ on ∂B_1. Therefore $\Phi_i \in \mathrm{Diff}^\infty(\overline{B_1}; \overline{B_1})$, by Theorem 29. Moreover $\Phi_i(x_i) = x_{i+1}$. Then the diffeomorphism $\Phi_\epsilon := \Phi_{N-1} \circ \cdots \circ \Phi_1$ has all the required properties. ■

Before stating the main result of this section, we need some notations and elementary properties of pullbacks and connected components.

Notation 21 *Let $\Omega \subset \mathbb{R}^n$ be open and bounded. If $f \in C^0(\overline{\Omega})$, we adopt the following notations*

$$F^+ = f^{-1}((0,\infty)) \quad \textit{and} \quad F^- = f^{-1}((-\infty,0)).$$

Moreover, if $x \in F^\pm$ then

$$F_x^\pm \quad \textit{is the connected component of } F^\pm \textit{ containing } x.$$

In the following lemma we state, without proof, some basic properties of pullbacks.

Lemma 22 (Properties of pullbacks) *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $f \in C^0(\overline{\Omega})$, $\Phi \in \mathrm{Diff}^1(\overline{\Omega}; \overline{\Omega})$ with $\det \nabla\Phi > 0$, $x \in F^+$, $y \in F^-$. Letting $\widetilde{f} := \Phi^*(f)$, we have*

$$\Phi^{-1}(F_x^+) = \widetilde{F}^+_{\Phi^{-1}(x)}, \quad \Phi^{-1}(F_y^-) = \widetilde{F}^-_{\Phi^{-1}(y)}$$

and, for any open $U \subset \Omega$,

$$\int_U f = \int_{\Phi^{-1}(U)} \Phi^*(f).$$

In particular, if $\Phi = \mathrm{id}$ on ∂U, the following holds

$$\int_U f = \int_U \Phi^*(f).$$

Moreover, if $\Phi_1, \Phi_2 \in C^1(\mathbb{R}^n; \mathbb{R}^n)$, then

$$(\Phi_1 \circ \Phi_2)^* = \Phi_2^* \circ \Phi_1^*. \tag{32}$$

The following one is a trivial result about the cardinality of the connected components of super (sub) level sets of continuous functions and we state it for the sake of completeness.

Lemma 23 *Let $f \in C^0(\overline{B_1})$. Let $\{F^+_{x_i}\}_{i\in I^+}$ and $\{F^-_{y_j}\}_{j\in I^-}$ be the connected components of F^+ respectively of F^-. Then I^+ and I^- are at most countable. Moreover, if $|I^+| = \infty$ or $|I^-| = \infty$, then*

$$\lim_{k\to\infty} \operatorname{meas}(F^+ \setminus \bigcup_{i=1}^{k} F^+_{x_i}) = 0 \quad \textit{or} \quad \lim_{k\to\infty} \operatorname{meas}(F^- \setminus \bigcup_{j=1}^{k} F^-_{y_j}) = 0$$

respectively.

One of the key lemmas in our proof of the main theorem is the following, which allows to concentrate the mass and to distribute it uniformly.

Lemma 24 (Uniform concentration of mass) *Let $k \geq 1$ be an integer, $f \in C^k(\overline{B_1})$ and $z \in F^+$. Suppose that A_1 and A_2 are two closed sets with non-empty interior such that*

$$A_1 \subset \operatorname{int}(A_2) \subset A_2 \subset F^+_z \cap B_1 \,.$$

Then, for every small $\epsilon > 0$, there exists $\Phi_{\epsilon,f,A_1,A_2} \in \operatorname{Diff}^k(\overline{B_1};\overline{B_1})$ (which will be simply denoted Φ_ϵ) satisfying the following properties

$$\operatorname{supp}(\Phi_\epsilon - \operatorname{id}) \subset F^+_z \cap B_1 \quad \textit{and} \quad \int_{F^+_z} \Phi^*_\epsilon(f) = \int_{F^+_z} f$$

$$\|\Phi^*_\epsilon(f)\|_{C^0} \textit{ is uniformly bounded with respect to } \epsilon \tag{33}$$

$$\Phi^*_\epsilon(f) = C_\epsilon \quad \textit{in } A_1\,, \quad C_\epsilon \textit{ constant} \tag{34}$$

$$0 < \Phi^*_\epsilon(f) \leq C_\epsilon \quad \textit{in } A_2 \setminus A_1 \tag{35}$$

$$\lim_{\epsilon\to 0} \Phi^*_\epsilon(f)(x) = \begin{cases} \int_{F^+_z} f/\operatorname{meas}(A_1) & x \in A_1 \\ 0 & x \in (F^+_z \cap B_1)\setminus A_1 \\ f(x) & \textit{elsewhere} \end{cases} \tag{36}$$

$$C_\epsilon \operatorname{meas}(A_1) \leq \int_{F^+_z} f \quad \textit{and} \quad \lim_{\epsilon\to 0}\int_{A_1} \Phi^*_\epsilon(f) = \int_{F^+_z} f \tag{37}$$

$$\int_0^1 s^{n-1}(1_{F^+_z\setminus A_2}\Phi^*_\epsilon(f))(s\frac{x}{|x|})ds \leq \epsilon, \quad x \neq 0. \tag{38}$$

Remark 25 *A similar result holds true if $A_1, A_2 \subset F^-_y$. The changes are straightforward. In particular, (35), (37) and (38) are replaced by*

$$C_\epsilon \leq \Phi^*_\epsilon(f) < 0 \quad \textit{in } A_2 \setminus A_1\,,$$

$$C_\epsilon \operatorname{meas}(A_1) \geq \int_{F^-_y} f \quad \textit{and} \quad \lim_{\epsilon\to 0}\int_{A_1} \Phi^*_\epsilon(f) = \int_{F^-_y} f$$

and

$$\int_0^1 s^{n-1}(1_{F_y^-\setminus A_2}\Phi_\epsilon^*(f))(s\frac{x}{|x|})ds \geq -\epsilon, \quad x \neq 0,$$

respectively.

Proof. We split the proof into two steps.

Step 1 (simplification). Using Corollary 14, it is sufficient to prove the existence of $f_\epsilon \in C^k(\overline{B_1})$, such that

$$\begin{cases} f_\epsilon > 0 \quad \text{in } F_z^+ \\ \operatorname{supp}(f - f_\epsilon) \subset F_z^+ \cap B_1 \\ \int_{F_z^+} f_\epsilon = \int_{F_z^+} f \end{cases}$$

satisfying also (33)-(38) with $\Phi_\epsilon^*(f)$ replaced by f_ϵ .

Step 2 (definition of f_ϵ). In the following we adopt the following notations:

$$M := \sup_{B_1} |f|, \quad m := \int_{F_z^+} f \quad \text{and} \quad k := \frac{1}{2\max\{1, M, 2m/\operatorname{meas}(A_1)\}}\,.$$

Let $0 < \epsilon_1 \leq 1/4$ be such that $A_1 + B_{\epsilon_1} \subset \operatorname{int}(A_2)$ and let $\eta_\epsilon \in C^\infty(\overline{B_1}; [0,1])$, $0 < \epsilon \leq \epsilon_1$, satisfy

$$\eta_\epsilon = 1 \quad \text{in } A_1 \quad \text{and} \quad \operatorname{supp} \eta_\epsilon \subset A_1 + B_\epsilon\,.$$

We claim that there exists a family of closed sets K_ϵ , such that

$$A_2 \subset K_\epsilon \subset F_z^+ \cap B_1 \tag{39}$$

$$K_\epsilon \subset K_{\epsilon'} \quad \text{if } \epsilon' < \epsilon \tag{40}$$

$$\bigcup_{\epsilon>0} K_\epsilon = F_z^+ \cap B_1 \tag{41}$$

$$f\,|_{(F_z^+\cap B_{1-k\epsilon})\setminus K_\epsilon} \leq k\epsilon. \tag{42}$$

In fact since $f = 0$ in $\partial F_z^+ \cap B_1$ and f is uniformly continuous, for every $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that

$$|f(y)| \leq k\epsilon \quad \forall y \in \left[\left(\partial F_z^+ \cap B_1\right) + B_\delta\right] \cap \overline{B_1}\,.$$

Then it is clear that there exists a family of closed sets $\{K_\epsilon\}$ satisfying (39), (40) and (41) and

$$\left(F_z^+ \cap B_{1-k\epsilon}\right) \setminus K_\epsilon \subset \left[\left(\partial F_z^+ \cap B_1\right) + B_\delta\right] \cap \overline{B_1}\,,$$

which implies (42).

Let f_ϵ , ϵ small, be defined as follows:

$$f_\epsilon := \begin{cases} \eta_\epsilon C_\epsilon + (1-\eta_\epsilon)k\epsilon & \text{in } A_2 \\ \xi_\epsilon k\epsilon + (1-\xi_\epsilon)f & \text{elsewhere} \end{cases}$$

where $\xi_\epsilon \in C^\infty(\overline{B_1};[0,1])$ is such that

$$\xi_\epsilon = 1 \text{ in } K_\epsilon\,, \quad \operatorname{supp}\xi_\epsilon \subset F_z^+ \cap B_1$$

and C_ϵ is the constant which guarantees that

$$\int_{F_z^+} f_\epsilon = \int_{F_z^+} f.$$

We claim that f_ϵ has all the required properties. Obviously $f_\epsilon \in C^k(\overline{B_1})$, $\operatorname{supp}(f - f_\epsilon) \subset F_z^+ \cap B_1$ and (34) holds. Using

$$\lim_{\epsilon\to 0} \eta_\epsilon = 1_{A_1} \quad \text{and} \quad \lim_{\epsilon\to 0} \xi_\epsilon = 1_{F_z^+\cap B_1}\,,$$

(the last one holding by (40) and (41)) the definition of C_ϵ and the dominated convergence theorem, we get

$$\lim_{\epsilon\to 0} C_\epsilon = m/\operatorname{meas}(A_1) \tag{43}$$

and thus

$$\lim_{\epsilon\to 0} f_\epsilon = \begin{cases} m/\operatorname{meas}(A_1) & x \in A_1 \\ 0 & x \in (F_z^+ \cap B_1) \setminus A_1 \\ f & \text{elsewhere} \end{cases}$$

and (36) follows.

Let us prove (33) and (35). From (43), we can find $\epsilon_2 \le \epsilon_1$ such that for every $\epsilon \le \epsilon_2$,

$$k\epsilon \le \epsilon \le m/(2\operatorname{meas}(A_1)) \le C_\epsilon \le 2m/\operatorname{meas}(A_1).$$

Then, (35) follows by the very definition of f_ϵ, and, for every $\epsilon \le \epsilon_2$, we get

$$f_\epsilon > 0 \quad \text{in } F_z^+ \quad \text{and} \quad \|f_\epsilon\|_{C^0} \le \max\{M, 2m/\operatorname{meas}(A_1)\} \tag{44}$$

and (33) follows.

The properties in (37) are easily implied by (33), (36) and $f_\epsilon > 0$ in F_z^+. To prove (38), first notice that, by definition of f_ϵ, $f_\epsilon = k\epsilon$ in $K_\epsilon \setminus A_2$. Then, using the definition of f_ϵ and (42), we get that

$$f_\epsilon\,|_{(F_z^+\cap B_{1-k\epsilon})\setminus A_2} \le k\epsilon\,.$$

This inequality, together with (44), implies that, for every $\epsilon \le \epsilon_2$ and every $x \ne 0$,

$$\begin{aligned}\int_0^1 s^{n-1}(1_{F_z^+\setminus A_2} f_\epsilon)(s\frac{x}{|x|})ds &\le \int_0^{1-k\epsilon} (1_{F_z^+\setminus A_2} f_\epsilon)(s\frac{x}{|x|})ds + \int_{1-k\epsilon}^1 (1_{F_z^+\setminus A_2} f_\epsilon)(s\frac{x}{|x|})ds \\ &\le \int_0^{1-k\epsilon} k\epsilon ds + \int_{1-k\epsilon}^1 \max\{M, 2m/\operatorname{meas}(A_1)\} ds \\ &\le k\epsilon + \max\{M, 2m/\operatorname{meas}(A_1)\}k\epsilon \le \epsilon\end{aligned}$$

and (38) follows. ■

7 Positive radial integration

In this section we show how to modify the distribution of mass of $f \in C^k(\overline{B_1})$ satisfying $\int_{B_1} f > 0$, in order to have strictly positive integrals on every radius. This is the central part of our argument.

Lemma 26 (Positive radial integration) *Let $k \geq 1$ be an integer and $f \in C^k(\overline{B_1})$ be such that*

$$\int_{B_1} f > 0. \tag{45}$$

Then there exists $\Phi \in \operatorname{Diff}^\infty(\overline{B_1}; \overline{B_1})$ such that $\Phi^*(f)(0) > 0$, $\operatorname{supp}(\Phi - \operatorname{id}) \subset B_1$ and

$$\int_0^r s^{n-1} \Phi^*(f)(s\frac{x}{|x|})ds > 0, \quad \text{for every } x \neq 0 \text{ and } r \in (0, 1]. \tag{46}$$

Remark 27 *(i) If $f \geq 0$, the proof is straightforward; it already ends after Step 1.*

(ii) If f_1 satisfies (46) with a certain Φ as in the lemma, then every $f \geq f_1$ satisfies (46) with the same Φ. Indeed,

$$\Phi^*(f_1)(x) = f_1(\Phi(x))\underbrace{\det \nabla\Phi(x)}_{>0} \leq f(\Phi(x)) \det \nabla\Phi(x) = \Phi^*(f)(x).$$

(iii) If, in addition to (45), $f > 0$ on ∂B_1, we can find with a similar argument (see [8] for details) Φ satisfying in addition

$$\int_r^1 s^{n-1} \Phi^*(f)(s\frac{x}{|x|})ds \geq 0, \quad \text{for every } x \neq 0 \text{ and } r \in [0, 1] .$$

Proof. Since the proof is rather long, we divide it into nine steps. The following three facts will be crucial.

(a) For fixed $a, b \in B_1$, there exists, from Lemma 20, $\Phi \in \operatorname{Diff}^\infty(\overline{B_1}; \overline{B_1})$ such that $\Phi(a) = b$. This will be used in Steps 1 and 5.

(b) From Lemma 24, we concentrate the mass contained in connected components of F^+ and F^- in balls or sectors of cones. This will be used in Steps 6 and 8.

(c) From Remark 27 (ii), it is sufficient to prove the result for a function $f_1 \leq f$. This will be used in Steps 2, 3 and 7.

Step 1. Without loss of generality, we can assume $f(0) > 0$. In fact, suppose that $f(0) \leq 0$. We prove that there exists a diffeomorphism Φ_1 such that $\Phi_1^*(f)(0) > 0$. Since $\int_{B_1} f > 0$, there exists $a \in B_1$ such that $f(a) > 0$. By Lemma 20, there exists $\Phi_1 \in \operatorname{Diff}^\infty(\overline{B_1}; \overline{B_1})$ such that

$$\operatorname{supp}(\Phi_1 - \operatorname{id}) \subset B_1 \quad \text{and} \quad \Phi_1(0) = a.$$

Since $\Phi_1^*(f)(0) = f(a)\det\nabla\Phi_1(0) > 0$, we have the claim. From now on, we write f in place of $\Phi_1^*(f)$. Moreover, we assume $F^- \neq \emptyset$, otherwise the proof is already done.

Step 2. We show that we can assume that $f \in C^\infty(\overline{B_1})$. First extend f so that $f \in C^k(\mathbb{R}^n)$ and let $f_\epsilon = f * \varphi_\epsilon$, where φ is a positive mollifier. For every $\sigma > 0$ there exists $\epsilon_0(\sigma)$ such that

$$|f_\epsilon(x) - f(x)| < \sigma \quad \text{for every } \epsilon \leq \epsilon_0(\sigma) \text{ and every } x \in \overline{B_1}\,. \tag{47}$$

Define $h_\sigma \in C^\infty(\overline{B_1})$ by

$$h_\sigma := f_{\epsilon_0(\sigma)} - \sigma.$$

Using Step 1 and (47), there exists $\sigma > 0$ such that h_σ verifies

$$\int_{B_1} h_\sigma > 0, \quad h_\sigma(0) > 0 \quad \text{and} \quad h_\sigma \leq f.$$

Using Remark 27 (ii) we have the assertion. For now on we write f instead of h_σ.

Step 3. We now show that we can assume that

$$\int_{B_1\setminus F_0^+} f > 0, \tag{48}$$

where, we recall, F_0^+ is the connected component of F^+ containing 0. In fact, by Steps 1 and 2 and (45), if $\delta_1 > 0$ is small enough we have that $B_{4\delta_1} \subset F_0^+$ and

$$\int_{B_1\setminus B_{4\delta_1}} f > 0. \tag{49}$$

Let $\eta \in C^\infty([0,1];[0,1])$ be such that

$$\eta\,(r) = \begin{cases} 1 & \text{if } r \leq \delta_1 \text{ or } 4\delta_1 \leq r \leq 1 \\ 0 & \text{if } 2\delta_1 \leq r \leq 3\delta_1\,. \end{cases}$$

If H_0^+ is the connected component containing 0 of

$$H^+ := \{x \in B_1 \,:\, \eta\,(|x|)\, f\,(x) > 0\},$$

we have that $B_{\delta_1} \subset H_0^+ \subset B_{2\delta_1}$. Using (49), we get

$$\int_{B_1\setminus H_0^+} (\eta f) \geq \int_{B_1\setminus B_{4\delta_1}} (\eta f) = \int_{B_1\setminus B_{4\delta_1}} f > 0.$$

Since $\eta f \leq f$, we may, according to Remark 27 (ii), proceed replacing f with ηf.

Step 4 (choice of N connected components of $F^+ \setminus F_0^+$). Let $F_{x_i}^+$, $i \in I^+$, $x_i \in B_1\setminus F_0^+$, be the pairwise disjoint connected components of $F^+\setminus F_0^+$. Notice that I^+ is not empty by Step 3 and it is at most countable, see Lemma 23. We claim that there exists $N \in \mathbb{N}$ such that

$$\int_{\cup_{i=1}^N F_{x_i}^+} f + \int_{F^-} f > 0. \tag{50}$$

In fact, suppose that I^+ is infinite (otherwise the assertion is trivial because of (48)) and let, using (48), $\epsilon > 0$ be such that

$$\int_{B_1 \setminus F_0^+} f > \epsilon. \tag{51}$$

Then, since f is bounded, there exists $N \in \mathbb{N}$ such that (see Lemma 23)

$$\int_{F^+ \setminus \cup_{i=1}^N F_{x_i}^+} f - \int_{F_0^+} f < \epsilon. \tag{52}$$

Combining (51) and (52), we deduce that (50) holds true.

Step 5. In this step we move the N connected components selected in the previous step, in order that they contain sectors of cone having the same axis. Choose $y \in F^-$, let $F_{x_1}^+, \cdots, F_{x_N}^+$ be the connected components of F^+ defined in the previous step and let $\rho > 0$ be such that $B_\rho \subset F_0^+$.

Step 5.1 (displacement of the points x_i). Applying $N+1$ times Lemma 20, it is easy to define $\Phi_2 \in \mathrm{Diff}^\infty(\overline{B_1}; \overline{B_1})$, with

$$\mathrm{supp}(\Phi_2 - \mathrm{id}) \subset B_1 \setminus \overline{B_\rho},$$

such that

$$\widetilde{x}_i := \Phi_2^{-1}(x_i), \quad 1 \le i \le N \quad \text{and} \quad \widetilde{y} := \Phi_2^{-1}(y)$$

satisfying

$$\rho < |\widetilde{x}_1| < \cdots < |\widetilde{x}_N| < |\widetilde{y}| < 1 \quad \text{and} \quad \frac{\widetilde{x}_i}{|\widetilde{x}_i|} = \frac{\widetilde{y}}{|\widetilde{y}|}, \quad 1 \le i \le N.$$

To be complete, we also define $x_0 = \widetilde{x}_0 = 0$.

Step 5.2 (definition of the sectors of cone). If $\delta > 0$ let K_δ be the closed cone having aperture δ, vertex 0 and axis $\mathbb{R}_+ \widetilde{y}$ and define

$$\widetilde{f} := \Phi_2^*(f).$$

Since

$$\widetilde{f}(\widetilde{x}_i) > 0, \ 0 \le i \le N \quad \text{and} \quad \widetilde{f}(\widetilde{y}) < 0,$$

then there exists $\delta > 0$ small enough such that

$$|\widetilde{x}_{i+1}| - |\widetilde{x}_i| > 4\delta, \ 0 \le i \le N-1 \quad \text{and} \quad |\widetilde{y}| - |\widetilde{x}_N| > 4\delta$$

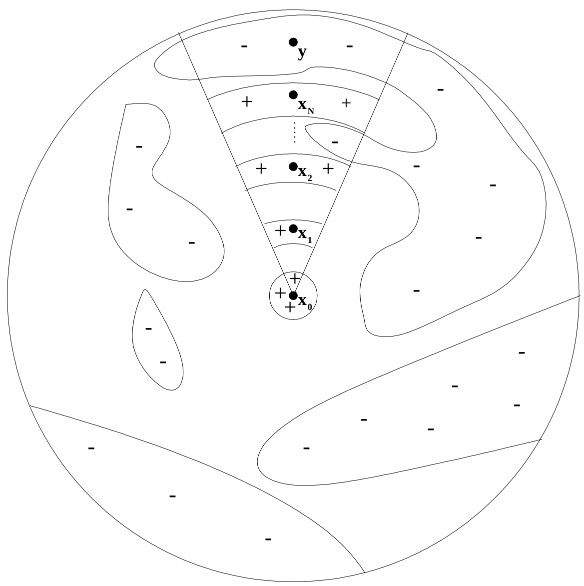
with

$$\begin{cases} \overline{B_{3\delta}} \subset \widetilde{F}_0^+ \\ K_{2\delta} \cap \left(\overline{B_{|\widetilde{x}_i|+2\delta}} \setminus B_{|\widetilde{x}_i|-\delta}\right) \subset \widetilde{F}^+_{\widetilde{x}_i}\,, \quad 1 \le i \le N \\ K_{2\delta} \cap \left(\overline{B_{|\widetilde{y}|+2\delta}} \setminus B_{|\widetilde{y}|-\delta}\right) \subset \widetilde{F}^-_{\widetilde{y}}\,. \end{cases}$$

Using Lemma 22 and (50), we get that $\widetilde{f}$ satisfies

$$\int_{\cup_{i=1}^N \widetilde{F}^+_{\widetilde{x}_i}} \widetilde{f} + \int_{\widetilde{F}^-} \widetilde{f} > 0. \tag{53}$$

From now on we write f, x_i and y in place of $\widetilde{f} = \Phi_2^*(f)$, $\widetilde{x}_i$ and $\widetilde{y}$, respectively (see the figure below).



Step 6 (concentration of the positive mass in the cone sectors). From now on, if $\sigma \in (-\delta/2, \delta]$ we use the following notations

$$\begin{cases} S_0^\sigma := \overline{B_{2\delta+\sigma}} \\ S_i^\sigma := K_{\delta+\sigma} \cap \left(\overline{B_{|x_i|+\delta+\sigma}} \setminus B_{|x_i|-\sigma}\right), \quad 1 \le i \le N \\ S^\sigma := K_{\delta+\sigma} \cap \left(\overline{B_{|y|+\delta+\sigma}} \setminus B_{|y|-\sigma}\right). \end{cases}$$

For the sake of simplicity, if $\sigma = 0$ we write S_0 , S_i and S in place of S_0^0 , S_i^0 and S^0, respectively. Let

$$\Phi_{3,\epsilon} := \Phi_{\epsilon,f,S_0,S_0^\delta} \circ \Phi_{\epsilon,f,S_1,S_1^\delta} \circ \cdots \circ \Phi_{\epsilon,f,S_N,S_N^\delta}\,,$$

where $\Phi_{\epsilon,f,S_i,S_i^\delta}$, $i = 0, \cdots, N$, is the C^∞ diffeomorphism obtained by Lemma 24 applied to f, $F^+_{x_i}$, $A_1 = S_i$, $A_2 = S_i^\delta$. Notice that $\operatorname{supp}(\Phi_{3,\epsilon} - \mathrm{id}) \subset B_1$.

By (34), (37) and (53), there exists $\widetilde{\epsilon}$ such that the constants $C_{i,\widetilde{\epsilon}}$ satisfy the inequality

$$\sum_{i=1}^{N} C_{i,\widetilde{\epsilon}} \operatorname{meas}(S_i) + \int_{F^-} f > 0.$$

Denoting $h := \Phi^*_{3,\widetilde{\epsilon}}(f)$ we have that h satisfies

$$h = f \quad \text{in } \overline{B_1} \setminus \cup_{i=0}^{N} F^+_{x_i}\,, \quad H^- = F^-,$$

$$F^+_{x_i} = H^+_{x_i}\,, \quad \int_{F^+_{x_i}} f = \int_{F^+_{x_i}} h, \quad 0 \le i \le N,$$

$$h \equiv C_{i,\widetilde{\epsilon}} > 0 \quad \text{in } S_i\,, \quad 0 \le i \le N, \tag{54}$$

$$\int_{\cup_{i=1}^{N} S_i} h + \int_{H^-} h > 0. \tag{55}$$

From now on, we write f, Φ_3 and C_i in place of $h, \Phi_{3,\widetilde{\epsilon}}$ and $C_{i,\widetilde{\epsilon}}$, $0 \le i \le N$.

Step 7 (modification of f in order to have F^- connected). Extend f so that $f \in C^\infty(\mathbb{R}^n)$, define $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$,

$$\tilde{f}(x) := \min\{f(x), 0\}$$

and let $\tilde{f}_\epsilon = \tilde{f} * \varphi_\epsilon$, where φ is a positive mollifier. By continuity of $\tilde{f}$, for every $\sigma > 0$ there exists $\epsilon_0(\sigma)$ such that

$$|\tilde{f}_\epsilon(x) - \tilde{f}(x)| < \sigma \quad \text{for every } \epsilon \le \epsilon_0(\sigma) \text{ and every } x \in \overline{B_1}\,. \tag{56}$$

Defining $h_\sigma \in C^\infty(\overline{B_1})$, $h_\sigma = \tilde{f}_{\epsilon_0(\sigma)} - \sigma$, we have, using (56), that

$$h_\sigma(x) < \tilde{f}(x) = \min\{f(x), 0\} \le f(x).$$

For every $\sigma \in (0, \delta/8)$, let $\xi_\sigma \in C^\infty(\overline{B_1}; [0,1])$ be such that

$$\xi_\sigma \equiv 1 \text{ in } \cup_{i=0}^{N} (S_i^\sigma \setminus S_i^{-\sigma}) \quad \text{and} \quad \operatorname{supp} \xi_\sigma \subset \cup_{i=0}^{N} (S_i^{2\sigma} \setminus S_i^{-2\sigma})$$

and

$$\{x \in \overline{B_1} \setminus \cup_{i=0}^{N} S_i \,:\, \xi_\sigma(x) < 1\} \quad \text{is connected.} \tag{57}$$

Moreover let $f_\sigma : \overline{B_1} \to \mathbb{R}$ be defined as

$$f_\sigma(x) := \begin{cases} (1 - \xi_\sigma(x)) f(x) & \text{if } x \in \cup_{i=0}^{N} S_i \\ (1 - \xi_\sigma(x)) h_\sigma(x) & \text{if } x \in (\cup_{i=0}^{N} S_i)^c. \end{cases} \tag{58}$$

It is easy to verify that f_σ is of class C^∞ and that it satisfies the following properties:

$$f_\sigma(x) \begin{cases} = h_\sigma(x) < \min\{f(x), 0\} \le f(x) & \text{if } x \in \overline{B_1} \setminus \cup_{i=0}^N S_i^{2\sigma}, \\ \le 0 < f(x) & \text{if } x \in \cup_{i=0}^N (S_i^{2\sigma} \setminus S_i^{\sigma}), \\ = 0 < f(x) & \text{if } x \in \cup_{i=0}^N (S_i^{\sigma} \setminus S_i^{-\sigma}), \\ \le f(x) & \text{if } x \in \cup_{i=0}^N (S_i^{-\sigma} \setminus S_i^{-2\sigma}), \\ = f(x) = C_i & \text{if } x \in S_i^{-2\sigma}, \text{ for some } i \in \{0, \cdots, N\}, \end{cases}$$

where C_i are as in (54); in particular, $f_\sigma \le f$. We moreover have

$$\begin{aligned} F_\sigma^- &= \{x \in \overline{B_1} : f_\sigma(x) < 0\} = \{x \in \overline{B_1} \setminus \cup_{i=0}^N S_i : f_\sigma(x) < 0\} \\ &= \{x \in \overline{B_1} \setminus \cup_{i=0}^N S_i : (1 - \xi_\sigma(x)) h_\sigma(x) < 0\} \\ &= \{x \in \overline{B_1} \setminus \cup_{i=0}^N S_i : \xi_\sigma(x) < 1\}, \end{aligned}$$

which is a connected set by (57); we thus have that

$$F_\sigma^- \subset \overline{B_1} \setminus \cup_{i=0}^N S_i \quad \text{and} \quad F_\sigma^- \text{ is connected.}$$

Notice that (55), (56) and (58) imply that we can choose σ such that

$$\sum_{i=1}^N C_i \operatorname{meas}(S_i^{-2\sigma}) + \int_{F_\sigma^-} f_\sigma = \int_{\cup_{i=1}^N S_i^{-2\sigma}} f_\sigma + \int_{F_\sigma^-} f_\sigma > 0, \tag{59}$$

since

$$\lim_{s \to 0^+} \left\{ \int_{\cup_{i=1}^N S_i^{-2s}} f_s + \int_{F_s^-} f_s \right\} = \int_{\cup_{i=1}^N S_i} f + \int_{F^-} f > 0.$$

From now on, we write f in place of f_σ , since $f_\sigma \le f$ and Remark 27 (ii) holds.

Step 8 (concentration of the negative mass). We finally concentrate the negative mass around y.

Step 8.1 (preliminaries). Let $\tau \in (0, \sigma/2]$. Using Remark 25 (with $A_1 = S^{-2\sigma-\tau}$ and $A_2 = S^{-2\sigma}$) and recalling that, by Step 7, $F_y^- = F^-$, we have, for ϵ small enough, $\Phi_{4,\epsilon}^\tau \in \operatorname{Diff}^\infty(\overline{B_1}; \overline{B_1})$ satisfying the following properties.

$$\operatorname{supp}(\Phi_{4,\epsilon}^\tau - \mathrm{id}) \subset F^- \cap B_1$$

$$\int_{F^-} (\Phi_{4,\epsilon}^\tau)^*(f) = \int_{F^-} f$$

$$(\Phi_{4,\epsilon}^\tau)^*(f) = C_\epsilon^\tau < 0 \quad \text{in } S^{-2\sigma-\tau}$$

and

$$C_\epsilon^\tau \le (\Phi_{4,\epsilon}^\tau)^*(f) < 0 \quad \text{in } S^{-2\sigma} \setminus S^{-2\sigma-\tau} \tag{60}$$

$$\lim_{\epsilon\to 0}(\Phi^{\tau}_{4,\epsilon})^*(f)(x) = \begin{cases} \int_{F^-} f/\operatorname{meas}(S^{-2\sigma-\tau}) & \text{if } x \in S^{-2\sigma-\tau} \\ 0 & \text{if } x \in (F^- \cap B_1)\setminus S^{-2\sigma-\tau} \\ f(x) & \text{elsewhere} \end{cases} \tag{61}$$

$$C^{\tau}_{\epsilon} \operatorname{meas}(S^{-2\sigma-\tau}) \geq \int_{F^-} f \tag{62}$$

$$\int_0^1 s^{n-1}(1_{F^-\setminus S^{-2\sigma}}(\Phi^{\tau}_{4,\epsilon})^*(f))(s\frac{x}{|x|})ds \geq -\epsilon, \text{ for every } x \neq 0. \tag{63}$$

Step 8.2 (choice of ϵ and τ). We first choose $\widetilde{\epsilon}$ small enough in order to have

$$\int_0^{2\delta-2\sigma} s^{n-1}C_0\, ds - \widetilde{\epsilon} > 0. \tag{64}$$

We claim that there exists $\widetilde{\tau}$ such that

$$\sum_{i=1}^{N} C_i \operatorname{meas}(S_i^{-2\sigma}) + C^{\widetilde{\tau}}_{\widetilde{\epsilon}} \operatorname{meas}(S^{-2\sigma}) > 0. \tag{65}$$

In fact, for every $\lambda \in (0,1)$ there exists $\tau \in (0, \sigma/2]$ such that

$$\frac{\operatorname{meas}\left(S^{-2\sigma}\right)}{\operatorname{meas}\left(S^{-2\sigma-\tau}\right)} \leq \frac{1}{1-\lambda}. \tag{66}$$

Using (62), we have that, for every $\lambda \in (0,1)$ and $\tau = \tau(\lambda)$ as in (66),

$$\begin{aligned} C^{\tau}_{\widetilde{\epsilon}} \operatorname{meas}(S^{-2\sigma}) &= C^{\tau}_{\widetilde{\epsilon}} \operatorname{meas}(S^{-2\sigma-\tau})\frac{\operatorname{meas}\left(S^{-2\sigma}\right)}{\operatorname{meas}\left(S^{-2\sigma-\tau}\right)} \\ &\geq C^{\tau}_{\widetilde{\epsilon}} \frac{1}{1-\lambda}\operatorname{meas}(S^{-2\sigma-\tau}) \geq \frac{1}{1-\lambda}\int_{F^-} f. \end{aligned}$$

By this inequality and (59), choosing λ sufficiently small, we have that there exists $\widetilde{\tau}$ such that (65) holds true. From now on we write f, ϵ, Φ_4 and C_- in place of $\left(\Phi^{\widetilde{\tau}}_{4,\widetilde{\epsilon}}\right)^*(f), \widetilde{\epsilon}, \Phi^{\widetilde{\tau}}_{4,\widetilde{\epsilon}}$ and $C^{\widetilde{\tau}}_{\widetilde{\epsilon}}$.

Step 8.3 (summary). Using (54), (60), (63), (64), (65) f satisfies the following properties

$$f \equiv C_0 > 0 \quad \text{in} \quad S_0^{-2\sigma} = \overline{B_{2\delta-2\sigma}} \tag{67}$$

$$f \equiv C_i > 0 \text{ in } S_i^{-2\sigma} \quad 1 \leq i \leq N \tag{68}$$

$$f \equiv C_- \text{ in } S^{-2\sigma-\widetilde{\tau}} \quad \text{and} \quad C_- \leq f < 0 \text{ in } S^{-2\sigma}\setminus S^{-2\sigma-\widetilde{\tau}} \tag{69}$$

$$\sum_{i=1}^{N}\int_{S_i^{-2\sigma}} f + \int_{S^{-2\sigma}} f \geq \sum_{i=1}^{N} C_i \operatorname{meas}(S_i^{-2\sigma}) + C_- \operatorname{meas}(S^{-2\sigma}) > 0 \tag{70}$$

$$\int_0^{2\delta-2\sigma} s^{n-1}C_0\, ds + \int_0^1 s^{n-1}\left(1_{F^-\setminus S^{-2\sigma}}f\right)(s\frac{x}{|x|})ds > 0. \tag{71}$$

Step 9 (conclusion). Let

$$\Phi = \Phi_1 \circ \Phi_2 \circ \Phi_3 \circ \Phi_4 \,.$$

Note that by construction $\operatorname{supp}(\Phi - \mathrm{id}) \subset B_1$. Because of all successive replacements of f in Steps 1-8 by new f, the lemma has to be proved for $\Phi = \mathrm{id}$. From (67), we have $f(0) > 0$. We finally show (46). We split into three parts.

Step 9.1. If $r \le 2\delta - 2\sigma$, (67) implies directly the assertion.

Step 9.2. Now, suppose that either $x \notin K_{\delta-2\sigma}$ and $r \in (2\delta - 2\sigma, 1]$ or $x \in K_{\delta-2\sigma}$ and $r \in (2\delta - 2\sigma, |y| + 2\sigma]$. Then (71) implies

$$\begin{aligned}\int_0^r s^{n-1} f(s\frac{x}{|x|})ds &\ge \int_0^r s^{n-1}(1_{F_0^+} f)(s\frac{x}{|x|})ds + \int_0^r s^{n-1}(1_{F^-} f)(s\frac{x}{|x|})ds \\ &= \int_0^r s^{n-1}(1_{F_0^+} f)(s\frac{x}{|x|})ds + \int_0^r s^{n-1}(1_{F^-\setminus S^{-2\sigma}} f)(s\frac{x}{|x|})ds \\ &\ge C_0 \int_0^{2\delta-2\sigma} s^{n-1} ds + \int_0^1 s^{n-1}(1_{F^-\setminus S^{-2\sigma}} f)(s\frac{x}{|x|})ds > 0.\end{aligned}$$

Step 9.3. It remains to consider the case $x \in K_{\delta-2\sigma}$, $r \in (|y| + 2\sigma, 1]$. Under these assumptions, we have

$$\begin{aligned}&\int_0^r s^{n-1} f(s\frac{x}{|x|})ds \\ &= \int_0^r s^{n-1}(1_{F_0^+} f)(s\frac{x}{|x|})ds + \int_0^r s^{n-1}(1_{\cup_{i=1}^N F_{x_i}^+} f)(s\frac{x}{|x|})ds + \int_0^r s^{n-1}(1_{F^-} f)(s\frac{x}{|x|})ds \\ &\ge \left\{ C_0 \int_0^{2\delta-2\sigma} s^{n-1} ds + \int_0^1 s^{n-1}(1_{F^-\setminus S^{-2\sigma}} f)(s\frac{x}{|x|})ds \right\} \\ &+ \left\{ \int_0^1 s^{n-1}(1_{\cup_{i=1}^N S_i^{-2\sigma}} f)(s\frac{x}{|x|})ds + \int_0^1 s^{n-1}(1_{S^{-2\sigma}} f)(s\frac{x}{|x|})ds \right\} > 0.\end{aligned}$$

In fact, the positivity of the first sum follows from (71). The second one is also positive, since, from (69)

$$\begin{aligned}&\int_0^1 s^{n-1}(1_{\cup_{i=1}^N S_i^{-2\sigma}} f)(s\frac{x}{|x|})ds + \int_0^1 s^{n-1}(1_{S^{-2\sigma}} f)(s\frac{x}{|x|})ds \\ &\ge \sum_{i=1}^N C_i \int_0^1 s^{n-1} 1_{S_i^{-2\sigma}}(s\frac{x}{|x|})ds + C_- \int_0^1 s^{n-1} 1_{S^{-2\sigma}}(s\frac{x}{|x|})ds\end{aligned}$$

and the positivity of the right hand side is guaranteed by (70) and the fact that $S_i^{-2\sigma}$ and $S^{-2\sigma}$ are sectors of a radial cone centered at 0. This concludes the proof. ∎

8 Appendix

We begin recalling some results on the topological degree (see e.g. [7] or [14] for further details).

Let Ω be a bounded open set of $\mathbb{R}^n$, $\Phi \in C^1(\overline{\Omega}; \mathbb{R}^n)$ and

$$Z_\Phi := \{x \in \overline{\Omega} : \det \nabla\Phi(x) = 0\}.$$

Then for every $p \in \mathbb{R}^n$ such that

$$p \notin \Phi(\partial\Omega) \cup \Phi(Z_\Phi),$$

we define the integer $\deg(\Phi, \Omega, p)$ as

$$\deg(\Phi, \Omega, p) := \sum_{x \in \Omega : \Phi(x) = p} \mathrm{sign}(\det \nabla\Phi(x)),$$

with the convention $\deg(\Phi, \Omega, p) = 0$ if $\{x \in \Omega : \Phi(x) = p\} = \emptyset$.

It is possible to extend the definition of $\deg(\Phi, \Omega, p)$ to $\Phi \in C^0(\overline{\Omega}; \mathbb{R}^n)$ and $p \notin \Phi(\partial\Omega)$, in particular using Sard theorem which states that

$$\mathrm{meas}(\Phi(Z_\Phi)) = 0. \tag{72}$$

In this framework, the following two properties hold.

(i) If $\Phi, \Psi \in C^0(\overline{\Omega}; \mathbb{R}^n)$ with $\Phi = \Psi$ on $\partial\Omega$, then for every $p \notin \Phi(\partial\Omega)$,

$$\deg(\Phi, \Omega, p) = \deg(\Psi, \Omega, p). \tag{73}$$

(ii) If $\Phi \in C^0(\overline{\Omega}; \mathbb{R}^n)$, $p \notin \Phi(\partial\Omega)$ and $\deg(\Phi, \Omega, p) \neq 0$, then there exists $x \in \Omega$ such that $\Phi(x) = p$.

In particular, if $\Phi \in C^0(\overline{\Omega}; \mathbb{R}^n)$ and $\Phi = \mathrm{id}$ on $\partial\Omega$, then

$$\Phi(\Omega) \supset \Omega \quad \text{and} \quad \Phi(\overline{\Omega}) \supset \overline{\Omega}. \tag{74}$$

As an application of these properties, we have the following lemma.

Lemma 28 *Let Ω be a bounded, connected and open set in $\mathbb{R}^n$ and let $\Phi \in C^0(\overline{\Omega}; \mathbb{R}^n)$ be one to one, such that $\Phi = \mathrm{id}$ on $\partial\Omega$. Then $\Phi \in \mathrm{Hom}(\overline{\Omega}; \overline{\Omega})$.*

Proof. By the boundedness of Ω and the continuity of Φ, if $F \subset \overline{\Omega}$ is closed then $\Phi(F)$ is closed, too. Since Φ is one to one, then

$$\Phi \in \mathrm{Hom}(\overline{\Omega}; \Phi(\overline{\Omega})).$$

Let us prove that $\Phi(\overline{\Omega}) = \overline{\Omega}$. Due to (74), it is enough to prove that $\Phi(\overline{\Omega}) \subset \overline{\Omega}$. By a classical result (see e.g. [7] Proposition 7.18) we have that $\Phi(\partial\Omega) = \partial(\Phi(\Omega))$. Thus, since $\Phi = \mathrm{id}$ on $\partial\Omega$, we get

$$\partial\Omega = \partial(\Phi(\Omega)) \quad \text{and} \quad \Phi(\Omega) \cap \partial\Omega = \emptyset. \tag{75}$$

Suppose by contradiction that $\Phi(x) \in (\overline{\Omega})^c$ for some $x \in \overline{\Omega}$. Since Φ is the identity map on $\partial\Omega$, we have that $x \in \Omega$. Let now consider $y \in \Omega$ such that $\Phi(y) \in \Omega$ (such a y surely exists by (74)) and let $c \in C^0([0,1];\Omega)$ be a path connecting x and y. Then, by continuity, there exists $t \in (0,1)$ such that $\Phi(c(t)) \in \partial\Omega$, contradicting (75). ■

We now provide a sufficient condition for the invertibility of functions in $C^1(\overline{\Omega};\mathbb{R}^n)$. A similar result can be found in Meisters-Olech [10].

Theorem 29 *Let Ω be a bounded open set in $\mathbb{R}^n$ and let $\Phi \in C^1(\overline{\Omega};\mathbb{R}^n)$ be such that*

$$\begin{cases} \det \nabla\Phi > 0 & \text{in } \Omega \\ \Phi = \mathrm{id} & \text{on } \partial\Omega. \end{cases}$$

Then $\Phi \in \mathrm{Diff}^1(\overline{\Omega};\overline{\Omega})$.

Remark 30 *Under the weaker hypotheses $\det \nabla\Phi \geq 0$, $\Phi = \mathrm{id}$ on $\partial\Omega$ and $Z_\Phi \cap \Omega$ does not have accumulation point, it can be proved that $\Phi \in C^1(\overline{\Omega};\overline{\Omega}) \cap \mathrm{Hom}(\overline{\Omega};\overline{\Omega})$, see [8].*

Proof. We divide the proof into two steps.

Step 1. We first prove that $\Phi(\Omega) = \Omega$. Using (74), we know that

$$\Phi(\Omega) \supset \Omega.$$

Let us show the reverse inclusion, i.e., $\Phi(\Omega) \subset \Omega$. We first prove that $\Phi(\Omega) \subset \overline{\Omega}$ and then conclude. By contradiction, let $x \in \Omega$ be such that $\Phi(x) \notin \overline{\Omega}$. By definition of the degree and (73), we get

$$0 < \deg(\Phi, \Omega, \Phi(x)) = \deg(\mathrm{id}, \Omega, \Phi(x)) = 0;$$

which is absurd.

To conclude, suppose that $x \in \Omega$ and $\Phi(x) \in \overline{\Omega} \setminus \Omega = \partial\Omega$. By the inverse function theorem, which can be applied since $\det \nabla\Phi(x) > 0$, there exists a neighborhood of x such that the restriction of Φ on this set is one to one and onto a neighborhood of $\Phi(x) \in \partial\Omega$. In particular, this implies the existence of $y \in \Omega$ such that $\Phi(y) \notin \overline{\Omega}$, which contradicts what has just been proved.

Step 2. Since $\Phi(\Omega) = \Omega$ and $\Phi = \mathrm{id}$ on $\partial\Omega$, we have that

$$\Phi(\overline{\Omega}) = \overline{\Omega}.$$

Moreover, $\Phi(\partial\Omega) \cap \Phi(\Omega) = \partial\Omega \cap \Omega = \emptyset$. Thus, it suffices to show that the restriction of Φ to Ω is one to one to conclude. We reason by contradiction. We assume that there exists $p \in \Omega$ which is the image of at least two elements in Ω. By (73), it follows that

$$2 \leq \deg(\Phi, \Omega, p) = \deg(\mathrm{id}, \Omega, p) = 1$$

which is the desired contradiction. ■

We also have a necessary condition for Φ to be a C^1 homeomorphism.

Proposition 31 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $\Phi \in C^1(\overline{\Omega};\overline{\Omega}) \cap \mathrm{Hom}(\overline{\Omega};\overline{\Omega})$ with $\Phi = \mathrm{id}$ on $\partial\Omega$. Then*

$$\det \nabla\Phi(x) \geq 0 \quad \text{in } \overline{\Omega} \quad \text{and} \quad \mathrm{int}(Z_\Phi) = \emptyset.$$

Proof. We split the proof into two steps.

Step 1. We show that $\det \nabla\Phi \geq 0$. By contradiction, suppose that there exists $y \in \overline{\Omega}$ such that $\det \nabla\Phi(y) < 0$. By continuity, without loss of generality, we can assume that $y \in \Omega$. In particular, $y \notin Z_\Phi$ and since Φ is one to one, we obtain

$$\Phi(y) \notin \Phi(Z_\Phi) \cup \Phi(\partial\Omega) = \Phi(Z_\Phi) \cup \partial\Omega.$$

By definition of $\deg(\Phi, \Omega, \Phi(y))$ and since $\Phi = \mathrm{id}$ on $\partial\Omega$, we have

$$1 = \deg(\Phi, \Omega, \Phi(y)) = \sum_{z\,:\,\Phi(z)=\Phi(y)} \mathrm{sign}(\det \nabla\Phi(z)).$$

Since $\mathrm{sign}(\det \nabla\Phi(y)) = -1$ the above equality implies that $\Phi^{-1}(\Phi(y))$ is not a singleton, which is absurd.

Step 2. We prove that $\mathrm{int}(Z_\Phi) = \emptyset$. By contradiction, suppose that $\mathrm{int}(Z_\Phi) \neq \emptyset$. By continuity of Φ^{-1}, we have

$$\Phi\left(\mathrm{int}(Z_\Phi)\right) = (\Phi^{-1})^{-1}(\mathrm{int}(Z_\Phi)) \neq \emptyset,$$

contradicting Sard theorem. ■

We conclude with some other necessary conditions.

Proposition 32 *Let Ω be a bounded open set in $\mathbb{R}^n$ and let $\Phi \in C^1(\overline{\Omega};\mathbb{R}^n)$ be such that*

$$\begin{cases} \det \nabla\Phi \geq 0 & \text{in } \Omega \\ \Phi = \mathrm{id} & \text{on } \partial\Omega. \end{cases} \tag{76}$$

Then

$$\mathrm{int}(\Phi(\Omega)) = \Omega. \tag{77}$$

Moreover, the following statement

$$\mathrm{int}(Z_\Phi) = \emptyset, \tag{78}$$

implies

$$\Phi(\overline{\Omega}) = \overline{\Omega}. \tag{79}$$

Finally, if (78) does not hold, then there exists one $\Phi \in C^1(\overline{\Omega};\mathbb{R}^n)$ such that $\Phi(\overline{\Omega}) \underset{\neq}{\supset} \overline{\Omega}$.

Proof. We divide the proof into three steps.

Step 1. We already know that $\Phi(\Omega) \supset \Omega$ and thus

$$\mathrm{int}(\Phi(\Omega)) \supset \Omega.$$

Let us show the reverse inclusion. We proceed by contradiction and assume that $\operatorname{int}(\Phi(\Omega)) \cap \Omega^c \neq \emptyset$. Therefore there exist y and ϵ such that

$$B_\epsilon(y) \subset \operatorname{int}(\Phi(\Omega)) \cap \left(\overline{\Omega}\right)^c \subset \Phi(\Omega) \cap \left(\overline{\Omega}\right)^c.$$

We also have, as in the proof of Theorem 29, that

$$\Phi(x) \in \Omega, \quad \text{if } x \notin Z_\Phi \cup \partial\Omega$$

which is equivalent to $\Phi\left((Z_\Phi \cup \partial\Omega)^c\right) \subset \Omega$. This implies

$$B_\epsilon(y) \subset \Phi(Z_\Phi)$$

which contradicts (72).

Step 2. Let us next show that (78) implies (79). If $x \in Z_\Phi \cap \Omega$, then there exists $x_\nu \notin Z_\Phi \cup \partial\Omega$ such that $x_\nu \to x$. Using Step 1, we also have $\Phi\left((Z_\Phi \cup \partial\Omega)^c\right) \subset \Omega$ and hence $\Phi(x_\nu) \in \Omega$, which leads to $\Phi(x) \in \overline{\Omega}$; and thus $\Phi(Z_\Phi) \subset \overline{\Omega}$. Hence we have shown that $\Phi(\overline{\Omega}) \subset \overline{\Omega}$. Since the reverse inclusion $\Phi(\overline{\Omega}) \supset \overline{\Omega}$ is always true, we have (77).

Step 3. We show that (79) may fail if (78) does not hold. Set $\Omega = B(0,1)$ and $n = 2$, consider

$$\Phi(x_1, x_2) := \rho(x_1^2 + x_2^2)(x_1, x_2) + \eta(x_1^2 + x_2^2)(x_1, 0)$$

where

$$\begin{cases} \rho, \eta \in C^\infty([0,1]; \mathbb{R}_+) \\ \operatorname{supp}\rho \subset (1/2, 1], \quad \operatorname{supp}\eta \subset (0, 1/2) \\ \rho' \geq 0 \text{ in } [0,1], \quad \rho \equiv 1 \text{ in } [3/4, 1] \\ \eta(1/4) = 4. \end{cases}$$

Let us verify the hypotheses of the proposition. Obviously, $\Phi \in C^1(\overline{\Omega}; \mathbb{R}^n)$ and $\operatorname{supp}(\Phi - \mathrm{id}) \subset B_1$. Let us now check that $\det \nabla\Phi \geq 0$. We separately consider two cases.

Case 1 $(1/2 \leq |x|^2 \leq 1)$. A straightforward computation implies that

$$\begin{aligned} \det \nabla\Phi(x) &= (2x_1^2\rho' + \rho)(2x_2^2\rho' + \rho) - 4x_1^2x_2^2\rho'^2 \\ &= 4x_1^2x_2^2\rho'^2 + 2|x|^2\rho\rho' + \rho^2 - 4x_1^2x_2^2\rho'^2 \\ &= 2|x|^2\rho\rho' + \rho^2 \geq 0. \end{aligned}$$

Case 2 $(0 \leq |x|^2 \leq 1/2)$. By definition of Φ it immediately follows that $\det \nabla\Phi = 0$. Thus, $\det \nabla\Phi \geq 0$.

We have the claim, since

$$\Phi(1/2, 0) = \eta(1/4)(1/2, 0) = (2, 0) \notin \overline{B_1}.$$

This concludes the proof of the proposition. ■

Acknowledgments. We have benefitted of several discussions with S. Bandyopadhyay. Long ago the second author has discussed the problem considered in this paper with I. Fonseca and L. Tartar, but with different arguments. Finally, we would like to thank D.Ye for several useful comments and for pointing out a mistake in an earlier version. The present article was essentially completed while the first author was visiting EPFL.

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