On the solvability of the equation $\operatorname{div} u = f$ in L^1 and in C^0.

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Abstract

We show that the equation $\operatorname{div} u = f$ has, in general, no Lipschitz (respectively $W^{1,1}$) solution if f is C^0 (respectively L^1).

1 Introduction

Consider a bounded open set $\Omega \subset \mathbb{R}^n$ and a vector field $u : \Omega \to \mathbb{R}^n$. Define the linear operator $L : X \to Y$ by

$$Lu\,(x) = \operatorname{div} u = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i}.$$

Usually this operator is coupled with some boundary conditions but we will be concerned here only with a local problem of regularity. It is well known that this operator is onto (as a direct consequence of classical regularity results on Laplace equation) in the following cases

$$X = C^{k+1,\alpha},\ Y = C^{k,\alpha} \text{ with } k \geq 0 \text{ and } 0 < \alpha < 1$$
$$X = W^{k+1,p},\ Y = W^{k,p} \text{ with } k \geq 0 \text{ and } 1 < p < \infty.$$

The aim of this report is to show that this operator is *not* onto when

$$X = C^1 \text{ (or } W^{1,\infty}), \ Y = C^0$$
$$X = W^{1,1}, \ Y = L^1.$$

It may seem that this follows at once from the known counterexamples for Laplace equation; this is not the case because the equation $\operatorname{div} u = f$ has other solutions than the one of the form $u = \operatorname{grad} v$.

After having solved the problem we have learnt that both questions have already been studied by several authors. The result concerning C^0 and L^∞ have been proved (to the best of our knowledge) by Preiss [6], Mc Mullen [4] and have been announced by Bourgain-Brézis in [2], who also mention the case of L^1.

2 The L^1 case: a first approach

Let

$$\psi(x_1, x_2) = x_1 x_2 V(|x|)$$

then

$$\psi_{x_1x_1} = \frac{x_1^3 x_2}{|x|^2} V''(|x|) + \frac{x_1 x_2}{|x|^3}\left(2x_1^2 + 3x_2^2\right) V'(|x|)$$
$$\psi_{x_2x_2} = \frac{x_2^3 x_1}{|x|^2} V''(|x|) + \frac{x_1 x_2}{|x|^3}\left(2x_2^2 + 3x_1^2\right) V'(|x|)$$
$$\psi_{x_1x_2} = \frac{x_1^2 x_2^2}{|x|^2} V''(|x|) + \frac{x_1^4 + x_1^2x_2^2 + x_2^4}{|x|^3} V'(|x|) + V(|x|).$$

Choosing $\Omega = \left\{x \in \mathbb{R}^2 : |x| < 1/2\right\}$ and for $0 < \alpha < 1$,

$$V(r) = |\log r|^\alpha$$

we get that

$$\psi_{x_1x_1}, \psi_{x_2x_2} \in C^0(\Omega), \ \psi_{x_1x_2} \notin L^\infty(\Omega).$$

- Let $\eta \in C_0^\infty(\Omega)$ and $\eta \equiv 1$ near $|x| = 0$. Define for N an integer

$$\psi^N(x) = \eta(x) x_1 x_2 V\left(\frac{1}{N} + |x|\right).$$

Observe that $\psi^N \in C_0^\infty(\Omega)$ and there exists a constant c_1 independent of N so that

$$\left|\psi^N_{x_1x_1}\right|_{L^\infty} + \left|\psi^N_{x_2x_2}\right|_{L^\infty} \le c_1$$

- We have furthermore, for $u = \left(u^1, u^2\right) \in W^{1,1}\left(\Omega; \mathbb{R}^2\right)$, that

$$\iint_\Omega \left(u^1_{x_1} + u^2_{x_2}\right) \psi^N_{x_1x_2} dx_1 dx_2 = \iint_\Omega \left(u^1_{x_2}\, \psi^N_{x_1x_1} + u^2_{x_1}\, \psi^N_{x_2x_2}\right) dx_1 dx_2$$

and thus there exists a constant $c_2 = c_2\left(c_1, |u|_{W^{1,1}}\right)$ independent of N so that

$$\left|\iint_{\Omega} \operatorname{div} u\, \psi^N_{x_1 x_2} dx_1 dx_2\right| \leq c_2.$$

- However if we choose

$$f(x) = \frac{1}{|x|^2 \left|\log |x|\right|^{\alpha+1}}$$

we get that $f \in L^1(\Omega)$ and, for ψ^N as above, we get by Fatou lemma that

$$\lim_{N\to\infty} \left|\iint_{\Omega} f\, \psi^N_{x_1 x_2} dx_1 dx_2\right| = \infty.$$

The combination of the above facts leads to the desired conclusion.

3 The L^1 case: a second approach

We start by recalling the definition of Lorentz spaces.

Let $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ (Ω a bounded open set) be measurable we then define the distribution function by

$$\lambda(s) = \operatorname{meas}\{x \in \Omega : |u(x)| \geq s\}$$

and the decreasing rearrangement of u by

$$u^*(t) = \inf\{s : \lambda(s) < t\},\ t \in [0, |\Omega|].$$

If $1 \leq p, q < \infty$ we define the Lorentz space $L^{p,q}(\Omega)$ to be the space of u such that

$$|u|_{L^{p,q}} = \left(\int_0^{|\Omega|} u^*(t)^q\, t^{\frac{q}{p}-1}\, dt\right)^{1/q} < \infty$$

and if $q = \infty$

$$|u|_{L^{p,\infty}} = \operatorname{ess\,sup}\left[u^*(t)\, t^{\frac{1}{p}}\right] < \infty.$$

In particular $L^{p,p}$ can be identified with L^p.

We now give an example that will be used below.

Proposition 1 *Let $\Omega = \{x \in \mathbb{R}^n : 0 < |x| < 1/2\}$. Let $\eta \in C^\infty(0, 1/2)$ be such that*

$$\eta(t) = \begin{cases} 1 & \text{near } t = 0 \\ 0 & \text{near } t = 1/2. \end{cases}$$

Let

$$V(r) = \int_{1/2}^{r} \frac{\rho^{1-n}}{\log \rho}\, d\rho,\ 0 < r < 1/2$$

$$\varphi(x) = \eta(|x|)\, V(|x|)$$

(note that, when $n=2$, $V(|x|)=\log|\log|x||-\log\log 2$). Call

$$f(x)=\Delta\varphi(x)=\begin{cases}\left(|x|^n\log^2|x|\right)^{-1} & \text{near } |x|=0\\ 0 & \text{near } |x|=1/2\end{cases}.$$

Then $f\in L^1(\Omega)$ and φ solves, in the sense of distributions,

$$\begin{cases}\Delta\varphi=f & \text{in } \Omega\\ \varphi=0 & \text{on } \partial\Omega.\end{cases}\tag{1}$$

Note however that $\nabla\varphi\notin L^{\frac{n}{n-1},1}(\Omega)$ and that, in the case $n=2$, $\varphi\notin L^\infty(\Omega)$.

Remark 2 *More refined examples show that solutions of (1) have their gradients in $L^{\frac{n}{n-1},\infty}(\Omega)$ but not in $L^{\frac{n}{n-1},q}(\Omega)$ for every $q<\infty$.*

Proof. Clearly $f\in L^1(\Omega)$ and $\varphi\notin L^\infty(\Omega)$ when $n=2$. We therefore only check that $\nabla\varphi\notin L^{\frac{n}{n-1},1}(\Omega)$. We have

$$\nabla\varphi(x)=\left[\eta(|x|)V'(|x|)+\eta'(|x|)V(|x|)\right]\frac{x}{|x|}$$

and hence the result will follow if we can show that $\psi\notin L^{\frac{n}{n-1},1}(0,r_0)$, for $r_0>0$ sufficiently small, where (ω_n denoting the measure of the unit ball)

$$\psi(t)=V'\left(\left(\frac{t}{\omega_n}\right)^{1/n}\right)=\frac{\left(\frac{t}{\omega_n}\right)^{\frac{1-n}{n}}}{\frac{1}{n}\log\left(\frac{t}{\omega_n}\right)}.$$

We therefore have

$$|\psi|_{L^{\frac{n}{n-1},1}}\equiv\int_0^{r_0}|\psi(t)|\,t^{-\frac{1}{n}}dt=n(\omega_n)^{\frac{n-1}{n}}\int_0^{r_0}\frac{dt}{t(\log t-\log\omega_n)}=\infty.$$

■

The combination of the preceding counterexample and the following proposition gives the result for the L^1 case.

Proposition 3 *Let $\Omega\subset\mathbb{R}^n$ be the unit ball and let $u\in W^{1,1}(\Omega;\mathbb{R}^n)$. Then there exists a solution, in the sense of distributions, of*

$$\begin{cases}\Delta\varphi=\operatorname{div}u & \text{in } \Omega\\ \varphi=0 & \text{on } \partial\Omega.\end{cases}\tag{2}$$

Furthermore $\nabla\varphi\in L^{\frac{n}{n-1},1}(\Omega)$ and hence in particular, when $n=2$, φ is continuous.

Proof. We just sketch the main ingredients of the proof.

- The first fact is that a more refined version of the Sobolev imbedding theorem gives that $u\in W^{1,1}$ implies $u\in L^{\frac{n}{n-1},1}$, cf. [9].

- Using the Green function $G=G(x,y)$ (cf. [3]) and applying the divergence theorem we can write the solution in terms of singular integrals, namely

$$\varphi(x)=\int_\Omega \operatorname{div} u(y)\; G(x,y)\; dy=-\int_\Omega \langle u(y);\nabla_y G(x,y)\rangle\; dy.$$

- The estimate on the gradient can be obtained as follows. Let $Tu=\nabla\varphi=\left(\frac{\partial\varphi}{\partial x_1},...,\frac{\partial\varphi}{\partial x_n}\right)$. Standard results on singular integrals (cf. Theorem 3 of Chapter II page 39 in [7]), show that for every $1<p<\infty$ we can find a constant c_p such that

$$|Tu|_{L^p}\le c_p\,|u|_{L^p}\ .$$

- Since $u\in L^{\frac{n}{n-1},1}$ we can use Marcinkiewicz interpolation theorem (see Theorem 5.3.2 page 113 in [1] or Theorem 3.15 of Chapter V page 197 in [8]) to find a constant $c'_{n/(n-1)}$ such that

$$|\nabla\varphi|_{L^{\frac{n}{n-1},1}}=|Tu|_{L^{\frac{n}{n-1},1}}\le c'_{\frac{n}{n-1}}\,|u|_{L^{\frac{n}{n-1},1}}\ .$$

The result then follows. ■

Remark 4 *It is interesting to compare the two arguments that have been used in this section and in the preceding one.*

The second method only uses the fact that $W^{1,1}\subset L^{\frac{n}{n-1},1}$ *and shows that not all functions of* L^1 *are divergences of functions in* $L^{\frac{n}{n-1},1}$*. It essentially uses the convolution by the elementary solution of the Laplacian, which has a singularity of the form* r^{2-n} *(or* $\log r$ *if* $n=2$*). One easily generalizes this fact. Note first that if* a *has derivatives that belong to* $L^{n,\infty}$ *(so that* $a\in BMO$*) then* $\operatorname{div} u * a=\sum u^j * a_{x_j}$ *(after truncation) is continuous. However* $f*a$ *cannot be continuous for all* $f\in L^1$ *unless* a *is bounded.*

The first method uses a larger class of functions a *(the* $\psi_{x_1x_2}$ *of the counterexample), those that satisfy* $a_{x_j}\in W^{-1,\infty}$ *for all* j *(note that since* $W_0^{1,1}$ *is dense in* $L^{\frac{n}{n-1},1}$ *we have* $L^{n,\infty}\subset W^{-1,\infty}$*). Indeed if* $f=\operatorname{div} u$ *with* $u\in W_0^{1,1}$ *then* $\langle f;a\rangle=-\sum\left\langle u^j;a_{x_j}\right\rangle$ *is well defined.*

4 The continuous case

We recall an example due to Ornstein [5] (Mc Mullen uses the more abstract version of Ornstein theorem to prove his result).

Let $N\in\mathbb{N}$ and $\Omega=(0,1)^2$ then there exists $\psi^N=\psi^N(x_1,x_2)\in C_0^\infty(\Omega)$ such that

$$\left|\psi^N_{x_1x_1}\right|_{L^1}+\left|\psi^N_{x_2x_2}\right|_{L^1}=1$$
$$N\left(\left|\psi^N_{x_1x_1}\right|_{L^1}+\left|\psi^N_{x_2x_2}\right|_{L^1}\right)\le\left|\psi^N_{x_1x_2}\right|_{L^1}\ .$$

Note that, for $u=\left(u^1,u^2\right)\in W^{1,\infty}\left(\Omega;\mathbb{R}^2\right)$,

$$\iint_\Omega\left(u^1_{x_1}+u^2_{x_2}\right)\,\psi^N_{x_1x_2}dx_1dx_2=\iint_\Omega\left(u^1_{x_2}\,\psi^N_{x_1x_1}+u^2_{x_1}\,\psi^N_{x_2x_2}\right)dx_1dx_2$$

and hence

$$\left| \iint_{\Omega} \operatorname{div} u \, \psi_{x_1 x_2}^N dx_1 dx_2 \right| \leq |u|_{W^{1,\infty}} \ .$$

Since $\lim_{N\to\infty} \left|\psi_{x_1 x_2}^N\right|_{L^1} = \infty$, using Banach-Steinhaus we can find $f \in C^0$ such that

$$\lim_{N\to\infty} \left| \iint_{\Omega} f \, \psi_{x_1 x_2}^N dx_1 dx_2 \right| = \infty.$$

Combining the above facts we have even shown that there is $f \in C^0$ such that no vector field $u \in W^{1,\infty}\left(\Omega; \mathbb{R}^2\right)$ can satisfy $\operatorname{div} u = f$.

Of course this result immediately extends to higher dimensions.

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