Existence and regularity of solutions of $d\omega = f$ with Dirichlet boundary conditions

B. Dacorogna
Department of mathematics, EPFL, 1015 Lausanne, Switzerland.
E-mail : Bernard.Dacorogna@epfl.ch

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Abstract

This article is dedicated to Olga A. Ladyzhenskaya for her 80th birthday and in admiration for her mathematical achievements.

Given a bounded open set $\Omega \subset \mathbb{R}^n$, a k+1 form f satisfying some compatibility conditions, we solve the problem (in Hölder spaces)

$$\begin{cases} d\omega = f \text{ in } \Omega \\ \omega = 0 \text{ on } \partial\Omega. \end{cases}$$

We consider, in particular, the divergence and the curl operators.

Contents

1 Introduction

The aim of the present article is to study

$$\begin{cases} d\omega = f \text{ in } \Omega \\ \omega = 0 \text{ on } \partial\Omega. \end{cases}$$

or its dual version

$$\begin{cases} \delta\omega = g \text{ in } \Omega \\ \omega = 0 \text{ on } \partial\Omega. \end{cases}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded smooth convex set, ω is a k form ($1 \leq k \leq n-1$) and d (respectively δ) denote the exterior derivative (respectively the codifferential). We will look for solutions in the Hölder class $C^{r,\alpha}$, completely analogous results holding in Sobolev spaces $W^{r,p}$.

In fact we have in mind two important cases. The first one is

$$\begin{cases} \operatorname{div} \omega = f \text{ in } \Omega \\ \omega = 0 \text{ on } \partial\Omega \end{cases} \tag{1}$$

where div is the usual divergence operator. We will assume that $f \in C^{r,\alpha}\left(\overline{\Omega}\right)$ and satisfy the compatibility condition

$$\int_\Omega f(x)\, dx = 0.$$

We will then find $\omega \in C^{r+1,\alpha}\left(\overline{\Omega}; \mathbb{R}^n\right)$ satisfying the above equations.

The second case is with $n = 3$, $f \in C^{r,\alpha}\left(\overline{\Omega}; \mathbb{R}^3\right)$ so that (denoting the scalar product by $\langle .; . \rangle$)

$$\operatorname{div} f = 0 \text{ in } \Omega \text{ and } \langle f; \nu \rangle = 0 \text{ on } \partial\Omega.$$

We will prove that there exists $\omega \in C^{r+1,\alpha}\left(\overline{\Omega}; \mathbb{R}^3\right)$ satisfying

$$\begin{cases} \operatorname{curl} \omega = f \text{ in } \Omega \\ \omega = 0 \text{ on } \partial\Omega. \end{cases} \tag{2}$$

The general problem under consideration is well known in algebraic topology since the classical work of De Rham (see for example [10]). However usually, either only manifolds without boundaries are considered or the forms have compact support. Moreover the question of regularity of the solution is not an issue that is discussed.

The particular case of the divergence (including the question of regularity), because of its relevancy to applications, has received special attention by many analysts. We quote only the few of them that we have been able to trace: Bogovski [1], Borchers-Sohr [2], Dacorogna-Moser [4] (cf. also Dacorogna [3]), Dautray-Lions [5], Galdi [7], Girault-Raviart [9], Kapitanskii-Pileckas [12], Ladyzhenskaya [14], Ladyzhenskaya-Solonnikov [15], Necas [19], Tartar [20], Von Wahl [21], [22].

The case of the curl in dimension 3, which is also useful for applications, has been considered in particular by Borchers-Sohr [2], Dautray-Lions [5], Griesinger [11], Von Wahl [21], [22].

We present here a different proof that is in the spirit of Dacorogna-Moser [4] and that applies to the general case of k forms. Of course the ingredients are also very similar to those of, for example, Ladyzhenskaya [14] or Von Wahl [21],

[22]. They differ essentially in the way we fix the boundary data. The proof is self contained up to the important result on elliptic systems (cf. Theorem 8) that finds its origins in Duff-Spencer [6] and Morrey [17], [18]. As quoted here the result is due to Kress [13].

We finally comment on possible generalizations of the results that we have obtained.

(1) A completely similar analysis can be carried over to inhomogeneous boundary data.

(2) At the end of Section 7 we will explain how one can deal with non convex sets. It should be immediately noted that, with no change, we could have assumed that the set $\Omega \subset \mathbb{R}^n$ is star shaped or more generally contractible. Moreover we should observe that for the particular case of the divergence no other condition than connectedness is assumed.

(3) The smoothness of the boundary $\partial\Omega$ can also be relaxed but require finer regularity results.

(4) As mentioned earlier, analogous results can be obtained by this method for Sobolev spaces instead of Hölder ones.

The article is organized as follows. For the sake of exposition we first discuss the problems (1) and (2), although both results are particular cases of the general ones contained in Section 7.

2 A preliminary lemma

We start with this elementary lemma whose proof can be found in Dacorogna and Moser [4]. This lemma and its consequences established in Section 6 will be used to fix the boundary data.

Lemma 1 *Let $r \geq 1$ be an integer and $0 < \alpha < 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with orientable $C^{r+2,\alpha}$ boundary consisting of finitely many connected components (ν denote the outward unit normal). Let $c \in C^{r,\alpha}\left(\overline{\Omega}\right)$ then there exists $b \in C^{r+1,\alpha}\left(\overline{\Omega}\right)$ satisfying*

$$\operatorname{grad} b = c\,\nu \text{ on } \partial\Omega.$$

Proof. If one is not interested in the sharp regularity result, a solution of the problem is given by

$$b\,(x) = -c\,(x)\,\zeta\,(\operatorname{dist}(x, \partial\Omega))$$

where $\operatorname{dist}(x, \partial\Omega)$ stands for the distance from x to the boundary and ζ is a smooth function so that $\zeta\,(0) = 0$, $\zeta'\,(0) = 1$ and $\zeta \equiv 0$ outside a small neighborhood of 0.

To construct a smoother solution we proceed as follows. First find a $C^{r+1,\alpha}\left(\overline{\Omega}\right)$ solution of (cf. Gilbarg and Trudinger [8] or Ladyzhenskaya and Uraltseva [16])

$$\begin{cases} \Delta d = \frac{1}{\operatorname{meas}\Omega} \int_{\partial\Omega} c\, d\sigma \text{ in } \Omega \\ \\ \frac{\partial d}{\partial \nu} = c \text{ on } \partial\Omega. \end{cases}$$

We then let $\chi \in C^{\infty}(\mathbb{R})$ be such that $\chi(0) = 1$, $\chi'(0) = 0$ and $\chi \equiv 0$ outside a small neighborhood of 0 and define

$$b(x) = d(x) - \chi(\operatorname{dist}(x, \partial\Omega))\, d(\psi(x)).$$

where $\psi(x) = x - \operatorname{dist}(x, \partial\Omega) \operatorname{grad}(\operatorname{dist}(x, \partial\Omega))$.

It remains to check that b has the claimed property. Indeed if $x \in \partial\Omega$ (note that $\psi(x) = x$ on $\partial\Omega$) then

$$\begin{aligned}
\operatorname{grad} b(x) &= \operatorname{grad} d(x) - \operatorname{grad} d(\psi(x))\, D\psi(x) \\
&= \operatorname{grad} d(x) - \operatorname{grad} d(x)\left[I - \operatorname{grad}(\operatorname{dist}(x, \partial\Omega)) \otimes \operatorname{grad}(\operatorname{dist}(x, \partial\Omega))\right] \\
&= \operatorname{grad} d(x)\left[\nu \otimes \nu\right] = \frac{\partial d}{\partial \nu}\,\nu \\
&= c\,\nu.
\end{aligned}$$

■

3 The case of the divergence in $\mathbb{R}^n$

Theorem 2 *Let $r \geq 0$ be an integer and $0 < \alpha < 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open set with orientable $C^{r+3,\alpha}$ boundary consisting of finitely many connected components (ν denote the outward unit normal). The following conditions are then equivalent.*

(i) $f \in C^{r,\alpha}\left(\overline{\Omega}\right)$ satisfies

$$\int_{\Omega} f(x)\, dx = 0.$$

(ii) There exists $\omega \in C^{r+1,\alpha}\left(\overline{\Omega}; \mathbb{R}^n\right)$ verifying

$$\begin{cases} \operatorname{div} \omega = f \ \textit{in}\ \Omega \\ \omega = 0 \ \textit{on}\ \partial\Omega \end{cases}$$

where $\operatorname{div} \omega = \sum_{i=1}^{n} \frac{\partial \omega_i}{\partial x_i}$.

Remark 3 *If the set Ω is disconnected, then the result holds true if the compatibility condition is understood on each connected component.*

Proof. (ii) $\Rightarrow$ (i) This implication is just the divergence theorem.

(i) $\Rightarrow$ (ii) We split the proof into two steps.

Step 1. We first find $a \in C^{r+2,\alpha}$ (cf. Gilbarg and Trudinger [8] or Ladyzhenskaya and Uraltseva [16]) satisfying

$$\begin{cases} \Delta a = f \text{ in } \Omega \\ \frac{\partial a}{\partial \nu} = 0 \text{ on } \partial\Omega. \end{cases}$$

Step 2. We then write

$$\omega = curl^* v + \operatorname{grad} a$$

where $v = (v_{ij})_{1 \le i < j \le n} \in \mathbb{R}^{n(n-1)/2}$,

$$curl^* v = ((curl^* v)_1, ..., (curl^* v)_n)$$

and

$$(curl^* v)_i = \sum_{j=1}^{i-1} \frac{\partial v_{ji}}{\partial x_j} - \sum_{j=i+1}^{n} \frac{\partial v_{ij}}{\partial x_j}.$$

Since $\operatorname{div} curl^* v = 0$ it remains to find $v \in C^{r+2,\alpha}$ such that

$$curl^* v = -\operatorname{grad} a \text{ on } \partial\Omega.$$

An easy computation shows that a solution of this problem is given by

$$\operatorname{grad} v_{ij} = \left(\frac{\partial a}{\partial x_i} \nu_j - \frac{\partial a}{\partial x_j} \nu_i \right) \nu \text{ on } \partial\Omega$$

whose solvability is ensured by Lemma 1. This achieves the proof of the theorem.

In order to clarify the link with the more abstract framework of differential forms, we rewrite the proof in this terminology. We consider ω as a 1 form and therefore the problem we want to solve is

$$\begin{cases} \delta\omega = f \text{ in } \Omega \\ \omega = 0 \text{ on } \partial\Omega. \end{cases}$$

We write

$$\omega = da + \delta v$$

(where a is a 0 form and v is a 2 form). This leads to

$$f = \delta\omega = \delta da = \Delta a$$

since $\delta\delta v = 0$, $\Delta a = \delta da + d\delta a$ and $\delta a = 0$, a being a 0 form. (The fact that $\Delta a = \delta da$ makes easier the case of 1 forms ω in comparison with k forms $k \ge 2$).

We also observe that

$$\delta_\nu (da) = \langle \operatorname{grad} a; \nu \rangle = \frac{\partial a}{\partial \nu}$$

which leads to our choice in Step 1.

Now in order to have $\omega = 0$ on the boundary it remains to solve (cf. Step 2)

$$\delta v = -da \text{ on } \partial\Omega.$$

The idea is then to find a solution, via Lemma 1, of

$$\operatorname{grad} v_{ij} = -[d_\nu da]_{ij} \nu = \left(\frac{\partial a}{\partial x_i} \nu_j - \frac{\partial a}{\partial x_j} \nu_i \right) \nu \text{ on } \partial\Omega$$

and then to check (as in Lemma 9 and 10) that such a v satisfies $\delta v = -da$ on $\partial\Omega$. ■

4 The case of the curl in $\mathbb{R}^3$

Theorem 4 *Let $r \geq 1$ be an integer and $0 < \alpha < 1$. Let $\Omega \subset \mathbb{R}^3$ be a bounded convex set with $C^{r+3,\alpha}$ boundary, ν denote the outward unit normal. The following conditions are then equivalent.*

(i) $f \in C^{r,\alpha}\left(\overline{\Omega};\mathbb{R}^3\right)$ verifies

$$\operatorname{div} f = 0 \text{ in } \Omega \text{ and } \langle f;\nu\rangle = 0 \text{ on } \partial\Omega.$$

(ii) There exists $\omega \in C^{r+1,\alpha}\left(\overline{\Omega};\mathbb{R}^3\right)$ satisfying

$$\begin{cases} \operatorname{curl}\omega = f \text{ in } \Omega \\ \omega = 0 \text{ on } \partial\Omega \end{cases}$$

where if $\omega = (\omega_1,\omega_2,\omega_3)$ then $\operatorname{curl}\omega = \left(\frac{\partial\omega_3}{\partial x_2} - \frac{\partial\omega_2}{\partial x_3}, \frac{\partial\omega_1}{\partial x_3} - \frac{\partial\omega_3}{\partial x_1}, \frac{\partial\omega_2}{\partial x_1} - \frac{\partial\omega_1}{\partial x_2}\right)$.

Proof. (ii) $\Rightarrow$ (i) The fact that $\operatorname{div} f = 0$ is obvious. We now show that $\langle f;\nu\rangle = 0$ on $\partial\Omega$. For this purpose we let $\psi \in C^2\left(\overline{\Omega}\right)$ be an arbitrary function. The integration by parts formula and the facts that $\omega = 0$ on $\partial\Omega$ and $\operatorname{div} f = 0$ lead to

$$\int_\Omega \langle \operatorname{grad}\psi; f\rangle\, dx = \int_\Omega \langle \operatorname{grad}\psi; \operatorname{curl}\omega\rangle\, dx = 0$$

$$\int_\Omega \langle \operatorname{grad}\psi; f\rangle\, dx = \int_{\partial\Omega} \psi\, \langle f;\nu\rangle\, d\sigma.$$

Combining these two equations and the arbitrariness of ψ, we have indeed obtained that $\langle f;\nu\rangle = 0$ on $\partial\Omega$.

(i) $\Rightarrow$ (ii) We divide the proof into two steps.

Step 1.We first find $u \in C^{r+1,\alpha}$, using Theorem 8, that solves the following system (denoting the vectorial product by $u \times \nu$)

$$\begin{cases} \operatorname{curl} u = f \text{ in } \Omega \\ \operatorname{div} u = 0 \text{ in } \Omega \\ u \times \nu = 0 \text{ on } \partial\Omega. \end{cases}$$

In terms of the notations of the next sections we are solving in fact (considering u as a 1 form and f as a 2 form)

$$\begin{cases} du = \widetilde{f} \text{ in } \Omega \\ \delta u = 0 \text{ in } \Omega \\ d_\nu u = 0 \text{ on } \partial\Omega \end{cases}$$

where if $f = (f_{12}, f_{13}, f_{23})$ then

$$\widetilde{f} = (f_{23}, -f_{13}, f_{12}).$$

The compatibility conditions for solving this problem being exactly

$$d\,\widetilde{f} = \operatorname{div} f = 0 \text{ in } \Omega \text{ and } d_\nu \widetilde{f} = 0 \Leftrightarrow \langle f;\nu\rangle = 0 \text{ on } \partial\Omega.$$

Step 2. We then let

$$\omega = u + \operatorname{grad} v$$

where $v \in C^{r+2,\alpha}$ solves on $\partial\Omega$

$$\operatorname{grad} v = -u.$$

Indeed this is possible by Lemma 1 and by the fact that $u \times \nu = 0$. ■

5 Notations and general results on differential forms

We will let $\Omega \subset \mathbb{R}^n$ be a bounded open set with sufficiently smooth orientable boundary and we will denote by ν the outward unit normal. We next let $0 \leq k \leq n$ and consider a k form (we will often identify, by abuse of notations, the form with the vector of $\mathbb{R}^{\binom{n}{k}}$ whose components are those of the form) $\omega : \Omega \to \mathbb{R}^{\binom{n}{k}}$

$$\omega = \sum_{i_1 < \ldots < i_k} \omega_{i_1 \ldots i_k} \, dx_{i_1} \wedge \ldots \wedge dx_{i_k}$$

(if $k = 0$ then ω is just a function).

We then define for such a form the following notions.

1) The *exterior derivative*, denoted $d\omega$, which is a $k+1$ form, is given by

$$d\omega = \sum_{i_1 < \ldots < i_{k+1}} \left(\sum_{\gamma=1}^{k+1} (-1)^{\gamma-1} \frac{\partial \omega_{i_1 \ldots i_{\gamma-1} i_{\gamma+1} \ldots i_{k+1}}}{\partial x_{i_\gamma}} \right) dx_{i_1} \wedge \ldots \wedge dx_{i_{k+1}}$$

(if $k = n$ we let $d\omega = 0$).

2) The *codifferential*, denoted $\delta\omega$, which is a $k-1$ form, and is defined as

$$\delta\omega = \sum_{i_1 < \ldots < i_{k-1}} \left(\sum_{j=1}^{n} \varepsilon_{i_1 \ldots i_{k-1}}^{j} \frac{\partial \omega_{(i_1 \ldots i_{k-1} j)}}{\partial x_j} \right) dx_{i_1} \wedge \ldots \wedge dx_{i_{k-1}}$$

(if $k = 0$ then as usual $\delta\omega = 0$); where we have denoted by $(i_1 \ldots i_{k-1} j)$ the k index rearranged increasingly and

$$\varepsilon_{i_1 \ldots i_{k-1}}^{j} = \begin{cases} 0 \text{ if } j \in \{i_1, \ldots, i_{k-1}\} \\ (-1)^{\gamma-1} \text{ if } i_{\gamma-1} < j < i_\gamma \end{cases} .$$

(if $k = 1$ then $\varepsilon^j = 1$).

3) The *tangential part* (denoted by $d_\nu\omega$) on the boundary $\partial\Omega$ is a $k+1$ form defined by

$$d_\nu\omega = \sum_{i_1 < \ldots < i_{k+1}} \left(\sum_{\gamma=1}^{k+1} (-1)^{\gamma-1} \nu_{i_\gamma} \omega_{i_1 \ldots i_{\gamma-1} i_{\gamma+1} \ldots i_{k+1}} \right) dx_{i_1} \wedge \ldots \wedge dx_{i_{k+1}}$$

(if $k=n$ then $d_\nu \omega = 0$).

4) The *normal part* (denoted by $\delta_\nu \omega$) on $\partial\Omega$ is a $k-1$ form given by

$$\delta_\nu \omega = \sum_{i_1<...<i_{k-1}} \left(\sum_{j=1}^{n} \varepsilon^{j}_{i_1 \dots i_{k-1}} \nu_j \, \omega_{(i_1 \dots i_{k-1} j)} \right) dx_{i_1} \wedge ... \wedge dx_{i_{k-1}}$$

(if $k=0$ we then let $\delta_\nu \omega = 0$).

Remark 5 *(1) One can define the operator δ, equivalently, by duality as*

$$\delta\omega = (-1)^{k(n-k)} * d\,(*\omega)\,.$$

(2) Our definition of the operator δ may differ from the one of some textbooks by a minus sign. Our choice is motivated (cf. (3) below) by the fact that we define for $\omega : \Omega \to \mathbb{R}$ the Laplacian by

$$\Delta\omega = \sum_{i=1}^{n} \frac{\partial^2 \omega}{\partial x_i^2}.$$

Those textbooks that have an opposite sign for the definition of δ define therefore the Laplacian with the opposite sign.

(3) Our definition of the tangential and normal parts of a k form ω is not the usual one; we have adopted here the definition of Kress [13]. For example Duff-Spencer [6] and Morrey [17], [18] write the tangential part $t\omega$ and the normal part $n\omega$ both as forms of the same degree as ω so that $\omega = t\omega + n\omega$. However our definitions and theirs carry similar informations as the following example shows. Indeed if $\Omega = \{x \in \mathbb{R}^n : x_n > 0\}$ and ω is a k form then, on $x_n = 0$ we have

$$\begin{aligned} d_\nu \omega = 0 \Leftrightarrow t\omega = 0 \Leftrightarrow \omega_{i_1 \dots i_k}(x_1, ..., x_{n-1}, 0) = 0 \ \textit{if}\ 1 \le i_1 < ... < i_k < n \\ \delta_\nu \omega = 0 \Leftrightarrow n\omega = 0 \Leftrightarrow \omega_{i_1 \dots i_k}(x_1, ..., x_{n-1}, 0) = 0 \ \textit{if}\ i_k = n. \end{aligned}$$

The terminology "tangential" and "normal", which will induce those ("Dirichlet" and "Neumann") used in Theorem 8 and Section 7, is not always appropriate. It is only adequate for 1 forms (and totally inadequate for $(n-1)$ forms) since then $d_\nu \omega$ can be identified with $\omega \times \nu$ (i.e. the vectorial product), while $\delta_\nu \omega$ is the scalar product $\langle \omega; \nu \rangle$.

The following can then be established.

Proposition 6 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with orientable C^1 boundary and ν denote the outward unit normal to $\partial\Omega$. Let $1 \le k \le n-1$ and $\omega \in C\left(\overline{\Omega}; \mathbb{R}^{\binom{n}{k}}\right)$ be a k form.*

(i) If in addition ω is C^2 then

$$dd\omega = 0,\ \delta\delta\omega = 0 \ \textit{and}\ \Delta\omega = d\delta\omega + \delta d\omega. \tag{3}$$

(ii) The following identity is valid on $\partial\Omega$

$$d_\nu \delta_\nu \omega + \delta_\nu d_\nu \omega = \omega. \tag{4}$$

(iii) If in addition ω is C^1 then the following version of the divergence theorem holds

$$\int_\Omega d\omega\, dx = \int_{\partial\Omega} d_\nu \omega\, d\sigma \text{ and } \int_\Omega \delta\omega\, dx = \int_{\partial\Omega} \delta_\nu \omega\, d\sigma.$$

(iv) The integration by parts formula holds namely

$$\int_\Omega \langle \psi; d\varphi \rangle\, dx + \int_\Omega \langle \delta\psi; \varphi \rangle\, dx = \int_{\partial\Omega} \langle \psi; d_\nu \varphi \rangle\, d\sigma = \int_{\partial\Omega} \langle \delta_\nu \psi; \varphi \rangle\, d\sigma$$

where $\psi \in C^1\left(\overline{\Omega}; \mathbb{R}^{\binom{n}{k}}\right)$, $\varphi \in C^1\left(\overline{\Omega}; \mathbb{R}^{\binom{n}{k-1}}\right)$ and the scalar product of two k forms α and β is defined by

$$\langle \alpha; \beta \rangle = \sum_{i_1 < \ldots < i_k} \alpha_{i_1 \ldots i_k}\, \beta_{i_1 \ldots i_k}.$$

We next give some examples which correspond to the two particular cases considered at the beginning of the present article.

Example 7 *(1) Consider the case of a 1 form ω*

$$\omega = \omega_1 dx_1 + \ldots + \omega_n dx_n$$

then

$$d\omega = \sum_{1 \leq i < j \leq n} \left(\frac{\partial \omega_j}{\partial x_i} - \frac{\partial \omega_i}{\partial x_j} \right) dx_i \wedge dx_j$$

which when $n = 3$ leads in terms of components to $\operatorname{curl} \omega$ (up to the sign and the order of the components). Similarly

$$\delta\omega = \operatorname{div} \omega = \sum_{i=1}^{n} \frac{\partial \omega_i}{\partial x_i}$$

$$d_\nu \omega = \sum_{1 \leq i < j \leq n} (\omega_j \nu_i - \omega_i \nu_j)\ dx_i \wedge dx_j$$

$$\delta_\nu \omega = \langle \nu; \omega \rangle = \sum_{i=1}^{n} \nu_i \omega_i.$$

In particular when $n = 2$ we have

$$d\omega = \left(\frac{\partial \omega_2}{\partial x_1} - \frac{\partial \omega_1}{\partial x_2} \right) dx_1 \wedge dx_2$$

$$\delta\omega = \operatorname{div}\omega = \frac{\partial\omega_1}{\partial x_1} + \frac{\partial\omega_2}{\partial x_2}$$

(the combination of the operators d and δ lead then to the anti Cauchy-Riemann operator).

(2) Consider the case of a 2 form ω in $\mathbb{R}^3$

$$\omega = \omega_{12}\,dx_1 \wedge dx_2 + \omega_{13}\,dx_1 \wedge dx_3 + \omega_{23}\,dx_2 \wedge dx_3$$

then

$$d\omega = \left(\frac{\partial\omega_{12}}{\partial x_3} - \frac{\partial\omega_{13}}{\partial x_2} + \frac{\partial\omega_{23}}{\partial x_1}\right) dx_1 \wedge dx_2 \wedge dx_3$$

$$\delta\omega = \left(-\frac{\partial\omega_{12}}{\partial x_2} - \frac{\partial\omega_{13}}{\partial x_3}\right) dx_1 + \left(\frac{\partial\omega_{12}}{\partial x_1} - \frac{\partial\omega_{23}}{\partial x_3}\right) dx_2 + \left(\frac{\partial\omega_{13}}{\partial x_1} + \frac{\partial\omega_{23}}{\partial x_2}\right) dx_3$$

$$d_\nu\omega = (\nu_3\omega_{12} - \nu_2\omega_{13} + \nu_1\omega_{23})\, dx_1 \wedge dx_2 \wedge dx_3$$

$$\delta_\nu\omega = (-\nu_2\omega_{12} - \nu_3\omega_{13})\, dx_1 + (\nu_1\omega_{12} - \nu_3\omega_{23})\, dx_2 + (\nu_1\omega_{13} + \nu_2\omega_{23})\, dx_3.$$

More generally if ω is a 2 form over $\mathbb{R}^n$ then

$$\delta\omega = \sum_{i=1}^{n}\left(\sum_{j=1}^{i-1}\frac{\partial\omega_{ji}}{\partial x_j} - \sum_{j=i+1}^{n}\frac{\partial\omega_{ij}}{\partial x_j}\right) dx_i\,.$$

The following result, for the existence part, is due to Kress [13] (cf. also Morrey [18] Section 7.7 and 7.8). The regularity then follows from standard arguments cf. Morrey [18].

Theorem 8 *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex set with $C^{r+1,\alpha}$ boundary ($r \geq 1$ being an integer and $0 < \alpha < 1$) and ν denote the outward unit normal. Let $1 \leq k \leq n-1$, $f \in C^{r,\alpha}\left(\overline{\Omega};\mathbb{R}^{\binom{n}{k+1}}\right)$ and $g \in C^{r,\alpha}\left(\overline{\Omega};\mathbb{R}^{\binom{n}{k-1}}\right)$ be such that*

$$df = 0 \text{ and } \delta g = 0 \text{ in } \Omega. \tag{5}$$

***Dirichlet problem**. If in addition to (5)*
- *either $1 \leq k \leq n-2$ and $d_\nu f = 0$ on $\partial\Omega$,*
- *or $k = n-1$ and $\int_\Omega f = 0$,*

then there exists $u \in C^{r+1,\alpha}\left(\overline{\Omega};\mathbb{R}^{\binom{n}{k}}\right)$ satisfying

$$\begin{cases} du = f \text{ in } \Omega \\ \delta u = g \text{ in } \Omega \\ d_\nu u = 0 \text{ on } \partial\Omega. \end{cases}$$

***Neumann problem**. If in addition to (5)*
- *either $2 \leq k \leq n-1$ and $\delta_\nu g = 0$ on $\partial\Omega$,*
- *or $k = 1$ and $\int_\Omega g = 0$,*

then there exists $v \in C^{r+1,\alpha}\left(\overline{\Omega}; \mathbb{R}^{\binom{n}{k}}\right)$ *satisfying*

$$\begin{cases} dv = f \text{ in } \Omega \\ \delta v = g \text{ in } \Omega \\ \delta_\nu v = 0 \text{ on } \partial\Omega. \end{cases}$$

6 A generalization of the preliminary lemma

We have two generalizations of Lemma 1.

Lemma 9 *Let* $r \geq 1$, $1 \leq k \leq n-1$ *be integers and* $0 < \alpha < 1$. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded open set with orientable* $C^{r+2,\alpha}$ *boundary consisting of finitely many connected components (*ν *denote the outward unit normal). Let* $c \in C^{r,\alpha}\left(\overline{\Omega}; \mathbb{R}^{\binom{n}{k}}\right)$ *such that*

$$d_\nu c = 0 \text{ on } \partial\Omega.$$

Then there exists $b \in C^{r+1,\alpha}\left(\overline{\Omega}; \mathbb{R}^{\binom{n}{k-1}}\right)$ *satisfying*

$$db = c \text{ on } \partial\Omega.$$

Proof. First we solve by Lemma 1 (note that when $k = 1$ Lemma 1 and the present lemma are the same) the problem

$$\operatorname{grad} b_{i_1 \dots i_{k-1}} = [\delta_\nu c]_{i_1 \dots i_{k-1}} \nu.$$

We then claim that

$$db = c \text{ on } \partial\Omega.$$

Observe first that the definition of b implies that

$$db = d_\nu \delta_\nu c.$$

We combine the above fact with the hypothesis $d_\nu c = 0$ and with the identity (4) to get

$$d_\nu \delta_\nu c + \delta_\nu d_\nu c = d_\nu \delta_\nu c = c,$$

which is the claimed result. ■

The second generalization is the dual of the preceding one and is proved by duality (replacing d by δ and conversely).

Lemma 10 *Let* $r \geq 1$, $1 \leq k \leq n-1$ *be integers and* $0 < \alpha < 1$. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded open set with orientable* $C^{r+2,\alpha}$ *boundary consisting of finitely many connected components (*ν *denote the outward unit normal). Let* $c \in C^{r,\alpha}\left(\overline{\Omega}; \mathbb{R}^{\binom{n}{k}}\right)$ *such that*

$$\delta_\nu c = 0 \text{ on } \partial\Omega.$$

Then there exists $b \in C^{r+1,\alpha}\left(\overline{\Omega};\mathbb{R}^{\binom{n}{k+1}}\right)$ *satisfying*

$$\delta b = c \text{ on } \partial\Omega.$$

7 The main result

Theorem 11 *Let $r \geq 1$, $1 \leq k \leq n-1$ be integers and $0 < \alpha < 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded convex set with $C^{r+3,\alpha}$ boundary and ν denote the outward unit normal. Let f be a $k+1$ form. The following two conditions are then equivalent.*

(i) • *Either $1 \leq k \leq n-2$ and $f \in C^{r,\alpha}\left(\overline{\Omega};\mathbb{R}^{\binom{n}{k+1}}\right)$ with*

$$df = 0 \text{ in } \Omega \text{ and } d_\nu f = 0 \text{ on } \partial\Omega;$$

• *or $k = n-1$ and $f \in C^{r,\alpha}\left(\overline{\Omega}\right)$ satisfies*

$$\int_\Omega f(x)\,dx = 0.$$

(ii) There exists $\omega \in C^{r+1,\alpha}\left(\overline{\Omega};\mathbb{R}^{\binom{n}{k}}\right)$ satisfying

$$\begin{cases} d\omega = f \text{ in } \Omega \\ \omega = 0 \text{ on } \partial\Omega. \end{cases} \tag{6}$$

Remark 12 *The above theorem is trivially valid for $k = 0$, i.e.*

$$\begin{cases} \operatorname{grad}\omega = f \text{ in } \Omega \\ \omega = 0 \text{ on } \partial\Omega. \end{cases}$$

Observe however that in the sufficiency part of the proof we cannot invoke anymore Theorem 8. A straightforward integration leads immediately to the result. Note also that in this case the solution is unique and regularity holds in C^r spaces as well.

Proof. (i) ⇒ (ii) We divide this proof into two steps.

Step 1. We start by applying Theorem 8 to get $u \in C^{r+1,\alpha}\left(\overline{\Omega};\mathbb{R}^{\binom{n}{k}}\right)$ solving the system

$$\begin{cases} du = f \text{ in } \Omega \\ \delta u = 0 \text{ in } \Omega \\ d_\nu u = 0 \text{ on } \partial\Omega. \end{cases}$$

Note that this is possible, in view of the compatibility conditions on f.

Step 2. We then use Lemma 9 to find $v \in C^{r+2,\alpha}\left(\overline{\Omega};\mathbb{R}^{\binom{n}{k}}\right)$ so that (note that this is possible since $d_\nu u = 0$ on $\partial\Omega$)

$$dv = -u \text{ on } \partial\Omega.$$

Finally write

$$\omega = u + dv$$

to obtain the result.

(ii) $\Rightarrow$ (i) We start by discussing the case $k = n-1$. Combining the divergence theorem and (6) we get

$$\int_\Omega f = \int_\Omega d\omega = \int_{\partial\Omega} d_\nu \omega = 0.$$

We next consider the case $1 \le k \le n-2$. The first condition, $df = 0$, is obvious, so it remains to prove that $d_\nu f = 0$ on $\partial\Omega$. To this aim we let ψ be any smooth $k+2$ form. We then use the integration by parts formula, (6) and the fact that $\delta\delta\psi = 0$ to get

$$\int_\Omega \langle \delta\psi; f\rangle\, dx = \int_\Omega \langle \delta\psi; d\omega\rangle\, dx + \int_\Omega \langle \delta\delta\psi; \omega\rangle\, dx = \int_{\partial\Omega} \langle \delta_\nu \delta\psi; \omega\rangle\, d\sigma = 0. \quad (7)$$

We again invoke the integration by parts formula and the fact that $df = 0$ to get

$$\int_\Omega \langle \delta\psi; f\rangle\, dx = \int_\Omega \langle \delta\psi; f\rangle\, dx + \int_\Omega \langle \psi; df\rangle\, dx = \int_{\partial\Omega} \langle \psi; d_\nu f\rangle\, d\sigma. \quad (8)$$

Combining (7), (8) and the arbitrariness of ψ, we have indeed obtained that $d_\nu f = 0$ on $\partial\Omega$, which is the claimed result. ■

The dual version of the preceding theorem is the following.

Theorem 13 *Let $r \ge 1$, $1 \le k \le n-1$ be integers and $0 < \alpha < 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded convex set with $C^{r+3,\alpha}$ boundary and ν denote the outward unit normal. Let f be a $k-1$ form. The following two conditions are then equivalent.*

(i) • *Either $2 \le k \le n-1$ and $f \in C^{r,\alpha}\left(\overline{\Omega}; \mathbb{R}^{\binom{n}{k-1}}\right)$ is such that*

$$\delta f = 0 \text{ in } \Omega \text{ and } \delta_\nu f = 0 \text{ on } \partial\Omega;$$

• *or $k = 1$ and $f \in C^{r,\alpha}\left(\overline{\Omega}\right)$ satisfies*

$$\int_\Omega f(x)\, dx = 0.$$

(ii) There exists $\omega \in C^{r+1,\alpha}\left(\overline{\Omega}; \mathbb{R}^{\binom{n}{k}}\right)$ satisfying

$$\begin{cases} \delta\omega = f \text{ in } \Omega \\ \omega = 0 \text{ on } \partial\Omega. \end{cases} \quad (9)$$

Remark 14 *The case $k = n$ requires a different treatment, cf. the preceding remark.*

We conclude the article with some comments on the results when Ω is not necessarily convex. We will assume that $\Omega \subset \mathbb{R}^n$ is a bounded connected set with orientable smooth boundary (ν will then denote the outward unit normal) consisting of finitely many connected components.

Definition 15 *Let $0 \leq k \leq n$ and ψ be a k form over Ω. The set of k harmonic fields with Dirichlet, respectively Neumann, boundary condition is defined as the vector space*

$$D_k(\Omega) = \left\{ \begin{array}{c} \psi \in C^0\left(\overline{\Omega}; \mathbb{R}^{\binom{n}{k}}\right) \cap C^1\left(\Omega; \mathbb{R}^{\binom{n}{k}}\right) : \\ d\psi = 0,\ \delta\psi = 0 \text{ in } \Omega \text{ and } d_\nu \psi = 0 \text{ on } \partial\Omega \end{array} \right\}$$

respectively

$$N_k(\Omega) = \left\{ \begin{array}{c} \psi \in C^0\left(\overline{\Omega}; \mathbb{R}^{\binom{n}{k}}\right) \cap C^1\left(\Omega; \mathbb{R}^{\binom{n}{k}}\right) : \\ d\psi = 0,\ \delta\psi = 0 \text{ in } \Omega \text{ and } \delta_\nu \psi = 0 \text{ on } \partial\Omega \end{array} \right\}.$$

Remark 16 *Note that we always have, for Ω as above,*

$$N_n(\Omega) \simeq D_0(\Omega) \simeq \{0\} \text{ and } D_n(\Omega) \simeq N_0(\Omega) \simeq \mathbb{R}.$$

Furthermore if $1 \leq k \leq n-1$ and if the set Ω is convex, or more generally contractible, then

$$N_k(\Omega) = D_k(\Omega) \simeq \{0\} \subset \mathbb{R}^{\binom{n}{k}},$$

while for general sets we have

$$\dim D_k(\Omega) = B_{n-k} \text{ and } \dim N_k(\Omega) = B_k$$

where B_k are the Betti numbers of Ω (cf. Duff-Spencer [6] and Kress [13]).

Theorem 11, respectively Theorem 13, remains valid for such general sets if we add the following necessary condition

$$\int_\Omega \langle f; \psi \rangle \, dx = 0, \, \forall \psi \in D_{k+1}(\Omega)$$

respectively

$$\int_\Omega \langle f; \psi \rangle \, dx = 0, \, \forall \psi \in N_{k-1}(\Omega).$$

Observe finally that when $k = n-1$ in Theorem 11, or $k = 1$ in Theorem 13, we therefore have no new condition. This explains why in Theorem 2 we do not assume that Ω is convex.

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