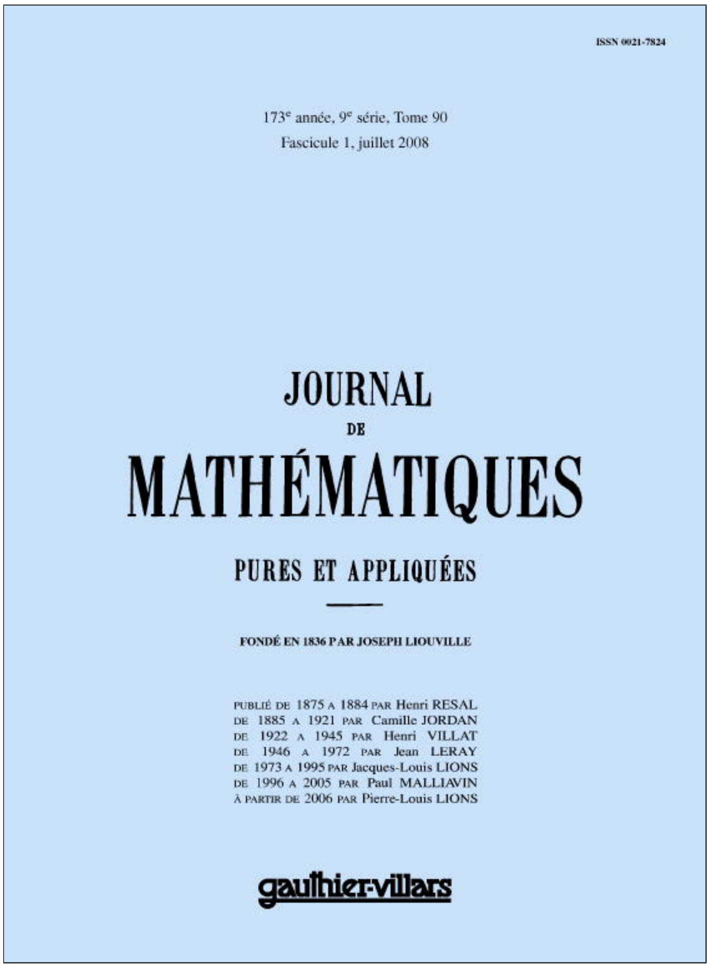
ISSN 0021-7824
173e année, 9e série, Tome 90
Fascicule 1, juillet 2008
JOURNAL
DE
MATHÉMATIQUES
PURES ET APPLIQUÉES
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J. Math. Pures Appl. 90 (2008) 66–81

JOURNAL DE MATHÉMATIQUES PURES ET APPLIQUÉES

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Lipschitz-continuous local isometric immersions: rigid maps and origami

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Received 1 January 2008

Available online 29 February 2008

Abstract

A *rigid map* $u:\Omega\subset\mathbb{R}^n\to\mathbb{R}^m$ is a *Lipschitz-continuous map* with the property that at every $x\in\Omega$ where u is differentiable then its gradient $Du(x)$ is an *orthogonal* $m\times n$ *matrix*. If Ω is convex, then u is globally a *short map*, in the sense that $|u(x)-u(y)|\leqslant|x-y|$ for every $x,y\in\Omega$; while locally, around any point of continuity of the gradient, u is an *isometry*. Our motivation to introduce Lipschitz-continuous local isometric immersions (versus maps of class C^1) is based on the possibility of solving Dirichlet problems; i.e., we can impose boundary conditions. We also propose an approach to the analytical theory of origami, the ancient Japanese art of paper folding. An *origami* is a piecewise C^1 rigid map $u:\Omega\subset\mathbb{R}^2\to\mathbb{R}^3$ (plus a condition which exclude self intersections). If $u(\Omega)\subset\mathbb{R}^2$ we say that u is a *flat origami*. In this case (and in general when $m=n$) we are able to describe the singular set Σ_u of the gradient Du of a piecewise C^1 rigid map: it turns out to be the boundary of the union of convex disjoint polyhedra, and some facet and edge conditions (*Kawasaki condition*) are satisfied. We show that these necessary conditions are also sufficient to recover a given singular set; i.e., we prove that every polyhedral set Σ which satisfies the Kawasaki condition is in fact the *singular set* Σ_u of a map u, which is uniquely determined once we fix the value $u(x_0)\in\mathbb{R}^n$ and the gradient $Du(x_0)\in O(n)$ at a single point $x_0\in\Omega\setminus\Sigma$. We use this characterization to solve a class of *Dirichlet problems* associated to some *partial differential systems* of *implicit type*.

Résumé

Une application $u:\Omega\subset\mathbb{R}^n\to\mathbb{R}^m$ est dite *rigide* si elle est *lipschitzienne* et si pour tout $x\in\Omega$ où u est différentiable, son gradient $Du(x)$ est une *matrice* $m\times n$ *orthogonale*. Une telle application u est une *contraction* dans Ω, c'est-à-dire que $|u(x)-u(y)|\leqslant|x-y|$ pour tous $x,y\in\Omega$; alors que localement, au voisinage d'un point où le gradient est continu, u est une *isométrie*. La nécessité de considérer des immersions localement isométriques lipschitziennes (au lieu d'applications de classe C^1) vient du fait que nous voulons résoudre un problème de Dirichlet. Cette formulation permet, de plus, une approche analytique de la théorie des origamis, l'ancien art japonais du pliage d'une feuille de papier. Un *origami* est alors vu comme une application rigide et C^1 par morceaux $u:\Omega\subset\mathbb{R}^2\to\mathbb{R}^3$ (auquel il faut ajouter une condition d'injectivité). Si $u(\Omega)\subset\mathbb{R}^2$, nous dirons que u est un *origami plat*. Dans ce cas (ou plus généralement quand $m=n$) nous montrerons que l'ensemble Σ_u, où le gradient Du de l'application rigide C^1 par morceaux est discontinu, est le bord d'une union de polyèdres convexes disjoints pour lesquelles les faces et arrêtes satisfont une certaine condition appelée *condition de Kawasaki*. Nous montrons, par ailleurs, que cette condition nécessaire est aussi suffisante pour recouvrer une application rigide, étant donné un ensemble singulier ; plus précisément on démontre que

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doi:10.1016/j.matpur.2008.02.011

tout ensemble polyédral Σ qui satisfait la condition de Kawasaki est en fait l'*ensemble singulier* Σ_u d'une application rigide u, qui est déterminée uniquement une fois fixés la valeur de l'application $u(x_0) \in \mathbb{R}^n$ et de son gradient $Du(x_0) \in O(n)$ en un seul point $x_0 \in \Omega \setminus \Sigma$. Nous utilisons cette caractérisation pour résoudre certains *problèmes de Dirichlet* associés à des *systèmes d'équations aux dérivées partielles de type implicite*.

Keywords: Lipschitz isometric immersions; Origami; Dirichlet problem

1. Introduction

J. Nash [23] in 1954 introduced the study of isometric imbeddings of class C^1; his result was improved by N.H. Kuiper [20]. They proved that every abstract n-dimensional manifold can be imbedded in $\mathbb{R}^m$ for $m \geqslant n+1$. An important reference is [15].

We briefly recall some well known and simple facts that we use below: (i) if $u : \mathbb{R}^n \to \mathbb{R}^m$ is a C^1-isometric immersion, then for every $x \in \mathbb{R}^n$ its gradient $Du(x)$ is an *orthogonal* $m \times n$ matrix, i.e., $Du^t Du = \mathrm{I}$ (here Du^t denotes the *transpose* matrix of Du, while I is the *identity matrix*). For $x \in \mathbb{R}^n$ we write $Du(x) \in O(n,m)$ ($O(n)$ if $m=n$). (ii) If $m<n$ then $Du^t Du = \mathrm{I}$ is not possible (there are not $m+1$ independent vectors in $\mathbb{R}^m$). Therefore we consider isometric maps $u : \mathbb{R}^n \to \mathbb{R}^m$ only when $m \geqslant n$. (iii) If $m=n$, then any $C^1(\mathbb{R}^n)$-isometric map u is *affine*, i.e., it can be represented under the form $u(x) = Ax + b$ for some matrix $A \in O(n) \subset \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$ and for every $x \in \mathbb{R}^n$.

Although we also consider in Section 3 the *strict* immersion from $\mathbb{R}^n$ to $\mathbb{R}^m$ with $m > n$, which is the most treated case in the mathematical literature, we mainly study in this paper the limiting case $m=n$. However, because of property (iii) above, we need the extension of the concept of C^1*-isometric* maps to *Lipschitz-continuous isometric immersions*. We explain here briefly the reasons.

Let $m=n$. When associated with a *boundary condition* posed on the boundary $\partial\Omega$ of a bounded open set $\Omega \subset \mathbb{R}^n$, then the request that u is a map of class C^1 is too strict. In fact, the *Dirichlet problem*:

$$\begin{cases} \text{find } u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n, \ u \text{ isometric map}, \\ \text{such that } u(x) = 0 \text{ for every } x \in \partial\Omega, \end{cases} \tag{1}$$

lacks a solution in the class of maps $u \in C^1(\Omega; \mathbb{R}^n)$.

On the contrary, if we look for *isometric immersions among Lipschitz-continuous maps*, then it is possible to get existence of solutions. A more convenient formulation of the *Dirichlet problem* to be considered in this more general framework is:

$$\begin{cases} \text{find } u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n \text{ Lipschitz-continuous} \\ \text{such that its gradient } Du(x) \text{ is orthogonal for almost every } x \in \Omega, \\ \text{and } u(x) = \varphi(x) \text{ for every } x \in \partial\Omega. \end{cases} \tag{2}$$

For the sake of illustration, the Dirichlet problem (2) for $n=1$, when $\Omega = (-1,1)$ and $\varphi = 0$ has solution, for instance, given by $u(x) = 1 - |x|$. A generalization of this simple example gives rise to the *Eikonal equation* $|Du| = 1$ for maps $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ (i.e., $m=1$) and the corresponding Dirichlet problem $|Du| = 1$ in Ω, $u = \varphi$ on $\partial\Omega$, can be solved (at least when the set Ω is convex and when the boundary datum φ satisfies a proper compatibility condition) with the theory of *viscosity solutions* (see, for instance, Crandall and Lions [9], Crandall, Ishii, and Lions [8]).

The study of the differential problem (2) is more recent. In fact, if $n>1$ the viscosity method does not apply, essentially due to the lack of *maximum principle* for *systems* of PDEs. For existence results in this vector-valued context we refer to the article [11] and the monograph [12] by Dacorogna and Marcellini, by mean of the *Baire category method*: finding *almost everywhere solutions* of differential systems of *implicit type*. We also refer to *convex integration* by Gromov [15] as in Müller and Šverák [22]. These methods are not constructive, i.e., they give existence of solutions but they do not give a way to compute them.

A differential problem of the type of (2) has been considered by Cellina and Perrotta [5], who studied a 3×3 *system* of PDEs of *implicit type* and proposed an explicit solution for the associated Dirichlet problem. Recently Dacorogna, Marcellini, and Paolini gave a contribution in [13], which can be considered a starting approach to the work presented here. See also [17].

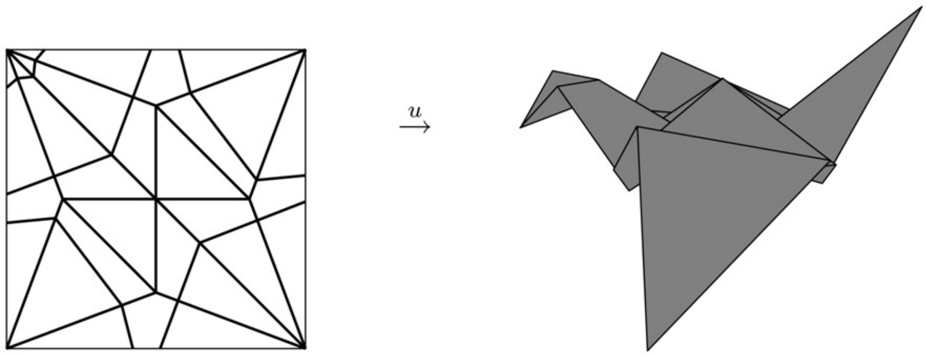

Fig. 1. On the right: the crane is the most famous origami. On the left: the corresponding singular set.

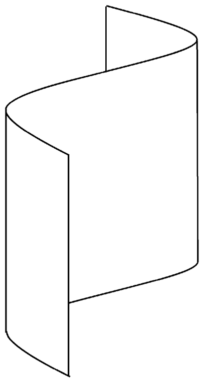

Fig. 2. This sheet of paper is bended but not folded. The corresponding singular set is empty.

In this paper we consider *Lipschitz-continuous maps* $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ whose differential (gradient) is almost everywhere an *orthogonal matrix*. Then, fixed $x \in \Omega$ where the map u is differentiable, the gradient $A = Du(x)$, being a $m \times n$ orthogonal matrix, represents a linear isometric immersion $A : \mathbb{R}^n \to \mathbb{R}^m$ for $n \leqslant m$. In correspondence the map u is a *Lipschitz-continuous isometric immersion*. We briefly call such maps *rigid maps*.

Therefore we say that a map $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ is *rigid* if u is Lipschitz-continuous in Ω and its gradient Du is orthogonal at almost every $x \in \Omega$; i.e., $Du^t Du = I$ (Definition 2.1). Such maps preserve the inner product; hence they preserve the length of curves and the geodesic distance. In particular they are *globally short*, in the sense that $|u(x) - u(y)| \leqslant |x - y|$ for every $x, y \in \Omega$, if Ω is convex (Proposition 3.4).

Rigid maps are widely studied in *plate theory*, since such maps represent a deformation of a thin material which has no elasticity but can be bended. A very common example of such a material is a sheet of paper. It can be bended, folded, or crumpled but cannot be compressed or stretched (see [6,7,19]). In particular isometric immersions are a good model for *origami*, the ancient Japanese art of paper folding. One of the aims of this paper is to propose a mathematical framework to treat origami.

As a matter of fact we can define an origami to be an injective *rigid* map $u : \mathbb{R}^2 \to \mathbb{R}^3$ which has the sheet of paper as domain $\Omega \subset \mathbb{R}^2$ and the 3-space as co-domain. With this example in mind, the singular set Σ_u of the points where the map u is not differentiable corresponds to the *crease pattern* in origami terminology. If we unfold the origami we see the crease pattern impressed in the sheet of paper.

Clearly the singular set Σ_u is *uniquely determined* by the map u. In the case of strict immersions (i.e., $m > n$) many rigid maps u can have the same singular set. For example, the singular sets shown in Figs. 1 and 2, correspond to many different rigid maps.

On the contrary we will see that, if $m = n$, then there is a great deal of rigidity in the reconstruction of u from Σ_u.

In fact, among others, a main result presented in this paper is the *Recovery theorem* (Theorem 4.9), where we show the possibility to uniquely (up to a rigid motion) reconstruct a rigid map from a given set of singularities; i.e., from a given singular set. A fundamental ingredient in this reconstruction is a necessary and sufficient compatibility condition on the geometry of the singular set, which we describe here in this introduction, just for the sake of exposition, in the two-dimensional case, but which holds, and we consider it below in this paper, in the general n-dimensional case. Following the terminology that can be found in the not numerous mathematical literature on origami (see for instance [3,16]), we call it *Kawasaki condition*.

Let $n = m = 2$ and let $\Sigma \subset \Omega$ be the union of a (locally) finite number of arcs (called edges) which meet in a (locally) finite number of points (called vertices). We will prove (Theorems 4.8 and 4.9) that Σ is the *singular set* of a *piecewise C^1 rigid map* (cf. Section 2) if and only if its edges are straight segments and the following *Kawasaki condition* holds at every vertex V of Σ: let $\alpha_1, \dots, \alpha_N$ be the amplitude of the consecutive angles determined by the N edges of Σ meeting in the vertex V; then N is even and

$$\alpha_1 + \alpha_3 + \cdots + \alpha_{N-1} = \alpha_2 + \alpha_4 + \cdots + \alpha_N = \pi.$$

In the general n-dimensional case we prove that every $(n-1)$-dimensional polyhedral set Σ which satisfies the Kawasaki condition is the singular set Σ_u of some rigid map u; moreover the map u is uniquely determined once we fix the value $u(x_0) \in \mathbb{R}^n$ and the differential $Du(x_0) \in O(n)$ at a single point $x_0 \in \Omega \setminus \Sigma$.

Going back to the *Dirichlet problems* (1) and (2), in Sections 5, 6 we will use Recovery Theorem 4.9 to find rigid maps with prescribed linear boundary conditions, respectively, in two and three dimensions. In particular for $n = 2$ we consider any linear, contraction map φ; as an extension to the result presented in [13], we will be able to find a rectangle $\Omega \subset \mathbb{R}^2$ and a rigid map $u : \bar{\Omega} \to \mathbb{R}^2$ such that $u = \varphi$ on $\partial\Omega$.

2. Rigid maps, origami and flat origami

In this section we present the definition of *rigid map* which is considered throughout the paper. As a byproduct we give a definition of *origami* and *flat origami* to show how it is possible to give an *analytical* definition of such a geometrical object. Some references on the usual *geometrical* approach to origami are [1–3,16,18,21].

Definition 2.1 *(Rigid map).* Let $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$. We say that u is a *rigid map* if u is Lipschitz-continuous, and $Du(x) \in O(n,m)$ (Du orthogonal, i.e. $Du^t Du = \mathrm{I}$) for a.e. $x \in \Omega$. We call *singular set* of the rigid map u the set of points $\Sigma_u \subset \Omega$ where u is not differentiable.

Definition 2.2 *(Piecewise C^1 rigid map).* We say that a rigid map u is *piecewise C^1*, if in addition the following conditions hold:

(i) Σ_u is closed in Ω;
(ii) u is C^1 on every connected component of $\Omega \setminus \Sigma_u$;
(iii) for every compact set $K \subset \Omega$ the number of connected components of $\Omega \setminus \Sigma_u$ which intersect K is finite.

Rigid maps can be used to define what we will call *origami*. In Fig. 1 is represented one of the most known origami (the *crane*) together with its singular set Σ_u. Fig. 2 represents a non-trivial rigid map (with $m = 2$, $n = 3$) which is C^1 (hence the singular set is empty).

To get a realistic physical model of origami we need to exclude self intersections. To be precise overlappings are allowed in the map but only if the configuration is *reachable* by means of non-intersecting (injective) maps. For example, the map $u(x, y) = (|x|, y, 0)$ is not injective but can be obtained as the limit as $t \to 0$ of the injective maps $u_t(x, y) = (|x| \cos t, y, x \sin t)$ which represent the actual folding process along time. On the other hand, the rigid map presented in Fig. 3, cannot be approximated by injective maps (see [3]).

Definition 2.3 *(Origami).* Let $\Omega \subset \mathbb{R}^2$. We say that $u : \Omega \to \mathbb{R}^3$ is an *origami* if u is a piecewise C^1 rigid map and there exists a sequence of maps $u_k : \Omega \to \mathbb{R}^3$ which are Lipschitz continuous and injective and such that $u_k \to u$ in the uniform convergence.

Definition 2.4 *(Flat origami).* We say that $u : \Omega \to \mathbb{R}^3$ is a *flat origami* if it is an origami and $u(\Omega)$ is contained in a plane. That is, up to an isometry, u can be represented as a map $\Omega \to \mathbb{R}^2$.

If u is a (flat) origami, it is possible to discriminate between *mountain folds* and *valley folds* in its singular set. The singular set, equipped with the information about mountain/valley folds is usually called *crease pattern*. We note that

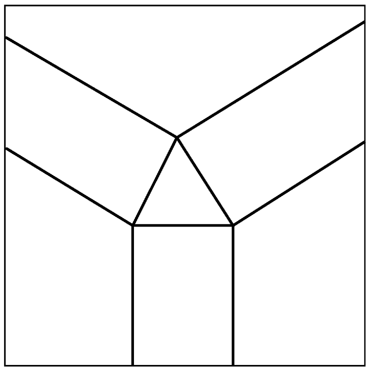

Fig. 3. A singular set which correspond to a rigid map which is not an origami. This gives rise to self-intersections when trying to actually fold with paper. This is "mathematical origami" but not a physically realizable origami.

the above definition could be extended to also take into account the difference between mountain and valley folds. In fact one could distinguish two different origami corresponding to the same rigid map by means of the approximating sequence.

To some degree the crease pattern can be used to reconstruct a flat origami. However there is no simple condition on the singular set to guarantee the existence of a corresponding flat origami.

We will see, in the sequel, that the correspondence between singular sets and piecewise C^1 rigid maps is instead very tight.

As we said before, the interpenetration problem arising in the definition of origami is only marginally described in this paper. Our approach is to consider a rigid map as a "mathematical origami". For instance, we solve the Dirichlet differential problem (6) by means of rigid maps. However the solutions represented in Fig. 7 are, in fact, "true" origami (we are able to fold the corresponding paper).

To our knowledge origami are mainly studied in two areas: algebraic and combinatorial.

In the algebraic setting the paper folding is used to construct algebraic numbers. Some elementary origami rules (Huzita–Hatori axioms, see [1]) are identified and used to construct a crease pattern which, in this case, is the union of straight lines. With this respect it is found that origami constructions are more powerful than constructions with rule and compass. In this setting there is no distinction between origami and rigid maps, since only the properties of the singular set are studied, without requiring the actual origami to be folded.

In the geometrical setting the compenetration problem is taken into account. It is shown that the Kawasaki condition is not enough to reconstruct an origami. Also more involved conditions are considered, which take into account also the mountain/valley distinction on the crease pattern. Anyway it is proved that the problem of deciding if a singular set is the crease pattern of an origami is hard (see [3]). Other mathematical papers study geometrical methods and algorithms to develop more and more complex and realistic origami models, as in [21].

3. Properties of rigid maps

It might be interesting to briefly inspect the definition of rigid maps in the general case $m \geqslant n$ before restricting our study to the case $m = n$. In the case $m > n$, the map is much less rigid, the gradient can vary smoothly. For example, given the arc-length parameterization $\gamma : \mathbb{R} \to \mathbb{R}^2$ of any curve in $\mathbb{R}^2$, the map $u(x, y) = (\gamma(x), y) \in \mathbb{R}^3$ is a rigid map whose image is the cylinder projecting on the curve γ. The corresponding singular set is empty (see Fig. 2). In Fig. 4 we have depicted another example.

However we have some rigidity also in this case. For example, it is not possible to obtain a spherical surface out of a sheet of paper: the Gauss curvature is always zero because the surface maintains the flat nature of the domain $\Omega \subset \mathbb{R}^n$. This is a consequence of the following result:

Lemma 3.1. *Let Ω be an open subset of $\mathbb{R}^n$. Suppose $u \in C^1(\Omega, \mathbb{R}^m)$, is an injective rigid map. Then $u(\Omega) \subset \mathbb{R}^m$ endowed by the geodesic distance induced by $\mathbb{R}^m$ is an n-dimensional Riemann surface and $u : \Omega \to u(\Omega)$ is an isometry.*

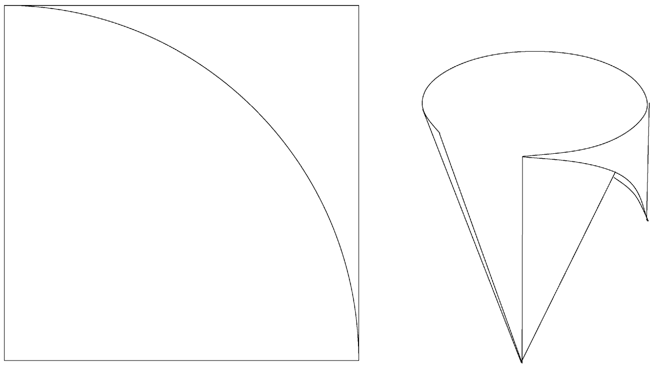

Fig. 4. A rigid map in the case $m = 2$, $n = 3$. The singular set is a curved line, and the image of the map is the union of two pieces of cones.

Proof. Since $Du(x_0)$ is orthogonal we know that the rank of $Du(x_0)$ is n. Hence, by the local invertibility theorem, the inverse map $u^{-1} : u(\Omega) \to \Omega$ is C^1 and hence u is a diffeomorphism. We also notice that Du being orthogonal we have

$$\langle Du(x)v, Du(x)w\rangle = \langle Du(x)^t Du(x)v, w\rangle = \langle v, w\rangle,$$

i.e., u preserves the Riemann structure and hence is an isometry between Riemann surfaces. $\square$

C^1-rigid maps are isometric immersions. The Nash–Kuiper [20,23] C^1-imbedding theorem asserts, in particular, that the map 0 can be uniformly approximated by such maps. In the present work, however, we are mostly interested in the case $m = n$ which is trivial for C^1-maps. Also we are interested in approximating a given map by means of a rigid map, but with precise Dirichlet conditions.

We recall some classical results on (global) isometric maps.

Theorem 3.2 *(Liouville). Let Ω be an open, connected set in $\mathbb{R}^n$, $u \in C^1(\Omega, R^n)$ and $Du \in O(n)$. Then u is affine.*

Theorem 3.3 *(Cartan–Dieudonné). Let $\Omega \subset \mathbb{R}^n$ be an open connected set and $u : \Omega \to \mathbb{R}^m$ be an isometry, i.e.,*

$$\big|u(x) - u(y)\big| = |x - y|, \quad \forall x, y \in \Omega.$$

Then $m \geqslant n$, u is affine, $Du \in O(m, n)$. Hence u is an affine rigid map. Also, u can be written as the composition of at most $n + 1$ affine symmetries.

Proposition 3.4 *(Shortness). Let u be a rigid map defined on a convex set Ω. Then u is short, that is, $|u(x) - u(y)| \leqslant |x - y|$ for every $x, y \in \Omega$, being also possible that $u(x) = u(y)$ for some $x \neq y$.*

Proof. Given any $x, y \in \Omega$, for every $\varepsilon > 0$ it is possible to find a Lipschitz curve $\gamma : [0, 1] \to \Omega$ such that $\gamma(0) = x$, $\gamma(1) = y$, $\ell(\gamma) = \int_0^1 |\gamma'| \leqslant |x - y| + \varepsilon$ and such that u is differentiable in the points $\gamma(t)$ for a.e. $t \in [0, 1]$. So, recalling that Du is an orthogonal matrix, we have:

$$\big|u(x) - u(y)\big| = \big|u\big(\gamma(1)\big) - u\big(\gamma(0)\big)\big| \leqslant \int_0^1 \left|\frac{d}{dt} u\big(\gamma(t)\big)\right| dt = \int_0^1 \big|Du\big(\gamma(t)\big)\gamma'(t)\big|\, dt$$

$$= \int_0^1 \big|\gamma'(t)\big|\, dt \leqslant |x - y| + \varepsilon.$$

We let $\varepsilon \to 0$ to conclude the proof. $\square$

4. Structure of the singular set in the case: $m = n$

We start with the study of the singular set $\Sigma = \Sigma_u$. We will see that there is a lot of rigidity on this set, when $m = n$. In the following we consider a piecewise C^1 rigid map $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ and let $\Sigma = \Sigma_u$ be its singular set.

Lemma 4.1 *(Facet rigidity). Let $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ be a piecewise C^1 rigid map. Then Du is constant* (*or equivalently u is affine*) *on every connected component of $\Omega \setminus \Sigma$.*

Proof. By restricting the map to a connected component of $\Omega \setminus \Sigma$ we might reduce ourselves to the case when Ω is connected and Σ_u is empty. The result then follows at once by Theorem 3.2. □

Definition 4.2 *(Polyhedral set).* We say that a set F in $\mathbb{R}^n$ is a k-dimensional *convex polyhedral facet* if F is bounded, nonempty, closed and there exists an affine k-dimensional plane Π and a finite number $H_1, \dots, H_N$ of open affine half-spaces such that

$$F = \overline{\Pi \cap H_1 \cap \cdots \cap H_N}.$$

The plane Π is the *supporting plane* of F.

We say that a set Σ in $\mathbb{R}^n$ is a k-dimensional *polyhedral set* if Σ is the union of a finite number of k-dimensional convex polyhedral facets.

We say that a set Σ is a *locally finite k-dimensional polyhedral set* in Ω if given any point $x \in \Sigma$ there exists a neighborhood U of x in Ω, such that $\Sigma \cap \overline{U}$ is a *polyhedral set*.

Lemma 4.3 *(Polyhedron condition). Suppose u is a piecewise C^1 rigid map defined on a open set Ω. Then Σ_u is a locally finite $(n-1)$-dimensional polyhedral set. Moreover, if Ω is convex then every connected component of $\Omega \setminus \Sigma$ is a convex set.*

Proof. We first prove that if Ω is convex, then every connected component is convex.

Consider a connected component A of $\Omega \setminus \Sigma$. On A the map u can be written as $u(x) = Jx + q$ for some $J \in O(n)$ and $q \in \mathbb{R}^n$. Take any two points $x_1, x_2 \in A$ and $t \in [0,1]$. Consider the point $x = tx_1 + (1-t)x_2$. Then, since J is orthogonal and since u is short,

$$\begin{aligned} |x_1 - x_2| &= \bigl|u(x_1) - u(x_2)\bigr| \leqslant \bigl|u(x_1) - u(x)\bigr| + \bigl|u(x) - u(x_2)\bigr| \\ &\leqslant |x - x_1| + |x - x_2| = |x_1 - x_2|. \end{aligned}$$

So all inequalities are equalities and also

$$\bigl|u(x) - u(x_1)\bigr| = |x - x_1|, \qquad \bigl|u(x) - u(x_2)\bigr| = |x - x_2|.$$

This means that $u(x) = Jx + q$. This is true for every x in the convex hull of A and hence u is differentiable on every point of the convex hull of A. Hence we conclude that A is convex because the singular set is outside its convex hull.

Now we suppose for a while that Ω is an open cube and we also suppose that $\Omega \setminus \Sigma$ has only a finite number of connected components. We claim that in this case $\overline{\Sigma}$ is a polyhedral set (notice that Σ is closed in Ω, but with $\overline{\Sigma}$ we denote the closure of Σ in $\mathbb{R}^n$). Since Ω is convex we know that each connected component is convex. Let us fix a connected component A of $\Omega \setminus \Sigma$. Given any other connected component A', since A and A' are convex, we know that there exists an affine plane Π which separates A from A'. We let $\Pi_1, \dots, \Pi_N$ be the planes which separate A from all other connected components and we let $\Pi_{N+1}, \dots, \Pi_M$ be all the planes containing the facets of the cube Ω (actually $M = N + 2n$). Then we let $H_1, \dots, H_M$ be the half-spaces such that $\partial H_i = \Pi_i$ and such that $H_i \supset A$ (this is always possible since $A \cap \Pi_i = \emptyset$ and A is connected). By definition the set $K = \bigcap_i H_i$ is an n-dimensional polyhedral set and its boundary ∂K is an $(n-1)$-dimensional polyhedral set. We claim that $K = \overline{A}$. By definition of Π_i and H_i we know that $A \subset K$ and hence also $\overline{A} \subset K$. On the other hand, let $x \in \mathbb{R}^n \setminus A$ be any point. If $x \in \mathbb{R}^n \setminus \Omega$ clearly $x \in \mathbb{R}^n \setminus \overline{K}$ because $K \subset \overline{\Omega}$. If else $x \in \Omega \setminus A$ then there exists a connected component A' different from A such that $x \in \overline{A}$ because Σ has nonempty interior. Since for some i we have $A' \subset \overline{\mathbb{R}^n \setminus H_i}$ we conclude that $x \in A' \subset \overline{\mathbb{R}^n \setminus K}$ and hence $\mathbb{R}^n \setminus A \subset \overline{\mathbb{R}^n \setminus K}$. Together with $A \subset K$ this gives $\overline{A} = K$.

So we have proved that if Ω is a cube then every connected component of $\Omega \setminus \Sigma$ is a n-dimensional polyhedral set. Hence the boundary ∂A is a $(n-1)$-dimensional polyhedral set and also $\overline{\Sigma}$ is, since $\Sigma \cup \partial\Omega$ is the union of all the boundaries of the connected components of $\Omega \setminus \Sigma$ and $\partial\Omega$ is a polyhedral set itself.

In the general case, when Ω is any open set, and u is any piecewise C^1-rigid map, we take any point $x \in \Sigma$ and consider a cubic neighborhood U of x which intersect a finite number of connected components of $\Omega \setminus \Sigma$. We know that $\overline{\Sigma \cap U} = \Sigma \cap \overline{U}$ is polyhedral and hence Σ itself is a locally finite polyhedral set. □

Definition 4.4 *(The integer n_Σ).* Let Σ be a closed set in Ω. Given a point $x \in \Omega$ we define $n_\Sigma(x)$ as the number of connected components of $\Omega \setminus \Sigma$ which include x in their closure.

By the definition of piecewise C^1 we are assuming that $n_\Sigma(x)$ is always finite. Clearly $n_\Sigma(x)$ is also positive.

Lemma 4.5 *(Facet condition). One has $n_\Sigma(x) = 1$ if and only if $x \in \Omega \setminus \Sigma$.*

Proof. Clearly if $x \in \Omega \setminus \Sigma$ then $n_\Sigma(x) = 1$ because every connected component of $\Omega \setminus \Sigma$ is open (recall that Σ is closed by hypothesis).

Consider now a point x with $n_\Sigma(x) = 1$. This means that there exists a neighborhood U of x such that $U \setminus \Sigma$ is contained in a single connected component of $\Omega \setminus \Sigma$. Hence, by Lemma 4.1, u is affine on $U \setminus \Sigma$. By definition u is Lipschitz on U and hence $U \setminus \Sigma$ is dense in U and being u continuous on the whole U it turns out that u is affine on U. Hence u is differentiable on U and $\Sigma \cap U = \emptyset$. $\square$

In the next lemma we consider a point which lies in the intersection of exactly two components. We prove that such intersection is indeed planar, without the assumption on the convexity of Ω. We also notice that once the map u is assigned on a connected component of $\Omega \setminus \Sigma$, its value is consequently assigned on the neighboring components.

Lemma 4.6 *(Edge condition). If $n_\Sigma(x_0) = 2$ then there exists a connected neighborhood U of x_0 such that the set $\Sigma \cap U = \Pi \cap U$ where Π is a $(n-1)$-dimensional plane $\Pi \ni x_0$. The map u is affine on the two components U_1 and U_2 of $U \setminus \Pi$ and if we let L_1 and L_2 be the two affine maps defining u in the two regions we have,*

$$L_1 = L_2 S, \qquad L_2 = L_1 S,$$

where S is the affine symmetry with respect to the plane Π. If J_i is the linear part of L_i (hence J_i is the gradient Du on the region U_i) we have,

$$J_1 = J_2 S', \qquad J_2 = J_1 S',$$

where S' is the linear part of S. In particular, $\det J_1 = -\det J_2$. *Notice also that $J_2 - J_1 = J_2(I - S')$ has rank one since $I - S' = 2v \otimes v$ where v is an orthonormal vector to Π.*

Proof. Let U be a connected neighborhood of x_0 which meets only two components of $\Omega \setminus \Sigma$. Let U_1 and U_2 be the intersection of these two components with U and let J_1 and J_2 be the (constant) value assumed by $Du(x)$ on the respective component. Notice that $J_1 \neq J_2$ otherwise u (which is continuous) would be differentiable everywhere in U.

We claim that $\Sigma \cap U \subset \overline{U_1} \cap \overline{U_2}$. To prove the claim consider any point $x \in \Sigma \cap U$. By Rademacher Theorem we know that Σ has no interior, hence every neighborhood of x contains points of $U_1 \cup U_2$. If there were a neighborhood U' of x such that $U' \setminus \Sigma \subset U_1$ then we would notice that in U' our map u is almost everywhere equal to an affine map with gradient J_1. Being also continuous, we would find that u is differentiable everywhere in U' against the hypothesis $x \in \Sigma$. Hence every neighborhood of x contains points of both U_1 and U_2 and the claim is proven.

Since we know that $u(x) = L_i(x) = u(x_0) + J_i(x - x_0)$ on U_i for $i = 1, 2$, by the previous claim and the continuity of u we conclude that the two affine maps L_i coincide on $\Sigma \cap U$. Since $J_1 \neq J_2$ we conclude that Σ is contained in the $(n-1)$-dimensional plane $\Pi = x_0 + V$ with $V = \operatorname{Ker}(J_1 - J_2) = \{w \in \mathbb{R}^n\colon\ (J_1 - J_2)w = 0\}$. Moreover $\Sigma \cap U = \Pi \cap U$ because if a single point of $(x_0 + V) \cap U$ were not in Σ, then $U_1 \cup U_2$ would be connected.

Consider now the map $S' = J_2^{-1} J_1$. Since $J_1 v = J_2 v$ on V, we know that $S' = \mathrm{I}$ on V. Moreover S' is an orthogonal matrix too. So if we consider a unit vector v which is orthogonal to V, the image $S'v$ is again a normal vector orthogonal to V. We have only two possibilities: either $S'v = v$ or $S'v = -v$. In the first case we have $S' = \mathrm{I}$ and hence $J_1 = J_2$ which is not possible. So we conclude that $Sv = -v$, i.e., $S' = \mathrm{I} - 2v \otimes v$, S' is the symmetry with respect to V (and S is the symmetry with respect to $x_0 + V$). $\square$

Definition 4.7 *(Kawasaki condition).* Let P be $(n-2)$-dimensional facet of a polyhedral set Σ and let $E_1, \dots, E_N$ be the $(n-1)$-dimensional facets of Σ which meet in P, ordered consecutively around P. Let $\alpha_1, \dots, \alpha_N$ be the angles determined by the facets E_i in P. We say that the *Kawasaki condition* holds in P if N is even and

$$\alpha_1 + \alpha_3 + \cdots + \alpha_{N-1} = \alpha_2 + \alpha_4 + \cdots + \alpha_N = \pi.$$

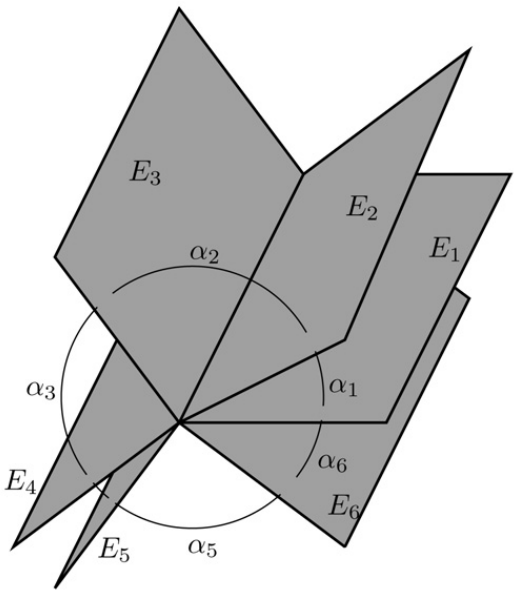

Fig. 5. The Kawasaki condition in $3D$.

Now we prove that Σ_u satisfies the Kawasaki condition. This property is known in the origami setting, for $n = 2$ [18].

Theorem 4.8 *(Necessary condition). Let u be a piecewise C^1 rigid map $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$. Let P be an $(n-2)$-dimensional facet of the corresponding polyhedral set $\Sigma = \Sigma_u$. Then the Kawasaki condition holds in P.*

Proof. Around the facet P we find a finite number of connected components of $\Omega \setminus \Sigma$. We enumerate them $A_1, \dots, A_N$ so that A_{i+1} is next to A_i. Let $L_1, \dots, L_N$ be respectively the affine maps defined by u in the corresponding regions. Then by Lemma 4.6 we know that $L_{i+1} = L_i S_i$ where S_i is the symmetry with respect to the plane containing $\overline{A_i} \cap \overline{A_{i+1}}$. By making a complete loop around the facet P we find the compatibility condition:

$$L_1 = L_1 S_1 S_2 \cdots S_{N-1} S_N.$$

Since every isometry S_i has negative determinant while $S_1 \cdots S_N = I$ has positive determinant, we conclude that N is even. Notice also that the composition of the two symmetries S_i and S_{i+1} is a rotation R_i of an angle $2\alpha_i$ around the facet P, where α_i is the angle determined by the planes of symmetry of S_i and S_{i+1}. Hence we have,

$$I = S_1 S_2 S_3 S_4 \cdots S_{N-1} S_N = R_1 R_3 \cdots R_{N-1},$$

which means that $2\alpha_1 + 2\alpha_3 + \cdots + 2\alpha_{N-1} = 2\pi$ and hence $\alpha_1 + \alpha_3 + \cdots + \alpha_{N-1} = \pi$. Since the sum $\alpha_1 + \alpha_2 + \cdots + \alpha_N = 2\pi$ we also have $\alpha_2 + \alpha_4 + \cdots + \alpha_N = \pi$. $\square$

Theorem 4.9 *(Recovery theorem). Let Ω be a simply connected open subset of $\mathbb{R}^n$. Let $\Sigma \subset \Omega$ be a locally finite polyhedral set satisfying the Kawasaki condition on every $(n-2)$-dimensional facet. Then there exists a rigid map u such that $\Sigma = \Sigma_u$ is the singular set of u. Moreover u is uniquely determined once we fix the value $y_0 = u(x_0)$ and the Jacobian $J_0 = Du(x_0)$ in a point $x_0 \in \Omega \setminus \Sigma$.*

Proof. We consider the class Γ of all continuous curves $\gamma : [0,1] \to \Omega$ with the following properties:

(1) $n_\Sigma(\gamma(t)) \leqslant 2$ for every $t \in [0,1]$;
(2) $\{t\colon n_\Sigma(\gamma(t)) = 2\}$ is finite and $n_\Sigma(\gamma(0)) = 1$, $n_\Sigma(\gamma(1)) = 1$;
(3) if $n_\Sigma(\gamma(t_0)) = 2$ for some $t_0 \in [0,1]$, then $\gamma(t)$ lies in different connected components of $\Omega \setminus \Sigma$ for $t < t_0$ and $t > t_0$ in a neighborhood of t_0 ($\gamma(t)$ crosses the edge).

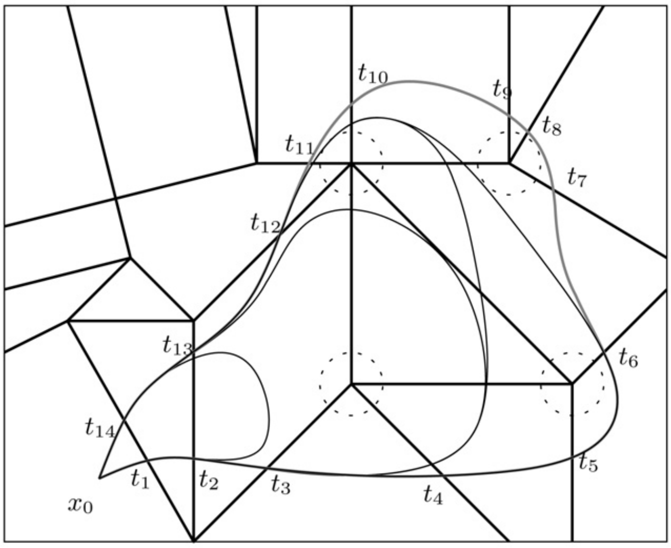

Fig. 6. The retraction of a closed path.

Given such a curve $\gamma \in \Gamma$ let $0 < t_1 < t_2 < \cdots < t_N < 1$ be the points where $n_\Sigma(\gamma(t)) = 2$, i.e., where the curve passes through an $(n-1)$-dimensional facet $F_j \ni \gamma(t_j)$ of the polyhedral set Σ. We then define S_j for $j = 1, \ldots, N$ to be the symmetry with respect to the plane containing F_j. Then we define $A_\gamma = S_1 S_2 \cdots S_{N-1} S_N$ the composition of all these isometries.

Notice that if a rigid map u exists with singular set Σ and if u coincides with the affine map L_0 in the component containing $\gamma(0)$, then necessarily (by Lemma 4.6) one has $u(\gamma(1)) = L_0 A_\gamma \gamma(1)$. We want to use this property to reconstruct u. To achieve this we want to prove that A_γ does depend only on the endpoints $\gamma(0)$ and $\gamma(1)$ but not on the path through these point. Equivalently it is enough to prove that $A_\gamma = I$ whenever γ is closed: $\gamma(1) = \gamma(0)$.

Clearly, if $\gamma \equiv x_0$ is constant then $A_\gamma = I$. In general, since Ω is simply connected, every closed curve $\gamma(t)$ can be retracted to the constant curve $\gamma_0(t) \equiv x_0$ by means of a continuous homotopy φ: $[0,1] \times [0,1]$ such that $\varphi(0,t) = x_0$, $\varphi(1,t) = \gamma(t)$, $\varphi(s,t) \in \Omega$ for all $s, t \in [0,1] \times [0,1]$. While we retract our curve γ, if the $(n-1)$-dimensional facets of Σ crossed by γ remain the same, by definition we have that A_γ does not vary. On the other hand, when the retraction makes γ cross an $(n-2)$-dimensional facet P of Σ, we notice that A_γ is multiplied by $S_1^P S_2^P \cdots S_N^P$ where the S_k^P are the symmetries with respect to the $(n-1)$-dimensional planes joining in the $(n-2)$-dimensional facet P. But the Kawasaki condition assures that this product is, actually, the identity map.

The retraction could, in principle, also cross an $(n-3)$-dimensional or lower-dimensional facets of Σ. In this case, however, we can tilt the retraction so that such a lower-dimensional facet is missed.

So we have proved that the isometry A_γ depends only on $\gamma(0)$ and $\gamma(1)$ and hence given $x \in \Omega \setminus \Sigma$ we can define $u(x) = L_0 A_\gamma x$ where γ is any admissible curve with end-points x_0 and x, L_0 is defined by $L_0 x = y_0 + J_0 x$ where y_0 and J_0 are given. We notice that $u(x)$ can be extended by continuity to the whole Ω. In fact on every $(n-1)$-dimensional facet of Σ the affine functions defining A differ by a symmetry which leaves fixed the $(n-1)$-dimensional plane. This is also true on the lower-dimensional facets of Σ which all live in the intersection of $(n-1)$-dimensional planes.

Hence $u(x): \Omega \to \mathbb{R}^n$ is a rigid map which has Σ as singular set and satisfies $Du(x_0) = J_0$, $u(x_0) = y_0$. Moreover, by construction, u is the unique rigid map with these properties. $\square$

5. The Dirichlet problem

A *Dirichlet problem* associated to a given Lipschitz continuous boundary datum $\varphi: \overline{\Omega} \subset \mathbb{R}^n \to \mathbb{R}^m$ and to a subset $E \subset \mathbb{R}^{m\times n}$ of $m \times n$ matrices can be formulated as follows: find a Lipschitz continuous map $u: \overline{\Omega} \subset \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\begin{cases} Du \in E & \text{a.e. in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases} \tag{3}$$

The boundary datum φ must satisfy a natural compatibility condition. In the simplest case—the scalar case $m = 1$—the compatibility condition on φ (see Theorem 2.10 in [12]; the existence result in this form is due to Dacorogna and Marcellini [10,11], cf. also Bressan and Flores [4] and De Blasi and Pianigiani [14]), requires that

$$D\varphi(x) \in E \cup \operatorname{int}\operatorname{co} E, \quad \text{a.e. in } \Omega,$$

where $\operatorname{int}\operatorname{co} E$ is the *interior* of the *convex hull* of the set E.

In the vector-valued case $m > 1$ we limit ourselves here to state the compatibility condition on φ only in the context of this paper. To this aim we consider the case $m = n \geqslant 2$ and φ affine map and we denote by $\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A)$, with $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$, the *singular values* of a matrix $A \in \mathbb{R}^{n\times n}$. We consider the Dirichlet problem (3) when the set E is given by:

$$E = \left\{A \in \mathbb{R}^{n\times n}\colon\ \lambda_i(A) = 1,\ i = 1, \ldots, n\right\} = O(n)$$

and we require the compatibility condition on the boundary value φ:

$$\lambda_n(D\varphi) < 1. \tag{4}$$

Then there exists a Lipschitz continuous map $u : \overline{\Omega} \subset \mathbb{R}^n \to \mathbb{R}^n$, i.e., $u \in W^{1,\infty}(\Omega; \mathbb{R}^n)$, such that

$$\begin{cases} Du \in E = O(n) & \text{a.e. in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases} \tag{5}$$

The result proved in [12] (see in particular Theorem 7.28 and Remark 7.29) guarantees existence but does not give a rule to build a solution. In [13] we recently proposed a method to compute a solution following some ideas (as described in the introduction) considered in a similar context by Cellina and Perrotta [5].

In this section we aim to extend the results of [13] by finding an explicit solution $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ to the system of implicit partial differential equations:

$$\begin{cases} Du \in O(2) & \text{a.e. in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{6}$$

where φ is an *affine map* and Ω is a well chosen rectangle (depending on φ). We emphasize that, as a by-product, we obtain existence of solutions in the class of *piecewise* C^1 rigid maps (more precisely in the class of *origami*) and not only in the wider class of *generic* Lipschitz continuous maps. We also observe that problem (6) cannot be solved by a piecewise C^1 map with finitely many pieces (unless φ is itself a solution).

Therefore, we consider the Dirichlet problem (6), where φ is an *affine map* with linear part $A = D\varphi \in \mathbb{R}^{2\times 2}$.

We can consider, without loss of generality, L to be diagonal with entries $\alpha, \beta \geqslant 0$,

$$A = \operatorname{diag}(\alpha, \beta) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}.$$

The case of a general affine contraction $\varphi(x) = Ax + b$, follows by the decomposition $A = RDQ$ with $R, Q \in O(2)$ and $D = \operatorname{diag}(\alpha, \beta)$ with $\alpha = \lambda_1(A)$ and $\beta = \lambda_2(A)$ the singular values of A (i.e., the square root of the eigenvalues of $A^t A$).

Notice that if both $\alpha = 1$ and $\beta = 1$, then $A \in O(2)$ and hence φ is itself a solution to (6). If $\alpha > 1$ or $\beta > 1$, then the system (6) has no solutions, because every solution has to be short while φ is not (this is also stated in (4)). On the other hand if $\alpha < 1$ or $\beta = 1$, the system does not have any solution as shown in Example 5.1.

Example 5.1. Consider the square domain $\Omega = (-1, 1) \times (-1, 1) \subset \mathbb{R}^2$ and the map $\varphi : \overline{\Omega} \to \mathbb{R}^2$,

$$\varphi(x, y) = (\alpha x, y)$$

with $\alpha \in [0, 1)$. The only 1-Lipschitz continuous map $u : \Omega \to \mathbb{R}^2$ which satisfies the boundary condition $u = \varphi$ on $\partial\Omega$ is φ itself. As a consequence, since $D\varphi$ is not orthogonal, there is no map u with boundary condition φ which has orthogonal gradient.

Indeed let u be a 1-Lipschitz continuous map with boundary condition φ. Fix $x \in (-1, 1)$. Notice that $|u(x, -1) - u(x, 1)| = |(\alpha x, -1) - (\alpha x, 1)| = 2$ is the maximum possible difference for a 1-Lipschitz map. Hence $u(x, \cdot)$ is linear, and hence $u(x, y) = \varphi(x, y)$.

We now define a rigid map which will be the base module to construct the solution of the Dirichlet problem.

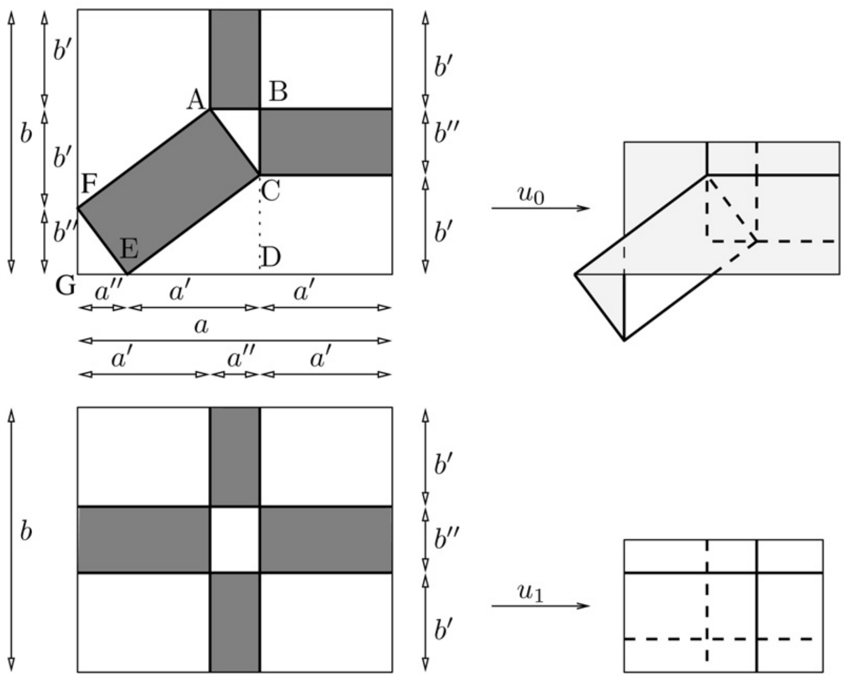

Fig. 7. The singular set of the base module used in Lemma 5.2.

Lemma 5.2 *(Base module). Let φ be the diagonal linear map $\varphi(x, y) = (\alpha x, \beta y)$ with $\alpha, \beta \in (0, 1)$. Let $a, b > 0$ satisfy the relation*:

$$\frac{b^2}{a^2} = \frac{1-\alpha^2}{1-\beta^2} \tag{7}$$

and consider the domain $R = [0, a] \times [0, b] \subset \mathbb{R}^2$. Define $a' = a(1+\alpha)/4$, $a'' = a(1-\alpha)/2$, $b' = b(1+\beta)/4$, $b'' = b(1-\beta)/2$ so that $a = 2a' + a''$, $b = 2b' + b''$. Then the two singular sets depicted in Fig. 7 satisfy Kawasaki condition. Also, up to an isometry, the corresponding maps u_0 and u_1 agree with φ on the four vertices of the rectangle R.

Proof. We consider the first singular set in Fig. 7. We claim that the triangles ABC and CDE are similar. In fact we have:

$$\frac{CD}{DE} \Big/ \frac{AB}{BC} = \frac{b'}{a'} \Big/ \frac{a''}{b''} = \frac{b(1+\beta)}{a(1+\alpha)} \Big/ \frac{a(1-\alpha)}{b(1-\beta)} = \frac{b^2(1-\beta^2)}{a^2(1-\alpha^2)} = 1$$

by condition (7). As a consequence angles ECD and ACB are complementary and hence the angle ECA is right. Since the triangles ABC and EGF are congruent, also the angle FEC is right and the quadrilateral $ACEF$ is a rectangle. So it is easy to check that Kawasaki condition holds in the internal vertices A, B and C and by Theorem 4.8 we know that there exists a (unique) rigid map $u_0 : R \to \mathbb{R}^2$ which has the singular set represented in Fig. 7 and also satisfies the conditions $u_0(0, 0) = (0, 0)$ and $Du_0(0, 0) = -I$. In particular we easily check that the map has the following values:

$$\begin{aligned}
&u_0(0,0) = (0,0), && u_0(a'',0) = (-a'',0),\\
&u_0(a,0) = (2a' - a'',0) = (\alpha a,0), && u_0(a,b') = (\alpha a,b'),\\
&u_0(a,b'+b'') = (\alpha a,b'-b''), && u_0(a,b) = (\alpha a,2b'-b'') = (\alpha a,\beta b),\\
&u_0(0,b'') = (0,-b''), && u_0(0,b) = (0,2b'-b'') = (0,\beta b),\\
&u_0(a',b) = (a',\beta b), && u_0(a'+a'',b) = (a'-a'',b).
\end{aligned}$$

We define u_1 following the second singular set in Fig. 7. The resulting map has the following values:

$$\begin{aligned} &u_1(0,0)=(0,0), && u_1(a',0)=(a',0),\\ &u_1(a'+a'',0)=(a'-a'',0), && u_1(a,0)=(2a'-a'',0)=(\alpha a,0),\\ &u_1(0,b')=(0,b'), && u_1(0,b'+b'')=(0,b'-b''),\\ &u_1(0,b)=(0,2b'-b'')=(\beta b,0). && \end{aligned}$$

The verification of the claims are then straightforward. □

Theorem 5.3 *(Dirichlet problem). Let $\varphi(x,y)=(\alpha x,\beta y)$ be a diagonal linear map with $\alpha,\beta\in(0,1)$, let $a,b>0$ satisfy relation* (7) *and $\Omega=(-a,a)\times(-b,b)$. Then there exists a piecewise C^1 rigid map $u:\bar{\Omega}\to\mathbb{R}^2$ with singular set Σ_u as in Fig.* 8*, such that $u=\varphi$ on $\partial\Omega$.*

Proof. We divide Ω into infinitely many rectangles homothetic to Ω as in Fig. 8. Then we put the base pattern u_1 (see Lemma 5.2) on the rectangles in the diagonal and the base pattern u_0 on the other rectangles to compose a singular set Σ. The base patterns have to be rescaled, translated and mirrored to fit the net, as shown in figure. As was proved in Lemma 5.2, in every vertex of the singular set two right angles meet. Hence it is clear that the Kawasaki condition holds on the resulting singular set Σ. We conclude that there exists a rigid map $u:\Omega\to\mathbb{R}^2$ which has the assigned singular set Σ. By the construction of the base modules u_0 and u_1 it is easily checked that this map is equal to the linear datum φ on every vertex of the pattern. Since the boundary $\partial\Omega$ is contained in the closure of the set of vertices and u is continuous, then $u\equiv\varphi$ on $\partial\Omega$. □

6. A 3-dimensional flat origami

In Section 2 we proposed definitions of origami as applications either from $\mathbb{R}^2\to\mathbb{R}^3$ or from $\mathbb{R}^2\to\mathbb{R}^2$ (flat case). Of course, mathematically, these definitions make sense in the more general framework of $n\geqslant 2$, $m\geqslant 2$. Here we give an example of a piecewise C^1 rigid map from $\mathbb{R}^3\to\mathbb{R}^3$ which, as a natural extension of the previous definitions, could be considered a *3-dimensional mathematical flat origami*, being a rigid map from $\mathbb{R}^3\to\mathbb{R}^3$.

Our aim is to construct a solution to the Dirichlet problem (5) in the case $n=3$, $\varphi=0$, $\Omega=(0,1)^3$. An explicit solution to this problem was given in [5]; here we present an alternative construction based on Recovery Theorem 4.9.

Theorem 6.1 *(3D Dirichlet problem). There exists a piecewise C^1 rigid map $u:[0,1]^3\to\mathbb{R}^3$ such that $u=0$ on the boundary. The base module of the singular set Σ_u is represented in Fig.* 9.

Proof. Step 1. We consider the cube $Q_1=[0,1]^3=\overline{\Omega}\subset\mathbb{R}^3$. We will use the coordinates $(x,y,z)\in\mathbb{R}^3$. First we find a rigid map $u_1:Q_1\to Q_1$ such that the six sides of ∂Q_1 all go into the side $\{x=0\}$ in ∂Q_1. To achieve this it is enough to fold Q_1 along the four planes $y=x$, $y=1-x$, $z=x$, $z=1-x$. In other words we consider the singular set $\Sigma_1=\{y=x\}\cup\{y=1-x\}\cup\{z=x\}\cup\{z=1-x\}$. This set satisfies the Kawasaki condition (every union of hyper-planes has this property) and hence there exists a unique map $u_1:Q_1\to\mathbb{R}^3$ which has Σ_1 as singular set and which is equal to the identity on the facet $Q_1\cap\{x=0\}$. The resulting map u_1 folds the whole cube Q_1 over the pyramid $Q_1\cap\{x<y, x<1-y, x<z, x<1-z\}$. So we can consider u_1 as a map $u_1:Q_1\to Q_1$ and we notice that $u_1(\partial Q_1)\subset\{x=0\}$ as claimed.

Step 2. We consider the long parallelepiped $Q_2=[0,4]\times[0,1]\times[0,1]$. Our aim is now to find a rigid map $u_2:Q_2\to\mathbb{R}^3$ such that $u_2(0,y,z)=(0,0,0)$ for every $y,z\in[0,1]$. Since $Q_1\subset Q_2$, and $u_1(\partial Q_1)\subset\{x=0\}$, the composition $u=u_2\circ u_1$ will be a map $u:Q_1\to\mathbb{R}^3$ and will satisfy the Dirichlet condition $u(x,y,z)=(0,0,0)$ for every $(x,y,z)\in\partial Q_1$.

To define u_2 we are going to consider a fractal singular set $\Sigma\subset Q_2$. We start with the polyhedral set Σ_2 represented in Fig. 9. This set is composed by the union of the two planes $\{x=y+2\}$, $\{x=z+3\}$ and four half planes $\{y=1/2,\ x\leqslant 5/2\}$, $\{x=5/2,\ y\leqslant 1/2\}$, $\{z=1/2,\ x<7/2\}$, $\{x=7/2,\ z\leqslant 1/2\}$. These planes meet in seven segments contained in five different lines. In these segments the Kawasaki condition is satisfied since the angles are

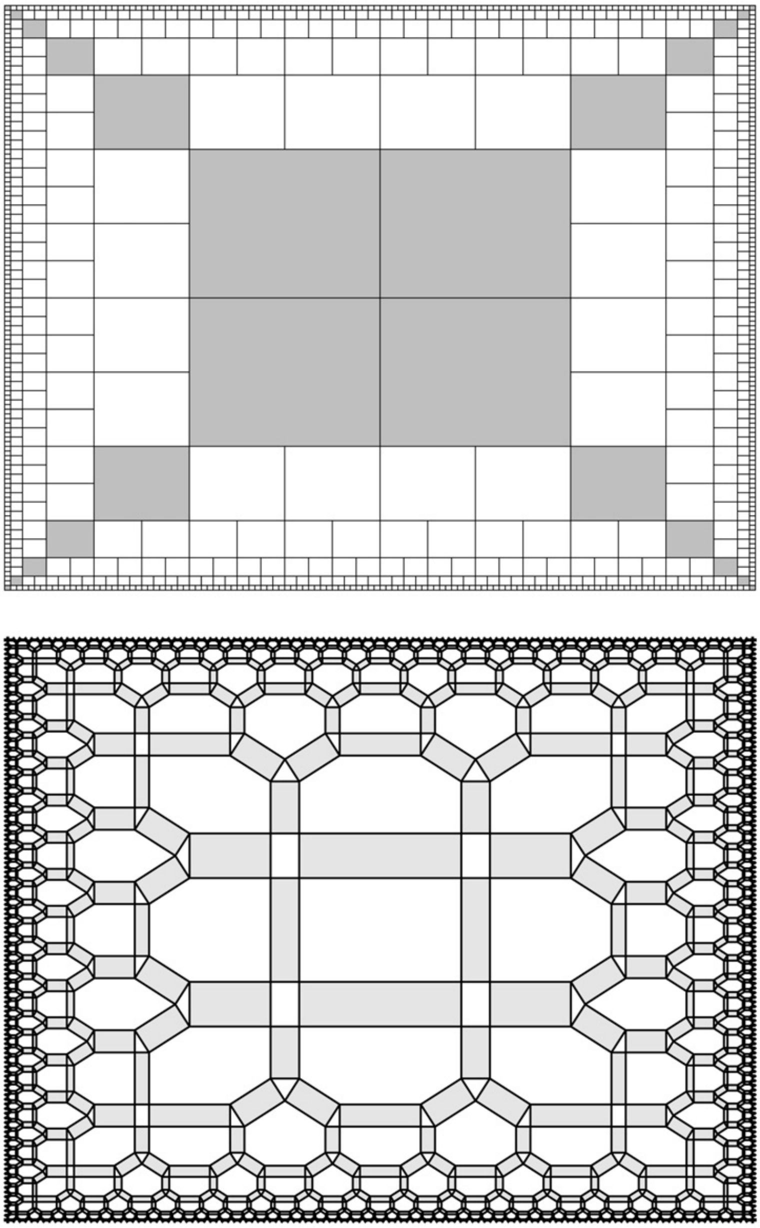

Fig. 8. Top. The fractal net of rectangles used to reproduce the singular set of Theorem 5.3. The colored rectangles will host the second module of Fig. 7 and the white rectangles will host the first module. Bottom. The resulting singular set Σ_u of the map u constructed in Theorem 5.3. The colored rectangles are the regions where $\det Du = -1$. Each vertex of the singular set is shared by two rectangles, hence the Kawasaki condition holds.

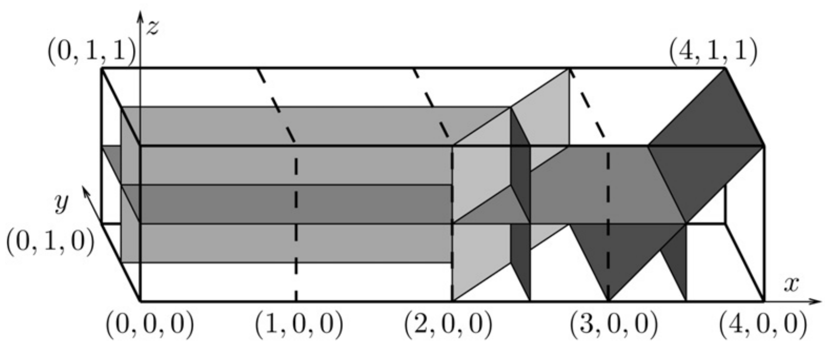

Fig. 9. The singular set Σ_2 which is the base module of the construction of a 3-dimensional solution to the Dirichlet problem.

either $\pi/2 + \pi/2 + \pi/2 + \pi/2$ or $\pi/4 + \pi/4 + 3\pi/4 + 3\pi/4$. We are going to compose this set Σ_2 with mirrored and rescaled copies of itself. We consider the four contractions $T_i : Q_2 \to Q_2$ defined by:

$$T_1(x, y, z) = (x, y, z)/2, \qquad T_2(x, y, z) = (x, 2 - y, z)/2,$$
$$T_3(x, y, z) = (x, y, 2 - z)/2, \qquad T_4(x, y, z) = (x, 2 - y, 2 - z)/2.$$

Given any set X we construct a replicated set $T(X)$ with the four rescaled and mirrored copies of X

$$T(X) = T_1(X) \cup T_2(X) \cup T_3(X) \cup T_4(X).$$

Notice that $T(Q_2) = [0, 2] \times [0, 1] \times [0, 1]$ and $T(Q_2 \setminus \partial Q_2) \cap \Sigma_2 = \emptyset$. This means that the rescaled copies of Σ_2 can only meet on the boundaries.

Finally we define the fractal set Σ by:

$$\Sigma = \bigcup_{k=0}^{\infty} T^k(\Sigma_2) = \Sigma_2 \cup T(\Sigma_2) \cup T\big(T(\Sigma_2)\big) \cup \cdots.$$

The resulting set $\Sigma \subset Q_2$ is a locally finite polyhedral set which satisfies the Kawasaki condition. In fact the Kawasaki condition is satisfied on the internal edges of every rescaled polyhedral set. If we take an edge on the boundary of these rescaled sets, we notice that on such an edge there meet half planes from 2 rescaled sets which are one the mirror of the other, and the mirror plane itself belongs to a bigger polyhedral rescalation of Σ_2. Hence the angles of the half planes on the given edge, repeat twice mirrored, and the Kawasaki condition holds automatically. Hence, by Recovery theorem, a map $u_2 : Q_2 \to \mathbb{R}^3$ exists which has Σ as singular set and such that $u_2(0, 0, 0) = (0, 0, 0)$.

Step 3. To conclude the statement, we are going to prove that $u_2(0, y, z) = (0, 0, 0)$ for every $y, z \in [0, 1]$. To achieve this we claim that for each integer $k = 0, 1, \ldots$ the image $u_2(X_k)$ of the square $X_k = Q_2 \cap \{x = 2/2^k\}$ has a diameter at most $\sqrt{2}/2^{k+1}$. As a consequence the map $u_2(0, y, z)$ is constant (recall the u_2 is continuous) and hence has value $(0, 0, 0)$.

Since the set Σ_2 contains the two planes of symmetry $y = 1/2$ and $z = 1/2$ for $x \leqslant 2$, and since Σ coincides with Σ_2 for $x > 2$, the resulting map u_2 has the property $u_2(2, y, z) = u_2(2, 1 - y, z) = u_2(2, y, 1 - z) = u_2(2, 1 - y, 1 - z)$ if $y, z \in [0, 1/2]$. Hence the image of any point $(2, y, z)$ for $y, z \in [0, 1]$ is also the image of a point with $y, z \in [0, 1/2]$. In general we notice that the image of a point $(2/2^k, y, z)$ for $y, z \in [0, 1]$ is also the image of a point with $y, z \in [0, 1/2^{k+1}]$ because the map u_2 for $x \in [1/2^{k+1}, 1/2^k]$ is obtained joining together four rescaled copies of the same map u_2 in the interval $[1/2^k, 1/2^{k-1}]$ with scaling factor $1/2$ and an appropriate rotation, mirroring and translation.

Hence the image of the points $(2/2^k, y, z)$ is contained in the image of a square of side $1/2^{k+1}$. Since the map u_2 is short, the diameter of such an image is not greater than the diameter of the square, which is $\sqrt{2}/2^{k+1}$, as claimed. $\square$

Acknowledgements

We would like to thank the referee for a thorough reading of the manuscript and for many helpful suggestions.

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