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QUALITATIVE PROPERTIES OF LIPSCHITZ SOLUTIONS OF EIKONAL TYPE SYSTEMS

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Abstract. We study the geometric properties of Lipschitz functions whose gradient takes only a finite number of vectorial values. A typical example is the so-called eikonal system. Functions whose gradient takes only a finite number of values are notably encountered in non convex problems of the calculus of variations and in applications to problems of potential wells type.

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1 Introduction

The aim of this report is to study geometric properties of Lipschitz functions (more precisely in $W^{1,\infty}(\Omega)$) $u:\Omega\subset\mathbb{R}^n\to\mathbb{R}$ with prescribed gradient, namely

$$Du(x)\in E \quad \text{a.e. } x\in\Omega,$$

where $E\subset\mathbb{R}^n$ is a fixed set containing only a finite number of points and $\Omega\subset\mathbb{R}^n$ is a bounded open set. This question is usually coupled with the prescription of a boundary datum. One typical example is the so-called eikonal system

$$\begin{cases} |u_{x_i}|=1 & \text{a.e. in } \Omega,\ i=1,...,n, \\ u=0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

This type of problems arise in several different contexts such as: non convex problems of the calculus of variations, implicit partial differential equations, phase transitions and problems of potential wells type; we refer to [3] for a large bibliography. We should also draw the attention that the problem under consideration is also a challenging problem in numerical analysis, cf. [2] and the references therein.

In order to describe our results, let $\Omega\subset\mathbb{R}^n$ be an open set and $u:\Omega\to\mathbb{R}$ be a function belonging to $W^{1,\infty}(\Omega)$. We fix $E\subset\mathbb{R}^n$ containing only a finite number of vectors, $E=\{\epsilon_1,\ldots,\epsilon_N\}$. We define the "singular set" of u

$$\Sigma_E := \{x\in\Omega : \text{either } u \text{ is not differentiable at } x \text{ or } Du(x)\notin E\}, \tag{1.2}$$

and the "regular set" of u

$$\Omega\setminus\Sigma_E := \{x\in\Omega : u \text{ is differentiable at } x \text{ and } Du(x)\in E\}$$

We assume that

$$Du(x)\in E \text{ for a.e. } x\in\Omega. \tag{1.3}$$

Therefore $|\Sigma_E|=0$. It is possible to construct an example in dimension $n=1$ for $\Omega=(0,1)$, with $E=\{-1,1\}$, of a function $u\in W^{1,\infty}(0,1)$ such that $|u'|=1$ almost everywhere in $(0,1)$ and Σ_E is a Cantor set with zero Lebesgue measure.
Since $E=\{\epsilon_1,\ldots,\epsilon_N\}$, we define the Borel sets

$$\Omega_i := \{x\in\Omega : Du(x)=\epsilon_i\}, \tag{1.4}$$

so that $\Omega_i\cap\Omega_j=\emptyset$ for $i,j\in\{1,\ldots,N\}$, $i\neq j$, and $\Omega\setminus\Sigma_E=\cup_j(\Omega_j\setminus\Sigma_E)$. Note that Ω_j is not necessarily a finite perimeter set in Ω.

Our first result (Theorem 2.1) deals with the behaviour of u near a "singular" point $x_0\in\Sigma_E$: there is $j\in\{1,\ldots,N\}$ such that Ω_j has neither density 0 nor 1 at x_0. This means that around x_0 there are "many" points x and "many" points y for which $Du(x)=\epsilon_j$ and $Du(y)\neq\epsilon_j$. This density result guarantees that

$$\Sigma_E\subset\Omega\cap\cup_{j=1}^N\partial^*\Omega_j$$

where $\partial^* A$ is the essential boundary of A. Our second result (Theorem 2.6) deals with the behaviour of u near a "regular" point $x_0 \in \Omega \setminus \Sigma_E$: there is $j \in \{1, \ldots, N\}$ such that Ω_j has density 1 at x_0. This means that "nearly all" points x around x_0 satisfy $Du(x) = \epsilon_j$. This second result requires some restrictions on $E = \{\epsilon_1, \ldots, \epsilon_N\}$, precisely

$$|\epsilon_i|^2 - \langle \epsilon_i, \epsilon_j \rangle > 0, \qquad i \neq j. \tag{1.5}$$

Condition (1.5) on E rules out the possibility that $0 \in E$; moreover, it links up ϵ_i with ϵ_j when their angle is less than $\pi/2$. Our third result (Theorem 2.8) is global; it no longer requires condition (1.5) but it asks for convexity of Ω; it says that the graph of u is in between two pyramids whose faces are related to $\epsilon_1, \ldots, \epsilon_N$.

Now let us restrict the attention to the eikonal system (1.1) in the two dimensional case $n = 2$ with $N = 4$, and $E = \{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$, where $\epsilon_i = (\pm 1/\sqrt{2}, \pm 1/\sqrt{2})$. In this case we are able to prove (Theorem 3.1) that, if the graph of u touches one face of the previous pyramids, then it must be "flat" for a "while".

Define

$$\Omega_E := \text{ interior of } (\Omega \setminus \Sigma_E) \tag{1.6}$$

Our last result (Theorem 3.2) deals with the local behaviour of u near "very bad" points $x_0 \in \Omega \setminus \overline{\Omega_E}$, namely those points that are not limits of points having a neighbourhood where u is differentiable and the gradient lies in E: the length of Σ_E is infinite near x_0. It turns out that

$$\mathcal{H}^1(\Sigma_E \cap B(x_0, r)) = +\infty \quad \forall r > 0$$

for every ball $B(x_0, r)$ with radius $r > 0$ around x_0. Note that condition "$Du(x) \in E$ for a.e. $x \in \Omega$" guarantees that $|\Sigma_E| = 0$: this and the previous result tell us that the singular set Σ_E is small but "very twisted" around "very bad" points $x_0 \in \Omega \setminus \overline{\Omega_E}$. This means that, if we know, a priori, that the length of Σ_E (or the perimeter of Ω_i) is finite, then there are not "very bad" points at all: $\Omega \setminus \overline{\Omega_E} = \emptyset$.

Next section contains precise statements and proofs in the n dimensional case; third section is devoted to the eikonal system in dimension $n = 2$.

2 Functions with prescribed gradient

Let us introduce some notation. In the sequel $n \geq 1$, $\Omega \subset \mathbb{R}^n$ is an open set, $x_0 \in \Omega$ and $r > 0$ is sufficiently small so that the open ball $B(x_0, r)$ centered at x_0 with radius r is contained in Ω. $|\cdot|$ denotes the Lebesgue measure and $\mathcal{H}^k$ the Hausdorff measure. We set $|B(x_0, r)| = \omega_n r^n$ and $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$. The set $E \subset \mathbb{R}^n$ consists of finitely many distinct vectors $\{\epsilon_i\}_{i=1}^N$, $N \geq 2$. u will be always a function in $W^{1,\infty}(\Omega)$. We next let, for $\xi \in \mathbb{R}^n$,

$$\mu(x_0, r, \xi) := \frac{|\{x \in B(x_0, r) : Du(x) = \xi\}|}{\omega_n r^n}$$

$$\nu(x_0, r, \xi) := \frac{|\{x \in B(x_0, r) : Du(x) \neq \xi\}|}{\omega_n r^n}.$$

Note that

$$\mu\left(x_0,r,\xi\right)+\nu\left(x_0,r,\xi\right)=1, \tag{2.1}$$

and if u satisfies (1.3) then

$$\sum_{i=1}^{N}\mu(x_0,r,\epsilon_i)=1. \tag{2.2}$$

If we indicate by Ω_ξ the set of all points $x\in\Omega$ where $Du(x)=\xi$, then the $\lim_{r\to 0^+}\mu(x_0,r,\xi)$ (if it exists) is usually denoted by $\Theta_n(\Omega_\xi,x_0)$ and is called the n-dimensional density of Ω_ξ at x_0. For simplicity we set $\Omega_i=\Omega_{e_i}$.

The first result deals with the behaviour of u near a point $x_0\in\Sigma_E$.

Theorem 2.1 *Let $u\in W^{1,\infty}(\Omega)$ satisfy (1.3) and let $x_0\in\Sigma_E$. Then there exists $\epsilon_i\in E$ such that $\Theta_n(\Omega_i,x_0)\notin\{0,1\}$.*

Note that the conclusion of Theorem 2.1 is: either the limit of $\mu(x_0,r,e_i)$ as $r\to 0^+$ does not exist, or the density of Ω_i at x_0 belongs to $(0,1)$.

To prove Theorem 2.1 we need the following lemma.

Lemma 2.2 *Let $x_0\in\Omega$, $\xi\in\mathbb{R}^n$ and $u\in W^{1,\infty}(\Omega)$. Then*

$$\lim_{r\to 0^+}\mu\left(x_0,r,\xi\right)=1 \implies u \text{ is differentiable at } x_0 \text{ and } Du\left(x_0\right)=\xi.$$

Proof. From (2.1), the assumption $\lim_{r\to 0^+}\mu\left(x_0,r,\xi\right)=1$ is equivalent to

$$\lim_{r\to 0^+}\nu\left(x_0,r,\xi\right)=0. \tag{2.3}$$

Set $R\left(x\right):=u\left(x\right)-u\left(x_0\right)-\langle\xi,x-x_0\rangle$ for any $x\in\Omega$. We have to show that (2.3) implies

$$\lim_{x\to x_0}\frac{\left|R\left(x\right)\right|}{\left|x-x_0\right|}=0. \tag{2.4}$$

Observing that R is differentiable almost everywhere with gradient $DR=Du-\xi$, for any $p\in[1,+\infty)$ we have

$$\begin{aligned}\frac{1}{\omega_n r^n}\int_{B(x_0,r)}\left|DR\right|^p\,dx &= \frac{1}{\omega_n r^n}\int_{B(x_0,r)}\left|Du-\xi\right|^p\,dx\\ &= \frac{1}{\omega_n r^n}\int_{B(x_0,r)\cap\{Du\neq\xi\}}\left|Du-\xi\right|^p dx\\ &\le \left\|Du-\xi\right\|^p_{L^\infty(B(x_0,r))}\,\nu\left(x_0,r,\xi\right).\end{aligned} \tag{2.5}$$

We now choose $p>n$ and use the continuity of u and Morrey's inequality (see [4, pag. 143]) to have, for every $x\in\Omega$ and $r:=\left|x-x_0\right|$,

$$\left|R\left(x\right)\right|=\left|R\left(x\right)-R\left(x_0\right)\right|\le cr\Big[\frac{1}{\omega_n r^n}\int_{B(x_0,r)}\left|DR\right|^p\,dy\Big]^{1/p}, \tag{2.6}$$

where c is a positive constant depending only on p and n. Combining (2.6), (2.5) and the fact that $u \in W^{1,\infty}(\Omega)$ we find, for a constant C independent of x_0 and $r := |x - x_0|$,

$$0 \leq \frac{|R(x)|}{|x - x_0|} \leq C\left[\nu(x_0, r, \xi)\right]^{1/p}$$

which, together with (2.3), implies (2.4). ■

We now conclude the proof of Theorem 2.1

Proof. (Theorem 2.1). Let $x_0 \in \Sigma_E$. Assume by contradiction that for any $i \in \{1, \ldots, N\}$ there exists $\lim_{r \to 0^+} \mu(x_0, r, e_i) \in \{0, 1\}$. From equality (2.2) we deduce that $\sum_{i=1}^{N} \lim_{r \to 0^+} \mu(x_0, r, e_i) = 1$. Hence there is $j \in \{1, \ldots, N\}$ such that $\Theta_n(\Omega_j, x_0) = 1$ and $\Theta_n(\Omega_i, x_0) = 0$ for any $i \in \{1, \ldots, N\}$, $i \neq j$. Therefore, from Lemma 2.2 it follows $Du(x_0) = e_j$, which contradicts $x \in \Sigma_E$. ■

Remark 2.3 Note that Theorem 2.1 gives that there are at least two distinct $\epsilon_i \in E$ for which the density of Ω_i at x_0 is neither zero nor one.

We denote by $\partial^* A$ the essential boundary of the $\mathcal{H}^n$-measurable set $A \subset \mathbb{R}^n$, i.e., the collection of all points $x_0 \in \mathbb{R}^n$ such that A has neither density 1 nor density 0 at x_0 [1]. We denote by $P(A, \Omega)$ the (possibly infinite) perimeter of the Borel set A in Ω.

Corollary 2.4 *Let $u \in W^{1,\infty}(\Omega)$ satisfy* (1.3) *Then*

$$\Sigma_E \subset \Omega \cap \Big(\bigcup_{j=1}^{N} \partial^* \Omega_j\Big), \tag{2.7}$$

and

$$\mathcal{H}^{n-1}(\Sigma_E) \leq \sum_{j=1}^{N} P(\Omega_j, \Omega). \tag{2.8}$$

Proof. Inclusion (2.7) follows from Theorem 2.1. The inequality follows from (2.7) and the equality $\mathcal{H}^{n-1}(\Omega \cap \partial^* A) = P(A, \Omega)$, valid for any Borel set $A \subset \mathbb{R}^n$. ■
Now we turn our attention to regular points $x_0 \in \Omega \setminus \Sigma_E$.

The following example shows that, in order that at some $x_0 \in \Omega \setminus \Sigma_E$ one of the Ω_j has density one, some restrictions on ϵ_i have to be assumed.

Example 2.5 Choose $E := \{\epsilon_1 = -1, \epsilon_2 = 0, \epsilon_3 = 1\} \subset \mathbb{R}$ and define the one-Lipschitz function $u : (-1, 1) \to \mathbb{R}$ as follows. For every $k \in \mathbb{N}$ we set $u = 0$ in $\left(\frac{1}{k+1}, \frac{1}{2}\left(\frac{1}{k+1} + \frac{1}{k}\right)\right)$. Let us divide the interval $\left(\frac{1}{2}\left(\frac{1}{k+1} + \frac{1}{k}\right), \frac{1}{k}\right)$ into two equal parts I_k^-, I_k^+: in I_k^- the function u increases with slope 1 from the value zero, and in I_k^+ it decreases with slope -1 from its value at the right extremum of I_k^-. Then $0 \leq u(x) \leq x^2$ for every $x \in (0, 1)$, and therefore u can be continuously defined at $x = 0$ by setting $u(0) = 0$. Now we extend u by parity to $(-1, 0)$. It turns out that $u'(0) = 0 = \epsilon_2$, hence $x_0 = 0 \in \Omega \setminus \Sigma_E$. On the other hand $\Theta_1(\Omega_2, 0) = \frac{1}{2}$.

Theorem 2.6 *Assume condition (1.5), and let $u \in W^{1,\infty}(\Omega)$ satisfy (1.3). Let $x_0 \in \Omega \diagdown \Sigma_E$ and take $\epsilon_{\bar{i}} \in E$ such that $x_0 \in \Omega_{\bar{i}}$. Then*

$$\Theta_n(\Omega_{\bar{i}}, x_0) = 1. \tag{2.9}$$

Proof. From (1.5) it follows that $\epsilon_{\bar{i}} \neq 0$. We then choose unit vectors $z_2, ..., z_n$ so that $\{\epsilon_{\bar{i}}, z_2, ..., z_n\}$ are mutually orthogonal. If $s := (s_2, ..., s_n)$ and $r > 0$ we have

$$u\Big(x_0 \pm r\epsilon_{\bar{i}} + \sum_{i=2}^{n} s_i z_i\Big) = u(x_0) + \Big\langle Du(x_0), \pm r\epsilon_{\bar{i}} + \sum_{i=2}^{n} s_i z_i\Big\rangle + R_\pm(r, s)$$

where $\frac{R_\pm(r,s)}{r+|s|} \to 0$ as $r + |s| \to 0$. Therefore, since $Du(x_0) = \epsilon_{\bar{i}}$, if we set

$$R(r, s) := u\Big(x_0 + r\epsilon_{\bar{i}} + \sum_{i=2}^{n} s_i z_i\Big) - u\Big(x_0 - r\epsilon_{\bar{i}} + \sum_{i=2}^{n} s_i z_i\Big) - 2r\,|\epsilon_{\bar{i}}|^2 \tag{2.10}$$

it follows that

$$\frac{\sup_{|s|<r} |R(r, s)|}{r} \to 0 \text{ as } r \to 0. \tag{2.11}$$

Define

$$\begin{aligned} Q_r^{n-1} &:= (-r, r)^{n-1}, \\ Q_r &:= (-r, r)^n, \\ Q(x_0, r) &:= \Big\{x \in \mathbb{R}^n : x = x_0 + t\epsilon_{\bar{i}} + \sum_{i=2}^{n} s_i z_i,\ (t, s) \in Q_r\Big\} \\ Q^i(x_0, r) &:= \{x \in Q(x_0, r) : Du(x) = \epsilon_i\},\ i = 1, ..., N. \end{aligned} \tag{2.12}$$

Note that

$$(2r)^n = |Q(x_0, r)|\,|\epsilon_{\bar{i}}|^{-1} = |\epsilon_{\bar{i}}|^{-1} \sum_i |Q^i(x_0, r)| \tag{2.13}$$

We have for $u \in \mathcal{C}^1(\Omega)$,

$$\begin{aligned} &\int_{Q_r^{n-1}} \Big[u\Big(x_0 + r\epsilon_{\bar{i}} + \sum_{i=2}^{n} s_i z_i\Big) - u\Big(x_0 - r\epsilon_{\bar{i}} + \sum_{i=2}^{n} s_i z_i\Big)\Big] ds \\ &= \int_{Q_r^{n-1}} ds \int_{-r}^{r} \frac{d}{dt}\Big[u\Big(x_0 + t\epsilon_{\bar{i}} + \sum_{i=2}^{n} s_i z_i\Big)\Big] dt \\ &= \int_{Q_r} \langle Du\Big(x_0 + t\epsilon_{\bar{i}} + \sum_{i=2}^{n} s_i z_i\Big), \epsilon_{\bar{i}}\rangle\, dt ds = \int_{Q(x_0,r)} \langle Du(y), \epsilon_{\bar{i}}\rangle |\epsilon_{\bar{i}}|^{-1}\, dy. \end{aligned} \tag{2.14}$$

By approximation using $\mathcal{C}^1(\Omega)$ functions, we have that the left hand side on (2.15) equals the right hand side also for any $u \in W^{1,\infty}(\Omega)$. Then we have

$$\begin{aligned}
&\int_{Q_r^{n-1}} \left[u\big(x_0 + r\epsilon_{\bar{i}} + \sum_{i=2}^{n} s_i z_i\big) - u\big(x_0 - r\epsilon_{\bar{i}} + \sum_{i=2}^{n} s_i z_i\big)\right] ds \\
=&\sum_{i=1}^{N} \langle \epsilon_i, \epsilon_{\bar{i}} \rangle \left| Q^i\left(x_0, r\right)\right| |\epsilon_{\bar{i}}|^{-1}.
\end{aligned} \tag{2.15}$$

Now define

$$R(r) := \int_{Q_r^{n-1}} R(r, s)\, ds.$$

From (2.10) and (2.15) we deduce

$$R(r) = -\left(2r\right)^n |\epsilon_{\bar{i}}|^2 + \sum_{i=1}^{N} \langle \epsilon_i, \epsilon_{\bar{i}} \rangle \left| Q^i\left(x_0, r\right)\right| |\epsilon_{\bar{i}}|^{-1}. \tag{2.16}$$

Inserting (2.13) in (2.16) yields

$$\begin{aligned}
R(r) &= |\epsilon_{\bar{i}}|^{-1} \sum_{i=1}^{N} \left[\langle \epsilon_i, \epsilon_{\bar{i}} \rangle - |\epsilon_{\bar{i}}|^2\right] \left| Q^i\left(x_0, r\right)\right| \\
&= |\epsilon_{\bar{i}}|^{-1} \sum_{i \neq \bar{i}} \left[\langle \epsilon_i, \epsilon_{\bar{i}} \rangle - |\epsilon_{\bar{i}}|^2\right] \left| Q^i\left(x_0, r\right)\right|,
\end{aligned}$$

which, combined with (1.5), leads to

$$0 \leq c |\epsilon_{\bar{i}}|^{-1} \sum_{i \neq \bar{i}} \left| Q^i\left(x_0, r\right)\right| \leq -R(r)\ , \tag{2.17}$$

for a suitable positive constant c. Using (2.11) we also infer that

$$\frac{R(r)}{r^n} \to 0 \text{ as } r \to 0. \tag{2.18}$$

Therefore, from (2.17) and (2.18) it follows

$$\lim_{r \to 0} \frac{\left| Q^i\left(x_0, r\right)\right|}{\left| Q\left(x_0, r\right)\right|} = \begin{cases} 0 & \text{if } i \neq \bar{i}, \\ 1 & \text{if } i = \bar{i}. \end{cases}$$

Since $\Theta_n(\Omega_{\bar{i}}, x_0) = 1$ if and only if $\lim_{r\to 0} \frac{|Q^{\bar{i}}(x_0,r)|}{|Q(x_0,r)|} = 1$, the result follows. ■

Remark 2.7 Recall [4, Section 1.7] that if B is a Borel set, then B has density one at almost every $x \in B$. Theorem 2.6 asserts that any point of $\Omega \setminus \Sigma_E$ is of density one for one of the Ω_i.

The next result tells us that condition $Du(x) \in E$ forces the graph of u to remain between two pyramids whose faces are determined by $\epsilon_1, \ldots, \epsilon_N$. Note that we require Ω to be convex.

Theorem 2.8 *Let $\Omega \subset \mathbb{R}^n$ be an open convex set and let $u \in W^{1,\infty}(\Omega)$ satisfy (1.3). Then for every $x \in \Omega$*

$$\operatorname{graph}(u) \subset \pi_{(x,u(x))},$$

where

$$\pi_{(x,u(x))} := \left\{ (y,z) \in \mathbb{R}^n \times \mathbb{R} : \min_{1 \leq i \leq N} \langle \epsilon_i, y - x \rangle \leq z - u(x) \leq \max_{1 \leq i \leq N} \langle \epsilon_i, y - x \rangle \right\}.$$

Proof. The claim is equivalent to showing that, for every $x, y \in \Omega$,

$$\min_{1 \leq i \leq N} \langle \epsilon_i, x - y \rangle \leq u(x) - u(y) \leq \max_{1 \leq i \leq N} \langle \epsilon_i, x - y \rangle. \tag{2.19}$$

Fix $x, y \in \Omega$ with $x \neq y$ and choose $z_1, ..., z_{n-1} \in \mathbb{R}^n$ independent unit vectors orthogonal to $x - y$. Since Ω is open and convex, we can choose $\delta > 0$ so that

$$tx + (1-t)y + \sum_{i=1}^{n-1} s_i z_i \in \Omega, \qquad (t, s_1, ..., s_{n-1}) \in A := (0,1) \times (-\delta, \delta)^{n-1}.$$

Setting $s = (s_1, ..., s_{n-1})$, we have

$$\begin{aligned} &\int_{(-\delta,\delta)^{n-1}} \Big[u\big(x + \sum_{j=1}^{n-1} s_j z_j\big) - u\big(y + \sum_{j=1}^{n-1} s_j z_j\big) \Big] ds \\ &= \int_A \langle Du\big(tx + (1-t)y + \sum_{j=1}^{n-1} s_j z_j\big), x - y \rangle \, dt ds. \end{aligned}$$

Define

$$A_i := \left\{ (t,s) \in A : Du\big(tx + (1-t)y + \sum_{j=1}^{n-1} s_j z_j\big) = \epsilon_i \right\}, \quad i = 1, ..., N.$$

Using (1.3) we deduce

$$\int_{(-\delta,\delta)^{n-1}} \Big[u\big(x + \sum_{j=1}^{n-1} s_j z_j\big) - u\big(y + \sum_{j=1}^{n-1} s_j z_j\big) \Big] ds = \sum_{i=1}^{N} \langle \epsilon_i, x - y \rangle |A_i|.$$

Hence

$$\begin{aligned} \min_{1 \leq i \leq N} \langle \epsilon_i, x - y \rangle &\leq \frac{1}{(2\delta)^{n-1}} \int_{(-\delta,\delta)^{n-1}} \Big[u\Big(x + \sum_{j=1}^{n-1} s_j z_j\Big) - u\Big(y + \sum_{j=1}^{n-1} s_j z_j\Big) \Big] ds \\ &\leq \max_{1 \leq i \leq N} \langle \epsilon_i, x - y \rangle. \end{aligned}$$

Letting $\delta \to 0$ and using the continuity of the function u, (2.19) follows. ■

3 Functions with prescribed gradient in 2D

In this section we confine ourselves to the case $n = 2$, $N = 4$, and

$$\epsilon_1 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), \quad \epsilon_2 = (\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), \quad \epsilon_3 = -\epsilon_1, \quad \epsilon_4 = -\epsilon_2.$$

Note that $|\epsilon_i|^2 - \langle \epsilon_i, \epsilon_j \rangle = 1$ for any $i \neq j$, and thus (1.5) is satisfied.
Let us consider the union $\partial^0 \pi_{(x,u(x))}$ of the eight "flat" parts of the boundary of $\pi_{(x,u(x))}$:

$$\partial^0 \pi_{(x,u(x))} := \Big\{ (y,z) \in \mathbb{R}^2 \times \mathbb{R} : y_1 \neq x_1, y_2 \neq x_2, z - u(x) = \max_{j=1,\dots,4} \langle \epsilon_j, y - x \rangle \text{ or } z - u(x) = \min_{j=1,\dots,4} \langle \epsilon_j, y - x \rangle \Big\}. \tag{3.1}$$

The following result tells that if graph(u) touches $\partial^0 \pi_{(x,u(x))}$ at some $(y, u(y))$, then graph(u) over a suitable rectangle $\mathcal{R}$ must be flat.

Theorem 3.1 *Let $\Omega \subset \mathbb{R}^2$ be a convex open set. Let $u \in W^{1,\infty}(\Omega)$ satisfy* (1.3). *If*

$$x, y \in \Omega, \qquad (y, u(y)) \in \partial^0 \pi_{(x,u(x))} \tag{3.2}$$

then

$$\epsilon := \frac{1}{\sqrt{2}} \text{sign}(u(y) - u(x))(\text{sign}(y_1 - x_1), \text{sign}(y_2 - x_2)) \ \in E,$$

and

$$\Omega \cap \mathcal{R} \neq \emptyset, \qquad Du(\xi) = \epsilon \quad \forall \xi \in \Omega \cap \mathcal{R}, \tag{3.3}$$

where

$$\mathcal{R} := \Big\{ \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 : \quad \min\{x_1, y_1\} < \xi_1 < \max\{x_1, y_1\}, \quad \min\{x_2, y_2\} < \xi_2 < \max\{x_2, y_2\} \Big\}.$$

Proof. Assumption (3.2) and the definition of $\partial^0 \pi_{(x,u(x))}$ guarantee that $x_1 \neq y_1$, $x_2 \neq y_2$, thus $\mathcal{R}$ is nonempty. In addition there exists a unique $i \in \{1, \dots, 4\}$ such that $y - x$ belongs to the open quarter Q_i among the four determined by the standard axes of $\mathbb{R}^2$, and

$$\epsilon_i = \frac{1}{\sqrt{2}}(\text{sign}(y_1 - x_1), \text{sign}(y_2 - x_2)),$$
$$\max_{j \in \{1,\dots,4\}} \langle \epsilon_j, y - x \rangle = \langle \epsilon_i, y - x \rangle > 0,$$
$$\min_{j \in \{1,\dots,4\}} \langle \epsilon_j, y - x \rangle = -\langle \epsilon_i, y - x \rangle.$$

The convexity of Ω implies that $tx + (1 - t)y \in \Omega \cap \mathcal{R}$ for every $t \in (0, 1)$ (in particular $\Omega \cap \mathcal{R}$ is nonempty). Take any $\xi \in \Omega \cap \mathcal{R}$: it turns out that $\xi - x \in Q_i$ and $\xi - y \in Q_{i+2}$

where Q_{i+2} is the open quarter opposite to Q_i. Thus

$$\max_{j\in\{1,\dots,4\}} \langle \epsilon_j, \xi - x\rangle = \langle \epsilon_i, \xi - x\rangle, \tag{3.4}$$
$$\min_{j\in\{1,\dots,4\}} \langle \epsilon_j, \xi - x\rangle = -\langle \epsilon_i, \xi - x\rangle, \tag{3.5}$$
$$\max_{j\in\{1,\dots,4\}} \langle \epsilon_j, \xi - y\rangle = -\langle \epsilon_i, \xi - y\rangle, \tag{3.6}$$
$$\min_{j\in\{1,\dots,4\}} \langle \epsilon_j, \xi - y\rangle = \langle \epsilon_i, \xi - y\rangle. \tag{3.7}$$

From Theorem 2.8 it follows that

$$(\xi, u(\xi)) \in \pi_{(x,u(x))} \tag{3.8}$$

and

$$(\xi, u(\xi)) \in \pi_{(y,u(y))}. \tag{3.9}$$

Inclusion (3.8) reads as

$$\min_{j\in\{1,\dots,4\}} \langle \epsilon_j, \xi - x\rangle \le u(\xi) - u(x) \le \max_{j\in\{1,\dots,4\}} \langle \epsilon_j, \xi - x\rangle, \tag{3.10}$$

thus (3.4), (3.5) give

$$u(x) - \langle \epsilon_i, \xi - x\rangle \le u(\xi) \le u(x) + \langle \epsilon_i, \xi - x\rangle. \tag{3.11}$$

Inclusion (3.9) reads as

$$\min_{j\in\{1,\dots,4\}} \langle \epsilon_j, \xi - y\rangle \le u(\xi) - u(y) \le \max_{j\in\{1,\dots,4\}} \langle \epsilon_j, \xi - y\rangle \tag{3.12}$$

thus (3.6), (3.7) give

$$u(y) + \langle \epsilon_i, \xi - y\rangle \le u(\xi) \le u(y) - \langle \epsilon_i, \xi - y\rangle. \tag{3.13}$$

Assumption (3.2) gives rise to the alternative

$$u(y) = u(x) - \langle \epsilon_i, y - x\rangle \tag{3.14}$$

or

$$u(y) = u(x) + \langle \epsilon_i, y - x\rangle \tag{3.15}$$

In the first case (3.14) we use the left hand side of (3.11) and the right hand side of (3.13) to deduce

$$\begin{aligned} & u(x) - \langle \epsilon_i, \xi - x\rangle \le u(\xi) \le u(y) - \langle \epsilon_i, \xi - y\rangle \\ = \; & u(x) - \langle \epsilon_i, y - x\rangle - \langle \epsilon_i, \xi - y\rangle = u(x) - \langle \epsilon_i, \xi - x\rangle \end{aligned} \tag{3.16}$$

and therefore

$$u(\xi) = u(x) + \langle -\epsilon_i, \xi - x\rangle \qquad \forall \xi \in \Omega \cap \mathcal{R}. \tag{3.17}$$

Note that, in this first case,

$$-\epsilon_i = [\mathrm{sign}(u(y) - u(x))]\frac{1}{\sqrt{2}}(\mathrm{sign}(y_1 - x_1), \mathrm{sign}(y_2 - x_2)). \tag{3.18}$$

In the second case (3.15) we use the right hand side of (3.11) and the left hand side of (3.13)

$$\begin{aligned} & u(x) + \langle \epsilon_i, \xi - x\rangle \geq u(\xi) \geq u(y) + \langle \epsilon_i, \xi - y\rangle \\ = \; & u(x) + \langle \epsilon_i, y - x\rangle + \langle \epsilon_i, \xi - y\rangle = u(x) + \langle \epsilon_i, \xi - x\rangle \end{aligned} \tag{3.19}$$

and therefore

$$u(\xi) = u(x) + \langle \epsilon_i, \xi - x\rangle \qquad \forall \xi \in \Omega \cap \mathcal{R}. \tag{3.20}$$

Note that, in this second case,

$$\epsilon_i = [\mathrm{sign}(u(y) - u(x))]\frac{1}{\sqrt{2}}(\mathrm{sign}(y_1 - x_1), \mathrm{sign}(y_2 - x_2)). \tag{3.21}$$

This concludes the proof of the theorem. ■

In the next theorem we estimate the length of Σ_E near $x_0 \in \Omega \setminus \overline{\Omega_E}$.

Theorem 3.2 *Let $\Omega \subset \mathbb{R}^2$ be a convex open set. Let $u \in W^{1,\infty}(\Omega)$ satisfy* (1.3). *Then*

$$x_0 \in \Omega \setminus \overline{\Omega_E} \quad \Longrightarrow \quad \mathcal{H}^1(\Sigma_E \cap B(x_0, r)) = +\infty \qquad \forall r > 0. \tag{3.22}$$

Proof. Since $x_0 \in \Omega \setminus \overline{\Omega_E}$ and $\Omega \setminus \overline{\Omega_E}$ is open, we can assume that $r > 0$ is such that $B(x_0, r) \subset \Omega \setminus \overline{\Omega_E}$. Take $\rho \in (0, r)$ and $\delta \in (0, r - \rho)$, so that $0 < \rho < \rho + \delta < r$. For each $n \in \mathbb{S}^1$ we define I_n to be the piece of radius in the n direction between ρ and $\rho + \delta$:

$$I_n := \{x \in \mathbb{R}^2 : \exists s \in (0, \delta) : x = x_0 + (\rho + s)n\} \tag{3.23}$$

and $\widetilde{I}_n^u$ to be the graph of u restricted to I_n:

$$\widetilde{I}_n^u = \{(x, z) \in \mathbb{R}^2 \times \mathbb{R} : x \in I_n, z = u(x)\}. \tag{3.24}$$

Define the unit vertical cylinder C around x_0

$$C := \{(x, z) \in \mathbb{R}^2 \times \mathbb{R} : |x - x_0| = 1\}$$

and the four planes α_i

$$\alpha_i := \{(x, z) \in \mathbb{R}^2 \times \mathbb{R} : z = u(x_0) + \langle \epsilon_i, x - x_0\rangle\}, \qquad i = 1, \ldots, 4.$$

Set also

$$C_E := C \cap \bigcup_{i=1}^{4} \alpha_i.$$

Finally we intersect C_E with the half-plane

$$\{(x, z) \in \mathbb{R}^2 \times \mathbb{R} : \exists \lambda > 0 : x = x_0 + \lambda n\}$$

and we get at most four points

$$\widetilde{n}_i := (n + x_0, \langle \epsilon_i, n\rangle + u(x_0)) \in \mathbb{R}^2 \times \mathbb{R}, \qquad i = 1, \dots, 4. \tag{3.25}$$

Now we consider the line segment $L(\widetilde{n}_i) \subset \mathbb{R}^2 \times \mathbb{R}$ starting from $(x_0 + \rho n, u(x_0 + \rho n))$ defined as

$$L(\widetilde{n}_i) := \{(x_0 + \rho n, u(x_0 + \rho n)) + s(\widetilde{n}_i - (x_0, u(x_0)) : s \in (0, \delta)\} \tag{3.26}$$

The proof of the theorem is achieved in four steps.

Step 1.

$$\forall i = 1, \dots, 4 \qquad \widetilde{I_n^u} \neq L(\widetilde{n}_i) \implies I_n \cap \Sigma_E \neq \emptyset. \tag{3.27}$$

If I_n contains a point x where u is not differentiable, then $x \in \Sigma_E$ and (3.27) is true. Therefore we can assume that u is differentiable at each point of I_n. Since for any $i = 1, \dots, 4$ we have $\widetilde{I_n^u} \neq L(\widetilde{n}_i)$, the mean value theorem guarantees that for every $i = 1, \dots, 4$ there exists $s_i \in (0, \delta)$ such that

$$\frac{d}{ds} u\big(x_0 + (\rho + s)n\big)_{|s=s_i} \neq \langle \epsilon_i, n\rangle. \tag{3.28}$$

Now we recall that, for differentiable (not necessarily $\mathcal{C}^1$) functions $f : [a, b] \to \mathbb{R}$, f' takes all values between $f'(a)$ and $f'(b)$. Therefore we can choose $s_* \in (0, \delta)$ such that (3.28) holds with $s_i = s_*$ for every $i = 1, \dots, 4$. Thus

$$Du(x_0 + (\rho + s_*)n) \neq \epsilon_i \qquad \forall i = 1, \dots, 4,$$

so that $x_0 + (\rho + s_*)n \in I_n \cap \Sigma_E$.

Step 2.

$$\begin{aligned} &\text{for infinitely many } n \in \mathbb{S}^1 \quad \exists i = i_n \in \{1, \dots, 4\} : \widetilde{I_n^u} = L(\widetilde{n}_i) \\ &\qquad\qquad\qquad \Downarrow \\ &\mathcal{H}^1(\Sigma_E \cap B(x_0, r)) \;=\; +\infty. \end{aligned} \tag{3.29}$$

Since we are dealing with infinitely many n's, we are allowed to get rid of eight of them: the four $n = \epsilon_i$ for $i = 1, \dots, 4$ and the four n's that divide the angle between ϵ_i and ϵ_{i+1} into two equal parts. Due to the symmetry of the problem, it suffices to consider the case

$$n = (\cos\theta)\epsilon_1 + (\sin\theta)\epsilon_2, \quad 0 < \theta < \frac{\pi}{4}. \tag{3.30}$$

Therefore we assume that for infinitely many n for which (3.30) holds there exists $i \in \{1, \dots, 4\}$ such that $\widetilde{I_n^u} = L(\widetilde{n}_i)$. This implies

$$\frac{d}{ds} u(x_0 + (\rho + s)n) = \langle \epsilon_i, n\rangle \qquad \forall s \in (0, \delta). \tag{3.31}$$

If $i = 1$ in (3.31), because of (3.30), the point $(x_0 + (\rho + s)n, u(x_0 + (\rho + s)n))$ belongs to the upper flat part of the boundary of $\pi_{(x_0+\rho n, u(x_0+\rho n))}$. Thus Theorem 3.1 guarantees the existence of an open set $\mathcal{R}$ such that $B(x_0,r)\cap\mathcal{R}\neq\emptyset$ and $Du=\epsilon_1$ on $B(x_0,r)\cap\mathcal{R}$, and this contradicts the assumption $B(x_0,r)\subset\Omega\setminus\overline{\Omega_E}$.

If $i=3$ in (3.31), the point $(x_0+(\rho+s)n, u(x_0+(\rho+s)n))$ belongs to the lower flat part of the boundary of $\pi_{(x_0+\rho n, u(x_0+\rho n))}$, thus Theorem 3.1 gives a contradiction.

Therefore we have $i=2$ or $i=4$ in (3.31). We assume $i=2$, the other case is similar. For every $s\in(0,\delta)$, $\delta_1>0$ and $\tau\in(0,\delta_1)$ we use Theorem 2.8 and we get

$$u(x_0+(\rho+s)n+\tau\epsilon_2)-u(x_0+(\rho+s)n)\leq\tau. \tag{3.32}$$

We claim that the strict inequality holds in (3.32). If this is not the case, the point $(x_0+(\rho+s)n+\tau\epsilon_2, u(x_0+(\rho+s)n+\tau\epsilon_2))$ belongs to the upper flat part of the boundary of $\pi_{(x_0+(\rho+s)n, u(x_0+(\rho+s)n))}$, thus Theorem 3.1 gives a contradiction.
For every $s\in(0,\delta)$ either $x_0+(\rho+s)n\in\Sigma_E$ or u is differentiable at $x_0+(\rho+s)n$ with $Du(x_0+(\rho+s)n)\in E$, thus $Du(x_0+(\rho+s)n)=\epsilon_2$ because of (3.31) and (3.30). In this last case we get

$$\tau-o(\tau)\leq u(x_0+(\rho+s)n+\tau\epsilon_2)-u(x_0+(\rho+s)n)<\tau. \tag{3.33}$$

Now we have the alternative: either there is $\tau\in(0,\delta_1)$ such that u is not differentiable at $x_0+(\rho+s)n+\tau\epsilon_2$, or for every $\tau\in(0,\delta_1)$ u is differentiable at $x_0+(\rho+s)n+\tau\epsilon_2$, thus (3.33) and mean value theorem imply that there is $\tau\in(0,\delta_1)$ such that

$$0<\frac{d}{dt}u(x_0+(\rho+s)n+t\epsilon_2)_{|t=\tau}<1. \tag{3.34}$$

In both cases $x_0+(\rho+s)n+\tau\epsilon_2\in\Sigma_E$. We have proved that

$$\forall s\in(0,\delta)\ \ \exists\tau\in[0,\delta_1):\quad x_0+(\rho+s)n+\tau\epsilon_2\in\Sigma_E. \tag{3.35}$$

Define

$$I_{n,\delta_1}:=\{x_0+(\rho+s)n+\tau\epsilon_2:\quad s\in(0,\delta),\quad \tau\in(-\delta_1,\delta_1)\}$$

and

$$F(x_0+(\rho+s)n+\tau\epsilon_2):=x_0+(\rho+s)n. \tag{3.36}$$

It turns out that F is Lipschitz and

$$\text{Lip } F\leq\sqrt{1+(\tan\theta)^2}\leq\sqrt{2}, \tag{3.37}$$

where we used (3.30). Employing (3.35) and (3.36) gives

$$I_n=F(\Sigma_E\cap I_{n,\delta_1}), \tag{3.38}$$

so that

$$\delta=\mathcal{H}^1(I_n)=\mathcal{H}^1(F(\Sigma_E\cap I_{n,\delta_1}))\leq(\text{ Lip } F)\mathcal{H}^1(\Sigma_E\cap I_{n,\delta_1}).$$

Taking into account (3.37) we get

$$\delta/\sqrt{2}\leq\mathcal{H}^1(\Sigma_E\cap I_{n,\delta_1}). \tag{3.39}$$

We are now ready to prove (3.29). We have infinitely many $n \in \mathbb{S}^1$ such that there exists $i \in \{1, \dots, 4\}$ for which $\widetilde{I}_n^u = L(\widetilde{n}_i)$ and (3.30) holds true. For every $m \in \mathbb{N}$ we consider exactly m of them, say $n^1, \dots, n^m$. We choose $\delta_1 > 0$ so small in such a way that $I_{n^\beta,\delta_1} \cap I_{n^\gamma,\delta_1} = \emptyset$ for $\beta \neq \gamma$ and $I_{n^\beta,\delta_1} \subset B(x_0, r)$ for every $\beta = 1, \dots, m$. Then

$$\mathcal{H}^1(\Sigma_E \cap B(x_0, r)) \geq \mathcal{H}^1(\Sigma_E \cap \cup_{\beta=1}^m I_{n^\beta,\delta_1}) = \sum_{\beta=1}^m \mathcal{H}^1(\Sigma_E \cap I_{n^\beta,\delta_1}) \geq m\delta/\sqrt{2}.$$

Since m can be arbitrarily large, we get $\mathcal{H}^1(\Sigma_E \cap B(x_0, r)) = +\infty$.

Step 3.

$$\begin{aligned} &\text{only for a finite number of } n \in \mathbb{S}^1 \quad \exists i = i_n \in \{1, \dots, 4\} : \widetilde{I}_n^u = L(\widetilde{n}_i) \\ &\qquad\qquad \Downarrow \\ &\mathcal{H}^1\left(\Sigma_E \cap (B(x_0, \rho + \delta) \setminus B(x_0, \rho))\right) \geq 2\pi\rho. \end{aligned}$$

Let $S_* \subset \mathbb{S}^1$ be finite the set of $n \in \mathbb{S}^1$ for which there exists $i = i_n \in \{1, \dots, 4\}$ such that $\widetilde{I}_n^u = L(\widetilde{n}_i)$. For every $n \in \mathbb{S}^1 \setminus S_*$ we have $\widetilde{I}_n^u \neq L(\widetilde{n}_i)$ for every $i \in \{1, \dots, 4\}$, thus *step 1* implies $\Sigma_E \cap I_n \neq \emptyset$. Let us summarize as follows:

$$n \in \mathbb{S}^1 \setminus S_* \implies \Sigma_E \cap I_n \neq \emptyset. \tag{3.40}$$

Let us consider the projection G of $B(x_0, \rho + \delta) \setminus B(x_0, \rho)$ onto $\partial B(x_0, \rho)$:

$$G(x) := x_0 + \rho \frac{x - x_0}{|x - x_0|}$$

It turns out that G is Lipschitz with Lip $G \leq 1$ and

$$G(x_0 + (\rho + s)n) = x_0 + \rho n \qquad \forall n \in \mathbb{S}^1,\ \forall s \in (0, \delta). \tag{3.41}$$

Using (3.40) we get

$$\partial B(x_0, \rho) \setminus \{x_0 + \rho n : n \in S^*\} \subset G(\Sigma_E \cap (B(x_0, \rho + \delta) \setminus B(x_0, \rho))).$$

Since we are assuming $\mathcal{H}^0(S^*) < +\infty$, we deduce

$$\begin{aligned} 2\pi\rho &= \mathcal{H}^1(\partial B(x_0, \rho)) = \mathcal{H}^1(\partial B(x_0, \rho) \setminus \{x_0 + \rho n : n \in S^*\}) \\ &\leq \mathcal{H}^1(G(\Sigma_E \cap (B(x_0, \rho + \delta) \setminus B(x_0, \rho)))) \\ &\leq (\text{Lip } G)\mathcal{H}^1(\Sigma_E \cap (B(x_0, \rho + \delta) \setminus B(x_0, \rho)))). \end{aligned}$$

Since Lip $G \leq 1$, *step 3* is proved.

Step 4. Let us consider two sequences $\{\rho_k\}$, $\{\delta_k\}$ such that

$$r/2 = \rho_1, \quad \rho_k < \rho_k + \delta_k < \rho_{k+1} < r \qquad \forall k \in \mathbb{N}.$$

Let I_n^k, $\widetilde{I}_n^{uk}$ and $L^k(\widetilde{n}_i)$ be the sets corresponding to $\rho = \rho_k$, $\delta = \delta_k$ in (3.23), (3.24) and (3.26). If there exists $k \in \mathbb{N}$ such that

$$\text{for infinitely many } n \in \mathbb{S}^1 \ \exists i = i_n \in \{1, \dots, 4\} : \widetilde{I}_n^{uk} = L^k(\widetilde{n}_i) \tag{3.42}$$

then *step 2* gives the equality $\mathcal{H}^1(\Sigma_E \cap B(x_0, r)) = +\infty$ and the proof is concluded. Therefore we can assume that for every $k \in \mathbb{N}$ formula (3.42) is false. Then *step 3* gives the estimate

$$\mathcal{H}^1\left(\Sigma_E \cap (B(x_0, \rho_k + \delta_k) \setminus B(x_0, \rho_k))\right) \geq 2\pi\rho_k. \tag{3.43}$$

The sets $B(x_0, \rho_k + \delta_k) \setminus B(x_0, \rho_k)$ are pairwise disjoint since $\rho_k + \delta_k < \rho_{k+1}$. Moreover $B(x_0, \rho_k + \delta_k) \subset B(x_0, r)$ and $\rho_k \geq r/2$. Then

$$\begin{aligned}
& \mathcal{H}^1\big(\Sigma_E \cap B(x_0, r)\big) \geq \mathcal{H}^1\big(\cup_{k=1}^{+\infty} (\Sigma_E \cap (B(x_0, \rho_k + \delta_k) \setminus B(x_0, \rho_k)))\big) \\
= & \sum_{k=1}^{+\infty} \mathcal{H}^1\big(\Sigma_E \cap (B(x_0, \rho_k + \delta_k) \setminus B(x_0, \rho_k))\big) \\
\geq & \sum_{k=1}^{+\infty} 2\pi\rho_k \geq \sum_{k=1}^{+\infty} 2\pi r/2 = +\infty.
\end{aligned}$$

This concludes the proof. ■

If we know a priori that $P(\Omega_i, \Omega)$ is finite for any i, then we use (2.8) so that the $\mathcal{H}^1(\Sigma_E)$ is finite too. Thus (3.22) tells us that $\Omega \setminus \overline{\Omega_E}$ is empty: this is summarized in the next corollary.

Corollary 3.3 *Let $\Omega \subset \mathbb{R}^2$ be a convex open set. Let $u \in W^{1,\infty}(\Omega)$ satisfy* (1.3). *Assume that $P(\Omega_i, \Omega) < +\infty$ for all $i = 1, \ldots, 4$. Then $\mathcal{H}^1(\Sigma_E) < +\infty$ and*

$$\Omega \setminus \overline{\Omega_E} = \emptyset.$$

Then next result claims that $\Omega_E \cap \Omega_i$ is open.

Theorem 3.4 *Let $\Omega \subset \mathbb{R}^2$ be a convex open set. Let $u \in W^{1,\infty}(\Omega)$ satisfy* (1.3). *If $B(x_0, r) \subset \Omega \setminus \Sigma_E$ for some $r > 0$, then there exists $i \in \{1, \ldots, 4\}$ such that*

$$Du(x) = \epsilon_i \quad \forall x \in B(x_0, r).$$

Moreover $\Omega_E \cap \Omega_j$ is open for all $j \in \{1, \ldots, 4\}$, and $\Omega_E = \cup_{j=1}^4 (\Omega_E \cap \Omega_j)$.

Proof. Since $x_0 \in \Omega \setminus \Sigma_E$, there exists $i \in \{1, \ldots, 4\}$ such that

$$Du(x_0) = \epsilon_i.$$

Take $x \in B(x_0, r)$ with $x \neq x_0$. Since $tx + (1-t)x_0 \in B(x_0, r) \subset \Omega \setminus \Sigma_E$ for every $t \in [0, 1]$, we have that u is differentiable at $tx + (1-t)x_0$ and there exists $j(t) \in \{1, \ldots, 4\}$ such that $Du(tx + (1-t)x_0) = \epsilon_{j(t)}$. In particular, there exists $j \in \{1, \ldots, 4\}$ such that $Du(x) = \epsilon_j$. If $\epsilon_j = \epsilon_i$, we are done. Let us assume that $\epsilon_j \neq \epsilon_i$ and define $G(t) := u(tx + (1-t)x_0)$. For every $t \in [0, 1]$ the derivative $G'(t)$ exists and $G'(t) = \langle Du(tx + (1-t)x_0), x - x_0 \rangle$. Moreover G' takes all intermediate values between $G'(0) = \langle Du(x_0), x - x_0 \rangle = \langle \epsilon_i, x - x_0 \rangle$ and $G'(1) = \langle Du(x), x - x_0 \rangle = \langle \epsilon_j, x - x_0 \rangle$.
We now observe that $G'(0) = G'(1)$ if and only if $\langle \epsilon_i - \epsilon_j, x - x_0 \rangle = 0$. Since $\epsilon_j \neq \epsilon_i$, we have $\epsilon_i - \epsilon_j \in \{\epsilon_i - \epsilon_{i+1}, \epsilon_i - \epsilon_{i+2}, \epsilon_i - \epsilon_{i+3}\}$, giving rise to the three lines

$$L_s = \{y \in \mathbb{R}^2 : \langle \epsilon_i - \epsilon_s, y \rangle = 0\} \qquad s = i+1, i+2, i+3. \tag{3.44}$$

Assume that $x \in B(x_0,r) \setminus \cup_{s=i+1}^{i+3}(x_0+L_s)$, where $x_0+L_s = \{x \in \mathbb{R}^2 : \langle \epsilon_i - \epsilon_s, x-x_0 \rangle = 0\}$. Then $\langle \epsilon_i - \epsilon_s, x - x_0 \rangle \neq 0$ for every $s = i+1, i+2, i+3$ so that $G'(0) \neq G'(1)$. Thus $G'(t)$ takes all the infinitely many intermediate values between $G'(0)$ and $G'(1)$: this is impossible since $G'(t) = \langle \epsilon_{j(t)}, x - x_0 \rangle$ and $\epsilon_{j(t)}$ takes the four values $\epsilon_1, \ldots, \epsilon_4$. We have proved that

$$x \in B(x_0,r) \setminus \cup_{s=i+1}^{i+3}(x_0+L_s) \Longrightarrow Du(x) = Du(x_0) = \epsilon_i. \tag{3.45}$$

Let us assume that $x \in B(x_0,r) \cap \cup_{s=i+1}^{i+3}(x_0+L_s)$ with $x \neq x_0$ and $Du(x) = \epsilon_j \neq \epsilon_i = Du(x_0)$. We can use Theorem 2.6 and we get

$$\lim_{\rho \to 0^+} \frac{\mathcal{H}^2(B(x,\rho) \cap \Omega_j)}{\mathcal{H}^2(B(x,\rho))} = 1. \tag{3.46}$$

Note that $0 < r - |x - x_0|$ and $B(x,\rho) \subset B(x_0,r)$ for every $\rho \in (0, r - |x - x_0|]$, thus, taking into account that $\mathcal{H}^2(x_0 + L_s) = 0$ and $Du = \epsilon_i \neq \epsilon_j$ on $B(x_0,r) \setminus \cup_{s=i+1}^{i+3}(x_0+L_s)$, we have

$$\begin{aligned} 0 &\leq \mathcal{H}^2\big(B(x,\rho) \cap \Omega_j\big) \leq \mathcal{H}^2\big(B(x_0,r) \cap \Omega_j\big) \\ &= \mathcal{H}^2\big([B(x_0,r) \setminus \cup_{s=i+1}^{i+3}(x_0+L_s)] \cap \Omega_j\big) = 0. \end{aligned}$$

The previous inequalities give $\mathcal{H}^2(B(x,\rho) \cap \Omega_j) = 0$ for every $\rho \in (0, r - |x - x_0|]$ and this contradicts (3.46). Thus

$$x \in B(x_0,r) \Longrightarrow Du(x) = Du(x_0) = \epsilon_i$$

and this concludes the proof. ■

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