Multiple Jacobian equations

B. DACOROGNA
Section de Mathématiques, EPFL
CH-1015 Lausanne, Switzerland
bernard.dacorogna@epfl.ch

O. KNEUSS
Department of Mathematics, University of California
Berkeley, CA, 94720, USA
olivier.kneuss@gmail.com

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Abstract

The existence, regularity and uniqueness of a local diffeomorphism φ satisfying

$$g_i(\varphi)\det\nabla\varphi = f_i \quad \text{for every } 1 \leq i \leq n$$

is discussed.

This paper is dedicated to the memory of M.I. Vishik and will appear in CPAA (Communications on Pure and Applied Analysis).

1 Introduction

Given functions g_i and f_i we wish to discuss the existence of a local diffeomorphism φ solving

$$g_i(\varphi)\det\nabla\varphi = f_i \quad \text{for every } 1 \leq i \leq n. \tag{1}$$

The case of one equation was first considered in the seminal paper of Moser [4]. It has received considerable attention (for a history see [1]). The optimal regularity in Hölder spaces, as well as the boundary value problem, was obtained in Dacorogna-Moser [3]. The problem (1) was raised to us by I. Ekeland. We prove here (cf. Theorem 1) the existence of a solution with optimal regularity, though not the expected one. We also obtain uniqueness of solutions, in sharp contrast with the classical case of one equation, once the solution is prescribed on a non-characteristic $(n-1)$ surface.

We give two ways of finding solutions. The first one (cf. Theorem 1) is constructive and uses the method of characteristics. The second one (cf. Theorem 3) uses the flow method of Moser. Finally in Theorem 5 we prove that these two methods, although seemingly very different, are in fact essentially the same.

2 Main result

Our main result is the following.

Theorem 1 *Let $n, r \geq 2$ be two integers, $x_0 \in \mathbb{R}^n$ and $g_i, f_i \in C^r(\mathbb{R}^n)$, $1 \leq i \leq n$, be such that $g_i(x_0), f_i(x_0) \neq 0$ for every $1 \leq i \leq n$,*

$$\operatorname{rank}\left[\begin{pmatrix} \nabla(g_2/g_1)(x_0) \\ \vdots \\ \nabla(g_n/g_1)(x_0) \end{pmatrix}\right] = \operatorname{rank}\left[\begin{pmatrix} \nabla(f_2/f_1)(x_0) \\ \vdots \\ \nabla(f_n/f_1)(x_0) \end{pmatrix}\right] = n-1 \qquad (2)$$

and

$$\frac{g_i}{g_1}(x_0) = \frac{f_i}{f_1}(x_0) \quad \text{for every } 2 \leq i \leq n. \qquad (3)$$

Part 1 (Existence and regularity). *There exist a neighbourhood U of x_0 and $\varphi \in \operatorname{Diff}^{r-1}(U; \varphi(U))$ such that $\varphi(x_0) = x_0$ and*

$$g_i(\varphi) \det \nabla\varphi = f_i \quad \text{in } U \text{ for every } 1 \leq i \leq n. \qquad (4)$$

The regularity is, in general, optimal.

Part 2 (Uniqueness). *Let $h \in C^{r-1}(\mathbb{R}^n)$ be such that $h(x_0) = 0$ and*

$$\det\left[\begin{pmatrix} \nabla h(x_0) \\ \nabla(f_2/f_1)(x_0) \\ \vdots \\ \nabla(f_n/f_1)(x_0) \end{pmatrix}\right] \neq 0. \qquad (5)$$

Let U be a neighbourhood of x_0, $\varphi \in \operatorname{Diff}^{r-1}(U; \varphi(U))$ and $\psi \in \operatorname{Diff}^{r-1}(U; \psi(U))$ be two solutions of (4) verifying

$$\varphi = \psi \quad \text{on} \quad \{x \in U : h(x) = 0\}.$$

Then, up to further restricting U,

$$\varphi \equiv \psi \quad \text{on } U.$$

Remark 2 (i) With exactly the same proof the same result holds in the Hölder spaces $C^{r,\alpha}$ with $0 < \alpha \leq 1$.

(ii) The fact that the regularity that we obtain is optimal, is, at first glance, surprising. It is moreover also optimal in Hölder spaces. More precisely it will be shown that, for any $r \geq 2$ and any $0 \leq \alpha \leq 1$, there exist g_i and f_i in $C^{r,\alpha}$ such that any solution φ of (4) is no more regular than $C^{r-1,\alpha}$.

(iii) The hypothesis (3) is obviously necessary to have $\varphi(x_0) = x_0$. The hypothesis (2) (although not necessary in general) is very reasonable: for example if $n = 2$ and $g_1 = g_2$ near x_0 (and thus $\nabla(g_2/g_1) = 0$) then obviously f_2 has to be equal to f_1 near x_0 to be able to solve (4) and vice versa.

(iv) The solution in Part 1 of the previous theorem is easily seen not to be unique. However (cf. Part 2) it becomes unique as soon as the value of the solution is prescribed not only at the point x_0 but on a $(n-1)$ surface (near x_0) compatible with the data.

(v) If we work in the Hölder spaces $C^{r,\alpha}$ with $0<\alpha<1$, it can easily be shown that we can assume with no loss of generality that $g_1 = f_1 = 1$. Indeed first find (cf. [1]) χ, ψ two local $C^{r+1,\alpha}$ diffeormorphisms near x_0 such that

$$g_1(\chi)\det\nabla\chi = 1, \quad f_1(\psi)\det\nabla\psi = 1 \quad \text{and} \quad \chi(x_0) = \psi(x_0) = x_0\,.$$

Define $\widetilde{g}_i = g_i(\chi)\det\nabla\chi$ and $\widetilde{f}_i = f_i(\psi)\det\nabla\psi$. Note that $\widetilde{g}_i, \widetilde{f}_i \in C^{r,\alpha}$ and $\widetilde{g}_1 = \widetilde{f}_1 = 1$. Then φ satisfies (4) if and only if $\widetilde{\varphi} = \chi^{-1}\circ\varphi\circ\psi$ verifies

$$\widetilde{g}_i(\widetilde{\varphi})\det\nabla\widetilde{\varphi} = \widetilde{f}_i \quad \text{for every } 1\le i\le n \quad \text{and} \quad \widetilde{\varphi}(x_0) = x_0\,.$$

Note also that (2) and (3) are individually satisfied if and only if their counterparts with g_i and f_i replaced by $\widetilde{g}_i$ and $\widetilde{f}_i$ are individually fulfilled.

(vi) There is nothing special in privileging the index 1 in the hypotheses of the previous theorem. Indeed for any permutation σ of the set $\{1,\cdots,n\}$ then (2) is equivalent to the same equation in (2) where g_i is replaced by $g_{\sigma(i)}$ and f_i is replaced by $f_{\sigma(i)}$. The same remark also holds for (3).

(vii) Using the notations of differential forms then (2) is equivalent to

$$d(g_2/g_1)(x_0)\wedge\cdots\wedge d(g_n/g_1)(x_0) \neq 0$$

and

$$d(f_2/f_1)(x_0)\wedge\cdots\wedge d(f_n/f_1)(x_0) \neq 0.$$

(viii) If we have to solve $1<s<n$ equations

$$g_i(\varphi)\det\nabla\varphi = f_i \quad \text{for every } 1\le i\le s,$$

we proceed similarly by adding $(n-s)$ equations so that (2) and (3) are satisfied. Of course the uniqueness established in Part 2 is then lost.

(ix) It is to be noted that the equivalent problem for 1 and 2 forms has already been considered in Chapter 15 of [1].

Proof With no loss of generality we can assume throughout the proof that $x_0 = 0$. We divide the proof into three steps.

Step 1 (Existence). Obviously (4) is equivalent to $\varphi(0) = 0$ and

$$\begin{cases} g_1(\varphi)\det\nabla\varphi = f_1 \\ \frac{g_i}{g_1}(\varphi) = \frac{f_i}{f_1} \qquad 2\le i\le n. \end{cases} \tag{6}$$

We claim that $\varphi = G^{-1}\circ F$ has all the desired properties where

$$G = \left(\langle a;x\rangle, \frac{g_2}{g_1},\cdots,\frac{g_n}{g_1}\right) \quad \text{and} \quad F = \left(u, \frac{f_2}{f_1},\cdots,\frac{f_n}{f_1}\right)$$

where $a \in \mathbb{R}^n$ and $u : \mathbb{R}^n \to \mathbb{R}$ are determined as follows. Using (2) we can select $a \in \mathbb{R}^n$ such that $\det \nabla G(0) \neq 0$ which immediately implies that $G \in \mathrm{Diff}^r(B_\epsilon; G(B_\epsilon))$ for $\epsilon > 0$ small enough (B_ϵ being the ball centered at 0 and of radius ϵ). Then, for any $u \in C^{r-1}$ with $u(0) = 0$, we have that $\varphi = G^{-1} \circ F$ satisfies, using (3), $\varphi(0) = 0$ and

$$\frac{g_i}{g_1}(\varphi) = \frac{f_i}{f_1} \quad \text{near } 0\text{, for every } 2 \leq i \leq n.$$

In view of the previous considerations, it only remains to find $u \in C^{r-1}$ such that $u(0) = 0$ and

$$g_1(\varphi) \det \nabla\varphi = f_1 \quad \text{near } 0 \tag{7}$$

(note that the above equation implies, in particular, that φ is a local diffeomorphism) or equivalently

$$g_1\left(G^{-1} \circ F\right) \cdot \left(\det \nabla \left(G^{-1}\right)\right)(F) \cdot \det \nabla F = f_1\,.$$

Let us investigate the terms in the left hand side of the last equation. Note that

$$g_1(G^{-1} \circ F)(x) \quad \text{and} \quad \left(\det \nabla \left(G^{-1}\right)\right)(F(x))$$

have, respectively, the form

$$\alpha(x, u(x)) \quad \text{and} \quad \beta(x, u(x))$$

where $\alpha \in C^r$ with $\alpha(0,0) \neq 0$ and $\beta \in C^{r-1}$ with $\beta(0,0) \neq 0$. Finally, using (2), we obtain that

$$\det \nabla F = \langle \nabla u; H \rangle$$

for some $H \in C^{r-1}(\mathbb{R}^n; \mathbb{R}^n)$ with $H(0) \neq 0$. We hence deduce that (7) can be written, near 0, as

$$\langle \nabla u(x); H(x) \rangle = \gamma(x, u(x)) = \frac{f_1(x)}{\alpha(x, u(x))\,\beta(x, u(x))}$$

where $\gamma \in C^{r-1}$. Using the method of characteristics, we can find, near 0, $u \in C^{r-1}$ verifying the last equation as well as $u(0) = 0$. This concludes the proof of the existence part.

Step 2 (Regularity). We now establish that the regularity $\varphi \in C^{r-1}$ (or $C^{r-1,\alpha}$) cannot be, in general, improved. Let

$$\begin{cases} g_1 = f_1 = 1 \\ g_i(x) = f_i(x) = 1 + x_i \quad \text{if } 2 \leq i \leq n-1 \\ g_n(x) = 1 + x_n \quad \text{and} \quad f_n(x) = f_n(x_n) \end{cases}$$

with $f_n \in C^{r,\alpha}$ but no more regular than $C^{r,\alpha}$ and such that $f_n(0) = 1$ and $f_n'(0) \neq 0$. Obviously g_i and f_i satisfy all the hypotheses in the statement of the theorem. We now claim that any solution of (4) is no more regular than

$C^{r-1,\alpha}$. Let φ be such a solution then (cf. the equivalent formulation (6)) we necessarily have

$$\det \nabla\varphi = 1, \quad \varphi^i(x) = x_i \text{ for } 2 \le i \le n-1 \quad \text{and} \quad \varphi^n(x) = f_n(x_n) - 1.$$

This immediately implies that

$$1 = \det \nabla\varphi(x) = \varphi^1_{x_1}(x) f_n'(x_n)$$

and hence

$$\varphi^1(x) = \frac{x_1}{f_n'(x_n)} + w(x_2, \cdots, x_n)$$

for some w. Since, by construction, f_n' is no more regular than $C^{r-1,\alpha}$, we immediately deduce that φ is no more regular than $C^{r-1,\alpha}$ for any choice of w. This shows the optimality of our regularity result.

Step 3 (Uniqueness). Let us assume that we have two solutions φ and ψ of (4). We already observed that this is equivalent to

$$\begin{cases} g_1(\varphi)\det\nabla\varphi = g_1(\psi)\det\nabla\psi = f_1 \\ \frac{g_i}{g_1}(\varphi) = \frac{g_i}{g_1}(\psi) = \frac{f_i}{f_1} \quad \text{for } 2 \le i \le n. \end{cases}$$

Setting, as before with an appropriate $a \in \mathbb{R}^n$,

$$G = \left(\langle a; x\rangle, \frac{g_2}{g_1}, \cdots, \frac{g_n}{g_1}\right)$$

and then

$$\widetilde{\varphi} = G \circ \varphi \quad \text{and} \quad \widetilde{\psi} = G \circ \psi$$

we get

$$\widetilde{\varphi}^i = \widetilde{\psi}^i = \frac{f_i}{f_1} \quad \text{for } 2 \le i \le n.$$

In order to prove that $\varphi = \psi$ it is enough to establish that $\widetilde{\varphi} = \widetilde{\psi}$ and it therefore remains to prove that $\widetilde{\varphi}^1 = \widetilde{\psi}^1$. As in Step 1, $\widetilde{\varphi}^1$ and $\widetilde{\psi}^1$satisfy a first order equation of the type

$$\left\langle \nabla\widetilde{\varphi}^1(x); H(x)\right\rangle = \gamma\left(x, \widetilde{\varphi}^1(x)\right)$$
$$\left\langle \nabla\widetilde{\psi}^1(x); H(x)\right\rangle = \gamma\left(x, \widetilde{\psi}^1(x)\right).$$

They moreover verify

$$\widetilde{\varphi}^1 = \widetilde{\psi}^1 \quad \text{on} \quad \{x \in U : h(x) = 0\}$$

with h as in (5) which reads with our new notations

$$\langle \nabla h(0); H(0)\rangle \neq 0.$$

The classical method of characteristics gives then uniqueness. ■

3 The flow method

We now present another proof of the main result under slightly more restrictive hypotheses.

Theorem 3 *Let $n \geq 2$, $r \geq 3$ be two integers and $x_0 \in \mathbb{R}^n$. Let, for every $1 \leq i \leq n$ and every $t \in [0,1]$,*

$$h_i \in C^r\left([0,1] \times \mathbb{R}^n\right), \quad h_i\left(t, x_0\right) \neq 0, \quad \frac{\partial}{\partial t}\left(h_i/h_1\right)\left(t, x_0\right) = 0 \tag{8}$$

and

$$\operatorname{rank}\left[\begin{pmatrix} \nabla_x\left(h_2/h_1\right)\left(t, x_0\right) \\ \vdots \\ \nabla_x\left(h_n/h_1\right)\left(t, x_0\right) \end{pmatrix}\right] = n-1. \tag{9}$$

Then there exist a neighbourhood U of x_0 and $\varphi\left(t, \cdot\right) \in \operatorname{Diff}^{r-2}(U; \varphi(U))$ for every $t \in [0,1]$ such that

$$h_i\left(t, \varphi\left(t, x\right)\right) \det \nabla_x \varphi\left(t, x\right) = h_i\left(0, x\right) \tag{10}$$

for every $1 \leq i \leq n$ and every $(t, x) \in [0,1] \times U$.

Remark 4 If the homotopy h_i is such that

$$h_i\left(0, x\right) = f_i\left(x\right) \quad \text{and} \quad h_i\left(1, x\right) = g_i\left(x\right)$$

we recover Theorem 1 with $\varphi\left(x\right) = \varphi\left(1, x\right)$. However the regularity is weaker. But we will see in Theorem 5 that a careful choice of the homotopy gives the optimal regularity.

Proof With no loss of generality we can assume that $x_0 = 0$.

Step 1. We claim (cf. Step 2) that there exist

$$w_i \in C^{r-2}\left([0,1] \times \mathbb{R}^n; \mathbb{R}^n\right)$$

such that, near $x = 0$,

$$\operatorname{div}_x\left(w_i\right) = -\frac{\partial}{\partial t} h_i\,, \quad \frac{w_1}{h_1} = \frac{w_i}{h_i} \quad \text{and} \quad w_i\left(t, 0\right) = 0$$

for every $t \in [0,1]$ and every $1 \leq i \leq n$. Assuming the claim and defining, near $x = 0$, $u \in C^{r-2}([0,1] \times \mathbb{R}^n; \mathbb{R}^n)$ by

$$u\left(t, x\right) = \frac{w_1\left(t, x\right)}{h_1\left(t, x\right)} = \frac{w_i\left(t, x\right)}{h_i\left(t, x\right)}$$

we immediately get, using the usual flow method (cf., for example, [1]) that, the flow $\varphi = \varphi\left(t, x\right)$, associated to the vector field u, is a local (near $x = 0$) C^{r-2} diffeomorphism and verifies $\varphi\left(t, 0\right) = 0$ as well as

$$h_i(t, \varphi\left(t, x\right)) \det \nabla_x \varphi\left(t, x\right) = h_i\left(0, x\right) \quad \text{for } t \in [0,1] \text{ and every } 1 \leq i \leq n.$$

Step 2. We now show that we can find w_i as in Step 1. In the rest of the proof we will use the notations of differential forms (cf., for example, [1]).

Step 2.1. We will construct the w_i (identifying, as usual, 1−forms with vector fields) as follows

$$w_1 = d_x\alpha + \delta_x\beta \quad \text{and} \quad w_i = \frac{h_i}{h_1} w_1 \quad 2 \leq i \leq n,$$

where $\alpha : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ satisfies $d_x\alpha \in C^{r-1}([0,1] \times \mathbb{R}^n)$,

$$\operatorname{div}_x (w_1) = \Delta_x \alpha = -\frac{\partial}{\partial t} h_1 \quad \text{and} \quad d_x\alpha(t,0) = 0 \quad \text{for } t \in [0,1]$$

and where $\beta : [0,1] \times \mathbb{R}^n \to \Lambda^2(\mathbb{R}^n)$ is a 2−form that has to be determined, with, in particular, $\delta_x \beta(t,0) = 0$. With such w_i the only two things left to check are, for every $2 \leq i \leq n$ and every $t \in [0,1]$,

$$\operatorname{div}_x(w_i) = -\frac{\partial}{\partial t} h_i \quad \text{and} \quad w_i(t,0) = 0$$

since all the other desired properties are trivially fulfilled. Hence, having in mind the form of the w_i, we have to find a 2−form β such that, for $t \in [0,1]$, $\delta_x\beta(t,0) = 0$ and, near 0 and for every $2 \leq i \leq n$,

$$\left\langle d_x\left(\frac{h_i}{h_1}\right); d_x\alpha + \delta_x\beta \right\rangle + \frac{h_i}{h_1}\Delta_x\alpha = -\frac{\partial}{\partial t} h_i$$

or equivalently

$$\left\langle d_x\left(\frac{h_i}{h_1}\right); \delta_x\beta \right\rangle = -\frac{\partial}{\partial t} h_i + \frac{h_i}{h_1}\frac{\partial}{\partial t} h_1 - \left\langle d_x\left(\frac{h_i}{h_1}\right); d_x\alpha \right\rangle =: r_i\,.$$

Note that $r_i \in C^{r-1}([0,1] \times \mathbb{R}^n)$. Moreover, since $d_x\alpha(t,0) = 0$ and (8) holds, we deduce that $r_i(t,0) = 0$ for every $2 \leq i \leq n$ and every $t \in [0,1]$. We therefore need to find β such that

$$\begin{cases} \left(d_x\left(\frac{h_i}{h_1}\right)\right) \wedge (d_x(*\beta)) = -(*r_i) & \text{for } 2 \leq i \leq n \\ \delta_x\beta(t,0) = 0 \quad \Leftrightarrow \quad d_x(*\beta)(t,0) = 0 & \text{for } t \in [0,1]\,. \end{cases} \tag{11}$$

Observe that (11) is invariant under the pullback of a local C^r diffeomorphism $\psi(t,\cdot)$ with $\psi(t,0) = 0$. Indeed calling

$$\gamma(t,\cdot) = *\left[(\psi(t,\cdot))^*(*\beta)\right]$$

we have

$$\begin{cases} \left(d_x\left(\psi(t,\cdot)^*\left(\frac{h_i}{h_1}\right)\right)\right) \wedge (d_x(*\gamma(t,\cdot))) = -\psi(t,\cdot)^*(*r_i) & \text{for } 2 \leq i \leq n \\ d_x(*\gamma)(t,0) = 0 \quad \Leftrightarrow \quad \delta_x\gamma(t,0) = 0 & \text{for } t \in [0,1]\,. \end{cases}$$

In Step 2.2 we will show that we can find $\psi(t,\cdot)$ a local diffeomorphism near $x=0$ such that $\psi(t,0)=0$ and

$$d_x\left(\psi(t,\cdot)^*\left(\frac{h_i}{h_1}\right)\right)=dx^i \quad \text{for every } 2\le i\le n.$$

So it is enough to solve (11) assuming that $h_i/h_1=x_i$. For this purpose we define $v\in C^{r-1}([0,1]\times\mathbb{R}^n;\Lambda^{n-2}(\mathbb{R}^n))$ by

$$v_{i_1\cdots i_{n-2}}(t,x)=0 \quad \text{if } i_1=1$$

and if $(i_1,\cdots,i_{n-2})=(2,\cdots,j-1,j+1,\cdots,n)$

$$v_{i_1\cdots i_{n-2}}(t,x)=(-1)^j\int_0^{x_1} r_j(t,s,x_2,\cdots,x_n)\,ds.$$

It is then easily seen that $d_xv(t,0)=0$ and

$$dx^i\wedge d_xv=-(*r_i), \quad \text{for } 2\le i\le n.$$

Finally $\beta=*v$ has all the claimed properties. This concludes the proof once Step 2.2 is established.

Step 2.2. Let us finally show that we can find $\psi(t,\cdot)$ a local diffeomorphism near $x=0$ such that $\psi(t,0)=0$ and

$$d_x\left(\psi(t,\cdot)^*\left(\frac{h_i}{h_1}\right)\right)=dx^i \quad \text{for every } 2\le i\le n.$$

Using (9) it is possible to find $A\in C^\infty([0,1];\mathbb{R}^n)$ such that

$$\det\left[\begin{pmatrix} A(t)\\ \nabla_x(h_2/h_1)(t,x_0)\\ \vdots\\ \nabla_x(h_n/h_1)(t,x_0)\end{pmatrix}\right]\neq 0 \quad \text{for every } t\in[0,1].$$

Let $\chi=\psi^{-1}\in C^r([0,1]\times\mathbb{R}^n;\mathbb{R}^n)$ be defined by

$$\chi(t,x)=\left(\langle A(t);x\rangle,\frac{h_2(t,x)}{h_1(t,x)}-\frac{h_2(0,0)}{h_1(0,0)},\cdots,\frac{h_n(t,x)}{h_1(t,x)}-\frac{h_n(0,0)}{h_1(0,0)}\right).$$

With our choice of A, the map $\chi(t,\cdot)$ is a local diffeomorphism near $x=0$, satisfies $\chi(t,0)=0$ (cf. (8)) and

$$\chi(t,\cdot)^*(dx^i)=d_x\left(\frac{h_i}{h_1}\right)(t,\cdot) \quad \text{for every } 2\le i\le n,$$

as wished. This proves the claim and concludes the proof. ■

We now show that in fact the solution found in Theorem 1 can also be obtained by the flow method.

Theorem 5 *Let $x_0 \in \mathbb{R}^n$, $f_i, g_i \in C^r(\mathbb{R}^n)$ satisfy all the hypotheses of Theorem 1 and in addition*

$$g_i(x_0)\ f_i(x_0) > 0 \quad \textit{for every } 1 \leq i \leq n.$$

Then there exist a homotopy h_i such that

$$h_i(0,x) = f_i(x) \quad \textit{and} \quad h_i(1,x) = g_i(x),$$

a neighbourhood U of x_0 and $\varphi(t,\cdot) \in \operatorname{Diff}^{r-1}(U;\varphi(U))$ for every $t \in [0,1]$ such that

$$\varphi(0,x) = x, \quad \varphi(t,x_0) = x_0$$

and

$$h_i(t,\varphi(t,x)) \det \nabla_x \varphi(t,x) = h_i(0,x) = f_i(x) \tag{12}$$

for every $1 \leq i \leq n$ and every $(t,x) \in [0,1] \times U$. In particular, $\varphi(1,\cdot)$ satisfy the conclusions of Theorem 1, namely $\varphi(1,\cdot) \in \operatorname{Diff}^{r-1}(U;\varphi(U))$, $\varphi(1,x_0) = x_0$ and

$$g_i(\varphi(1,x)) \det \nabla_x \varphi(1,x) = f_i(x) \quad \textit{for every } x \in U \textit{ and } 1 \leq i \leq n.$$

Remark 6 The proof will show that if φ is as in the conclusion of Theorem 1, then

$$\varphi(1,x) = \varphi(x).$$

Before starting with the proof we need to prove the following lemma.

Lemma 7 *Let $n, r \geq 1$ be two integers, $x_0 \in \mathbb{R}^n$ and $\varphi \in C^r(\mathbb{R}^n;\mathbb{R}^n)$ be such that*

$$\det \nabla\varphi(x_0) > 0 \quad \textit{and} \quad \varphi(x_0) = x_0.$$

Then there exist a neighbourhood U of x_0 and $H \in C^r([0,1] \times U;\mathbb{R}^n)$ such that

$$H(0,x) = x, \quad H(1,x) = \varphi(x) \quad \textit{for every } x \in U$$

$$H(t,x_0) = x_0 \quad \textit{for every } t \in [0,1]$$

$$H(t,\cdot) \in \operatorname{Diff}^r(U;H(t,U)) \quad \textit{for every } t \in [0,1]$$

$$\frac{\partial^m}{\partial t^m} H \in C^r([0,1] \times U;\mathbb{R}^n) \quad \textit{for every } m \in \mathbb{N}.$$

Proof With no loss of generality we can assume that $x_0 = 0$.

Step 1. Using a well-known result (cf. for example [2]) there exists $A \in C^\infty([0,1/2];\mathbb{R}^{n\times n})$ such that

$$A(0) = \mathrm{Id}, \quad A(1/2) = \nabla\varphi(0) \quad \text{and} \quad \det A(t) > 0 \quad \text{for } t \in [0,1/2].$$

Define $F : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$F(t,x) = \begin{cases} \nabla\varphi(0) \cdot x + (2t-1)[\varphi(x) - \nabla\varphi(0) \cdot x] & \text{if } t \in [1/2,1] \\ A(t) \cdot x & \text{if } t \in [0,1/2]. \end{cases}$$

It is elementary to see that F has all the required properties for U a neighbourhood of 0 small enough except that F is only C^0 in $t = 1/2$.

Step 2. Extend F to $\mathbb{R}^{n+1}$ by

$$F(t,x) = \begin{cases} F(0,x) & \text{if } t \leq 0 \\ F(1,x) & \text{if } t \geq 1. \end{cases}$$

Let $\rho_\epsilon(\cdot) = 1/\epsilon\rho(\cdot/\epsilon)$ where $\rho \in C^\infty(\mathbb{R})$ is an usual mollifier and define

$$F_\epsilon(t,x) = \int_{-\infty}^{+\infty} \rho_\epsilon(t-s)F(s,x)ds.$$

Taking ϵ small enough and U smaller if necessary, we obtain that H defined by

$$H(t,x) = F_\epsilon(t,x) + t(F(1,x) - F_\epsilon(1,x)) + (1-t)(F(0,x) - F_\epsilon(0,x))$$

has all the claimed properties. This concludes the proof. ■

We may now turn to the proof of Theorem 5.

Proof In the sequel U will denote a generic neighbourhood of x_0 that can be smaller from line to line.

Step 1. Using Theorem 1 there exists φ a C^{r-1} local diffeomorphism near x_0 verifying $\varphi(x_0) = x_0$ and, in a neighbourhood of x_0,

$$g_i(\varphi)\det\nabla\varphi = f_i \quad \text{for every } 1 \leq i \leq n.$$

Note, in particular, that $\det\nabla\varphi > 0$. Using Lemma 7 we can find a neighbourhood U of x_0 and $\varphi \in C^{r-1}([0,1] \times U; \mathbb{R}^n)$ such that

$$\varphi(0,x) = x, \quad \varphi(1,x) = \varphi(x) \quad \text{for every } x \in U,$$

$$\varphi(t,x_0) = x_0 \quad \text{for every } t \in [0,1],$$

$$\varphi(t,\cdot) \in \text{Diff}^{r-1}(U; \varphi(t,U)) \quad \text{for every } t \in [0,1],$$

$$\frac{\partial^m}{\partial t^m}\varphi \in C^{r-1}([0,1] \times U; \mathbb{R}^n) \quad \text{for every } m \in \mathbb{N}.$$

Step 2. For $1 \leq i \leq n$, let $h_i \in C^{r-2}([0,1] \times U)$ be defined by

$$h_i(t,\cdot) = f_i(\varphi^{-1}(t,\cdot))\det\nabla_x\varphi^{-1}(t,\cdot)$$

which clearly implies (12). Note that $h_i(1,\cdot) = g_i(\cdot)$ and $h_i(0,\cdot) = f_i(\cdot)$. The theorem is therefore proved.

Step 3. We conclude the proof with some remarks.

(i) Note that $\varphi(1,\cdot) = \varphi$ (which is the φ of Theorem 1).

(ii) Define the vector field $u \in C^{r-2}([0,1] \times U; \mathbb{R}^n)$ as

$$u(t,\cdot) = \frac{\partial}{\partial t}\varphi(t, \varphi^{-1}(t,\cdot))$$

which obviously leads to

$$\frac{\partial}{\partial t}\varphi(t,\cdot)=u(t,\varphi(t,\cdot)). \tag{13}$$

Thus $\varphi(t,\cdot)$ is indeed the flow associated to the vector field u.

(iii) Observe also that (according to (12), (13) and cf., for example, Proposition 10.5 in [1]) we have, in a neighbourhood of x_0,

$$\operatorname{div}_x[u(t,\cdot)\,h_i(t,\cdot)]=-\frac{\partial}{\partial t}h_i(t,\cdot)\quad\text{for every } t\in[0,1] \text{ and every } 1\leq i\leq n.$$

■

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