Calculus of variations, implicit partial differential equations and microstructure

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Received 14 May 2004, accepted 2 January 2005

Key words Calculus of variations, microstructure
MSC (2000) 74N15

We study existence of solutions for implicit partial differential equations of the form

$$\begin{cases} F(x,u,Du)=0 & \text{a.e. in } \Omega \\ u=\varphi & \text{on } \partial\Omega. \end{cases}$$

as well as minimization problems of the type

$$\inf\left\{\int_\Omega f(Du(x))\,dx : u=\varphi \text{ on } \partial\Omega\right\}.$$

We discuss several examples that are relevant for applications to geometry, non linear elasticity or optimal design. All these examples exhibit what can be called microstructures.

1 Introduction

We discuss the existence of Lipschitz solutions of *implicit* equations of the type

$$\begin{cases} F(x,u(x),Du(x))=0 & \text{a.e. } x\in\Omega \\ u(x)=\varphi(x) & x\in\partial\Omega. \end{cases}$$

where $\Omega\subset\mathbb{R}^n$ is an open set, $u:\Omega\to\mathbb{R}^m$ (if $m=1$ or, by abuse of language, if $n=1$, we will say that it is *scalar* valued while if $m,n\geq 2$, we will speak of the *vector* valued case) and Du denotes its Jacobian matrix, i.e. $Du=\left(\frac{\partial u_i}{\partial x_j}\right)$ seen as a real matrix of size $m\times n$, $F:\Omega\times\mathbb{R}^m\times\mathbb{R}^{m\times n}\to\mathbb{R}$ is continuous and φ is a given Lipschitz map.

To make simpler the notations and the results we will mostly assume that the function F does not depend explicitly on the variables x and u. Many extensions to the more general case are available, however they are more complicate to state and are, usually, not as sharp, we refer for a thorough discussion on these matters to Dacorogna-Marcellini [16]; we, however, give one such result in Theorem 23.

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Depending on the context it might be convenient to rewrite (in an equivalent way) the problem in terms of differential inclusions, more precisely

$$\begin{cases} Du(x) \in E & \text{a.e. } x \in \Omega \\ u(x) = \varphi(x) & x \in \partial\Omega \end{cases}$$

where $E = \{\xi \in \mathbb{R}^{m\times n} : F(\xi) = 0\}$. It will turn out that the sets that play the central role are the *convex hull* of E (denoted by $\operatorname{co} E$) in the scalar case and the *quasiconvex hull* (denoted $\operatorname{Qco} E$) in the vectorial case. These notions as well as the related concepts of *polyconvex* and *rank one convex hulls* (denoted respectively by $\operatorname{Pco} E$ and $\operatorname{Rco} E$) are defined in the next section.

We then show how this analysis applies to minimization of *non convex* (more precisely *non quasiconvex)* problems of the calculus of variations

$$\inf\left\{\int_\Omega f(Du(x))\,dx : u = \varphi \text{ on } \partial\Omega\right\}.$$

We do not discuss now the abstract results that will be precisely stated in the following sections, but we rather discuss several examples that are mathematically as well as physically relevant.

We start with the scalar case (here $m = 1$, while $n \geq 1$).

Example 1 (Convex Hamiltonian and the eikonal equation) *The most important example, and in some sense the prototype, is the eikonal equation*

$$\begin{cases} |Du(x)| = 1 & a.e.\ x \in \Omega \\ u(x) = \varphi(x) & x \in \partial\Omega. \end{cases}$$

In terms of the set E we have here that E is the unit sphere and its convex hull is thus the unit ball.

More generally, replacing the norm $|.|$ by a convex (and coercive) Hamiltonian F (in the particular case of the eikonal equation we have $F(\xi) = |\xi| - 1$) we have $E = \{\xi \in \mathbb{R}^n : F(\xi) = 0\}$ and its convex hull is

$$\operatorname{co} E = \{\xi \in \mathbb{R}^n : F(\xi) \leq 0\}.$$

There is a huge literature on the subject. The main tool for solving this type of problems is the "viscosity method" introduced by Hopf, Kruzkov, Crandall-Lions and others. This is a very powerful technique, but its drawback is that it is not very well adapted to (scalar) non convex Hamiltonians and to the vectorial case. Let us also say that, for applications, the existence of viscosity solutions, usually, prevents the formation of microstructures.

Example 2 (Non convex Hamiltonian and the eikonal system) *Consider, for $\xi \in \mathbb{R}^n$, the non convex Hamiltonian*

$$F(\xi) = \sum_{i=1}^{n}\left[(\xi_i)^2 - 1\right]^2$$

and the associated equation $F(Du) = 0$. *It is easily seen to be equivalent to the following system*

$$\begin{cases} \left|\frac{\partial u}{\partial x_i}\right| = 1,\ i = 1, ..., n & a.e.\ in\ \Omega \\ u = \varphi & on\ \partial\Omega\,. \end{cases}$$

Here E *is therefore composed of the vertices of the cube* $[-1,1]^n$ *and its convex hull is precisely the cube.*

It can be shown, cf. Cardaliaguet-Dacorogna-Gangbo-Georgy [4]*, that there is no viscosity solution and that, near the boundary, microstructure should occur. This problem has also been investigated numerically by Dacorogna-Glowinski-Pan* [11].

We now turn our attention to the vector valued case (i.e., $m, n \geq 2$). Before describing the results, let us emphasize, once more, that, in the vector valued case, the general rule is that microstructure will always occur.

Example 3 (Singular values) *This problem has been studied in Dacorogna-Tanteri* [21] *and in Dacorogna-Marcellini* [16]*; it is an important problem in geometry and in non linear elasticity. First let* $0 \leq \lambda_1(\xi) \leq ... \leq \lambda_n(\xi)$ *denote the singular values of a matrix* $\xi \in \mathbb{R}^{n\times n}$ *(cf. below) and let* $0 < a_1 \leq ... \leq a_n$*. The problem is to find a map* u *satisfying*

$$\begin{cases} \lambda_i(Du) = a_i,\ i = 1, ..., n & a.e.\ in\ \Omega \\ u = \varphi & on\ \partial\Omega\,. \end{cases}$$

Generalizations of this problem have been considered in Dacorogna-Tanteri [22]*, namely*

$$\lambda_i(Du) = a_i,\ i = 2, ..., n \quad and \quad \det Du = 1,$$

a problem that turns out to be important for incompressible materials, or in Dacorogna-Ribeiro [20] *where the problem is*

$$\lambda_i(Du) = a_i,\ i = 2, ..., n \quad and \quad \det Du \in \{\alpha, \beta\}\,.$$

This last problem in fact contains both previous cases (the first one if we set $\beta = -\alpha$*, the second one if we let* $\alpha = \beta = 1$*).*

The solving of these equations have direct applications in the calculus of variations, notably (cf. Dacorogna-Ribeiro [20]*) to*

$$\inf\left\{\int_\Omega g(\det Du(x))\,dx : u = \varphi\ on\ \partial\Omega\right\}$$

which turns out to have applications for phase transitions (see Dacorogna [8] *or* [10]*). It can also be applied to Saint Venant Kirchhoff materials (cf. Dacorogna-Marcellini* [12]*).*

In terms of sets we can write for the three cases under consideration, respectively

$$E = \left\{\xi \in \mathbb{R}^{n\times n} : \lambda_i(\xi) = a_i,\ i = 1, ..., n\right\}$$

$$E_1 = \left\{ \xi \in \mathbb{R}^{n\times n} : \lambda_i(\xi) = a_i,\ i = 2, ..., n,\ \det \xi = 1 \right\}$$
$$E_{\alpha,\beta} = \left\{ \xi \in \mathbb{R}^{n\times n} : \lambda_i(\xi) = a_i,\ i = 2, ..., n,\ \det \xi \in \{\alpha, \beta\} \right\}.$$

Under some natural compatibility conditions on the a_i, our result gives

$$\operatorname{co} E = \left\{ \xi \in \mathbb{R}^{n\times n} : \sum_{i=\nu}^{n} \lambda_i(\xi) \leq \sum_{i=\nu}^{n} a_i,\ \nu = 1, ..., n \right\}$$

$$\operatorname{Pco} E = \overline{\operatorname{Qco}} E = \operatorname{Rco} E = \left\{ \xi \in \mathbb{R}^{n\times n} : \prod_{i=\nu}^{n} \lambda_i(\xi) \leq \prod_{i=\nu}^{n} a_i,\ \nu = 1, ..., n \right\}.$$

$$\operatorname{Pco} E_1 = \operatorname{Rco} E_1 = \left\{ \xi \in \mathbb{R}^{n\times n} : \prod_{i=\nu}^{n} \lambda_i(\xi) \leq \prod_{i=\nu}^{n} a_i,\ \nu = 2, ..., n,\ \det \xi = 1 \right\}.$$

$$\operatorname{Pco} E_{\alpha,\beta} = \operatorname{Rco} E_{\alpha,\beta} = \left\{ \xi \in \mathbb{R}^{n\times n} : \prod_{i=\nu}^{n} \lambda_i(\xi) \leq \prod_{i=\nu}^{n} a_i,\ \nu = 2, ..., n,\ \det \xi \in [\alpha, \beta] \right\}.$$

Example 4 (Complex eikonal equation) *The problem has been introduced by Magnanini and Talenti* [28] *motivated by problems of geometrical optics with diffraction. Given $g \in \mathbb{R}$, the question is to find a complex function $w : \Omega \subset \mathbb{R}^n \to \mathbb{C}$ ($w(x) = u(x) + i\, v(x)$) satisfying*

$$\begin{cases} \sum_{i=1}^{n} w_{x_i}^2 = g^2 & a.e.\ in\ \Omega \\ w = \varphi & on\ \partial\Omega \end{cases}$$

where $w_{x_i} = \partial w / \partial x_i$. The problem is then equivalent to

$$\begin{cases} |Du|^2 = |Dv|^2 + g^2 & a.e.\ in\ \Omega, \\ \langle Dv; Du \rangle = 0 & a.e.\ in\ \Omega, \\ w = \varphi & on\ \partial\Omega. \end{cases}$$

This problem can be brought back to the previous analysis (see Dacorogna-Marcellini [16] *or Dacorogna-Tanteri* [22]*). In terms of the set E we have, setting $s = \sqrt{r^2 + g^2}$,*

$$E = \left\{ \xi = \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^{2\times n} \ :\ |a| = s,\ \ |b| = r \text{ and } \langle a; b \rangle = 0 \right\}.$$

Letting

$$A = \begin{pmatrix} 1/s & 0 \\ 0 & 1/r \end{pmatrix} \in \mathbb{R}^{2\times 2}$$

it can be proved that

$$\operatorname{Pco} E = \operatorname{Rco} E = \left\{ \xi \in \mathbb{R}^{2\times n} \ :\ \lambda_1(A\xi),\ \lambda_2(A\xi) \leq 1 \right\}$$

where $\lambda_1(A\xi)$, $\lambda_2(A\xi)$ are the singular values of the matrix $A\xi$.

Example 5 (Nematic elastomers) *This example is related to some work of DeSimone-Dolzmann* [24] *on nematic elastomers; we refer to this article for the description of the physical model. It turns out that it can also be treated as a problem of singular values. More precisely we let $r < 1$ (this is called the oblate case while $r > 1$ is called the prolate case and it can be handled similarly) and, as above, $0 \leq \lambda_1(\xi) \leq \ldots \leq \lambda_n(\xi)$ denote the singular values of a matrix $\xi \in \mathbb{R}^{n\times n}$. The problem is then to solve, and this was achieved by Dacorogna-Tanteri* [22],

$$\begin{cases} \lambda_1(Du(x)) = \ldots = \lambda_{n-1}(Du(x)) = r^{1/2n} & a.e.\ in\ \Omega, \\ \lambda_n(Du(x)) = r^{(1-n)/2n},\ \det Du(x) = 1 & a.e.\ in\ \Omega, \\ u = \varphi & on\ \partial\Omega. \end{cases}$$

In terms of sets we have

$$E = \left\{\xi : \lambda_\nu(\xi) = r^{\frac{1}{2n}},\ 1 \leq \nu \leq n-1,\ \lambda_n(\xi) = r^{\frac{(1-n)}{2n}},\ \det\xi = 1\right\}.$$

$$\begin{aligned} \operatorname{Pco} E = \operatorname{Rco} E &= \left\{\xi : \prod_{i=\nu}^{n} \lambda_i(\xi) \leq r^{(1-\nu)/2n},\ 2 \leq \nu \leq n,\ \det\xi = 1\right\} \\ &= \left\{\xi :\ \lambda_\nu(\xi) \in \left[r^{1/2n}, r^{(1-n)/2n}\right],\ 1 \leq \nu \leq n,\ \det\xi = 1\right\}. \end{aligned}$$

Example 6 (Optimal design) *The classical problem of optimal design studied by Kohn-Strang* [27] *which is mathematically formulated as (here $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$)*

$$\inf\left\{\int_\Omega f(Du(x))\,dx : u = \varphi \text{ on } \partial\Omega\right\}$$

where

$$f(\xi) = \begin{cases} 1 + |\xi|^2 & \text{if } \xi \neq 0 \\ 0 & \text{if } \xi = 0. \end{cases}$$

can also be treated, cf. Dacorogna-Marcellini [12] *and* [16]*, by the methods discussed in the next sections.*

The associated algebraic problem is (denoting the set of 2×2 symmetric matrices by $\mathbb{R}_s^{2\times 2}$)

$$E = \left\{\xi \in \mathbb{R}_s^{2\times 2} \ :\ \operatorname{trace}\xi \in \{0,1\},\ \det\xi \geq 0\right\},$$

$$\operatorname{Rco} E = \operatorname{co} E = \left\{\xi \in \mathbb{R}_s^{2\times 2} \ :\ 0 \leq \operatorname{trace}\xi \leq 1,\ \det\xi \geq 0\right\}.$$

Example 7 (Potential wells) *The general problem of potential wells has been intensively studied by many authors in conjunction with crystallographic models involving fine microstructures. The reference papers on the subject are Ball and James* [1], [2]*. The problem is very difficult and at the moment only the case of two potential wells in two dimensions has been fully resolved, cf. Müller-Sverak in* [30] *and Dacorogna-Marcellini in* [13] *and* [16] *for the case $\det A \neq \det B$ and cf. Müller-Sverak in* [31] *and Dacorogna-Tanteri in* [22] *for the case* $\det A = \det B$. *The problem can be formulated as, given two matrices* $A, B \in \mathbb{R}^{2\times 2}$ *with positive determinant, find a map* $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ *so that*

$$Du(x) \in E = SO(2)A \cup SO(2)B, \text{ a.e. } x \in \Omega$$

and with prescribed boundary value, $u = \varphi$ *on* $\partial\Omega$.

The different convex hulls have been computed by Sverak in [33].

2 Preliminaries

We recall the main notations that we will use throughout the article and we refer, if necessary, for more details to Dacorogna-Marcellini [16] and to Dacorogna [10].

In the remaining part of the article we will always assume that the sets $\Omega \subset \mathbb{R}^n$ are bounded and open. The boundedness is however not a real restriction, since all constructions are done locally.

Notation 8 *We will denote by*

- $W^{1,\infty}(\Omega;\mathbb{R}^m)$ *the space of maps* $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ *such that*

$$u \in L^\infty(\Omega;\mathbb{R}^m) \text{ and } Du = \left(\frac{\partial u^i}{\partial x_j}\right)_{1\le j\le n}^{1\le i\le m} \in L^\infty\left(\Omega;\mathbb{R}^{m\times n}\right);$$

- $W_0^{1,\infty}(\Omega;\mathbb{R}^m) = W^{1,\infty}(\Omega;\mathbb{R}^m) \cap W_0^{1,1}(\Omega;\mathbb{R}^m)$;

- $Aff_{piec}\left(\overline{\Omega};\mathbb{R}^m\right)$ *will stand for the subset of* $W^{1,\infty}(\Omega;\mathbb{R}^m)$ *consisting of piecewise affine maps;*

- $C^1_{piec}\left(\overline{\Omega};\mathbb{R}^m\right)$ *will denote the subset of* $W^{1,\infty}(\Omega;\mathbb{R}^m)$ *consisting of piecewise* C^1 *maps.*

We next define the main notions of convexity used throughout the article.

Definition 9 *(i) A function* $f : \mathbb{R}^{m\times n} \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ *is said to be polyconvex if*

$$f\left(\sum_{i=1}^{\tau+1} t_i\xi_i\right) \le \sum_{i=1}^{\tau+1} t_i f(\xi_i)$$

whenever $t_i \ge 0$ *and*

$$\sum_{i=1}^{\tau+1} t_i = 1,\ T\left(\sum_{i=1}^{\tau+1} t_i\xi_i\right) = \sum_{i=1}^{\tau+1} t_i T(\xi_i)$$

where for a matrix $\xi \in \mathbb{R}^{m\times n}$ *we let*

$$T(\xi) = (\xi, adj_2\xi, \ldots, adj_{m\wedge n}\xi)$$

where $adj_s\xi$ *stands for the matrix of all* $s \times s$ *subdeterminants of the matrix* ξ, $1 \leq s \leq m \wedge n = \min\{m, n\}$ *and where*

$$\tau = \tau(m,n) = \sum_{s=1}^{m \wedge n} \binom{m}{s}\binom{n}{s} \text{ and } \binom{m}{s} = \frac{m!}{s!\,(m-s)!}.$$

(ii) A Borel measurable function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ *is said to be quasiconvex if*

$$\int_U f(\xi + D\varphi(x))\, dx \geq f(\xi) \operatorname{meas}(U)$$

for every bounded domain $U \subset \mathbb{R}^n$, $\xi \in \mathbb{R}^{m\times n}$, *and* $\varphi \in W_0^{1,\infty}(U; \mathbb{R}^m)$.

(iii) A function $f : \mathbb{R}^{m\times n} \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ *is said to be rank one convex if*

$$f(t\xi_1 + (1-t)\xi_2) \leq t\, f(\xi_1) + (1-t)\, f(\xi_2)$$

for every ξ_1, ξ_2 *with* $rank\{\xi_1 - \xi_2\} = 1$ *and every* $t \in [0,1]$.

(iv) A Borel measurable function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ *is said to be quasiaffine (or equivalently polyaffine or rank one affine) if both* f *and* $-f$ *are quasiconvex*

(v) The different envelopes of a given function f *are defined as*

$$\begin{aligned} Cf &= \sup\{g \leq f : g \text{ convex}\}, \\ Pf &= \sup\{g \leq f : g \text{ polyconvex}\}, \\ Qf &= \sup\{g \leq f : g \text{ quasiconvex}\}, \\ Rf &= \sup\{g \leq f : g \text{ rank one convex}\}. \end{aligned}$$

As well known we have that the following implications hold

$$f \text{ convex} \Longrightarrow f \text{ polyconvex} \Longrightarrow f \text{ quasiconvex} \Longrightarrow f \text{ rank one convex}.$$

Remark 10 *It can be shown that a quasiaffine function is necessary of the form*

$$f(\xi) = \langle \alpha; T(\xi) \rangle + \beta$$

for some constants $\alpha \in \mathbb{R}^\tau$ *and* $\beta \in \mathbb{R}$ *and where* $\langle .;. \rangle$ *stands for the scalar product in* $\mathbb{R}^\tau$*; which in the case* $m = n = 2$ *reads as (*$\alpha = (\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}, \alpha_5) \in \mathbb{R}^5$*)*

$$f(\xi) = \sum_{i,j=1}^{2} \alpha_{ij}\xi_{ij} + \alpha_5 \det \xi + \beta.$$

We now give an important example that concerns singular values.

Example 11 *Let* $0 \leq \lambda_1(\xi) \leq ... \leq \lambda_n(\xi)$ *denote the singular values of a matrix* $\xi \in \mathbb{R}^{n\times n}$, *which are defined as the eigenvalues of the matrix* $(\xi\xi^t)^{1/2}$. *The functions*

$$\xi \to \sum_{i=\nu}^{n} \lambda_i(\xi) \quad \text{and} \quad \xi \to \prod_{i=\nu}^{n} \lambda_i(\xi), \; \nu = 1, ..., n,$$

are respectively convex and polyconvex (note that $\prod_{i=1}^n \lambda_i(\xi) = |\det \xi|$*). In particular the function* $\xi \to \lambda_n(\xi)$ *is convex and in fact is the operator norm.*

We finally recall the notations for various convex hulls of sets.

Notation 12 *We let, for* $E\subset\mathbb{R}^{m\times n}$*,*

$$\overline{\mathcal{F}}_E=\left\{f:\mathbb{R}^{m\times n}\to\overline{\mathbb{R}}=\mathbb{R}\cup\{+\infty\}:f|_E\leq 0\right\}$$
$$\mathcal{F}_E=\left\{f:\mathbb{R}^{m\times n}\to\mathbb{R}:f|_E\leq 0\right\}.$$

We then have respectively, the convex, polyconvex, rank one convex and (closure of the) quasiconvex hull defined by

$$\operatorname{co}E=\left\{\xi\in\mathbb{R}^{m\times n}:f(\xi)\leq 0,\ \textit{for every convex } f\in\overline{\mathcal{F}}_E\right\}$$
$$\operatorname{Pco}E=\left\{\xi\in\mathbb{R}^{m\times n}:f(\xi)\leq 0,\ \textit{for every polyconvex } f\in\overline{\mathcal{F}}_E\right\}$$
$$\operatorname{Rco}E=\left\{\xi\in\mathbb{R}^{m\times n}:f(\xi)\leq 0,\ \textit{for every rank one convex } f\in\overline{\mathcal{F}}_E\right\}$$
$$\overline{\operatorname{Qco}}E=\left\{\xi\in\mathbb{R}^{m\times n}:f(\xi)\leq 0,\ \textit{for every quasiconvex } f\in\mathcal{F}_E\right\}.$$

We should point out that by replacing $\overline{\mathcal{F}}_E$ by $\mathcal{F}_E$ in the definitions of $\operatorname{co}E$ and $\operatorname{Pco}E$ we get their closures denoted by $\overline{\operatorname{co}}E$ and $\overline{\operatorname{Pco}}E$. However if we do so in the definition of $\operatorname{Rco}E$ we get a larger set than the closure of $\operatorname{Rco}E$. We should also draw the attention that some authors call the set

$$\left\{\xi\in\mathbb{R}^{m\times n}:f(\xi)\leq 0,\ \text{for every rank one convex } f\in\overline{\mathcal{F}}_E\right\}$$

the lamination convex hull, while they reserve the name of rank one convex hull to the set

$$\left\{\xi\in\mathbb{R}^{m\times n}:f(\xi)\leq 0,\ \text{for every rank one convex } f\in\mathcal{F}_E\right\}.$$

We think however that our terminology is more consistent with the classical definition of convex hull.

In general we have, for any set $E\subset\mathbb{R}^{m\times n}$,

$$E\subset\operatorname{Rco}E\subset\operatorname{Pco}E\subset\operatorname{co}E$$

$$\overline{E}\subset\overline{\operatorname{Rco}}E\subset\overline{\operatorname{Qco}}E\subset\overline{\operatorname{Pco}}E\subset\overline{\operatorname{co}}E.$$

3 Implicit partial differential equations

3.1 Abstract results for first order equations

We start with the following definition introduced by Dacorogna-Marcellini in [15] (cf. also [16]), which is the key condition to get existence of solutions.

Definition 13 (Relaxation property) *Let* $E,K\subset\mathbb{R}^{m\times n}$*. We say that* K *has the relaxation property with respect to* E *if for every bounded open set* $\Omega\subset\mathbb{R}^n$*, for every affine function* u_ξ *satisfying*

$$Du_\xi(x)=\xi\in K,$$

there exist a sequence $u_\nu \in Aff_{piec}\left(\overline{\Omega};\mathbb{R}^m\right)$

$$u_\nu \in u_\xi + W_0^{1,\infty}\left(\Omega;\mathbb{R}^m\right),\ Du_\nu(x) \in E \cup K,\ \text{a.e. in } \Omega$$

$$u_\nu \overset{*}{\rightharpoonup} u_\xi \text{ in } W^{1,\infty},\ \int_\Omega \operatorname{dist}\left(Du_\nu(x);E\right)dx \to 0 \text{ as } \nu \to \infty.$$

Remark 14 *(i) It is interesting to note that in the scalar case ($n=1$ or $m=1$) then $K = \operatorname{int}\operatorname{co} E$ has the relaxation property with respect to E.*

(ii) In the vectorial case we have that, if K has the relaxation property with respect to E, then necessarily

$$K \subset \overline{\mathrm{Qco}}E.$$

Indeed first recall that the definition of quasiconvexity implies that, for every quasiconvex $f \in \mathcal{F}_E$,

$$f(\xi)\operatorname{meas}\Omega \le \int_\Omega f\left(Du_\nu(x)\right)dx.$$

Combining this last result with the fact that $\{Du_\nu\}$ is uniformly bounded, the fact that any quasiconvex function is continuous and the last property in the definition of the relaxation property, we get the inclusion $K \subset \overline{\mathrm{Qco}}E$.

The main theorem is then.

Theorem 15 *Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Let $E, K \subset \mathbb{R}^{m\times n}$ be such that E is compact and K is bounded. Assume that K has the relaxation property with respect to E. Let $\varphi \in Aff_{piec}\left(\overline{\Omega};\mathbb{R}^m\right)$ be such that*

$$D\varphi(x) \in E \cup K,\ \text{a.e. in } \Omega.$$

Then there exists (a dense set of) $u \in \varphi + W_0^{1,\infty}\left(\Omega;\mathbb{R}^m\right)$ such that

$$Du(x) \in E,\ \text{a.e. in } \Omega.$$

Remark 16 *(i) According to Chapter 10 in* [16]*, the boundary datum φ can be more general if we make the following extra hypotheses:*

- in the scalar case, if K is open, φ can be even taken in $W^{1,\infty}\left(\Omega;\mathbb{R}^m\right)$, with $D\varphi(x) \in E \cup K$ (cf. Corollary 10.11 in [16]*);*

- in the vectorial case, if the set K is open, φ can be taken in $C^1_{piec}\left(\overline{\Omega};\mathbb{R}^m\right)$ (cf. Corollary 10.15 or Theorem 10.16 in [16]*), with $D\varphi(x) \in E \cup K$. While if K is open and convex, φ can be taken in $W^{1,\infty}\left(\Omega;\mathbb{R}^m\right)$ provided*

$$D\varphi(x) \in C,\ \text{a.e. in } \Omega$$

where $C \subset K$ is compact (cf. Corollary 10.21 in [16]*).*

(ii) This theorem has been established by Dacorogna-Pisante [18]*. However it was first proved by Dacorogna-Marcellini in* [15] *(cf. also Theorem 6.3 in* [16]*) under the further hypothesis that*

$$E = \left\{ \xi \in \mathbb{R}^{m \times n} : F_i(\xi) = 0,\ i = 1, 2, ..., I \right\}$$

where $F_i : \mathbb{R}^{m \times n} \to \mathbb{R}, i = 1, 2, ..., I,$ *are quasiconvex. This hypothesis was later removed by Sychev in* [34] *(see also Müller and Sychev* [32]*). Kirchheim in* [26] *pointed out that using a classical result of function theory then the proof of Dacorogna-Marcellini was still valid without the extra hypothesis on* E*.*

We now give the proof of Dacorogna-Pisante [18], which is a combination of [15] and [26].

Proof. We let $\overline{V}$ be the closure in $L^\infty(\Omega; \mathbb{R}^m)$ of

$$V = \left\{ u \in Aff_{piec}\left(\overline{\Omega}; \mathbb{R}^m\right) : u = \varphi \text{ on } \partial\Omega \text{ and } Du(x) \in E \cup K \right\}.$$

Observe that V is non empty since $\varphi \in V$. Let, for $k \in \mathbb{N}$,

$$V^k = \operatorname{int} \left\{ u \in \overline{V} : \int_\Omega \operatorname{dist}(Du(x); E)\, dx \leq \frac{1}{k} \right\}.$$

We claim that V^k, in addition to be open, is dense in the complete metric space $\overline{V}$; this will be seen below. We can then conclude by Baire category theorem that

$$\bigcap_{k=1}^{\infty} V^k \subset \left\{ u \in \overline{V} : \operatorname{dist}(Du(x), E) = 0, \text{ a.e. in } \Omega \right\} \subset \overline{V}$$

is dense, and hence non empty, in $\overline{V}$. The result then follows, since E is compact.

We now show that V^k is dense in $\overline{V}$. So let $u \in \overline{V}$ and $\epsilon > 0$ be arbitrary. We wish to find $v \in V^k$ so that

$$\|u - v\|_{L^\infty} \leq \epsilon.$$

- We start by finding $\alpha \in \overline{V}$ a point of continuity of the operator D so that

$$\|u - \alpha\|_{L^\infty} \leq \epsilon/3.$$

This is always possible by virtue of Lemma 17.

- We next define the oscillation of D at u by

$$\omega_D(u) = \lim_{\delta \to 0} \sup_{\varphi, \psi \in B_\delta(u)} \|D\varphi - D\psi\|_{L^1}.$$

where $B_\delta(u) = \left\{ w \in \overline{V} : \|u - w\|_{L^\infty} < \delta \right\}$. It is easy to see that

(i) D is continuous at u if and only if $\omega_D(u) = 0$

(ii) The set $\{u \in \overline{V} : \omega_D(u) < \epsilon\}$ is an open set in $\overline{V}$.

- We next approximate $\alpha \in \overline{V}$ by $\beta \in V$ so that

$$\|\beta - \alpha\|_{L^\infty} \leq \epsilon/3 \text{ and } \omega_D(\beta) < 1/2k.$$

- Finally we use the relaxation property on every piece where $D\beta$ is constant and we then construct $v \in V$, by patching all the pieces together, such that

$$\|\beta - v\|_{L^\infty} \leq \epsilon/3,\ \omega_D(v) < \frac{1}{2k} \text{ and } \int_\Omega \operatorname{dist}(Dv(x);E)\,dx < \frac{1}{2k}.$$

Moreover, since $\omega_D(v) < 1/2k$, we can find $\delta = \delta(k, v) > 0$ so that

$$\|v - \psi\|_{L^\infty} \leq \delta \Longrightarrow \|D\psi - Dv\|_{L^1} \leq \frac{1}{2k}$$

and hence

$$\int_\Omega \operatorname{dist}(D\psi(x);E)\,dx \leq \int_\Omega \operatorname{dist}(Dv(x);E)\,dx + \|D\psi - Dv\|_{L^1} < \frac{1}{k}$$

for every $\psi \in B_\delta(v)$; which implies that $v \in V^k$.

Combining these facts we have indeed obtained the desired density result. ■

In the proof of our main theorem we have used the following lemma (for a proof see [18]) which is essentially a consequence of Baire theorem.

Lemma 17 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $u_0 \in W^{1,\infty}(\mathbb{R}^n)$. Let $V \subset u_0 + W_0^{1,\infty}(\Omega)$ be a non empty complete space with respect to the L^∞ metric. Then the set of points of continuity of the gradient operator $D : V \to L^p(\Omega; \mathbb{R}^n)$, where $1 \leq p < \infty$, is dense in V.*

To conclude this section we give a sufficient condition that ensures the relaxation property. In concrete examples this condition is usually much easier to check than the relaxation property. We start with a definition.

Definition 18 (Approximation property) *Let $E \subset K(E) \subset \mathbb{R}^{m\times n}$. The sets E and $K(E)$ are said to have the approximation property if there exists a family of closed sets E_δ and $K(E_\delta)$, $\delta > 0$, such that*

(1) $E_\delta \subset K(E_\delta) \subset \operatorname{int} K(E)$ for every $\delta > 0$;

(2) for every $\epsilon > 0$ there exists $\delta_0 = \delta_0(\epsilon) > 0$ such that $\operatorname{dist}(\eta; E) \leq \epsilon$ for every $\eta \in E_\delta$ and $\delta \in [0, \delta_0]$;

(3) if $\eta \in \operatorname{int} K(E)$ then $\eta \in K(E_\delta)$ for every $\delta > 0$ sufficiently small.

We therefore have the following theorem (cf. Theorem 6.14 in [16] and for a slightly more flexible one see Theorem 6.15).

Theorem 19 *Let $E \subset \mathbb{R}^{m\times n}$ be compact and $\operatorname{Rco} E$ have the approximation property with $K(E_\delta) = \operatorname{Rco} E_\delta$, then $\operatorname{int}\operatorname{Rco} E$ has the relaxation property with respect to E.*

3.2 Abstract results for general equations

For higher derivatives we will adopt the following notations.

Notation 20 - *Let $N, n, m \geq 1$ be integers. For $u : \mathbb{R}^n \to \mathbb{R}^m$ we write*

$$D^N u = \left(\frac{\partial^N u^i}{\partial x_{j_1} ... \partial x_{j_N}} \right)_{1 \leq j_1, ..., j_N \leq n}^{1 \leq i \leq m} \in \mathbb{R}_s^{m \times n^N}.$$

(The index s stands here for all the natural symmetries implied by the interchange of the order of differentiation). When $N = 1$ we have

$$\mathbb{R}_s^{m \times n} = \mathbb{R}^{m \times n}$$

while if $m = 1$ and $N = 2$ we obtain

$$\mathbb{R}_s^{n^2} = \mathbb{R}_s^{n \times n}$$

i.e., the usual set of symmetric matrices.

- *For $u : \mathbb{R}^n \to \mathbb{R}^m$ we let*

$$D^{[N]} u = \left(u, Du, ..., D^N u \right)$$

stand for the matrix of all partial derivatives of u up to the order N. Note that

$$D^{[N-1]} u \in \mathbb{R}_s^{m \times M_N} = \mathbb{R}^m \times \mathbb{R}^{m \times n} \times \mathbb{R}_s^{m \times n^2} \times ... \times \mathbb{R}_s^{m \times n^{(N-1)}},$$

where

$$M_N = 1 + n + ... + n^{(N-1)} = \frac{n^N - 1}{n - 1}.$$

Hence

$$D^{[N]} u = \left(D^{[N-1]} u, D^N u \right) \in \mathbb{R}_s^{m \times M_N} \times \mathbb{R}_s^{m \times n^N}.$$

We therefore have the following

Notation 21 *We will denote by*

- *$W^{N,\infty}(\Omega; \mathbb{R}^m)$ the space of maps $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ such that $D^{[N]} u \in L^\infty$;*

- *$W_0^{N,\infty}(\Omega; \mathbb{R}^m) = W^{N,\infty}(\Omega; \mathbb{R}^m) \cap W_0^{N,1}(\Omega; \mathbb{R}^m)$;*

- *$Aff_{piec}^N \left(\overline{\Omega}; \mathbb{R}^m \right)$ will stand for the subset of $W^{N,\infty}(\Omega; \mathbb{R}^m)$ so that $D^N u$ is piecewise constant;*

- *$C_{piec}^N \left(\overline{\Omega}; \mathbb{R}^m \right)$ will denote the subset of $W^{N,\infty}(\Omega; \mathbb{R}^m)$ so that $D^N u$ is piecewise continuous.*

In the present section we will extend the results of the preceding section. We first define the relaxation property in a more general context.

Definition 22 (Relaxation property) *Let $E, K \subset \mathbb{R}^n \times \mathbb{R}_s^{m \times M_N} \times \mathbb{R}_s^{m \times n^N}$. We say that K has the relaxation property with respect to E if for every bounded open set $\Omega \subset \mathbb{R}^n$, for every u_ξ, a polynomial of degree N with $D^N u_\xi(x) = \xi$, satisfying*

$$\left(x, D^{[N-1]} u_\xi(x), D^N u_\xi(x)\right) \in K,$$

there exists a sequence $u_\nu \in Aff_{piec}^N\left(\overline{\Omega}; \mathbb{R}^m\right)$ such that

$$u_\nu \in u_\xi + W_0^{N,\infty}(\Omega; \mathbb{R}^m), \ u_\nu \overset{*}{\rightharpoonup} u_\xi \text{ in } W^{N,\infty}$$

$$\left(x, D^{[N-1]} u_\nu(x), D^N u_\nu(x)\right) \in E \cup K, \text{ a.e. in } \Omega$$

$$\int_\Omega \operatorname{dist}\left(\left(x, D^{[N-1]} u_\nu(x), D^N u_\nu(x)\right); E\right) dx \to 0 \text{ as } \nu \to \infty.$$

In the sequel we will denote points of E by (x, s, ξ) with $x \in \mathbb{R}^n$, $s \in \mathbb{R}_s^{m \times M_N}$ and $\xi \in \mathbb{R}_s^{m \times n^N}$.

The following theorem, due to Dacorogna-Pisante [18], following earlier work of Dacorogna-Marcellini [16], is the main abstract existence theorem.

Theorem 23 *Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Let $E, K \subset \mathbb{R}^n \times \mathbb{R}_s^{m \times M_N} \times \mathbb{R}_s^{m \times n^N}$ be such that E is closed, K open and both E and K are bounded uniformly for $x \in \Omega$ and whenever s vary on a bounded set of $\mathbb{R}_s^{m \times M_N}$. Assume that K has the relaxation property with respect to E. Let $\varphi \in C_{piec}^N\left(\overline{\Omega}; \mathbb{R}^m\right)$ be such that*

$$\left(x, D^{[N-1]} \varphi(x), D^N \varphi(x)\right) \in E \cup K, \text{ a.e. in } \Omega;$$

then there exists (a dense set of) $u \in \varphi + W_0^{N,\infty}(\Omega; \mathbb{R}^m)$ such that

$$\left(x, D^{[N-1]} u(x), D^N u(x)\right) \in E, \text{ a.e. in } \Omega.$$

Remark 24 *(i) The boundedness of E (or of K) stated in the theorem should be understood as follows. For every $R > 0$, there exists $\gamma = \gamma(R)$ so that*

$$(x, s, \xi) \in E, \ x \in \Omega \text{ and } |x| + |s| \leq R \ \Rightarrow \ |\xi| \leq \gamma.$$

(ii) In this theorem we need that K is open (in the relative topology of $\mathbb{R}^n \times \mathbb{R}_s^{m \times M_N} \times \mathbb{R}_s^{m \times n^N}$) because of the presence of lower order terms.

(iii) As in the previous section, a theorem such as Theorem 19 is also available in the present context, but we do not discuss the details and we refer to Theorem 6.14 and Theorem 6.15 in [16].

3.3 Examples

We now return to the examples in the introduction and we give several existence theorems that follow from the abstract ones.

The first one concerns the scalar case (Example 1 and 2).

Theorem 25 *Let $\Omega \subset \mathbb{R}^n$ be bounded and open and $E \subset \mathbb{R}^n$. Let $\varphi \in W^{1,\infty}(\Omega)$ satisfies*

$$D\varphi(x) \in E \cup \operatorname{int}\operatorname{co} E, \text{ a.e. } x \in \Omega \tag{3.1}$$

(where $\operatorname{int}\operatorname{co} E$ *stands for the interior of the convex hull of E); then there exists (a dense set of) $u \in W^{1,\infty}(\Omega)$ such that*

$$\begin{cases} Du(x) \in E & \text{a.e. } x \in \Omega \\ u(x) = \varphi(x) & x \in \partial\Omega. \end{cases} \tag{3.2}$$

Remark 26 *(i) The theorem obviously applies to*

$$\begin{cases} F(Du(x)) = 0 & \text{a.e. } x \in \Omega \\ u(x) = \varphi(x) & x \in \partial\Omega. \end{cases}$$

It suffices to write $E = \{\xi \in \mathbb{R}^n : F(\xi) = 0\}$. If for example F is convex with $F(0) < 0$ and $\lim F(\xi) = +\infty$ *if $|\xi| \to +\infty$, as for the eikonal equation where $F(\xi) = |\xi| - 1$, then the compatibility condition (3.1) reads as follows*

$$F(D\varphi(x)) \le 0, \text{ a.e. } x \in \Omega.$$

(ii) The compatibility condition (3.1), when appropriately interpreted, is also necessary, cf. [16].

(iii) This theorem has a long history and we refer to [16] *for more historical background. Let us quote however some main results : Bressan-Flores* [3]*, Cellina* [5]*, Dacorogna-Marcellini* [16]*, De Blasi-Pianigiani* [23] *and Friesecke* [25].

The next one deals with the singular values case and, when properly read, it incorporates Example 3, Example 4 or Example 5.

Theorem 27 (Singular values) *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $\alpha < \beta$ and $0 < \gamma_2 \le ... \le \gamma_n$ be such that*

$$\gamma_2 \prod_{i=2}^{n} \gamma_i \ge \max\{|\alpha|, |\beta|\}.$$

Let $\varphi \in C^1_{piec}(\overline{\Omega}; \mathbb{R}^n)$ be such that, for almost every $x \in \Omega$,

$$\begin{cases} \alpha < \det D\varphi(x) < \beta, \\ \displaystyle\prod_{i=\nu}^{n} \lambda_i(D\varphi(x)) < \prod_{i=\nu}^{n} \gamma_i, \ \nu = 2, ..., n, \end{cases}$$

then there exists $u \in \varphi + W_0^{1,\infty}(\Omega; \mathbb{R}^n)$ so that

$$\begin{cases} \det Du \in \{\alpha, \beta\}, \text{ a.e. in } \Omega, \\ \lambda_\nu(Du) = \gamma_\nu, \ \nu = 2, ..., n, \text{ a.e. in } \Omega. \end{cases}$$

Remark 28 *(i) If $\alpha = -\beta < 0$ and if we set*

$$\gamma_1 = \beta \left[\prod_{i=2}^{n} \gamma_i \right]^{-1},$$

we recover the result of Dacorogna-Marcellini [16]*, namely that if*

$$\prod_{i=\nu}^{n} \lambda_i(D\varphi(x)) < \prod_{i=\nu}^{n} \gamma_i, \ \nu = 1, ..., n,$$

then there exists $u \in \varphi + W_0^{1,\infty}(\Omega; \mathbb{R}^n)$ so that

$$\lambda_\nu(Du) = \gamma_\nu, \ \nu = 1, ..., n, \ a.e. \ in \ \Omega.$$

(ii) If $\alpha = \beta \neq 0$ we can also prove, as in Dacorogna-Tanteri [22]*, that if*

$$\begin{cases} \det D\varphi(x) = \alpha, \\ \prod_{i=\nu}^{n} \lambda_i(D\varphi(x)) < \prod_{i=\nu}^{n} \gamma_i, \ \nu = 2, ..., n, \end{cases}$$

then there exists $u \in \varphi + W_0^{1,\infty}(\Omega; \mathbb{R}^n)$ so that

$$\lambda_\nu(Du) = \gamma_\nu, \ \nu = 2, ..., n \text{ and } \det Du = \alpha \ a.e. \ in \ \Omega.$$

(iii) The theorem remains valid if the γ_i are allowed to be continuous functions of the form $\gamma_i = \gamma_i(x, u)$.

We now discuss Example 7 (Example 6 will be discussed in the section on the calculus of variations).

Theorem 29 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $A, B \in \mathbb{R}^{2\times 2}$ with $\det B, \det A > 0$,*

$$E = SO(2)A \cup SO(2)B,$$

and

$$\xi \in \operatorname{int} \operatorname{Rco} E$$

then there exists $u \in W^{1,\infty}\left(\Omega; \mathbb{R}^2\right)$ such that

$$\begin{cases} Du(x) \in E, \ a.e. \ in \ \Omega \\ u(x) = \xi x, \ on \ \partial\Omega. \end{cases}$$

Remark 30 *(i) If $\det B > \det A > 0$, the set $\operatorname{Rco} E$, according to Sverak* [33]*, is given by*

$$\operatorname{Rco} E = \left\{ \begin{array}{c} \xi \in \mathbb{R}^{2\times 2} : \text{ there exist } 0 \leq \alpha \leq \frac{\det B - \det \xi}{\det B - \det A}, \ 0 \leq \beta \leq \frac{\det \xi - \det A}{\det B - \det A} \\ R, S \in SO(2), \text{ so that } \xi = \alpha RA + \beta SB \end{array} \right\}$$

while the interior is given by the same formulas with strict inequalities in the right hand side.

(ii) If $\det A = \det B > 0$.

$$\text{Rco}\, E = \left\{ \begin{array}{c} \xi \in \mathbb{R}^{2\times 2} : \text{ there exist } R, S \in SO(2)\,,\ 0 \leq \alpha, \beta \leq \alpha + \beta \leq 1, \\ \det \xi = \det A = \det B \text{ so that } \xi = \alpha R A + \beta S B \end{array} \right\}$$

and in this case the interior is understood relative to the manifold of matrices with constant determinant.

4 Calculus of variations

4.1 Abstract results

We consider here the minimization problem

$$(P) \qquad \inf \left\{ I(u) = \int_\Omega f(Du(x))\, dx : \ u \in \varphi + W_0^{1,\infty}(\Omega; \mathbb{R}^m) \right\}$$

where Ω is a bounded open set of $\mathbb{R}^n$, φ is affine, i.e. $D\varphi = \xi_0$ (in some cases we can also consider more general boundary data) and $f : \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}$ is a lower semicontinuous function.

The scalar case ($n = 1$ or $m = 1$) has been intensively studied by many authors including: Aubert-Tahraoui, Buttazzo-Ferone-Kawohl, Cellina, Cesari, Dacorogna, Ekeland, Friesecke, Klötzler, Marcellini, Mascolo, Mascolo-Schianchi, Olech and Raymond and we refer to [12] for precise references.

The vectorial case has been investigated for some special examples by Mascolo-Schianchi [29], Cellina-Zagatti [6] and Dacorogna-Ribeiro [20]. The only systematic studies are those of Dacorogna-Marcellini [12] and Dacorogna-Pisante-Ribeiro [19] that we discuss now.

The main tool for attacking the problem is the relaxation theorem of Dacorogna [9] (see also [10]).

Theorem 31 (Relaxation theorem) *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded open set,* $\varphi \in W^{1,\infty}(\Omega; \mathbb{R}^m)$ *and* $f : \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}$ *be locally bounded, non negative and lower semicontinuous function. Let* Qf *be the quasiconvex envelope of* f *and*

$$(QP) \qquad \inf \left\{ \overline{I}(u) = \int_\Omega Qf(Du(x))\, dx : \ u \in \varphi + W_0^{1,\infty}(\Omega; \mathbb{R}^m) \right\}.$$

Then for every $u \in \varphi + W_0^{1,\infty}(\Omega; \mathbb{R}^m)$*, there exists a sequence* $u_\nu \in \varphi + W_0^{1,\infty}(\Omega; \mathbb{R}^m)$ *such that*

$$u_\nu \to u \text{ in } L^\infty(\Omega; \mathbb{R}^m) \text{ as } \nu \to \infty$$

$$I(u_\nu) \to \overline{I}(u) \text{ as } \nu \to \infty$$

so that in particular

$$\inf(QP) = \inf(P).$$

In the scalar case the quasiconvex envelope is nothing else than the convex envelope (i.e. $Qf = Cf$) and we recover the classical result of L.C. Young, Ekeland and others.

It should be pointed out that Dacorogna formula in [10] for computing the quasiconvex envelope is particularly useful in this context; it reads as follows

$$Qf(\xi) = \inf \left\{ \frac{1}{\operatorname{meas}\Omega} \int_\Omega f(\xi + Du(x))\, dx : \ u \in W_0^{1,\infty}(\Omega;\mathbb{R}^m) \right\}.$$

With the help of the relaxation theorem we are now in a position to discuss some existence results for the problem (P) when the boundary datum is affine. The following lemma (cf. [12]) is elementary and gives a necessary and sufficient condition for existence of minima.

Lemma 32 *Let Ω, f and φ be as above, in particular $D\varphi = \xi_0$. The problem (P) has a solution if and only if there exists $\overline{u} \in \varphi + W_0^{1,\infty}(\Omega;\mathbb{R}^m)$ such that*

$$f(D\overline{u}(x)) = Qf(D\overline{u}(x)), \text{ a.e. } x \in \Omega \tag{4.1}$$

$$\int_\Omega Qf(D\overline{u}(x))\, dx = Qf(\xi_0) \operatorname{meas}\Omega. \tag{4.2}$$

Proof. By the relaxation theorem and since φ is affine, we have

$$\inf(P) = \inf(QP) = Qf(\xi_0) \operatorname{meas}\Omega.$$

Moreover, since we always have $f \geq Qf$ and we have a solution of (4.1) satisfying (4.2), we get that $\overline{u}$ is a solution of (P). The fact that (4.1) and (4.2) are necessary for the existence of a minimum for (P) follows in the same way. ■

The previous lemma explains why the set

$$K = \left\{ \xi \in \mathbb{R}^{m\times n} : Qf(\xi) < f(\xi) \right\}$$

plays a central role in the existence theorems that follow. In order to ensure (4.1) we will have to consider differential inclusions of the form studied in the previous section, namely: find $\overline{u} \in \varphi + W_0^{1,\infty}(\Omega;\mathbb{R}^m)$ such that

$$D\overline{u}(x) \in \partial K, \text{ a.e. } x \in \Omega.$$

In order to deal with the second condition (4.2) we will have to impose some hypotheses of the type "Qf is *quasiaffine* on K".

Before mentioning our more general existence theorem, we prefer to start with a corollary that is simpler to state.

Corollary 33 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $f : \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}$ a lower-semicontinuous function and let*

$$K_0 \subset K = \left\{ \xi \in \mathbb{R}^{m\times n} : Qf(\xi) < f(\xi) \right\}$$

be a connected component of K that is assumed to be bounded. Let $\xi_0 \in K_0$ ($\varphi(x) = \xi_0 x$) and assume that Qf is quasiaffine on $\overline{K}_0$, meaning that there exist $\alpha \in \mathbb{R}^\tau$ and $\beta \in \mathbb{R}$ such that

$$Qf(\xi) = \langle \alpha; T(\xi) \rangle + \beta, \ \forall \xi \in \overline{K}_0.$$

Then the problem

$$(P) \qquad \inf \left\{ I(u) = \int_\Omega f(Du(x))\, dx : \ u \in \varphi + W_0^{1,\infty}(\Omega; \mathbb{R}^m) \right\}$$

has a solution $\bar{u} \in \varphi + W_0^{1,\infty}(\Omega; \mathbb{R}^m)$.

While the corollary is nearly optimal in the scalar case, it is not so in the vectorial case for several reasons. The main ones are that:

1) it might be that there is no *bounded* connected component of K, as is exemplified by Theorem 35;

2) the function Qf is, in general, not quasiaffine on the whole of the connected component of K but only on a subset, as seen in Theorem 37.

We are therefore lead to refine the corollary, for example as in Dacorogna-Pisante-Ribeiro [19].

Theorem 34 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $\xi_0 \in \mathbb{R}^{m\times n}$, $f : \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}$ a lower-semicontinuous function and let*

$$K = \left\{ \xi \in \mathbb{R}^{m\times n} : Qf(\xi) < f(\xi) \right\}.$$

Assume that there exists $K_0 \subset K$ such that

- *$\xi_0 \in K_0$,*
- *K_0 is bounded and has the relaxation property with respect to $\overline{K}_0 \cap \partial K$,*
- *Qf is quasiaffine on $\overline{K}_0$.*

Let $\varphi(x) = \xi_0 x$. Then the problem

$$(P) \qquad \inf \left\{ I(u) = \int_\Omega f(Du(x))\, dx : \ u \in \varphi + W_0^{1,\infty}(\Omega; \mathbb{R}^m) \right\}$$

has a solution $\bar{u} \in \varphi + W_0^{1,\infty}(\Omega; \mathbb{R}^m)$.

Proof. Since $\xi_0 \in K_0$ and K_0 is bounded and has the relaxation with respect to $\overline{K}_0 \cap \partial K$, we can find, appealing to Theorem 15, a map $\bar{u} \in \varphi + W_0^{1,\infty}(\Omega; \mathbb{R}^m)$ satisfying

$$D\bar{u} \in \overline{K}_0 \cap \partial K, \text{ a.e. in } \Omega,$$

which means that (4.1) of Lemma 32 is satisfied. Moreover, since Qf is quasiaffine on $\overline{K}_0$, we have that (4.2) of Lemma 32 holds and thus the claim. ■

4.2 Examples

We will discuss here two examples and we refer to Dacorogna-Marcellini [12], Dacorogna-Ribeiro [20] and Dacorogna-Pisante-Ribeiro [19] for more examples.

We start by considering the minimization problem,

$$(P) \qquad \inf\left\{\int_\Omega g(\Phi(Du(x)))\,dx:\ u\in\varphi+W_0^{1,\infty}(\Omega;\mathbb{R}^m)\right\}$$

where Ω is a bounded open set of $\mathbb{R}^n$, $\varphi\in W^{1,\infty}(\Omega;\mathbb{R}^m)$ and

- $g:\mathbb{R}\longrightarrow\overline{\mathbb{R}}=\mathbb{R}\cup\{+\infty\}$ is a lower-semicontinuous non convex function,
- $\Phi:\mathbb{R}^{m\times n}\longrightarrow\mathbb{R}$ is quasiaffine and non constant.

We recall that in particular we can have, when $m=n$, $\Phi(\xi)=\det\xi$.

The relaxed problem is then

$$(QP) \qquad \inf\left\{\int_\Omega Cg(\Phi(Du(x)))\,dx:\ u\in\varphi+W_0^{1,\infty}(\Omega;\mathbb{R}^m)\right\},$$

where Cg is the convex envelope of g (here $f(\xi)=g(\Phi(\xi))$ and we get $Qf=Cg$, as established by Dacorogna [8]).

The existence result is the following.

Theorem 35 *Let $\Omega\subset\mathbb{R}^n$ be a bounded open set, $g:\mathbb{R}\longrightarrow\overline{\mathbb{R}}=\mathbb{R}\cup\{+\infty\}$ a lower-semicontinuous function such that*

$$\lim_{|t|\to+\infty}\frac{g(t)}{|t|}=+\infty \tag{4.3}$$

and $\varphi(x)=\xi_0 x$, with $\xi_0\in\mathbb{R}^{m\times n}$. Then there exists $\bar{u}\in\varphi+W_0^{1,\infty}(\Omega;\mathbb{R}^m)$ solution of (P).

Remark 36 *(i) The above result has first been established by Mascolo-Schianchi* [29] *and then by Cellina-Zagatti* [6]*, Dacorogna-Marcellini* [12] *and Dacorogna-Ribeiro* [20].

(ii) In the case $m=n$ and $\Phi(\xi)=\det\xi$, we can also handle non affine boundary datum φ using a result of Dacorogna-Moser [17].

Proof. We just suggest how to construct the set K_0 of Theorem 34. We also assume that $m=n$ and $\Phi(\xi)=\det\xi$; the general case can be handled in a similar way, cf. Dacorogna-Ribeiro [20]. Here the problem is sufficiently flexible, so that we can proceed in several ways. We choose here the one that uses Theorem 27 (this is clearly not the simplest way to proceed, but has the advantage to use an already existing theorem). We have

$$K=\left\{\xi\in\mathbb{R}^{n\times n}:Cg(\det\xi)<g(\det\xi)\right\}$$

which is, because of the hypotheses, a countable union of open intervals; let (α,β) be such one and with the property that $\alpha<\det\xi_0<\beta$. Note that here the function Qf $(Qf(\xi)=Cg(\det\xi))$ is quasiaffine on K; so that the problem is only the lack of boundedness of the set K. In order to remedy to this fact, we choose $0 < \gamma_2 \leq ... \leq \gamma_n$ be such that

$$\gamma_2 \prod_{i=2}^{n} \gamma_i \geq \max\{|\alpha|, |\beta|\} \quad \text{and} \quad \gamma_i > \lambda_i(\xi_0), \ i = 2, ..., n$$

and let

$$K_0 = \left\{ \xi \in \mathbb{R}^{n \times n} : \prod_{i=\nu}^{n} \lambda_i(\xi) < \prod_{i=\nu}^{n} \gamma_i, \ \nu = 2, ..., n, \ \det \xi \in (\alpha, \beta) \right\}.$$

The proof follows then easily and we do not discuss the details. ■

We now turn our attention to Example 6. We recall that the problem under consideration is

$$(P) \qquad \inf \left\{ \int_\Omega f(Du(x)) \, dx : \ u \in \varphi + W_0^{1,\infty}(\Omega; \mathbb{R}^2) \right\}$$

where Ω is a bounded open set of $\mathbb{R}^2$, $\varphi \in W^{1,\infty}(\Omega; \mathbb{R}^2)$ is affine, i.e. $D\varphi = \xi_0$, and

$$f(\xi) = \begin{cases} 1 + |\xi|^2 & \text{if } \xi \neq 0 \\ 0 & \text{if } \xi = 0. \end{cases}$$

It was shown by Kohn-Strang [27] that the quasiconvex envelope is then

$$Qf(\xi) = \begin{cases} 1 + |\xi|^2 & \text{if } |\xi|^2 + 2|\det \xi| \geq 1 \\ 2\left(|\xi|^2 + 2|\det \xi|\right)^{1/2} - 2|\det \xi| & \text{if } |\xi|^2 + 2|\det \xi| < 1. \end{cases}$$

The existence of minimizers for problem (P) was then established by Dacorogna-Marcellini in [12] and [16], namely

Theorem 37 *Let $\Omega \subset \mathbb{R}^2$, f and ξ_0 be as above. Then a necessary and sufficient condition for (P) to have a solution is that one of the following conditions hold:*
(i) $\xi_0 = 0$ or $|\xi_0|^2 + 2|\det \xi_0| \geq 1$, (ie. $f(\xi_0) = Qf(\xi_0)$)
(ii) $\det \xi_0 \neq 0$.

Proof. We do not discuss the details and in particular not the necessary part. We just point out how to define the set K_0 of Theorem 34. Assume that $f(\xi_0) > Qf(\xi_0)$ and $\det \xi_0 \neq 0$. Since the function f is invariant under rotations and symmetries, we can assume, without loss of generality, that

$$\xi_0 \in \mathbb{R}_s^{2\times 2}, \ \det \xi_0 > 0 \text{ and } \operatorname{trace} \xi_0 \in (0,1).$$

where $\mathbb{R}_s^{2\times 2}$ denotes the set of 2×2 symmetric matrices. We next define

$$K = \left\{ \xi \in \mathbb{R}^{2\times 2} : |\xi|^2 + 2|\det \xi| < 1 \right\}$$
$$K_0 = \left\{ \xi \in \mathbb{R}_s^{2\times 2} : \det \xi > 0 \text{ and } \operatorname{trace} \xi \in (0,1) \right\}.$$

It is clear that Qf is quasiaffine on K_0 ($Qf(\xi) = 2 \operatorname{trace} \xi - 2 \det \xi$), while it is not so on K. ■

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