Direct methods in the calculus of variations

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Preface

The present monograph is a revised and augmented edition of "Direct methods in the calculus of variations" [179] which is now out of print. The core and the structure of the present book is essentially the one of [179], although it has now almost doubled its size. While writing the present volume, it clearly appeared to me that a new subject has emerged and that it deserves to be called "*quasiconvex analysis*". This name, of course, refers to "convex analysis", although the new subject is still in its infancy when compared with the classical one.

The calculus of variations is an immense and very active field. It is therefore, when writing a book, necessary to make a severe selection. This was already the case for [179] and is even more so for this new edition. Rather than superficially covering a lot of materials, I preferred to privilege only some aspects of the field. Here are some main features of the book. I strongly emphasized the resemblances between convex and quasiconvex analysis as well as the "algebraic" aspect of the field, notably through the determinants and singular values. Besides the classical results on lower semicontinuity and relaxation, an important feature of the monograph is the emphasis on the existence of minimizers for non convex problems.

In doing so I missed several important aspects of the calculus of variations such as regularity theory, study of stationary points, existence and relaxation in BV spaces, minimal surfaces, Young measures and the mathematical study of microstructures, Γ convergence and homogenization. However there are already several excellent books on these subjects, some of them very classical, such as: Almgren [18], Ambrosio-Fusco-Pallara [25], Braides-Defranceschi [101], Buttazzo [112], Buttazzo-Giaquinta-Hildebrandt [117], Dal Maso [217], Dierkes-Hildebrandt-Küster-Wohlrab [248], Dolzmann [249], Ekeland [263], Ekeland-Temam [264], Evans [271], Fonseca-Leoni [284], Giaquinta [307], Giaquinta-Hildebrandt [309], Giaquinta-Modica-Soucek [312], Gilbarg-Trudinger [313], Giusti [315], [316], Ladyzhenskaya-Uraltseva [388], Mawhin-Willem [440], Morrey [455], Müller [462], Nitsche [476], Pedregal [492], Roubicek [517] or Struwe [546], [547]. I have also added in the bibliography several articles which present important developments that I did not discuss in the present monograph, but are still closely related.

For a reader not very familiar with the calculus of variations, it might be advisable to start with an introductory book such as [180], which could be considered as a companion to the present one. Nevertheless the present monograph, which is essentially a reference book on the subject of quasiconvex analysis, can be used, as was [179], for an advanced course on the calculus of variations.

I would next like to reiterate my thanks to the people that helped me while writing the earlier version [179], namely J.M. Ball, L. Boccardo, P. Ciarlet, I. Ekeland, J.C. Evard, B. Kawohl, P. Marcellini, J. Moser, C.A. Stuart, E. Zehnder and B. Zwahlen.

However since then, I have benefited of many other important discussions. Surely the most influential ones were with P. Marcellini with whom we have a long standing collaboration. We have written together several articles and a book [202], which helped me in writing Part III of the present monograph. I want also to recall fruitful discussions with E. Acerbi, J.J. Alibert, N. Ansini, G. Aubert, S. Bandyopadhyay, A.C. Barroso, H. Brézis, G. Buttazzo, P. Cardaliaguet, A. Cellina, G. Croce, G. Dal Maso, F. De Blasi, E. De Giorgi, O. Dosly, J. Douchet, A. Ferriero, I. Fonseca, N. Fusco, W. Gangbo, N. Georgy, F. Gianetti, J.P. Haeberly, H. Hartwig, S. Hildebrandt, T. Iwaniec, O. Kneuss, H. Koshigoe, P.L. Lions, J. Maly, P. Maréchal, A. Martinaglia, E. Mascolo, J. Matias, P. Metzener, G. Mingione, G. Modica, S. Müller, F. Murat, G. Pianigiani, G. Pisante, L. Poggiolini, A.M. Ribeiro, N. Rochat, C. Sbordone, K.D. Semmler, V. Sverak, M. Sychev, R. Tahraoui, C. Tanteri, L. Tartar, M. Troyanov and K. Zhang.

My thanks also go to Mme. G. Rime who typed the manuscript of [179] and to Mme. M.F. De Carmine who typed an earlier version of the present monograph. Finally M. Hägler and C. Hebeisen prepared for me all the pictures that are inserted in the book.

During the past several years I have benefited from grants from the "Fonds National Suisse" and the "Troisième Cycle Romand". Of course, particular thanks go to the "Section de Mathématiques" of the "Ecole Polytechnique Fédérale de Lausanne".

Chapter 1

Introduction

1.1 The direct methods of the calculus of variations

The main problem that we will be investigating throughout the present monograph is the following. Consider the functional

$$I(u) := \int_{\Omega} f(x, u(x), \nabla u(x))\, dx$$

where

- $\Omega \subset \mathbb{R}^n$, $n \geq 1$, is a bounded open set, a point in Ω is denoted by $x = (x_1, ..., x_n)$;

- $u : \Omega \to \mathbb{R}^N$, $N \geq 1$, $u = \left(u^1, \cdots, u^N\right)$, and hence

$$\nabla u = \left(\frac{\partial u^j}{\partial x_i}\right)_{1 \leq i \leq n}^{1 \leq j \leq N} \in \mathbb{R}^{N \times n};$$

- $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R}$, $f = f(x, u, \xi)$, is a given function.

We say that the problem under consideration is *scalar* if either $N = 1$ or $n = 1$, otherwise we speak of the *vectorial* case.

Associated to the functional I is the following minimization problem

$$(P) \quad m := \inf \{I(u) : u \in X\},$$

meaning that we wish to find $\overline{u} \in X$ such that

$$m = I(\overline{u}) \leq I(u), \text{ for every } u \in X.$$

Here X is the space of admissible functions (in most parts it is the Sobolev space $u_0 + W_0^{1,p}\left(\Omega; \mathbb{R}^N\right)$, where u_0 is a given function).

We now give several examples.

1) The classical calculus of variations dealt essentially with the case $n = N = 1$, where the most celebrated examples are *Fermat principle* in geometrical optics where

$$f(x, u, \xi) := g(x, u)\sqrt{1+\xi^2},$$

Newton problem where

$$f(x, u, \xi) = f(u, \xi) := 2\pi u \frac{\xi^3}{1+\xi^2}$$

or the *brachistochrone* problem where

$$f(x, u, \xi) = f(u, \xi) := \frac{\sqrt{1+\xi^2}}{\sqrt{2gu}}.$$

2) When turning our attention to the case $n > N = 1$ (in our terminology, it is still part of the scalar case), the *Dirichlet integral* surely plays a central role; we have there

$$f(x, u, \xi) = f(\xi) := \frac{1}{2}|\xi|^2.$$

A natural generalization is when $1 < p < \infty$ and

$$f(x, u, \xi) = f(\xi) := \frac{1}{p}|\xi|^p.$$

The *minimal surface in non parametric form* enters also in this framework; we have in this case

$$f(x, u, \xi) = f(\xi) := \sqrt{1+|\xi|^2}.$$

In geometrical terms the integral represents the area of the surface given by $(x, u(x)) \in \mathbb{R}^{n+1}$ when $x \in \Omega \subset \mathbb{R}^n$.

3) In the vectorial case $n, N \geq 2$, the first example is the case of *minimal surfaces in parametric form*, a geometrical framework more general than the preceding one. In this case we have $N = n+1$ and therefore the matrix $\xi \in \mathbb{R}^{(n+1)\times n}$. We denote by $\operatorname{adj}_n \xi \in \mathbb{R}^{n+1}$ the vector formed by all the $n \times n$ minors of the matrix ξ. Finally we let

$$f(x, u, \xi) = f(\xi) := |\operatorname{adj}_n \xi|$$

where $|.|$ stands for the Euclidean norm. In geometrical terms the integral represents the area of the surface given by $u(x) \in \mathbb{R}^{n+1}$ when $x \in \Omega \subset \mathbb{R}^n$; moreover $\operatorname{adj}_n \nabla u$ represents the normal to the surface.

Other important examples in the vectorial case are motivated by non linear elasticity, a particularly simple one is when $N = n$ and

$$f(x, u, \xi) = f(\xi) := g(\det \xi)$$

where $g : \mathbb{R} \to \mathbb{R}$ is a given function.

We do not discuss the history of the calculus of variations and we refer for this matter to the books of Dierkes-Hildebrandt-Küster-Wohlrab [248], Giaquinta-Hildebrandt [309], Goldstine [319] and Monna [449].

The first question that arises in conjunction with problem (P) is, of course, the existence of minimizers. This strongly depends on the choice of admissible functions, that we denoted by X. A natural choice would be a subspace of $C^1\left(\Omega;\mathbb{R}^N\right)$ or even $C^2\left(\Omega;\mathbb{R}^N\right)$, if we want to be able to write the differential equation naturally associated to the minimization problem and known as the *Euler-Lagrange equation*. This turns out to be a strategy too hard to implement in most problems, particularly those dealing with partial derivatives (i.e. $n > 1$). The essence of the *direct methods of the calculus of variations* is to split the problem into two parts. First to enlarge the space of admissible functions, for example by considering spaces such as the Sobolev spaces $W^{1,p}$, so as to get a general *existence* theorem and then to prove some *regularity* results that should satisfy any minimizer of (P). In the present book, we are essentially concerned only with the first problem. In most cases the space of admissible functions is

$$X = u_0 + W_0^{1,p}\left(\Omega;\mathbb{R}^N\right),$$

where u_0 is a given function and the notation $u \in X$ is a short cut for meaning that $u = u_0$ on $\partial\Omega$ and $u \in W^{1,p}\left(\Omega;\mathbb{R}^N\right)$.

The existence of minimizers in the above space relies in the fundamental property of (sequential) *weak lower semicontinuity*, meaning that

$$u_\nu \rightharpoonup \overline{u} \text{ in } W^{1,p} \;\Rightarrow\; \liminf_{\nu\to\infty} I\left(u_\nu\right) \geq I\left(\overline{u}\right) \tag{1.1}$$

where $\rightharpoonup$ stands for weak convergence. This property is thoroughly investigated, notably in Chapters 3 and 8.

It turns out that the property (1.1) is intimately related to the convexity of the function $\xi \to f\left(x,u,\xi\right)$ in the scalar case $N = 1$ or $n = 1$ and to the quasiconvexity (in the sense of Morrey) of the same function in the vectorial case.

This leads us to the study of *convex analysis* in Chapter 2 and to *quasiconvex analysis* in Chapters 5, 6 and 7.

We now discuss in more details the content of the monograph and we outline in every chapter some of the main results. We state them, most of the time, under slightly stronger hypotheses than needed, but we refer to the precise theorems at each step.

1.2 Convex analysis and the scalar case

We start with the scalar case where $n = 1$ or $N = 1$. The first one corresponds to the case of one single independent variable and is much easier to deal with, in particular from the point of view of regularity. It is discussed in the general framework of the scalar case in Chapter 3, but has also a special treatment in Chapter 4. The second case $n > N = 1$ involves partial derivatives and is considerably harder; it is discussed in Chapter 3. However, since both cases use in a significant way many results of convex analysis, we start with the study of this classical subject.

1.2.1 Convex analysis

In Chapter 2, we present the most important results of convex analysis. Even though many excellent books exist on the subject, we have decided, for the convenience of the reader, to state and to prove all the results that we need. Another motivation in the presentation of this chapter has been to stress both the similarities and the differences with quasiconvex analysis that is discussed in Part II.

Traditionally convex analysis start with the notion of convex set and then with the one of convex functions. This is also the path we have followed, in contrast with the quasiconvex case.

We start by recalling the notion of convex set. A set $E \subset \mathbb{R}^N$ is said to be *convex* if for every $x, y \in E$ and every $t \in [0, 1]$

$$tx + (1 - t)\, y \in E.$$

We then give several elementary properties concerning the interior, closure and boundary of convex sets. We next turn to two of the most useful results for convex sets, namely the *separation theorems* (cf. Corollary 2.11) and *Carathéodory theorem* (cf. Theorem 2.13). A typical separation theorem is for example the following.

Theorem 1.1 *Let $E \subset \mathbb{R}^N$ be convex and $\overline{x} \in \partial E$. Then there exists $a \in \mathbb{R}^N$, $a \neq 0$, so that*

$$\langle \overline{x}; a \rangle \leq \langle x; a \rangle\,, \text{ for every } x \in E$$

where $\langle .; . \rangle$ denotes scalar product in $\mathbb{R}^N$.

We also recall that the *convex hull* of a set $E \subset \mathbb{R}^N$ is the smallest convex set containing E and is denoted by $\operatorname{co} E$. Carathéodory theorem then states that

Theorem 1.2 *Let $E \subset \mathbb{R}^N$, then*

$$\operatorname{co} E = \left\{ x \in \mathbb{R}^N : x = \textstyle\sum_{i=1}^{N+1} \lambda_i x_i\,,\ x_i \in E,\ \lambda_i \geq 0 \text{ with } \sum_{i=1}^{N+1} \lambda_i = 1 \right\}.$$

We then conclude this brief account on convex sets by recalling the notion of extreme points of a convex set and the *Minkowski theorem* ensuring that if E is compact and E_{ext} denotes the set of extreme points of $\operatorname{co} E$, then

$$\operatorname{co} E = \operatorname{co} E_{ext}\,.$$

We next discuss the concept of convex function. We recall that a function $f : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ is said to be *convex* if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

for every $x, y \in \mathbb{R}^N$ and every $t \in [0,1]$. An important property of convex functions that take only finite values (i.e. $f : \mathbb{R}^N \to \mathbb{R}$) is that they are everywhere continuous (cf. Theorem 2.31).

The notions of convex set and function are related through the *indicator function* of a set E defined by

$$\chi_E(x) = \begin{cases} 0 & \text{if } x \in E \\ +\infty & \text{if } x \notin E. \end{cases}$$

Indeed the function χ_E is convex if and only if the set E is convex.

As we defined the notion of convex hull for a set, a natural concept is the *convex envelope* of a given function f, which is, by definition, the largest convex function below f and it is denoted by Cf. We can therefore write, for every $x \in \mathbb{R}^N$,

$$Cf(x) := \sup\{g(x) : g \leq f \text{ and } g \text{ convex}\}.$$

Of central importance in convex analysis is the concept of conjugate function (or *Legendre transform*). The *conjugate* of a function f is a function $f^* : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ defined by

$$f^*(x^*) := \sup_{x \in \mathbb{R}^N} \{\langle x; x^* \rangle - f(x)\}$$

which is a convex function, independently of the convexity of f. Iterating the process we define the *biconjugate* of f as $f^{**} : \mathbb{R}^N \to \mathbb{R} \cup \{\pm\infty\}$, it is given by

$$f^{**}(x) = \sup_{x^* \in \mathbb{R}^N} \{\langle x; x^* \rangle - f^*(x^*)\}.$$

It turns out that if f takes only finite values then (cf. Theorem 2.43)

$$Cf = f^{**}.$$

Finally we also investigate the differentiability of convex functions, discussing, in particular, the notion of *subgradient*.

1.2.2 Lower semicontinuity and existence results

The main result of Chapter 3 is the following (more general ones are found in Theorem 3.15 and Corollary 3.24).

Theorem 1.3 *Let $n, N \in \mathbb{N}$, $p \geq 1$, $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary, $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R}$ be a non negative continuous function and*

$$I(u) = \int_\Omega f(x, u(x), \nabla u(x))\, dx.$$

Part 1. If the function $\xi \to f(x,u,\xi)$ is convex, then I is (sequentially) weakly lower semicontinuous in $W^{1,p}$ (meaning that (1.1) is satisfied).

Part 2. Conversely if either $N = 1$ or $n = 1$ and I is (sequentially) weakly lower semicontinuous in $W^{1,p}$, then the function $\xi \to f(x,u,\xi)$ is convex.

We should emphasize that in the vectorial case, $n, N \geq 2$, Part 1 of the theorem is valid but the conclusion of Part 2 does not hold.

This theorem, in the scalar case, has as a first direct consequence that the functional is (sequentially) weakly continuous in $W^{1,p}$, meaning that

$$u_\nu \rightharpoonup \overline{u} \text{ in } W^{1,p} \;\Rightarrow\; \lim_{\nu\to\infty} I(u_\nu) = I(\overline{u}),$$

if and only if $\xi \to f(x,u,\xi)$ is affine. This result again strongly contrasts with the vectorial case.

The main implication of the lower semicontinuity theorem is on the existence of minimizers for the problem

$$(P) \quad \inf\left\{ I(u) : u \in u_0 + W_0^{1,p}\left(\Omega;\mathbb{R}^N\right)\right\} = m.$$

Indeed we have, as a special case of our general theorem (cf. Theorem 3.30), the following result.

Theorem 1.4 *Let Ω be a bounded open set of $\mathbb{R}^n$ with Lipschitz boundary. Let $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ be continuous and satisfying the* coercivity *condition*

$$f(x,u,\xi) \geq \alpha_1 |\xi|^p - \alpha_2, \ \forall\ (x,u,\xi) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n}$$

for some $\alpha_1 > 0$, $\alpha_2 \in \mathbb{R}$ and $p > 1$. Assume that $\xi \to f(x,u,\xi)$ is convex *and that $I(u_0) < \infty$. Then (P) has at least one minimizer.*

This theorem is also valid in the vectorial case, but can then be improved a great deal.

As well known, associated to any variational problem is the differential equation known as the *Euler-Lagrange equation.* Under appropriate regularity hypotheses on the function f and on a minimizer $\overline{u}$ of (P), we find that $\overline{u}$ should satisfy, for every $x \in \Omega$,

$$(E) \quad \sum_{\alpha=1}^{n} \frac{\partial}{\partial x_\alpha}\left[\frac{\partial f}{\partial \xi_\alpha^i}(x,\overline{u},\nabla\overline{u})\right] = \frac{\partial f}{\partial u^i}(x,\overline{u},\nabla\overline{u}), \ i = 1,\cdots,N.$$

The differential equation is a second order ordinary differential equation if $n = N = 1$, a system of such equations if $N > n = 1$, a single second order partial differential equation if $n > N = 1$ and a system of such equations when $n, N \geq 2$. In any case the convexity of the function $\xi \to f(x,u,\xi)$, ensures the *ellipticity* of the Euler-Lagrange equations. The prototype example being the Dirichlet integral where $n > N = 1$,

$$f(x,u,\xi) = f(\xi) := \frac{1}{2}|\xi|^2$$

and the associated equation being nothing else than the *Laplace equation*

$$\Delta u = 0.$$

1.2.3 The one dimensional case

In Chapter 4, we specialize to the case where $N = n = 1$, although most of the results are also valid if $N > n = 1$. We are therefore considering the problem

$$(P) \quad \inf \left\{ I(u) = \int_a^b f(x, u(x), u'(x))\, dx \,:\, u \in X \right\}$$

where $f : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $p \geq 1$ and

$$X = \left\{ u \in W^{1,p}(a, b),\, u(a) = \alpha,\ u(b) = \beta \right\}.$$

The Euler-Lagrange equation that should satisfy any minimizer $\overline{u}$ of (P) is then given by

$$(E) \quad \frac{d}{dx}\left[f_\xi\left(x, \overline{u}(x), \overline{u}'(x)\right)\right] = f_u\left(x, \overline{u}(x), \overline{u}'(x)\right),\ x \in [a, b]$$

where $f_\xi = \partial f / \partial \xi$ and $f_u = \partial f / \partial u$. When the function f does not depend explicitly on the variable x, one can find a first integral of (E) which is known as the *second form* of the Euler-Lagrange equation and can be written as

$$f\left(\overline{u}(x), \overline{u}'(x)\right) - \overline{u}'(x)\, f_\xi\left(\overline{u}(x), \overline{u}'(x)\right) = \text{constant},\ x \in [a, b].$$

At this stage it might be enlightening to see some examples that show that, even when $n = N = 1$, the hypotheses of the existence theorem (cf. Theorem 1.4) are essentially optimal. Indeed *non existence* of minimizers in Sobolev spaces occur in all the following cases.

1) Let (cf. Example 4.4) $f(\xi) = e^{-\xi^2}$ and

$$(P) \quad \inf \left\{ I(u) = \int_0^1 f(u'(x))\, dx : u \in X \right\}$$

where $X = W_0^{1,1}(0,1) = \left\{ u \in W^{1,1}(0,1) : u(0) = u(1) = 0 \right\}$. Here both the convexity and coercivity hypotheses of the theorem are violated.

2) Consider (cf. Example 4.5) the case $f(x, u, \xi) = f(u, \xi) = \sqrt{u^2 + \xi^2}$ and

$$(P) \quad \inf \left\{ I(u) = \int_0^1 f(u(x), u'(x))\, dx : u \in X \right\}$$

where $X = \left\{ u \in W^{1,1}(0,1) : u(0) = 0,\ u(1) = 1 \right\}$. In this case the coercivity condition holds with $p = 1$ (and not, as it should be, with $p > 1$).

3) The present example (cf. Example 4.6) is known as *Weierstrass example.* Let $f(x,u,\xi) = f(x,\xi) = x\xi^2$ and

$$(P) \quad \inf \left\{ I(u) = \int_0^1 f(x, u'(x))\, dx : u \in X \right\}$$

where $X = \left\{ u \in W^{1,2}(0,1) : u(0) = 1,\ u(1) = 0 \right\}$. The coercivity hypothesis is violated just at one point (namely at $x = 0$).

4) Let (the example is known as *Bolza example*, cf. Example 4.8)

$$f(x,u,\xi) = f(u,\xi) = \left(\xi^2 - 1\right)^2 + u^4$$

$$(P) \quad \inf \left\{ I(u) = \int_0^1 f(u(x), u'(x))\, dx : u \in W_0^{1,4}(0,1) \right\}.$$

Here it is the convexity assumption on the function $\xi \to f(x,u,\xi)$ which is not satisfied.

Another advantage of the case $N = n = 1$ is that, under appropriate conditions on f notably the convexity of $\xi \to f(x,u,\xi)$, the solutions of (E) are also solutions (and conversely), cf. Theorem 4.29, of the *Hamiltonian system*

$$(H) \quad \begin{cases} u'(x) = H_v(x, u(x), v(x)) \\ v'(x) = -H_u(x, u(x), v(x)) \end{cases}$$

where $v(x) = f_\xi(x, u(x), u'(x))$ and H is the Legendre transform of $\xi \to f(x,u,\xi)$, namely

$$H(x,u,v) = \sup_{\xi \in \mathbb{R}} \left\{ v\,\xi - f(x,u,\xi) \right\}.$$

In classical mechanics f is called the *Lagrangian* and H the *Hamiltonian.*

We conclude the study of Chapter 4 with a brief discussion on *Lavrentiev phenomenon.* We just study the following example (cf. Theorem 4.41) exhibited by Mania. We let

$$f(x,u,\xi) = \left(x - u^3\right)^2 \xi^6$$

$$I(u) = \int_0^1 f(x, u(x), u'(x))\, dx.$$

Consider the two different Sobolev spaces

$$\begin{aligned} \mathcal{W}_\infty &= \left\{ u \in W^{1,\infty}(0,1) : u(0) = 0,\ u(1) = 1 \right\} \\ \mathcal{W}_1 &= \left\{ u \in W^{1,1}(0,1) : u(0) = 0,\ u(1) = 1 \right\} \end{aligned}$$

and the corresponding minimization problems

$$\inf \left\{ I(u) : u \in \mathcal{W}_\infty \right\} = m_\infty \quad \text{and} \quad \inf \left\{ I(u) : u \in \mathcal{W}_1 \right\} = m_1 .$$

We prove that

$$m_\infty > m_1 = 0$$

and that $\overline{u}(x) = x^{1/3}$ is a minimizer of I over $\mathcal{W}_1$.

1.3 Quasiconvex analysis and the vectorial case

We next turn to the vectorial case $n, N \geq 2$, which is the heart of our book and deals with what we call *quasiconvex analysis*. The structure is similar to that of Part I, namely we develop the quasiconvex analysis in Chapters 5, 6 and 7 and then discuss lower semicontinuity and existence results in Chapter 8.

A first striking difference between our presentation of convex and quasiconvex analysis is the order in which we deal with sets and functions. In convex analysis we first defined, as essentially all other authors, the concept of convex sets and then of convex functions. In the present context, we do exactly the reverse. This has some historical reasons. The notion of quasiconvex function was introduced by Morrey in 1952, while the corresponding notion for sets appears almost fifty years after and is, in some sense, in its infancy.

The main motivation for introducing the notion of quasiconvexity is to generalize Theorem 1.3 to the vectorial case.

1.3.1 Quasiconvex functions

Unfortunately, when generalizing the notion of convex function to the vectorial case, several different concepts arise naturally. The notion of *quasiconvex* function arise, as already said, in conjunction with (sequential) weak lower semicontinuity of the corresponding integral. When dealing with the Euler-Lagrange equation, the right concept is the ellipticity and this leads to the definition of *rank one convex* function. Finally, when one wants to generalize the separation theorems, Carathéodory theorem or the notion of duality, one is driven to the concept of *polyconvexity*.

We now describe the content of Chapter 5 and we start with the following definitions.

Definition 1.5 *Let $f : \mathbb{R}^{N \times n} \to \mathbb{R}$.*

(i) The function f is said to be rank one convex *if*

$$f(\lambda \xi + (1-\lambda)\eta) \leq \lambda f(\xi) + (1-\lambda) f(\eta)$$

for every $\lambda \in [0,1]$, $\xi, \eta \in \mathbb{R}^{N \times n}$ with $\operatorname{rank}\{\xi - \eta\} \leq 1$.

(ii) If f is Borel measurable and locally integrable, then it is said to be quasiconvex *if*

$$f(\xi) \leq \frac{1}{\operatorname{meas} D} \int_D f(\xi + \nabla \varphi(x))\, dx$$

for every bounded open set $D \subset \mathbb{R}^n$, *for every* $\xi \in \mathbb{R}^{N \times n}$ *and for every* $\varphi \in W_0^{1,\infty}\left(D; \mathbb{R}^N\right)$.

(iii) The function f *is said to be* polyconvex *if there exists* $F : \mathbb{R}^{\tau(n,N)} \to \mathbb{R}$ *convex, such that*

$$f(\xi) = F(T(\xi))$$

where $T : \mathbb{R}^{N \times n} \to \mathbb{R}^{\tau(n,N)}$ *is such that*

$$T(\xi) = \left(\xi, \operatorname{adj}_2 \xi, \cdots, \operatorname{adj}_{n \wedge N} \xi\right).$$

In the previous definition, $\operatorname{adj}_s \xi$ *stands for the matrix of all* $s \times s$ *minors of the matrix* $\xi \in \mathbb{R}^{N \times n}$, $2 \leq s \leq n \wedge N = \min\{n, N\}$, *and*

$$\tau(n, N) = \sum_{s=1}^{n \wedge N} \sigma(s)$$

where

$$\sigma(s) = \binom{N}{s}\binom{n}{s} = \frac{N!n!}{(s!)^2 (N-s)!(n-s)!}.$$

(iv) A function f *is said to be* rank one affine*, respectively* quasiaffine*,* polyaffine*, if* f and $-f$ *are rank one convex, respectively quasiconvex, polyconvex.*

Remark 1.6 (i) Note that in the case $N = n = 2$, the notion of polyconvexity can be read as follows:

$$\begin{cases} \tau(n, N) = \tau(2,2) = 5 \ \ (\text{since } \sigma(1) = 4,\ \sigma(2) = 1) \\ T(\xi) = (\xi, \det \xi),\ f(\xi) = F(\xi, \det \xi). \end{cases}$$

(ii) The first and third definitions extend in a straightforward manner to functions $f : \mathbb{R}^{N \times n} \to \mathbb{R} \cup \{+\infty\}$. However this is not the case for quasiconvex functions. At the moment no good definition of quasiconvexity for such functions is available. This fact is a strong source of difficulty when dealing with the definition of quasiconvex sets. ♢

The main properties of these functions are now given (cf. Theorems 5.3 and 5.20).

Theorem 1.7 *Let* $f : \mathbb{R}^{N \times n} \to \mathbb{R}$.

(i) The following implications hold

$$f \text{ convex} \Rightarrow f \text{ polyconvex} \Rightarrow f \text{ quasiconvex} \Rightarrow f \text{ rank one convex.}$$

(ii) If $N = 1$ *or* $n = 1$, *then all these notions are equivalent.*

(iii) If $f \in C^2\left(\mathbb{R}^{N \times n}\right)$ *then rank one convexity is equivalent to* Legendre-Hadamard *condition (or* ellipticity *condition)*

$$\sum_{i,j=1}^{N} \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 f(\xi)}{\partial \xi_\alpha^i \partial \xi_\beta^j} \lambda^i \lambda^j \mu_\alpha \mu_\beta \geq 0$$

for every $\lambda \in \mathbb{R}^N, \mu \in \mathbb{R}^n, \xi = \left(\xi_\alpha^i\right)_{1\leq\alpha\leq n}^{1\leq i\leq N} \in \mathbb{R}^{N\times n}$.

(iv) The notions of rank one affine, quasiaffine and polyaffine are equivalent. Moreover any quasiaffine function is of the form

$$f(\xi) = \alpha + \langle\beta; T(\xi)\rangle$$

where $\alpha \in \mathbb{R}$ *and* $\beta \in \mathbb{R}^{\tau(n,N)}$ *and* $\langle .; . \rangle$ *denotes scalar product in* $\mathbb{R}^{\tau(n,N)}$.

We now give some significant examples. The first one (cf. Theorem 5.25) concerns quadratic forms and is one of the most important, since then the associated Euler-Lagrange equations are *linear*.

Theorem 1.8 *Let* M *be a symmetric matrix in* $\mathbb{R}^{(N\times n)\times(N\times n)}$. *Let*

$$f(\xi) = \langle M\xi; \xi\rangle$$

where $\xi \in \mathbb{R}^{N\times n}$ *and* $\langle \cdot; \cdot \rangle$ *denotes the scalar product in* $\mathbb{R}^{N\times n}$. *The following statements then hold.*

(i) f *is rank one convex if and only if* f *is quasiconvex.*

(ii) If $N = 2$ *or* $n = 2$, *then*

$$f \text{ polyconvex } \Leftrightarrow f \text{ quasiconvex } \Leftrightarrow f \text{ rank one convex.}$$

(iii) If $N, n \geq 3$, *then in general*

$$f \text{ rank one convex } \not\Rightarrow f \text{ polyconvex.}$$

We then turn to some more examples.

1) Let $N = n = 2$. The function

$$f(\xi) = \det \xi$$

is quasiaffine and thus polyconvex, quasiconvex or rank one convex, but not convex.

2) When $n \geq 2$ and $N \geq 3$, Sverak (cf. Theorem 5.50) produced an example of a function which is rank one convex but not quasiconvex; answering a long standing conjecture of Morrey. It is still not known if there are rank one convex but not quasiconvex functions in the case $N = n = 2$, or more generally $n \geq N = 2$.

3) Let $N = n = 2$. The function studied by Alibert-Dacorogna-Marcellini (cf. Theorem 5.51) and given by $f_\gamma : \mathbb{R}^{2\times 2} \to \mathbb{R}$, for $\gamma \in \mathbb{R}$, where

$$f_\gamma(\xi) = |\xi|^2 \left(|\xi|^2 - 2\gamma \det \xi \right)$$

is such that

$$\begin{array}{rcl} f_\gamma \text{ is convex} & \Leftrightarrow & |\gamma| \le \gamma_c = 2\sqrt{2}/3, \\ f_\gamma \text{ is polyconvex} & \Leftrightarrow & |\gamma| \le \gamma_p = 1, \\ f_\gamma \text{ is quasiconvex} & \Leftrightarrow & |\gamma| \le \gamma_q \text{ and } \gamma_q > 1, \\ f_\gamma \text{ is rank-one convex} & \Leftrightarrow & |\gamma| \le \gamma_r = 2/\sqrt{3}\,. \end{array}$$

It is not presently known if $\gamma_q = 2/\sqrt{3}\,$.

1.3.2 Quasiconvex envelopes

In Chapter 6, we define the convex Cf (already defined in Section 1.2.1) polyconvex Pf, quasiconvex Qf, rank one convex envelope Rf which are, respectively, defined as the largest convex, polyconvex, quasiconvex, rank one convex function below f. We therefore have, for every $\xi \in \mathbb{R}^{N\times n}$,

$$\begin{array}{rcl} Cf(\xi) & = & \sup\{g(\xi) : g \le f \text{ and } g \text{ convex}\} \\ Pf(\xi) & = & \sup\{g(\xi) : g \le f \text{ and } g \text{ polyconvex}\} \\ Qf(\xi) & = & \sup\{g(\xi) : g \le f \text{ and } g \text{ quasiconvex}\} \\ Rf(\xi) & = & \sup\{g(\xi) : g \le f \text{ and } g \text{ rank one convex}\}\,. \end{array}$$

Observe that Theorem 1.7 immediately implies

$$Cf \le Pf \le Qf \le Rf \le f.$$

Several representation formulas exist for computing these envelopes, we just give a formula for the quasiconvex envelope (cf. Theorem 6.9).

Theorem 1.9 *Let $f : \mathbb{R}^{N\times n} \to \mathbb{R}$ be locally bounded, non negative and Borel measurable. Then for every $\xi \in \mathbb{R}^{N\times n}$,*

$$Qf(\xi) = \inf\left\{\frac{1}{\operatorname{meas} D}\int_D f(\xi + \nabla\varphi(x))\,dx : \varphi \in W_0^{1,\infty}\left(D;\mathbb{R}^N\right)\right\}$$

where $D \subset \mathbb{R}^n$ is a bounded open set. In particular the infimum in the formula is independent of the choice of D.

We now give some examples.

1) Let $f : \mathbb{R}^{N\times n} \to \mathbb{R}$, $\Phi : \mathbb{R}^{N\times n} \to \mathbb{R}$ be quasiaffine not identically constant and $g : \mathbb{R} \to \mathbb{R}$ such that

$$f(\xi) = g(\Phi(\xi))\,.$$

Then (cf. Theorem 6.24)

$$Pf = Qf = Rf = Cg \circ \Phi$$

and in general $Qf > Cf$.

2) Recall the area type case, where $N = n + 1$. Let $f : \mathbb{R}^{(n+1)\times n} \to \mathbb{R}$ and $g : \mathbb{R}^{n+1} \to \mathbb{R}$ be such that

$$f(\xi) = g(\operatorname{adj}_n \xi).$$

Then (cf. Theorem 6.26)

$$Pf = Qf = Rf = Cg \circ \operatorname{adj}_n$$

and in general $Qf > Cf$.

3) An interesting problem in optimal design is the following. Let $N = n = 2$ and, for $\xi \in \mathbb{R}^{2\times 2}$,

$$f(\xi) = \begin{cases} 1 + |\xi|^2 & \text{if } \xi \neq 0 \\ 0 & \text{if } \xi = 0. \end{cases}$$

Then (cf. Theorem 6.28) $Pf = Qf = Rf$ and

$$Qf(\xi) = \begin{cases} 1 + |\xi|^2 & \text{if } |\xi|^2 + 2|\det \xi| \geq 1 \\ 2(|\xi|^2 + 2|\det \xi|)^{1/2} - 2|\det \xi| & \text{if } |\xi|^2 + 2|\det \xi| < 1. \end{cases}$$

We also have

$$Cf(\xi) = \begin{cases} 1 + |\xi|^2 & \text{if } |\xi| \geq 1 \\ 2|\xi| & \text{if } |\xi| < 1. \end{cases}$$

1.3.3 Quasiconvex sets

We have seen in Section 1.2.1 that the connection between convex functions and sets is made via the indicator function. We recall that for a set E, the indicator function is defined by

$$\chi_E(x) = \begin{cases} 0 & \text{if } x \in E \\ +\infty & \text{if } x \notin E. \end{cases}$$

Moreover the function χ_E is convex if and only if the set E is convex.

The aim of Chapter 7 is to extend the definition of convexity for sets, to polyconvexity, quasiconvexity and rank one convexity.

A natural way to define polyconvex, quasiconvex or rank one convex set E, would be by requiring that χ_E be polyconvex, quasiconvex or rank one convex. This is indeed so (cf. Proposition 7.5) for the first and third cases but not for quasiconvex sets, since, as we already said, we lack a good definition of quasiconvexity for functions that are allowed to take the value $+\infty$.

Before giving the definitions, let us introduce some notations. We let in this section $O(n)$ be the set of $n \times n$ orthogonal matrices,

$$D := (0, 1)^n \subset \mathbb{R}^n$$

and $W_{per}^{1,\infty}(D;\mathbb{R}^N)$ be the space of periodic functions in $W^{1,\infty}(D;\mathbb{R}^N)$, meaning that

$$u(x) = u(x + e_i), \text{ for every } x \in D \text{ and } i = 1, \cdots, n,$$

where $\{e_1, \cdots, e_n\}$ is the standard orthonormal basis of $\mathbb{R}^n$. Finally $\mathcal{W}_{per}$ denotes the subspace of functions in $W_{per}^{1,\infty}(D;\mathbb{R}^N)$, whose gradients take only a finite number of values.

We are now in a position to give the following definitions (cf. Definition 7.2).

Definition 1.10 *(i) We say that $E \subset \mathbb{R}^{N\times n}$ is* polyconvex *if there exists a convex set $K \subset \mathbb{R}^{\tau(N,n)}$ such that*

$$\left\{\xi \in \mathbb{R}^{N\times n} : T(\xi) \in K\right\} = E.$$

(ii) We say that $E \subset \mathbb{R}^{N\times n}$ is quasiconvex *if we have*

$$\left.\begin{array}{c} \xi + \nabla\varphi(x)R \in E, \textit{ a.e. } x \in D, \\ \textit{for some } R \in O(n) \textit{ and some } \varphi \in \mathcal{W}_{per} \end{array}\right\} \Rightarrow \xi \in E.$$

(iii) We say that $E \subset \mathbb{R}^{N\times n}$ is rank one convex *if for every $\lambda \in [0,1]$ and $\xi, \eta \in E$ such that* rank $\{\xi - \eta\} = 1$, *then*

$$\lambda\xi + (1-\lambda)\eta \in E.$$

The best definition for quasiconvex sets is unclear. Several definitions have already been considered by other authors. The one we propose here is consistent with known properties for functions and have most properties which are desirable as witnessed by the following theorem (cf. Theorem 7.7).

Theorem 1.11 *Let $E \subset \mathbb{R}^{N\times n}$. The following implications then hold*

$$E \textit{ convex } \Rightarrow E \textit{ polyconvex } \Rightarrow E \textit{ quasiconvex} \Rightarrow E \textit{ rank one convex.}$$

All counter implications are false, as soon as $N, n \geq 2$.

We should draw the attention to the last statement of the theorem. Surprisingly it is better than the corresponding one for functions, where the example of Sverak provides a rank one convex function which is not quasiconvex only when $n \geq 2$ and $N \geq 3$.

Before continuing one main difference between convex sets and generalized ones should be emphasized. A set can be polyconvex, and thus quasiconvex and rank one convex, and be *disconnected*. Indeed if $\xi, \eta \in \mathbb{R}^{N\times n}$ are such that rank $\{\xi - \eta\} \geq 2$, then $E = \{\xi, \eta\}$ is polyconvex.

We next point out a fact (the second one in the next proposition) strikingly different from the equivalent one for convex sets (cf. Proposition 7.24).

Proposition 1.12 *(i) Let $E \subset \mathbb{R}^{N\times n}$ be, respectively, a polyconvex, quasiconvex or rank one convex set. Then* $\operatorname{int} E$ *is also, respectively, polyconvex, quasiconvex or rank one convex.*

(ii) There exists a polyconvex and bounded set $E \subset \mathbb{R}^{2\times 2}$ such that $\overline{E}$ is not rank one convex (and hence neither quasiconvex, nor polyconvex).

We next define the *polyconvex*, *quasiconvex* and *rank one convex hull* of a set $E \subset \mathbb{R}^{N\times n}$ as the smallest polyconvex, quasiconvex and rank one convex set containing E; they are respectively denoted by $\operatorname{Pco} E$, $\operatorname{Qco} E$ and $\operatorname{Rco} E$.

We clearly have

$$E \subset \operatorname{Rco} E \subset \operatorname{Qco} E \subset \operatorname{Pco} E \subset \operatorname{co} E.$$

Other hulls are also defined in Chapter 7.

We finally conclude this section by giving an example. We first recall that the singular values of a given matrix $\xi \in \mathbb{R}^{n\times n}$, denoted by

$$0 \leq \lambda_1(\xi) \leq \cdots \leq \lambda_n(\xi),$$

are the eigenvalues of $\left(\xi\xi^t\right)^{1/2}$. Let $0 < \gamma_1 \leq \cdots \leq \gamma_n$ and consider the set

$$E = \left\{\xi \in \mathbb{R}^{n\times n} : \lambda_i(\xi) = \gamma_i,\ i = 1, \cdots, n\right\}.$$

We prove (cf. Theorem 7.43) that

$$\operatorname{co} E = \left\{\xi \in \mathbb{R}^{n\times n} : \textstyle\sum_{i=\nu}^{n} \lambda_i(\xi) \leq \sum_{i=\nu}^{n} \gamma_i\,,\ \nu = 1, \cdots, n\right\}$$

$$\operatorname{Pco} E = \operatorname{Qco} E = \operatorname{Rco} E = \left\{\xi \in \mathbb{R}^{n\times n} : \textstyle\prod_{i=\nu}^{n} \lambda_i(\xi) \leq \prod_{i=\nu}^{n} \gamma_i\,,\ \nu = 1, \cdots, n\right\}.$$

1.3.4 Lower semicontinuity and existence theorems

In Chapter 8, we extend the lower semicontinuity results (cf. Theorem 1.3) to the vectorial context. This is a delicate matter and we deal, in Chapter 8, with it in several steps. We now gather Theorems 8.1 and 8.11 to obtain the following result.

Theorem 1.13 *Let $1 \leq p < \infty$, $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and let*

$$f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R},\ f = f(x, u, \xi),$$

be a continuous function satisfying

$$0 \leq f(x, u, \xi) \leq g(x, u)\left(1 + |\xi|^p\right)$$

where

$$g : \Omega \times \mathbb{R}^N \to \mathbb{R},\ g = g(x, u),$$

is a non negative continuous function. Let

$$I(u) = \int_\Omega f(x, u(x), \nabla u(x))\, dx.$$

Then I *is (sequentially) weakly lower semicontinuous in* $W^{1,p}(\Omega; \mathbb{R}^N)$ *if and only if* $\xi \to f(x, u, \xi)$ *is quasiconvex, i.e.*

$$\frac{1}{\operatorname{meas} D} \int_D f(x_0, u_0, \xi_0 + \nabla\varphi(x))\, dx \geq f(x_0, u_0, \xi_0)$$

for every $(x_0, u_0, \xi_0) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n}$, *for every bounded open set* $D \subset \mathbb{R}^n$ *and for every* $\varphi \in W_0^{1,\infty}(D; \mathbb{R}^n)$.

This result has as an immediate corollary that I is (sequentially) weakly continuous in $W^{1,p}$ if and only if $\xi \to f(x, u, \xi)$ is quasiaffine, i.e. all minors of the matrix $\xi \in \mathbb{R}^{N\times n}$ are weakly continuous. We now restate, in a more convenient and more general way, this result in the case where $N = n = 2$ (see Theorem 8.20, Lemma 8.24 and Corollary 8.26). Let us start with the simple but fundamental observation that Jacobian determinants can be written in divergence form. More precisely if $u \in C^2(\Omega; \mathbb{R}^2)$, then letting

$$\operatorname{Det} \nabla u := \frac{\partial}{\partial x_1}\left(u^1 \frac{\partial u^2}{\partial x_2}\right) - \frac{\partial}{\partial x_2}\left(u^1 \frac{\partial u^2}{\partial x_1}\right),$$

we find that

$$\operatorname{Det} \nabla u(x) = \det \nabla u(x),\ \text{for every } x \in \Omega,$$

since we trivially have

$$\begin{aligned} \det \nabla u &= \frac{\partial u^1}{\partial x_1}\frac{\partial u^2}{\partial x_2} - \frac{\partial u^2}{\partial x_1}\frac{\partial u^1}{\partial x_2} \\ &= \frac{\partial}{\partial x_1}\left(u^1 \frac{\partial u^2}{\partial x_2}\right) - \frac{\partial}{\partial x_2}\left(u^1 \frac{\partial u^2}{\partial x_1}\right) = \operatorname{Det} \nabla u. \end{aligned}$$

The quantity $\operatorname{Det} \nabla u$ is called the *distributional Jacobian* of u. We can now state the theorem (cf. Theorem 8.20, Lemma 8.24, Corollary 8.26 and Example 8.28).

Theorem 1.14 *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded open set,* $1 < p < \infty$, *and let*

$$u_\nu \rightharpoonup u \text{ in } W^{1,p}(\Omega; \mathbb{R}^2).$$

Part 1. *If* $p > 2$, *then*

$$\det \nabla u_\nu \rightharpoonup \det \nabla u \text{ in } L^{p/2}(\Omega).$$

If $p = 2$ the result is false, but the following convergence holds

$$\det \nabla u_\nu \rightharpoonup \det \nabla u \text{ in } \mathcal{D}'(\Omega).$$

Part 2. *If $p \geq \frac{4}{3}$, then* $\operatorname{Det} \nabla u \in \mathcal{D}'(\Omega)$ *and if $p \geq 2$, then*

$$\operatorname{Det} \nabla u = \det \nabla u \text{ in } \mathcal{D}'(\Omega).$$

Part 3. *If $p > \frac{4}{3}$, then*

$$\operatorname{Det} \nabla u_\nu \rightharpoonup \operatorname{Det} \nabla u \text{ in } \mathcal{D}'(\Omega).$$

If $p \leq \frac{4}{3}$, the result is false.

Theorem 1.13 has also as a direct consequence the following existence theorem (cf. Theorem 8.29).

Theorem 1.15 *Let $p > 1$, $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Let $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R}$, $f = f(x, u, \xi)$, be a continuous function satisfying*

$$\xi \to f(x, u, \xi) \text{ is quasiconvex}$$

$$\alpha_1 |\xi|^p + \beta_1 \leq f(x, u, \xi) \leq \alpha_2 (|\xi|^p + 1)$$

for every $(x, u, \xi) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n}$, where $\alpha_2 \geq \alpha_1 > 0$, $\beta_1 \in \mathbb{R}$. Let

$$(P) \quad \inf \left\{ I(u) = \int_\Omega f(x, u(x), \nabla u(x))\, dx : u \in u_0 + W_0^{1,p}\left(\Omega; \mathbb{R}^N\right) \right\}$$

then (P) admits at least one minimizer.

Using Theorem 1.14, we can also prove some existence theorems for polyconvex functions (cf. Theorem 8.31).

1.4 Relaxation and non convex problems

In Part III we go back to the study of

$$(P) \quad \inf \left\{ I(u) = \int_\Omega f(x, u(x), \nabla u(x))\, dx : u \in u_0 + W_0^{1,p}\left(\Omega; \mathbb{R}^N\right) \right\}$$

where

- $\Omega \subset \mathbb{R}^n$, $n \geq 1$, is a bounded open set;
- $u : \Omega \to \mathbb{R}^N$, $N \geq 1$ and $u_0 \in W^{1,p}\left(\Omega; \mathbb{R}^N\right)$ is a given function;
- $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R}$, $f = f(x, u, \xi)$, is a given non convex (non quasiconvex in the vectorial case) function.

The direct methods (cf. Theorems 1.4 and 1.15) do not apply and the general rule is that (P) has no minimizers, as already pointed out in Section

1.2.3. However there is a way of defining generalized solutions of (P) via the so called *relaxed problem*

$$(QP) \quad \inf\left\{\overline{I}(u)=\int_\Omega Qf\left(x,u\left(x\right),\nabla u\left(x\right)\right)dx : u\in u_0+W_0^{1,p}\left(\Omega;\mathbb{R}^N\right)\right\}$$

where Qf is the quasiconvex envelope of f (with respect to the last variable ∇u), defined in Section 1.3.2.

The relaxed problem is useful, not only to define generalized solutions of (P), but also to show that in many cases, although the direct methods do not apply, the problem (P) do have minimizers.

1.4.1 Relaxation theorems

In Chapter 9, we prove the relaxation theorem (cf. Theorems 9.1 and 9.8) and we state it here, as usual under stronger hypotheses, in the case where f does not depend on lower order terms.

Theorem 1.16 *Let $\Omega\subset\mathbb{R}^n$ be a bounded open set. Let $f:\mathbb{R}^{N\times n}\to\mathbb{R}$ be a Borel measurable function satisfying, for $1\le p<\infty$,*

$$0\le f\left(\xi\right)\le\alpha\left(1+\left|\xi\right|^p\right),\ \textit{for every } \xi\in\mathbb{R}^{N\times n}$$

where $\alpha>0$ is a constant while for $p=\infty$ it is assumed that f is locally bounded. Let, for every $\xi\in\mathbb{R}^{N\times n}$,

$$Qf\left(\xi\right)=\sup\left\{g\left(\xi\right):g\le f \quad \textit{and} \quad g \ \textit{quasiconvex}\right\}$$

be the quasiconvex envelope of f.

Part 1. *Then*

$$\inf\left(P\right)=\inf(QP).$$

More precisely for every $u\in W^{1,p}\left(\Omega;\mathbb{R}^N\right)$, there exists a sequence $\{u_\nu\}_{\nu=1}^\infty\subset u+W_0^{1,p}\left(\Omega;\mathbb{R}^N\right)$ such that

$$u_\nu\to u \quad \textit{in} \quad L^p\left(\Omega;\mathbb{R}^N\right),\ \textit{as } \nu\to\infty$$

$$\int_\Omega f\left(\nabla u_\nu\left(x\right)\right)dx\to\int_\Omega Qf\left(\nabla u\left(x\right)\right)dx,\ \textit{as } \nu\to\infty.$$

Part 2. *Assume, in addition to the hypotheses of Part 1, that, if $1<p<\infty$, there exist $\alpha\ge\beta>0$, $\gamma\in\mathbb{R}$ such that*

$$\gamma+\beta\left|\xi\right|^p\le f\left(\xi\right)\le\alpha\left(1+\left|\xi\right|^p\right),\ \textit{for every } \xi\in\mathbb{R}^{N\times n},$$

then, in addition to the conclusions of Part 1, the following holds

$$u_\nu\rightharpoonup u \ \textit{in } W^{1,p}\left(\Omega\right),\ \textit{as } \nu\to\infty.$$

1.4.2 Some existence theorems for differential inclusions

When we apply, in Section 1.4.3, the relaxation theorems to get existence of minimizers for the problem (P), we need to find solutions of some *differential inclusions*. This is achieved in Chapter 10, where we deal with the problem of finding a map $u \in W^{1,\infty}\left(\Omega;\mathbb{R}^N\right)$ which solves

$$\begin{cases} \nabla u(x) \in E & \text{a.e. } x \in \Omega \\ u(x) = u_0(x) & x \in \partial\Omega \end{cases}$$

where $u_0 \in W^{1,\infty}\left(\Omega;\mathbb{R}^N\right)$ is a given map and $E \subset \mathbb{R}^{N\times n}$ is a given set.

In this introductory chapter, we do not give any general result but discuss only some significant examples.

The first one concerns the scalar case, where the result takes an almost optimal form (cf. Theorem 10.18).

Theorem 1.17 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $E \subset \mathbb{R}^n$. Let $u_0 \in W^{1,\infty}(\Omega)$ satisfy*

$$\nabla u_0(x) \in E \cup \operatorname{int}\operatorname{co} E, \ \text{a.e. } x \in \Omega \tag{1.2}$$

(where $\operatorname{int}\operatorname{co} E$ stands for the interior of the convex hull of E); then there exists $u \in u_0 + W_0^{1,\infty}(\Omega)$ such that

$$\nabla u(x) \in E, \ \text{a.e. } x \in \Omega.$$

The theorem has as immediate consequence the following result (cf. Corollary 10.20). If $F : \mathbb{R}^n \to \mathbb{R}$ is continuous and such that

$$\lim_{|\xi|\to\infty} F(\xi) = +\infty$$

and $u_0 \in W^{1,\infty}(\Omega)$ verifies

$$F(\nabla u_0(x)) \leq 0, \ \text{a.e. } x \in \Omega,$$

then there exists $u \in u_0 + W_0^{1,\infty}(\Omega)$ such that

$$F(\nabla u(x)) = 0, \ \text{a.e. } x \in \Omega.$$

The condition (1.2) is also necessary when the boundary datum is affine (cf. Theorem 10.24).

Theorem 1.18 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $E \subset \mathbb{R}^n$, u_0 be such that*

$$\nabla u_0 = \xi_0$$

for some $\xi_0 \in \mathbb{R}^n$. If $u \in u_0 + W_0^{1,\infty}(\Omega)$ solves

$$\nabla u(x) \in E, \ \text{a.e. } x \in \Omega,$$

then

$$\xi_0 \in E \cup \operatorname{int} \operatorname{co} E.$$

The next result, which is now a vectorial one, (cf. Theorem 10.25) should be related to the example given in Section 1.3.3.

Theorem 1.19 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $0 < a_1 \leq \cdots \leq a_n$ and $\xi_0 \in \mathbb{R}^{n \times n}$ be such that*

$$\prod_{i=\nu}^{n} \lambda_i(\xi_0) < \prod_{i=\nu}^{n} a_i \,, \;\; \nu = 1, \cdots, n.$$

If u_0 is an affine map such that $\nabla u_0 = \xi_0$, then there exists $u \in u_0 + W_0^{1,\infty}(\Omega; \mathbb{R}^n)$ so that, for almost every $x \in \Omega$,

$$\lambda_\nu(\nabla u\,(x)) = a_\nu \,, \;\; \nu = 1, \cdots, n.$$

1.4.3 Some existence results for non quasiconvex integrands

We now apply (cf. Chapter 11) the results of the two previous sections to prove existence of minimizers for

$$(P) \quad \inf \left\{ I\,(u) = \int_\Omega f\,(\nabla u\,(x))\,dx : u \in u_0 + W_0^{1,\infty}\left(\Omega; \mathbb{R}^N\right) \right\}$$

where u_0 is an affine map such that $\nabla u_0 = \xi_0 \in \mathbb{R}^{N \times n}$, but without assuming any convexity or quasiconvexity hypothesis on the integrand f. We could also treat integrands depending on lower order terms as well as boundary data that are not affine, but then only very few general results can be given and moreover they are often restricted to the scalar case.

Clearly if the integrand f were quasiconvex, because of the special form of the boundary datum, we would trivially have that u_0 is a minimizer of (P).

From the relaxation theorem, the following theorem easily follows (cf. Theorem 11.1).

Theorem 1.20 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, f a non negative, locally bounded and lower semicontinuous function and u_0 be as above, in particular $\nabla u_0 = \xi_0$. The problem (P) has a solution if and only if there exists $\overline{u} \in u_0 + W_0^{1,\infty}(\Omega; \mathbb{R}^N)$ such that*

$$f\,(\nabla \overline{u}\,(x)) = Qf\,(\nabla \overline{u}\,(x))\,, \;\; a.e. \;\, x \in \Omega$$

$$\int_\Omega Qf\,(\nabla \overline{u}\,(x))\;dx = Qf\,(\xi_0)\;\operatorname{meas}\Omega.$$

We do not continue here with general necessary and sufficient conditions for the existence of minimizers for (P), but we rather give several examples.

We start with the very elementary case where $n = N = 1$ (cf. Theorem 11.24). The result adapts in a straightforward manner to the case $N > n = 1$.

Theorem 1.21 *Let $f : \mathbb{R} \to \mathbb{R}$ be non negative, locally bounded and lower semicontinuous. Let $a < b$, $\alpha, \beta \in \mathbb{R}$ and*

$$(P) \quad \inf \left\{ I(u) = \int_a^b f(u'(x))\, dx : u \in X \right\}$$

where

$$X = \left\{ u \in W^{1,\infty}(a,b) : u(a) = \alpha,\ u(b) = \beta \right\}.$$

The two following statements are then equivalent:

(i) Problem (P) has a minimizer;

(ii) there exist $0 \leq \lambda \leq 1$ and $\gamma, \delta \in \mathbb{R}$ such that

$$Cf(\frac{\beta - \alpha}{b - a}) = \lambda f(\gamma) + (1 - \lambda) f(\delta) \quad \textit{and} \quad \frac{\beta - \alpha}{b - a} = \lambda\gamma + (1 - \lambda)\delta \qquad (1.3)$$

where $Cf = \sup\{g \leq f : g \text{ convex}\}$.

Furthermore if (1.3) is satisfied, then

$$\overline{u}(x) = \begin{cases} \gamma(x - a) + \alpha & \text{if } x \in [a, a + \lambda(b - a)] \\ \delta(x - a) + \lambda(\gamma - \delta)(b - a) + \alpha & \text{if } x \in (a + \lambda(b - a), b] \end{cases}$$

is a minimizer of (P).

Note that by Carathéodory theorem we always have

$$Cf(\frac{\beta - \alpha}{b - a}) = \inf\{\lambda f(\gamma) + (1 - \lambda) f(\delta) : \frac{\beta - \alpha}{b - a} = \lambda\gamma + (1 - \lambda)\delta\}. \qquad (1.4)$$

Therefore (1.3) states that a necessary and sufficient condition for existence of solutions is that the infimum in (1.4) be attained. Note also that if f is convex *or* f coercive (in the sense that $f(\xi) \geq a|\xi|^p + b$ with $p > 1$, $a > 0$) then the infimum in (1.4) is always attained. Hence, if $f(x, u, \xi) = f(\xi)$, counterexamples to existence must be non convex *and* non coercive, as the example already considered in Section 1.2.3, where $f(\xi) = e^{-\xi^2}$.

Of course if f depends explicitly on u, the example of Bolza (given in Section 1.2.3) shows that the theorem is then false.

We now give three examples in the vectorial case.

1) The first one (cf. Theorem 11.32) deals with the minimization problem

$$(P) \quad \inf \left\{ \int_\Omega g(\Phi(\nabla u(x)))\, dx : \ u \in u_0 + W_0^{1,\infty}(\Omega; \mathbb{R}^N) \right\}$$

where:

- $g : \mathbb{R} \to \mathbb{R}$ is a lower semicontinuous, locally bounded and non negative function,

- $\Phi : \mathbb{R}^{N \times n} \to \mathbb{R}$ is quasiaffine and non constant (in particular we can have, when $N = n$, $\Phi(\xi) = \det \xi$).

The relaxed problem is then (cf. Section 1.3.2)

$$(QP) \quad \inf\left\{\int_\Omega Cg(\Phi(\nabla u(x)))\,dx:\ u\in u_0+W_0^{1,\infty}(\Omega;\mathbb{R}^N)\right\},$$

where Cg is the convex envelope of g. The existence result is the following.

Theorem 1.22 *Let $\Omega\subset\mathbb{R}^n$ be a bounded open set, $g:\mathbb{R}\to\mathbb{R}$ as above and satisfying*

$$\lim_{|t|\to+\infty}\frac{g(t)}{|t|}=+\infty$$

and $u_0(x)=\xi_0 x$, with $\xi_0\in\mathbb{R}^{N\times n}$. Then there exists $\bar{u}\in u_0+W_0^{1,\infty}(\Omega;\mathbb{R}^N)$ a minimizer of (P).

2) The second example deals with integrands of area type (see Section 1.3.2), where $N=n+1$ and

$$f(\xi)=g(\operatorname{adj}_n\xi).$$

The minimization problem is then

$$(P) \quad \inf\left\{\int_\Omega g(\operatorname{adj}_n(\nabla u(x)))\,dx:\ u\in u_0+W_0^{1,\infty}(\Omega;\mathbb{R}^{n+1})\right\}$$

where Ω is a bounded open set of $\mathbb{R}^n$, $\nabla u_0=\xi_0$ and $g:\mathbb{R}^{n+1}\to\mathbb{R}$ is a non negative, lower semicontinuous and locally bounded non convex function.

From Section 1.3.2, we have

$$Qf(\xi)=Cg(\operatorname{adj}_n\xi).$$

We next set

$$S=\{y\in\mathbb{R}^{n+1}:Cg(y)<g(y)\}$$

and assume, in order to avoid the trivial situation, that $\operatorname{adj}_n\xi_0\in S$.

The existence result (cf. Theorem 11.36) is then given by

Theorem 1.23 *If S is bounded, Cg is affine in S and* $\operatorname{rank}\xi_0\geq n-1$, *then (P) has a solution.*

3) The third one is the problem of optimal design already discussed in Section 1.3.2, where

$$f(\xi)=\begin{cases}1+|\xi|^2 & \text{if } \xi\neq 0\\ 0 & \text{if } \xi=0.\end{cases}$$

Consider the problem

$$(P) \quad \inf\left\{\int_\Omega f(\nabla u(x))\,dx:\ u\in u_0+W_0^{1,\infty}(\Omega;\mathbb{R}^2)\right\},$$

where Ω is a bounded open set of $\mathbb{R}^2$ and $\nabla u_0=\xi_0$.

We then have the following (cf. Theorem 11.35).

Theorem 1.24 *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set, $f : \mathbb{R}^{2\times 2} \to \mathbb{R}$ be as above and $\xi_0 \in \mathbb{R}^{2\times 2}$. Then a necessary and sufficient condition for (P) to have a solution is that one of the following conditions hold:*

(i) $\xi_0 = 0$ *or* $|\xi_0|^2 + 2|\det \xi_0| \geq 1$, *(i.e.* $f(\xi_0) = Qf(\xi_0)$*)*

(ii) $\det \xi_0 \neq 0$.

1.5 Miscellaneous

In Part IV, we gather some notations and standard results on function spaces and on singular values. We also devote the last two chapters on results that play only a marginal role in our analysis, but have some interest on their own.

1.5.1 Hölder and Sobolev spaces

In Chapter 12, we only fix the notations concerning the main function spaces that we use, namely the *Hölder spaces* $C^{m,\alpha}\left(\overline{\Omega};\mathbb{R}^N\right)$ and the *Sobolev spaces* $W^{m,p}\left(\Omega;\mathbb{R}^N\right)$, where m is an integer, $0 < \alpha \leq 1$ and $1 \leq p \leq \infty$. We recall without proofs the most important results, as for example the Sobolev imbedding theorem, that we use throughout the book.

1.5.2 Singular values

We recall in Chapter 13, the definition and some elementary properties of the *singular values* of a matrix $\xi \in \mathbb{R}^{n\times n}$ (in the present introduction, we discuss only the case $N = n$, but in Chapter 13 we consider general matrices in $\mathbb{R}^{N\times n}$). We denote by

$$0 \leq \lambda_1(\xi) \leq \cdots \leq \lambda_n(\xi)$$

the eigenvalues of $\left(\xi\xi^t\right)^{1/2}$. The main result (cf. Theorem 13.3) is

Theorem 1.25 *Let $\xi \in \mathbb{R}^{n\times n}$ and $0 \leq \lambda_1(\xi) \leq \cdots \leq \lambda_n(\xi)$ be its singular values then there exist $R, Q \in O(n)$ (the set of* orthogonal *matrices $R \in \mathbb{R}^{n\times n}$ so that $R^t R = I$) such that*

$$R\xi Q = \operatorname{diag}(\lambda_1(\xi), \cdots, \lambda_n(\xi)) := \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}.$$

In some applications it might be better to replace the singular values $\lambda_1(\xi) \leq \cdots \leq \lambda_n(\xi)$ of a given matrix $\xi \in \mathbb{R}^{n\times n}$ by its *signed singular values*

$$0 \leq |\mu_1(\xi)| \leq \cdots \leq \mu_n(\xi),$$

defined by

$$\mu_1(\xi) = \lambda_1(\xi)\,\text{sign}(\det \xi) \quad \text{and} \quad \mu_j(\xi) = \lambda_j(\xi),\ j = 2, \cdots, n.$$

We then have the following inequality (cf. Theorem 13.10).

Theorem 1.26 *Let $\xi, \eta \in \mathbb{R}^{n\times n}$. Then*

$$\max_{Q,R\in SO(n)} \{\text{trace}(Q\xi R^t \eta^t)\} = \sum_{j=1}^{n} \mu_j(\xi)\,\mu_j(\eta)$$

and consequently,

$$\text{trace}(\xi\eta^t) \le \sum_{j=1}^{n} \mu_j(\xi)\,\mu_j(\eta).$$

1.5.3 Some underdetermined partial differential equations

In Chapter 14, we prove existence of solutions for three types of underdetermined partial differential equations, which are encountered in mechanics. All three problems bear in common that, in general, there are infinitely many solutions.

The first one concerns the divergence operator (cf. Theorem 14.2).

Theorem 1.27 *Let $m \ge 0$ be an integer and $0 < \alpha < 1$. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded connected open set. The following conditions are then equivalent.*

(i) $f \in C^{m,\alpha}\left(\overline{\Omega}\right)$ satisfies

$$\int_\Omega f(x)\,dx = 0.$$

(ii) There exists $u \in C^{m+1,\alpha}\left(\overline{\Omega};\mathbb{R}^n\right)$ verifying

$$\begin{cases} \operatorname{div} u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $\operatorname{div} u = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i}$.

The second one is related to the curl operator (cf. Theorem 14.4).

Theorem 1.28 *Let $m \ge 1$ be an integer and $0 < \alpha < 1$. Let $\Omega \subset \mathbb{R}^3$ be a smooth bounded convex set and ν denote the outward unit normal. The following conditions are then equivalent.*

(i) $f \in C^{m,\alpha}\left(\overline{\Omega};\mathbb{R}^3\right)$ verifies

$$\operatorname{div} f = 0 \text{ in } \Omega \text{ and } \langle f;\nu\rangle = 0 \text{ on } \partial\Omega.$$

(ii) There exists $u \in C^{m+1,\alpha}\left(\overline{\Omega};\mathbb{R}^3\right)$ satisfying

$$\begin{cases} \operatorname{curl} u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where if $u = \left(u^1, u^2, u^3\right)$ *then*

$$\operatorname{curl} u = \left(\frac{\partial u^3}{\partial x_2} - \frac{\partial u^2}{\partial x_3}, \frac{\partial u^1}{\partial x_3} - \frac{\partial u^3}{\partial x_1}, \frac{\partial u^2}{\partial x_1} - \frac{\partial u^1}{\partial x_2}\right).$$

Both cases are examples of a study of Dirichlet problems associated to $du = f$, where u is a k form and d is the exterior derivative.

The next theorem (cf. Theorem 14.6) is the nonlinear version of Theorem 1.27. In terms of fluid mechanics, the first one is in Eulerian coordinates, while the second one is in Lagrangian coordinates.

For $m \geq 1$ an integer, $0 < \alpha < 1$ and $\Omega \subset \mathbb{R}^n$ a bounded open set, with sufficiently regular boundary, we denote by $\operatorname{Diff}^{m,\alpha}\left(\overline{\Omega}\right)$ the set of diffeomorphisms $u : \overline{\Omega} \to \overline{\Omega}$ such that $u, u^{-1} \in C^{m,\alpha}\left(\overline{\Omega}; \mathbb{R}^n\right)$.

Theorem 1.29 *Let* $m \geq 0$ *be an integer and* $0 < \alpha < 1$. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded connected open set, with sufficiently smooth boundary. Let* $f \in C^{m,\alpha}\left(\overline{\Omega}\right)$, $f > 0$ *in* $\overline{\Omega}$ *and*

$$\int_\Omega f(x)\, dx = \operatorname{meas} \Omega.$$

Then there exists $u \in \operatorname{Diff}^{m+1,\alpha}\left(\overline{\Omega}\right)$ *satisfying*

$$\begin{cases} \det \nabla u(x) = f(x) & x \in \Omega \\ u(x) = x & x \in \partial\Omega. \end{cases}$$

The theorem can be applied in a straightforward manner to the following minimization problem (cf. Corollary 14.9)

$$(P) \quad \inf\left\{I(u) = \int_\Omega g(\det \nabla u(x))\, dx : u \in u_0 + W_0^{1,\infty}(\Omega; \mathbb{R}^n)\right\}$$

where $g : \mathbb{R} \to \mathbb{R}$ is convex and $\Omega \subset \mathbb{R}^n$ and u_0 satisfy appropriate smoothness conditions.

1.5.4 Extension of Lipschitz maps

In Chapter 15, we consider the problem of extending a Lipschitz map defined on a set to the full space so as to preserve the Lipschitz constant. This is a classical problem and is of particular importance in the calculus of variations, where it is known as Mac Shane lemma in the scalar case and Kirszbraun theorem in the vectorial case.

Let us describe more precisely the problem. We consider two Banach spaces $(E, \|.\|_E)$ and $(F, \|.\|_F)$. We ask whenever a map $u : D \subset E \to F$ satisfying

$$\|u(x) - u(y)\|_F \leq \|x - y\|_E, \ x, y \in D$$

can be extended to the whole of E so as to preserve the inequality.

We need the following three definitions.

Definition 1.30 *(i) We say that $u : E \to F$ is a* contraction on D *or u is* 1–Lipschitz on D *if*

$$\|u(x) - u(y)\|_F \leq \|x - y\|_E \quad \textit{for all } x, y \in D.$$

In this case, we write that $u \in \mathrm{Lip}_1(D, F)$.

(ii) When $u \in \mathrm{Lip}_1(E, F)$, we simply say that u is a contraction*.*

Definition 1.31 *(i) We say that $[E; F]$ has the* extension property for contractions on D *if every $u \in \mathrm{Lip}_1(D, F)$ has an extension $\widetilde{u} \in \mathrm{Lip}_1(E, F)$.*

(ii) If $[E; F]$ has the extension property for contractions for every $D \subset E$, we simply say that $[E; F]$ has the extension property for contractions*.*

Definition 1.32 *The unit sphere S^F (i.e. the set of $x \in F$ such that $\|x\|_F = 1$) is said to be* strictly convex *if it has no flat part, meaning that*

$$\|(1 - t)x + ty\|_F < (1 - t)\|x\|_F + t\|y\|_F = 1$$

for all $t \in (0, 1)$ and all $x, y \in S^F$ such that $x \neq y$.

A particularly interesting example is the *Hölder norms* $|x|_p$ over $\mathbb{R}^n$, $1 \leq p \leq \infty$, and they are defined as

$$|x|_p := \begin{cases} \left[\sum_{i=1}^n |x_i|^p\right]^{1/p} & \text{if } 1 \leq p < \infty \\ \max_{1 \leq i \leq n} \{|x_i|\} & \text{if } p = \infty. \end{cases}$$

When $n \geq 2$, the unit sphere for $|\cdot|_p$ is strictly convex if and only if $1 < p < \infty$.

We can now state our main theorems (cf. Theorems 15.11 and 15.12). Part (i) of Theorem 1.33 is known as Mac Shane lemma and the implication (i) $\Rightarrow$ (ii) in Theorem 1.34 is known as Kirszbraun theorem.

Theorem 1.33 *(i) Let $(E, \|.\|_E)$ be a normed space, then, $[E; \mathbb{R}]$ has the extension property for contractions.*

(ii) Let $(F, \|.\|_F)$ be a Banach space, then, $[\mathbb{R}; F]$ has the extension property for contractions.

We now turn our attention to the case where both E and F have dimension at least 2 and we give a theorem which characterizes the Banach spaces for which $[E, F]$ has the extension property for contractions.

Theorem 1.34 *Assume that $(E, \|.\|_E)$ and $(F, \|.\|_F)$ are Banach spaces such that $\dim E$, $\dim F \geq 2$ and that the unit sphere in F is strictly convex. Assume also that every closed set $D \subset E$ contains a countable set $D_c \subset D$ whose closure is D. Then, the three following properties are equivalent:*

(i) $\|.\|_E$ and $\|.\|_F$ are induced by an inner product;

(ii) $[E; F]$ has the extension property for contractions;

(iii) for every $\bar{x} \in E$ and every $S := \{x_1, x_2, x_3\} \subset E$, every $u \in \mathrm{Lip}_1(S, F)$ has an extension $\widetilde{u} \in \mathrm{Lip}_1(S \cup \{\bar{x}\}, F)$.

Several comments are in order.

(i) If S consists of only two points $x, y \in E$, $x \neq y$, then the extension to any third point is always possible.

(ii) If one drops the assumption that S^F is strictly convex, $[E; F]$ may have the extension property for contractions even if none of the norms is induced by an inner product; for example, if $F = \mathbb{R}^N$, $N \geq 2$, and $\|.\|_F = |\cdot|_\infty$, then $[E; F]$ has the extension property for any E.

(iii) In the case of Hölder norms with $(E = \mathbb{R}^n, |.|_p)$ and $\left(F = \mathbb{R}^N, |.|_q\right)$ with $n, N \geq 2$, $1 < q < \infty$ and $1 \leq p \leq \infty$, then $[E; F]$ has the extension property if and only if $p = q = 2$.