manuscripta math. 64, 253 - 260 (1989)

manuscripta mathematica

MONOTONICITY OF CERTAIN DIFFERENTIAL OPERATORS IN DIVERGENCE FORM

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I. Introduction

In this article we give necessary conditions for the monotonicity of the operator

$$A(u) = -\frac{d}{dx}\left(a(x,u(x),\frac{du}{dx})\right) \qquad (1)$$

in some special cases.

More precisely we shall prove the following theorems.

Theorem 1. Let $b \in C^1(\mathbb{R})$ and $b(s) \geq b_o > 0$ for every $s \in \mathbb{R}$. Let

$$A(u) = -(b(u)u')' .$$

Then A is monotone over $H_o^1(0,1)$ if and only if b is constant. (Here and throughout this article $H^1(0,1)$, $H_o^1(0,1)$, $W^{1,\infty}(0,1)$, $W_o^{1,\infty}(0,1)$ will denote the standard Sobolev spaces.)

Theorem 2. Let $b \in C^1(\mathbb{R})$, $c \in C^2(\mathbb{R})$ and

$$A(u) = -(b(u')+c(u))' .$$

Then the following hold

i) if A is monotone over $W_0^{1,\infty}(0,1)$, then $b'(\xi) \geq 0$ for every $\xi \in \mathbb{R}$ and $|c'(s)|$ is bounded for every $s \in \mathbb{R}$;

ii) there exist a monotone function b and a nonlinear function c such that A is monotone.

It is interesting to compare the above results with that of Boccardo-Dacorogna [B D] where it was shown that a necessary condition for the pseudo-monotonicity of A of the form (1) is that $a(x,s,\cdot)$ be monotone for every $(x,s) \in (0,1) \times \mathbb{R}$. This condition turns out to be not only necessary but also sufficient by the well known results of Brézis [B] and Lions [L]. Therefore the operator $A(u) = -(b(u)u')'$ is always pseudo monotone while it is monotone only if b is constant.

Note however that monotonicity of operators A of the form (1) is not necessary to get uniqueness of solutions for equations of the type $A(u) = g$, c.f. Artola [A], Chipot-Michaille [CM], Carillo-Chipot [CC], Carillo [C].

In proving the above theorems we use similar devices as those of Dacorogna [D] where the convexity of the following integral

$$I(u) = \int_0^1 f(x,u(x),u'(x))dx \tag{2}$$

is studied.

In particular one deduces immediately from [D]

Theorem 3. Let $b,c \in C^1(\mathbb{R})$ and

$$A(u) = -(b(u'))' + c(u)$$

then A is monotone over $W_0^{1,\infty}(0,1)$ if and only if

i) $b'(\xi) \geq 0$ for every $\xi \in \mathbb{R}$,

ii) $\pi^2 b'(\xi) + c'(s) \geq 0$ for every $(s,\xi) \in \mathbb{R}^2$.

The results obtained in Theorem 1 and 2 cannot, however, be deduced from those of [D] since there is no I as in (2) such that $A = I'$.

II. Proofs of the theorems

Proof of Theorem 1: ($\Leftarrow$) The sufficiency is trivial.

($\Rightarrow$) The monotonicity of A is equivalent to

$$\int_0^1 (b(u)u'-b(v)v')(u'-v')dx \geq 0$$

for every $u,v \in H_0^1(0,1)$. Writing $u = v + t\phi$ we have

$$f(t) \equiv \int_0^1 (b(v+t\phi)(v'+t\phi') - b(v)v')\phi' dx \geq 0$$

for every $v,\phi \in H_0^1(0,1)$, $t \geq 0$. Since $f(0) = 0$ and $f(t) \geq 0$ for $t \geq 0$ we deduce that

$$f'(0) = \int_0^1 [b(v)\phi'^2 + b'(v)v'\phi\phi']dx \geq 0 , \tag{1}$$

for every $v,\phi \in H_0^1(0,1)$. We now show that (1) implies that b is constant. We first show that $b' \geq 0$ and then show that $b' \leq 0$.

Let N be an even integer and let I_N and J_N be defined as follows

$$I_N = \bigcup_{\nu=2}^{N-3} \bigcup_{k=0}^{\frac{N^2}{2}-1} \left(\frac{\nu}{N}+\frac{2k}{N^3}, \frac{\nu}{N}+\frac{2k+1}{N^3}\right)$$

$$J_N = \bigcup_{\nu=2}^{N-3} \bigcup_{k=0}^{\frac{N^2}{2}-1} \left(\frac{\nu}{N}+\frac{2k+1}{N^3}, \frac{\nu}{N}+\frac{2(k+1)}{N^3}\right)$$

Note that $I_N \cap J_N = \phi$, $\bar{I}_N \cup \bar{J}_N = [\frac{2}{N}, 1-\frac{2}{N}]$ and meas I_N = meas $J_N = \frac{1}{2}-\frac{2}{N}$. We then define ϕ_N as

$$\phi_N(x) = \begin{cases} Nx & \text{if } x \in (0,\frac{1}{N}) \\ 1 & \text{if } x \in (\frac{1}{N}, \frac{2}{N}) \\ x+1-\frac{\nu}{N}-\frac{2k}{N^3} & \text{if } x \in I_N \\ -x+1+\frac{\nu}{N}+\frac{2(k+1)}{N^3} & \text{if } x \in J_N \\ 1 & \text{if } x \in (1-\frac{2}{N},1-\frac{1}{N}) \\ -Nx+N & \text{if } x \in (1-\frac{1}{N}, 1) \ . \end{cases}$$

Then $\phi_N \in W_o^{1,\infty}(0,1) \subset H_o^1(0,1)$ and

$$\begin{cases} |\phi_N(x)-1| \le \frac{1}{N^3} & \text{if } x \in (\frac{1}{N}, 1-\frac{1}{N}) \\ \phi_N'(x) = \begin{cases} 1 & \text{if } x \in I_N \\ -1 & \text{if } x \in J_N \ . \end{cases} \end{cases}$$

We now define v_N in an analogous way. Let $\alpha \in \mathbb{R}$ be fixed and

$$v_N(x) = \begin{cases} 0 & \text{if } x \in (0, \frac{1}{N}) \\ N\alpha x-\alpha & \text{if } x \in (\frac{1}{N},\frac{2}{N}) \\ N^2\left(x-\frac{\nu}{N}-\frac{2k}{N^3}\right)+\alpha & \text{if } x \in I_N \\ -N^2\left(x-\frac{\nu}{N}-\frac{2(k+1)}{N^3}\right)+\alpha & \text{if } x \in J_N \\ -N\alpha(x-1+\frac{1}{N}) & \text{if } x \in (1-\frac{2}{N},1-\frac{1}{N}) \\ 0 & \text{if } x \in (1-\frac{1}{N}, 1) \ . \end{cases}$$

Summarizing the properties of v_N we have $v_N \in W_o^{1,\infty}(0,1) \subset H_o^1(0,1)$ and

$$\begin{cases} |v_N(x)-\alpha| \le \frac{1}{N} & \text{if } x \in (\frac{2}{N},1-\frac{2}{N}) \\ v_N'(x) = \begin{cases} N^2 & \text{if } x \in I_N \\ -N^2 & \text{if } x \in J_N \ . \end{cases} \end{cases}$$

We now evaluate (1) with v_N and ϕ_N as above. We have

$$f'(0) = \int_0^1 [b(v_N)\phi_N'^2 + b'(v_N)v_N'\phi_N\phi_N']dx$$
$$= \int_0^{1/N} b(0)N^2 dx + \int_{2/N}^{1-2/N} [b(v_N) + b'(v_N)N^2\phi_N]dx$$
$$+ \int_{1-1/N}^1 b(0)N^2 dx \geq 0 \ .$$

Choosing $\varepsilon > 0$, we can find N sufficiently large so that

$$2b(0)N + (1-\frac{4}{N})(b(\alpha) + b'(\alpha)N^2) \geq -\varepsilon \ .$$

The above identity implies immediately that $b'(\alpha) \geq 0$.

To obtain that $b'(\alpha) \leq 0$, one proceeds exactly in the same way reversing the signs of ϕ_N' on I_N and J_N and keeping v_N as above. ∎

Remark: A similar argument as in Theorem 1 can be carried over to higher dimensions.

Proof of Theorem 2. i) The fact that $b'(\xi) \geq 0$ is well known and follows for example from [BD]. As in Theorem 1 we have

$$f(t) \equiv \int_0^1 [b(v'+ t\phi') - b(v') + c(v+ t\phi) - c(v)]\phi' dx \geq 0$$

for every $t \geq 0$ and every $v,\phi \in W_0^{1,\infty}(0,1)$. As before the monotonicity of A implies then that

$$f'(0) = \int_0^1 [b'(v')\phi'^2 + c'(v)\phi\phi']dx \geq 0 \qquad (1)$$

for every $v,\phi \in W_0^{1,\infty}(0,1)$. One can then perform an integration by parts in (1) and get

$$f'(0) = \int_0^1 [b'(v')\phi'^2 - \frac{1}{2}c''(v)v'\phi^2]dx \geq 0 \ . \qquad (2)$$

Now let N be an integer and define

$$\phi(x) = \begin{cases} 4x & \text{if } x \in (0,\frac{1}{4}) \\ 1 & \text{if } x \in (\frac{1}{4},\frac{1}{2}) \\ -4(x-\frac{3}{4}) & \text{if } x \in (\frac{1}{2},\frac{3}{4}) \\ 0 & \text{if } x \in (\frac{3}{4},1) \end{cases}$$

$$v_N(x) = \begin{cases} 0 & \text{if } x \in (0,\frac{1}{4}) \\ 4N(x-\frac{1}{4}) & \text{if } x \in (\frac{1}{4},\frac{1}{2}) \\ N & \text{if } x \in (\frac{1}{2},\frac{3}{4}) \\ -4N(x-1) & \text{if } x \in (\frac{3}{4},1) \end{cases} .$$

Using (2) we have

$$f'(0) = \int_0^{1/4} 16\, b'(0)\,dx - \frac{1}{2}\int_{1/4}^{1/2} c''(v)v'\,dx + \int_{1/2}^{3/4} 16\, b'(0)\,dx$$

$$= 8\, b'(0) - \frac{1}{2}\,[c'(v(\tfrac{1}{2})) - c'(v(\tfrac{1}{4}))]$$

$$= 8\, b'(0) - \frac{1}{2}\,c'(N) + \frac{1}{2}\,c'(0) \geq 0 .$$

Thus $\lim_{x\to+\infty} c'(x) < +\infty$. A similar construction with $\phi = 0$ in $(0,\frac{1}{4})$, $4(x-\frac{1}{4})$ in $(\frac{1}{4},\frac{1}{2})$, 1 in $(\frac{1}{2},\frac{3}{4})$, $-4(x-1)$ in $(\frac{3}{4},1)$ and $v_N = 4Nx$ in $(0,\frac{1}{4})$, N in $(\frac{1}{4},\frac{1}{2})$, $-4N(x-\frac{3}{4})$ in $(\frac{1}{2},\frac{3}{4})$, 0 in $(\frac{3}{4},1)$ give immediately that $\lim_{x\to+\infty} c'(x) > -\infty$. A completely analogous construction leads to $\lim_{x\to-\infty} c'(x)$ bounded. This establishes the first part of the theorem.

ii) In general one cannot conclude that $c'(s) \equiv$ constant for every $s \in \mathbb{R}$. (Note that if this is the case then A is always monotone.) To construct b and c satisfying the conclusion of the second part of the theorem, we choose any monotone b such that $b'(\xi) \geq b_0 > 0$, for every $\xi \in \mathbb{R}$. We then let

$$c(s) = b_0 \sin s \;,\; s \in \mathbb{R} .$$

We then claim that the corresponding operator A is monotone. From (1) it is clear that the monotonicity of A is equivalent to

$$f'(0) = \int_0^1 [b'(v')\phi'^2 + b_0 \cos v\ \phi\phi']dx \geq 0$$

for every $v,\phi \in W_0^{1,\infty}(0,1)$. Observe that

$$f'(0) \geq \int_0^1 [b'(v')\phi'^2 - b_0|\phi\phi'|]dx \geq b_0\{\int_0^1 [\phi'^2 - |\phi\phi'|]dx\} .$$

It then follows from Poincaré-Wirtinger inequality that

$$f'(0) \geq \frac{b_0}{2}\{\int_0^1 [\phi'^2 - 2|\phi\phi'| + \phi^2]dx\} \geq 0 ;$$

establishing therefore the monotonicity of A . ∎

Acknowledgements: The second author wishes to thank the I.A.C. in Rome where this research started and the E.T.H. in Zürich where it was completed.

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(Received January 31, 1989)