# Viscosity solutions, almost everywhere solutions and explicit formulas

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**Abstract**

Consider the differential inclusion $Du \in E$ in $\mathbb{R}^n$. We exhibit an explicit solution that we call *fundamental*. It turns out to be also a *viscosity solution*, when properly defining this notion. Finally we consider a Dirichlet problem associated to the differential inclusion and we give an iterative procedure for finding a solution.

## 1 Introduction

Existence of *almost everywhere* solutions of the first order Dirichlet problem related to *implicit differential equations* of the type

$$\begin{cases} F(Du(x)) = 0, & \text{a.e. } x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

has recently been extensively studied in the book [6] by the authors. Here $F : \mathbb{R}^n \to \mathbb{R}$ is a continuous function and we look for a Lipschitz-continuous solution $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$. A wide literature on this subject can be found in [6], not only for scalar problems as this one is, but also for *vector-valued solutions* of *first order systems* related to maps $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ and $F : \mathbb{R}^{m \times n} \to \mathbb{R}^N$, for some $m, N \geq 1$.

Existence of *viscosity solutions* of the Dirichlet problem (1) is now well established. It has been studied by many authors starting with Hopf, Lax, Kruzkov and Crandall-Lions; see for example [1] or [6] for more historical comments. One of the earliest and still one of the most complete monograph on the subject is [10] by P.L. Lions. The research in this field remains very active; in particular H. Ischii and P. Loreti [8], motivated by an optimization problem, recently gave an existence result of viscosity solutions of the Dirichlet problem (1). See also [2] and [9].

In this paper we give some existence results, either in the case of *almost everywhere* solutions, or, when possible, of *viscosity solutions*. One of our aims is to give some constructive explicit formulas (cf. Theorems 1 and 6). Moreover, if the geometry of the set $\Omega$ and the assumptions on the function $F$ make it possible, following [3] we give (cf. Corollary 8) an explicit formula for a viscosity solution of the Dirichlet problem (1), simply in terms of sup and inf. Otherwise, with general $F$ and $\Omega$, we propose, in Section 4, an iteration scheme for characterizing a solution.

In Section 2 we introduce the notion of viscosity solution of a *differential inclusion*; namely: given a closed set $E$, we say that a function $u$ is a viscosity solution of the differential inclusion

$$Du(x) \in E,\ x \in \Omega \tag{2}$$

if $u$ is a viscosity solution of the equation

$$F(Du(x)) = 0,\ x \in \Omega, \tag{3}$$

where $F(\xi) = \operatorname{dist}\{\xi, E\}$. We will prove in Theorem 6 that the function $L : \mathbb{R}^n \to \mathbb{R}$, defined by

$$L(x) = \max\{\langle \xi, x\rangle : \ \xi \in E\},$$

is a viscosity solution of the differential inclusion (2), i.e. it is a *fundamental solution* of the equation (3).

# 2 Fundamental solution and viscosity solutions of differential inclusions

We start by recalling some classical definitions and notations in convex analysis. We say that $\xi \in \mathbb{R}^n$ is an *extreme point* for a convex set $K \subset \mathbb{R}^n$ if the conditions

$$\begin{cases} \xi = t\xi_1 + (1-t)\xi_2 \\ \xi_1, \xi_2 \in K, \quad t \in (0,1) \end{cases}$$

imply that $\xi = \xi_1 = \xi_2$.

If $E$ is a set (not necessarily convex) of $\mathbb{R}^n$, we denote by $E_{\text{ext}}$ the set of extreme points of the convex hull of $E$ denoted by $\operatorname{co} E$ (note that $E_{\text{ext}} \subset E$).

We also recall that the domain of a convex function $L : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined as

$$\operatorname{dom} L = \{x \in \mathbb{R}^n : \ L(x) < +\infty\}.$$

The next Theorem 1 generalizes an analogous result obtained by Ischii and Loreti (see the proof of Theorem 2.2 in [8]) in the case that $E$ is the level set of a continuous, positively homogenous function of degree one, equal to zero only at the origin of $\mathbb{R}^n$.

**Theorem 1** *Let $E$ be a compact set of $\mathbb{R}^n$. For every $x \in \mathbb{R}^n$ let*

$$L(x) = \max\{\langle \xi, x\rangle : \xi \in E\}.$$

*Then*

$$DL(x) \in E \text{ a.e. } x \in \mathbb{R}^n.$$

**Remark 2** *(i) It should be noted that in fact the theorem is more precise, namely*

$$DL(x) \in \overline{E_{\text{ext}}} \subset E \cap \partial \operatorname{co} E \text{ a.e. } x \in \mathbb{R}^n.$$

*(ii) If $E$ is any set, not necessarily closed or bounded, then the proof gives (replacing* max *by* sup*) that*

$$DL(x) \in \overline{E} \text{ a.e. } x \in \operatorname{dom} L.$$

*(iii) In terms of convex analysis and anticipating on (4) we can say that L is the support function of* $\operatorname{co} E$.

Before proceeding with the proof it might be interesting to rewrite the theorem in terms of equations.

**Corollary 3** *Let $F : \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that*

$$E = \{\xi \in \mathbb{R}^n : F(\xi) = 0\}$$

*is a bounded set. Let $L(x) = \max\{\langle \xi, x\rangle : F(\xi) = 0\}$; then*

$$F(DL(x)) = 0 \text{ a.e. } x \in \mathbb{R}^n.$$

**Proof.** The following representation formula for $L$ holds (see Rockafellar [12] Theorem 32.2)

$$L(x) = \max\{\langle \xi, x\rangle : \xi \in \operatorname{co} E\} = \max\{\langle \xi, x\rangle : \xi \in E\}, \quad \forall x \in \mathbb{R}^n. \tag{4}$$

In fact one has the more precise result (see Rockafellar [12] Corollary 32.3.2)

$$L(x) = \max\{\langle \xi, x\rangle : \xi \in \operatorname{co} E\} = \max\{\langle \xi, x\rangle : \xi \in E_{\text{ext}}\}, \quad \forall x \in \mathbb{R}^n. \tag{5}$$

Let $\{\xi_h\}_{h\in\mathbb{N}}$ be a (finite or) countable dense subset of $E_{\text{ext}} \subset E$ and, analogously to (4), for every $h \in \mathbb{N}$ and for every $x \in \mathbb{R}^n$ let us define

$$L_h(x) = \max\{\langle \xi_1, x\rangle, \langle \xi_2, x\rangle, \ldots, \langle \xi_h, x\rangle\}.$$

Clearly the gradient $DL_h$ exists almost everywhere in $\mathbb{R}^n$ and

$$DL_h(x) \in \{\xi_1, \xi_2, \ldots, \xi_h\} \subset E_{\text{ext}}, \quad \text{a.e. } x \in \mathbb{R}^n. \tag{6}$$

For every $x \in \mathbb{R}^n$ the sequence $L_h(x)$ is increasing with respect to $h \in \mathbb{N}$ and we have

$$L(x) = \sup\{L_h(x) : h \in \mathbb{N}\} = \lim_{h \to +\infty} L_h(x).$$

For every $h \in \mathbb{N}$ the sequence $L_h(x)$ is convex with respect to $x \in \mathbb{R}^n$ and

$$\operatorname{dom} L_h = \operatorname{dom} L = \mathbb{R}^n.$$

Thus we can apply Lemma 4 and we obtain that, at every point where $L_h$ and $L$ are differentiable (i.e., almost everywhere in $\mathbb{R}^n$)

$$DL_h(x) \to DL(x).$$

Therefore, by (6), we get the conclusion

$$DL(x) \in \overline{E_{\text{ext}}} \subset E, \quad \text{a.e.}\, x \in \mathbb{R}^n.$$

■

In the proof of the previous Theorem 1 we used a result given in [11] (Lemma 5.9), that we recall here in a form more appropriate to the applications given in this paper.

**Lemma 4** *Let $\{L_h\}_{h\in\mathbb{N}}$ be a sequence of convex functions, defined on $\mathbb{R}^n$ with values on $\mathbb{R} \cup \{+\infty\}$, with pointwise limit $L : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. At every point $x \in \operatorname{int}[(\cap_{h\in\mathbb{N}} \operatorname{dom} L_h) \cap \operatorname{dom} L]$ where $L_h$ and $L$ are differentiable, the gradient $DL_h(x)$ converges in $\mathbb{R}^n$ to the gradient $DL(x)$.*

**Proof.** For every $h \in \mathbb{N}$ let $\operatorname{dom} L_h$ and $\operatorname{dom} L$ be the domains of $L_h$ and $L$. Then each $L_h$ is locally Lipschitz-continuous in $\operatorname{int} \operatorname{dom} L_h$ and $L$ is locally Lipschitz-continuous in $\operatorname{int} \operatorname{dom} L$. Therefore, for every $h \in \mathbb{N}$, there exists a set $N_h \subset \operatorname{dom} L_h \subset \mathbb{R}^n$ of zero measure such that $L_h$ is differentiable at every point of $\operatorname{dom} L_h \backslash N_h$. Analogously, there exists a set $N \subset \operatorname{dom} L$ of zero measure such that $L$ is differentiable at every point of $\operatorname{dom} L \backslash N$. Then the set of points $x \in \mathbb{R}^n$ where $L_h$ and $L$ are differentiable is (possibly empty and) given by

$$\left( \bigcap_{h\in\mathbb{N}} (\operatorname{dom} L_h \backslash N_h) \right) \bigcap (\operatorname{dom} L \backslash N)$$

and differs from the intersection of their domains $(\cap_{h\in\mathbb{N}} \operatorname{dom} L_h) \cap \operatorname{dom} L$ by (at most) a set $(\cup_{h\in\mathbb{N}} N_h) \cup N$ of zero measure. Let $x \in \operatorname{int}[(\cap_{h\in\mathbb{N}} \operatorname{dom} L_h) \cap \operatorname{dom} L]$ be a point of $\mathbb{R}^n$ where $L_h$ and $L$ are differentiable. Let $i \in \{1, 2, \ldots, n\}$ and $h \in \mathbb{N}$ be fixed. Then at $x = (x_1, \ldots, x_i, \ldots, x_n)$ the partial derivatives $\partial L_h / \partial x_i$ and $\partial L / \partial x_i$ are well defined. An elementary application of the convex inequality for the function $L_h$ gives the monotonicity of the difference quotient; precisely, if $t > 0$ is sufficiently small and if, as usual, we denote by $x \pm te_i$ the two points of $\mathbb{R}^n$ with coordinates respectively $(x_1, \ldots, x_{i-1}, x_i \pm t, x_{i+1}, \ldots, x_n)$, we have

$$\frac{L_h(x - te_i) - L_h(x)}{-t} \leq \frac{\partial L_h}{\partial x_i}(x) \leq \frac{L_h(x + te_i) - L_h(x)}{t}$$

and, in the limit as $h \to +\infty$,

$$\frac{L(x - te_i) - L(x)}{-t} \leq \liminf_{h \to +\infty} \frac{\partial L_h}{\partial x_i}(x) \leq \limsup_{h \to +\infty} \frac{\partial L_h}{\partial x_i}(x) \leq \frac{L(x + te_i) - L(x)}{t}.$$

Since $L$ is differentiable at $x$, as $t \to 0^+$ we obtain that $\partial L_h / \partial x_i(x)$ converges to $\partial L / \partial x_i$. The property being so for every $i \in \{1, 2, \ldots, n\}$, we have the conclusion, i.e., that the gradient $DL_h(x)$ converges in $\mathbb{R}^n$ to the gradient $DL(x)$. ■

**Remark 5** *With a slightly different proof, as in Lemma 5.9 in [11], we can give a compactness result. Precisely, we can show that from every locally bounded sequence* $\{L_h\}_{h\in\mathbb{N}}$ *of convex functions* ($\{L_h\}_{h\in\mathbb{N}}$ *uniformly bounded in* $L^\infty_{\mathrm{loc}}(\Omega)$, *with* $\Omega$ *open set in* $\mathbb{R}^n$) *it is possible to select a subsequence* $\{L_{h_k}\}_{k\in\mathbb{N}}$ *whose gradients* $\{DL_{h_k}\}_{k\in\mathbb{N}}$ *converge almost everywhere in* $\Omega$, *and at the same time* $\{L_{h_k}\}_{k\in\mathbb{N}}$ *converges in the strong topology of* $W^{1,q}_{\mathrm{loc}}(\Omega)$, *for every* $q \in [1, +\infty)$.

With the help of the above construction we can give a definition of what we mean by viscosity solutions of differential inclusions. Given a closed set $E$, we say that a function $u$ is a *viscosity solution of the differential inclusion*

$$Du(x) \in E,\ x \in \mathbb{R}^n$$

if $u$ is a viscosity solution of the equation

$$F(Du(x)) = 0,\ x \in \mathbb{R}^n,$$

where $F(\xi) = \operatorname{dist}\{\xi, E\}$. We therefore have the following result.

**Theorem 6** *Let $E$ be a compact set of $\mathbb{R}^n$. For every $x \in \mathbb{R}^n$ let*

$$L(x) = \max\{\langle \xi, x\rangle : \xi \in E\}.$$

*Then $L$ is a viscosity solution of*

$$DL(x) \in E,\ x \in \mathbb{R}^n.$$

**Proof.** The function $L$ being convex we have that $D^+L(x)$ (the superdifferential of $L$ at $x$, see [1] and [6] for the precise definition of this set) is either empty or reduced to $\{DL(x)\}$, i.e. $x$ is a point of differentiability of $L$ and we know by Theorem 1 that at such points $DL(x) \in E$. We therefore have that

$$F(p) = 0,\ \forall p \in D^+L(x)$$

which means that $L$ is a viscosity subsolution (see Proposition 4.7 of [6]) of $F(Du) = 0$.

Since $F \geq 0$ we deduce trivially that

$$F(p) \geq 0,\ \forall p \in D^-L(x),$$

where $D^-L(x)$ is the subdifferential of $L$ at $x$. This means that $L$ is a viscosity supersolution of $F(Du) = 0$.

Combining these two results we have indeed that $L$ is a viscosity solution of $F(Du) = 0$ and hence of $Du \in E$. ■

# 3 Fundamental solution and the boundary condition

We now want to discuss a Dirichlet problem in a bounded domain. We first fix the notations.

We let $\Omega \subset \mathbb{R}^n$ be a bounded open convex set and denote by $\nu(y)$ the outward unit normal at $y \in \partial\Omega$ (that exists at almost all points $y \in \partial\Omega$, since $\Omega$ is convex).

We next let $E \subset \mathbb{R}^n$ be a compact set with $0 \in \operatorname{int} \operatorname{co} E$. We then associate to $\operatorname{co} E$ its *gauge* $\rho$, which is a convex and positively homogeneous of degree one function, such that

$$\operatorname{co} E = \{\xi \in \mathbb{R}^n : \ \rho(\xi) \leq 1\}.$$

Recall also that

$$L(x) = \max\{\langle \xi, x\rangle : \ \xi \in E\}.$$

We should immediately note that, with our hypotheses on $E$ (and invoking (4)), the function $L$ is in fact the polar of $\rho$, denoted sometimes also by $\rho^0$.

We finally consider the Dirichlet problem

$$\begin{cases} Du(x) \in E, & \text{a.e. } x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases}$$

We could also consider the case of a more general boundary datum of class $C^1$ but the analysis can then be carried in a straightforward manner.

We have the following theorem, that is inspired by Cardaliaguet-Dacorogna-Gangbo-Georgy [3] (see also [6]).

**Theorem 7** *Let $\Omega$, $\nu$, $E$, $\rho$ and $L$ be as above and satisfy in addition*

$$\frac{-\nu(y)}{\rho(-\nu(y))} \in E, \ \text{a.e. } y \in \partial\Omega; \tag{7}$$

*then the function $u : \mathbb{R}^n \to \mathbb{R}$, defined by*

$$u(x) = \min\{L(x-y) : \ y \in \partial\Omega\}, \tag{8}$$

*solves the Dirichlet problem*

$$\begin{cases} Du(x) \in E, & \text{a.e. } x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{9}$$

As before we rewrite this theorem in terms of functions.

**Corollary 8** *Let $F : \mathbb{R}^n \to \mathbb{R}$ be continuous with $F(\xi) \to \infty$ as $|\xi| \to \infty$ and $F(0) < 0$. Set*

$$E = \{\xi \in \mathbb{R}^n : \ F(\xi) = 0\}.$$

*Let $\Omega$, $\nu$, $\rho$ and $L$ be as above. If*

$$F\left(\frac{-\nu(y)}{\rho(-\nu(y))}\right)=0 \text{ a.e. } y\in\partial\Omega \tag{10}$$

*then*

$$u(x)=\min\{L(x-y):\ y\in\partial\Omega\} \tag{11}$$

*solves*

$$\left\{\begin{array}{ll} F(Du(x))=0, & \text{a.e. } x\in\Omega,\\ u(x)=0, & x\in\partial\Omega\,. \end{array}\right. \tag{12}$$

*Furthermore if $E\subset\partial\operatorname{co}E$, then $u$ is a viscosity solution.*

**Remark 9** *(i) The first part of the Corollary follows immediately from the theorem. The fact that $u$ is a viscosity solution (when $E\subset\partial\operatorname{co}E$) has been established in [3].*

*(ii) Note that if, in addition, $\partial\operatorname{co}E\subset E$ (which happens if, for instance, $F$ is convex or more generally if the set $\{\xi:\ F(\xi)\leq 0\}$ is convex) then (10) is always satisfied. In fact, since $\rho$ is positively homogeneous of degree one,*

$$\rho\left(\frac{-\nu(y)}{\rho(-\nu(y))}\right)=1\ \Rightarrow\ \frac{-\nu(y)}{\rho(-\nu(y))}\in\partial\operatorname{co}E\subset E.$$

*Moreover, if $E=\partial\operatorname{co}E$, $u$ defined in (11) is the unique viscosity solution of (12).*

*(iii) According to Theorem 4.1 of Lions [10], the Dirichlet problem (12) has always a viscosity solution. However the solution given by (11) is not necessarily a viscosity solution; it is so when $E\subset\partial\operatorname{co}E$.*

We can now proceed with the proof of the theorem.

**Proof.** (Theorem 7) We recall the two following facts (the first one is just Hopf-Lax formula and the second one is Lemma 2.9 in [3] or Lemma 4.17 in [6]). We also use the standard notation $D^{+}u(x)$, respectively $D^{-}u(x)$, for the superdifferential, respectively the subdifferential, of $u$ at $x$ (see [6] for more details).

Fact 1: The function $u$ is the viscosity solution of

$$\left\{\begin{array}{ll} \rho(Du(x))=1, & x\in\Omega\,,\\ u(x)=0, & x\in\partial\Omega\,. \end{array}\right. \tag{13}$$

Fact 2: Let $y(x)\in\partial\Omega$ be such that

$$u(x)=L(x-y(x))\,.$$

Then, if $p\in D^{-}u(x)$ (i.e. $D^{-}u(x)$ is non empty), the outward unit normal $\nu(y(x))$ is well defined and there exists $\lambda(y(x))>0$ such that

$$p=-\lambda(y(x))\,\nu(y(x))\,. \tag{14}$$

Since we are interested in almost everywhere solutions we need only to consider points $x \in \Omega$ where $D^{+}u(x) = D^{-}u(x) = \{Du(x)\}$. Combining (13) and (14), with $p = Du(x)$, and the homogeneity of $\rho$, we get that $\lambda(y) = 1/\rho(-\nu(y))$ and hence

$$Du(x) = \frac{-\nu(y)}{\rho(-\nu(y))}.$$

The hypothesis (7) leads to the result $Du \in E$. ■

# 4 The iteration scheme

As above we let $\Omega \subset \mathbb{R}^n$ be a non empty bounded open set. We want to find, with the help of the previous construction, a solution $u \in W_0^{1,\infty}(\Omega)$ of the differential inclusion

$$Du(x) \in E, \text{ a.e. } x \in \Omega,$$

where $E \subset \mathbb{R}^n$ is a compact set with $0 \in \operatorname{int} \operatorname{co} E$. We let $\rho$ be the gauge associated to $\operatorname{co} E$.

We will find a sequence of disjoint convex open sets $\Omega_i \subset \Omega$ so that

$$\operatorname{meas}\left[\Omega \setminus \bigcup_{i=1}^{\infty} \Omega_i\right] = 0$$

and the function $u$ will be defined as

$$u(x) = \begin{cases} \inf\{L(x-y) : \ y \in \partial\Omega_i\} & x \in \Omega_i \\ 0 & x \in \Omega \setminus \bigcup_{i=1}^{\infty} \Omega_i . \end{cases}$$

Observe that $u$ is a viscosity solution of the Dirichlet problem $Du \in E$ in $\Omega_i$, $u = 0$ on $\partial\Omega_i$ for every $i$ (but not globally in $\Omega$).

Any Vitali covering by level sets of the function $L$ has all the above requirements. However we will choose, among them, one with some maximality properties. In particular we want that $\Omega_1 = \Omega$ if $\Omega$ is convex and $\frac{-\nu}{\rho(-\nu)} \in E$, a.e. on $\partial\Omega$, where $\nu$ is the outward unit normal to $\Omega$ (recall that this always happens if $E = \partial \operatorname{co} E$ or if $\Omega$ is the level set of the function $L$).

Before describing this construction we need to introduce some notations.

**Notation 10** *Let $x_0 \in \mathbb{R}^n$. We let $G_{x_0}$ be the set of all gauges centered at $x_0$. In other words this is the set of all convex functions $\gamma : \mathbb{R}^n \to \mathbb{R}$ satisfying*

$$\gamma(x_0) = 0, \ \gamma(x) > 0, \ \forall x \in \mathbb{R}^n \setminus \{x_0\},$$

$$\gamma(t(x - x_0) + x_0) = t\gamma(x), \ \forall x \in \mathbb{R}^n, \ \forall t > 0.$$

**Proposition 11** *(i) If $\gamma \in G_{x_0}$ is differentiable at $x \in \mathbb{R}^n$ (this happens at almost all points) then it is differentiable at any $x_t \in \mathbb{R}^n$ of the form $x_t = t(x - x_0) + x_0$, $t > 0$ and*

$$D\gamma(x_t) = D\gamma(x).$$

*In particular $\gamma$ is differentiable at almost all points of $\{x \in \mathbb{R}^n : \gamma(x) = 1\}$.*

*(ii) Let $C \subset \mathbb{R}^n$ be a non empty bounded open convex set and $x_0 \in C$. The gauge of $C$ centered at $x_0$ is defined as*

$$\gamma_{C,x_0}(x) = \inf\left\{\lambda \geq 0 : \ x_0 + \frac{x - x_0}{\lambda} \in C\right\}.$$

*Then $\gamma_{C,x_0} \in G_{x_0}$ and*

$$C = \{x \in \mathbb{R}^n : \ \gamma_{C,x_0}(x) < 1\},$$
$$\partial C = \{x \in \mathbb{R}^n : \ \gamma_{C,x_0}(x) = 1\}.$$

**Remark 12** *At almost every point $x \in \partial C$, $\gamma_{C,x_0}$ is differentiable and $D\gamma_{C,x_0}(x)$ is then an outward normal to $C$.*

We will proceed inductively to define $\Omega_i$. We start by choosing a sequence of points in $\Omega$, $\{x^N\}_{N=1}^{\infty}$, dense in $\Omega$. We set $\Omega_0 = \emptyset$ and assume that $\Omega_i$ has already been defined. If $\Omega \setminus \bigcup_{k=0}^{i} \overline{\Omega}_k = \emptyset$, then the procedure is already over. We then define, $N = N(i+1)$,

$$N(i+1) = \min\left\{N : \ x^N \in \Omega \setminus \bigcup_{k=0}^{i} \overline{\Omega}_k\right\}$$

and we label $x_{i+1} = x^{N(i+1)}$ (so that $x_1 = x^1$). We then choose $r_{i+1} > 0$ sufficiently small so that

$$\left\{x \in \mathbb{R}^n : \ l_{r_{i+1}}(x) \equiv \frac{L(x_{i+1} - x)}{r_{i+1}} < 1\right\} \subset \Omega \setminus \bigcup_{k=0}^{i} \overline{\Omega}_k,$$

where

$$L(x) = \max\{\langle \xi, x\rangle : \ \xi \in E\}.$$

This is always possible since $\Omega \setminus \bigcup_{k=0}^{i} \overline{\Omega}_k$ is an open set, $x_{i+1} \in \Omega \setminus \bigcup_{k=0}^{i} \overline{\Omega}_k$, $L(0) = 0$ and $L$ is locally Lipschitz.

We next define

$$\Gamma\left(x_{i+1}, \Omega \setminus \bigcup_{k=0}^{i} \overline{\Omega}_k\right) = \left\{\begin{array}{c} \gamma \in G_{x_{i+1}} : \ \frac{-D\gamma(x)}{\rho(-D\gamma(x))} \in E, \ \text{a.e. } x \in \mathbb{R}^n \\ \\ \left\{x \in \mathbb{R}^n : \ l_{r_{i+1}}(x) < 1\right\} \subset \{x \in \mathbb{R}^n : \ \gamma(x) < 1\} \subset \Omega \setminus \bigcup_{k=0}^{i} \overline{\Omega}_k \end{array}\right\}.$$

Note that $l_{r_{i+1}} \in \Gamma\left(x_{i+1}, \Omega \setminus \bigcup_{k=0}^{i} \overline{\Omega}_k\right)$, since, by Theorem 1, $DL \in E$ and $\rho(DL) = 1$. Observe also that if $\gamma \in \Gamma\left(x_{i+1}, \Omega \setminus \bigcup_{k=0}^{i} \overline{\Omega}_k\right)$ then

$$\gamma \leq l_{r_{i+1}}. \tag{15}$$

We now claim that there exists $\gamma_{i+1} \in \Gamma\left(x_{i+1}, \Omega \setminus \overset{i}{\underset{k=0}{\cup}} \overline{\Omega}_k\right)$ such that if

$$\Omega_{i+1} = \{x \in \mathbb{R}^n : \ \gamma_{i+1}(x) < 1\} \ ,$$

then

$$\operatorname{meas}(\Omega_{i+1}) = \sup_{\gamma \in \Gamma\left(x_{i+1}, \Omega \setminus \overset{i}{\underset{k=0}{\cup}} \overline{\Omega}_k\right)} \left[\operatorname{meas}\{x \in \mathbb{R}^n : \ \gamma(x) < 1\}\right].$$

Indeed let $\{\gamma^s\}$ be a maximizing sequence. From (15), we deduce that up to a subsequence, that we still label $\{\gamma^s\}$, the sequence converges to an element $\gamma_{i+1} \in \Gamma\left(x_{i+1}, \Omega \setminus \overset{i}{\underset{k=0}{\cup}} \overline{\Omega}_k\right)$. In fact all the conditions are easily checked. By Remark 5 we have

$$\frac{-D\gamma_{i+1}(x)}{\rho(-D\gamma_{i+1}(x))} \in E\,.$$

Let us prove, for example, that $\gamma_{i+1}(x) \neq 0$ if $x \neq x_{i+1}$. Assume for the sake of contradiction that there exists $y \neq x_{i+1}$ with $\gamma_{i+1}(y) = 0$. We would deduce that $\gamma_{i+1} \equiv 0$ on the half line $x_{i+1} + t(y - x_{i+1})$, $t \geq 0$, which contradicts, $\Omega$ being bounded, the inclusion $\{x \in \mathbb{R}^n : \ \gamma_{i+1}(x) < 1\} \subset \Omega$.

Since the measure is upper semicontinuous (in fact even continuous), cf. Proposition 14, with respect to the type of convergence under consideration we have the result.

Observe that, as wished, $\Omega_1 = \Omega$ if $\Omega$ is convex and $\frac{-\nu}{\rho(-\nu)} \in E$, a.e. on $\partial\Omega$ (because choosing $\omega$ the gauge of $\Omega$ centered at $x_1$ we would have $\omega \in \Gamma(x_1, \Omega)$).

Since we have, with this procedure, exhausted all elements of the sequence $\left\{x^N\right\}$, we have indeed

$$\operatorname{meas}\left[\Omega \setminus \overset{\infty}{\underset{i=1}{\cup}} \overline{\Omega}_i\right] = 0.$$

**Example 13** *Consider the case* $\Omega = (0,1)^2 \subset \mathbb{R}^2$, $u = u(x_1, x_2)$ *and*

$$\begin{cases} \left(\left(\frac{\partial u}{\partial x_1}\right)^2 - 1\right)^2 + \left(\left(\frac{\partial u}{\partial x_2}\right)^2 - 1\right)^2 = 0 & a.e. \ in \ \Omega \\ u = 0 & on \ \partial\Omega\,. \end{cases}$$

*Choosing the grid sequence* $\left\{x^N\right\}_{N=1}^{\infty}$ *in a suitable way, starting with* $x^1 = (0,0)$, *we find with our procedure*

$$\Omega^1 = \left\{x \in \mathbb{R}^2 : \ |x_1| + |x_2| \leq 1\right\} \ and \ u(x_1, x_2) = 1 - |x_1| - |x_2| \ \ in \ \Omega_1.$$

*Similarly for the* $\Omega_i$. *Our construction is compatible with the numerical computations of [4].*

We end up with an elementary convergence result that we used above.

**Proposition 14** *Let* $\{\gamma^s\}_{s\in\mathbb{N}}$ *and* $\gamma^\infty$ *be measurable functions defined on a bounded measurable set* $\Omega \subset \mathbb{R}^n$*. Let*

$$\Omega^s = \{x \in \Omega : \ \gamma^s(x) \leq 1\},$$
$$\Omega^\infty = \{x \in \Omega : \ \gamma^\infty(x) \leq 1\}.$$

*If* $\gamma^s \to \gamma^\infty$ *a.e. in* $\Omega$*, then*

$$\text{meas}(\Omega^\infty) \geq \limsup_{s\to\infty} \text{meas}(\Omega^s).$$

*If, in addition,* $\gamma^s$ *and* $\gamma^\infty$ *are gauges centered at* $x_0 \in \Omega$ *and* $\Omega$ *is open, then*

$$\text{meas}(\Omega^\infty) = \lim_{s\to\infty} \text{meas}(\Omega^s).$$

**Remark 15** *Note that if* $\gamma^s$ *and* $\gamma^\infty$ *are merely convex, then continuity does not hold as the following example shows. Let* $\Omega \subset \mathbb{R}^n$ *be a bounded open set containing the unit ball* $B_1$*. If*

$$\gamma^s(x) = \begin{cases} 1 & \text{if } |x| \leq 1 \\ \frac{1}{s}|x| + \frac{s-1}{s} & \text{if } |x| > 1, \end{cases}$$

*then* $\Omega^s = B_1$ *for every* $s \in \mathbb{N}$ *, while* $\Omega^\infty = \Omega$*.*

**Proof.** 1) Define

$$\chi^s(x) = \begin{cases} 0 & \text{if } x \in \Omega^s \\ 1 & \text{if } x \notin \Omega^s \end{cases}$$

and similarly for $\chi^\infty$. Note that because of the convergence of $\gamma^s$ to $\gamma^\infty$, we have that, at almost all points where $\chi^\infty(x) = 1$ (i.e. $\gamma^\infty(x) > 1$) and for large enough $s$, $\chi^s(x) = 1$ and thus

$$\lim_{s\to\infty} \chi^s(x) = \chi^\infty(x), \text{ a.e. } x \notin \Omega^\infty.$$

Moreover, trivially, $\liminf_{s\to\infty} \chi^s(x) \geq \chi^\infty(x) = 0$, a.e. $x \in \Omega^\infty$ and therefore

$$\liminf_{s\to\infty} \chi^s(x) \geq \chi^\infty(x), \text{ a.e. } x \in \Omega.$$

Therefore by Fatou's lemma

$$\begin{aligned} \liminf_{s\to\infty} [\text{meas}(\Omega) - \text{meas}(\Omega^s)] &= \liminf_{s\to\infty} \int_\Omega \chi^s(x)\, dx \\ &\geq \int_\Omega \chi^\infty(x)\, dx = \text{meas}(\Omega) - \text{meas}(\Omega^\infty) \end{aligned}$$

which gives the upper semicontinuity.

2) Let $B_\varepsilon = \{x \in \mathbb{R}^n : \ |x| \leq \varepsilon\}$ and for $A \subset \mathbb{R}^n$ define

$$A + B_\varepsilon = \{x \in \mathbb{R}^n : \ x = y + z \text{ with } y \in A \text{ and } |z| \leq \varepsilon\}.$$

The Hausdorff distance between two sets is then defined as

$$d(A,B) = \inf\{\varepsilon \geq 0 : \ A \subset B + B_\varepsilon ,\ B \subset A + B_\varepsilon\}.$$

Observe (see below) that since $\gamma^s$ and $\gamma^\infty$ are gauges then

$$d(\Omega^s, \Omega^\infty) \to 0, \text{ as } s \to \infty \tag{16}$$

and therefore (see Theorem 6.2.17 in [13])

$$\operatorname{meas}(\Omega^\infty) = \lim_{s\to\infty} \operatorname{meas}(\Omega^s).$$

We now establish (16). We will prove that for every $\varepsilon > 0$ we can find $s$ sufficiently large so that

$$\Omega^\infty \subset \Omega^s + B_\varepsilon ,\ \Omega^s \subset \Omega^\infty + B_\varepsilon . \tag{17}$$

Assume without loss of generality that $x_0 = 0$. Since $\Omega$ is bounded and $\gamma^s$ are gauges that converge almost everywhere to a gauge $\gamma^\infty$, the convergence is, in fact, uniform. Furthermore there exist $m, M > 0$ so that

$$m|x| \leq \gamma^s(x), \gamma^\infty(x) \leq M|x|, \forall x \in \Omega,$$

and, for $s$ sufficiently large,

$$|\gamma^s(x) - \gamma^\infty(x)| \leq \varepsilon^2, \forall x \in \Omega.$$

Let $x \in \Omega^\infty$, i.e. $\gamma^\infty(x) \leq 1$, and choose $\delta > 0$ such that

$$\frac{\varepsilon^2}{1+\varepsilon^2} \leq \delta \leq m\varepsilon$$

and observe that

$$\gamma^s((1-\delta)x) = (1-\delta)\gamma^s(x) \leq (1-\delta)\left(\gamma^\infty(x) + \varepsilon^2\right) \leq (1-\delta)\left(1+\varepsilon^2\right) \leq 1,$$

$$|\delta x| \leq \delta \frac{\gamma^\infty(x)}{m} \leq \frac{\delta}{m} \leq \varepsilon.$$

Therefore $x = (1-\delta)x + \delta x \in \Omega^s + B_\varepsilon$ which is the first inclusion in (17). The second one being proved in a similar manner we have the claim. ■

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