Exercice 1 (cf. Exercices 4.3.2 et 4.3.3). Soient $h \in L^2(\Omega)$ et $\overline{u} \in W^{1,2}(\Omega)$ satisfaisant

$$\int_\Omega \langle \nabla \overline{u}(x) ; \nabla \varphi(x) \rangle \, dx = \int_\Omega h(x)\, \varphi(x)\, dx, \quad \forall\, \varphi \in W_0^{1,2}(\Omega). \tag{1}$$

Soit $O \subset \overline{O} \subset \Omega$ un ouvert.

(i) Montrer qu'il existe $\gamma = \gamma(O, \Omega) > 0$ tel que

$$\|\nabla \overline{u}\|_{L^2(O)} \le \gamma \left(\|\overline{u}\|_{L^2(\Omega)} + \|h\|_{L^2(\Omega)} \right).$$

Indication : Choisir $\varphi = \rho^2 \overline{u}$ avec $\rho \in C_0^\infty(\Omega)$ telle que $\rho \equiv 1$ sur O.

(ii) Montrer que l'estimation obtenue au Théorème 4.8, à savoir que $\overline{u} \in W^{2,2}_{\text{loc}}(\Omega)$ est tel que

$$\|\overline{u}\|^2_{W^{2,2}(O)} \le \widetilde{\gamma} \left(\|\overline{u}\|^2_{W^{1,2}(\Omega)} + \|h\|^2_{L^2(\Omega)} \right) \tag{2}$$

peut être raffinée en

$$\|\overline{u}\|_{W^{2,2}(O)} \le \gamma \left(\|\overline{u}\|_{L^2(\Omega)} + \|h\|_{L^2(\Omega)} \right).$$

Indication : Considrer $O \subset \overline{O} \subset U \subset \overline{U} \subset \Omega$ et appliquer (2) d'abord $O \subset U$, puis $U \subset \Omega$.

(iii) Montrer que si, de plus, $u_0 = 0$ dans le Théorème 4.8, alors l'estimation est même de la forme

$$\|\overline{u}\|_{W^{2,2}(O)} \le \gamma \|h\|_{L^2(\Omega)}.$$

Indication : Bien choisir φ dans (1).

Exercice 2 (cf. Exercice 4.5.3). On montre ici que si $f \in C^0$, alors, en général, il n'existe pas de solution $u \in C^2$ de $\Delta u = f$. Soient $0 < \alpha < 1$, $\Omega = \left\{ x \in \mathbb{R}^2 : |x| < 1/2 \right\}$ et

$$u(x) = u(x_1, x_2) = \begin{cases} x_1 x_2 \left|\log|x|\right|^\alpha & \text{si } 0 < |x| < 1/2 \\ 0 & \text{si } x = 0. \end{cases}$$

Montrer que

$$u_{x_1 x_1},\ u_{x_2 x_2} \in C^0\left(\overline{\Omega}\right) \quad \text{et} \quad u_{x_1 x_2} \notin L^\infty(\Omega)$$

ce qui implique que $\Delta u = u_{x_1 x_1} + u_{x_2 x_2} \in C^0$, alors que $u \notin C^2$; en fait u n'est même pas $W^{2,\infty}$.

Corrigés de l'Exercice 1. *(i)* Let $\rho \in C_0^\infty(\Omega)$ be such that

$$0 \leq \rho \leq 1 \quad \text{and} \quad \rho \equiv 1 \text{ in } O$$

and choose $\varphi = \rho^2 \overline{u} \in W_0^{1,2}(\Omega)$ in the equation so that

$$\int_\Omega h\, \rho^2 \overline{u} = \int_\Omega \left\langle \nabla \overline{u}; \nabla \left[\rho^2 \overline{u}\right]\right\rangle = \int_\Omega \rho^2 \left|\nabla \overline{u}\right|^2 + 2 \int_\Omega \left\langle \nabla \overline{u}; \rho\, \overline{u}\, \nabla \rho \right\rangle .$$

In the following, all L^2 norms are understood in $L^2(\Omega)$ unless stated otherwise. On a donc, recalling that $0 \leq \rho \leq 1$, que

$$\begin{aligned} \left\|\rho\, \nabla \overline{u}\right\|_{L^2}^2 = \int_\Omega \rho^2 \left|\nabla \overline{u}\right|^2 &\leq \int_\Omega h\, \rho^2 \overline{u} - 2 \int_\Omega \left\langle \nabla \overline{u}; \rho\, \overline{u}\, \nabla \rho \right\rangle \\ &\leq \left\|\overline{u}\right\|_{L^2} \left\|h\right\|_{L^2} + 2 \int_\Omega \left|\rho\, \nabla \overline{u}\right| \left|\overline{u}\, \nabla \rho\right| \\ &\leq \left\|\overline{u}\right\|_{L^2} \left\|h\right\|_{L^2} + 2 \left\|\rho\, \nabla \overline{u}\right\|_{L^2} \left\|\overline{u}\, \nabla \rho\right\|_{L^2} . \end{aligned}$$

Fix $\epsilon > 0$. We can therefore find $\gamma_1 = \gamma_1(O, \Omega, \alpha, \beta) > 0$ so that

$$\left\|\rho\, \nabla \overline{u}\right\|_{L^2}^2 \leq \gamma_1 \left(\left\|\overline{u}\right\|_{L^2}^2 + \left\|h\right\|_{L^2}^2 + \epsilon \left\|\rho\, \nabla \overline{u}\right\|_{L^2}^2 + \frac{1}{\epsilon} \left\|\overline{u}\right\|_{L^2}^2 \right).$$

Choosing ϵ sufficiently small we have indeed obtained

$$\left\|\nabla \overline{u}\right\|_{L^2(O)} \leq \left\|\rho\, \nabla \overline{u}\right\|_{L^2(\Omega)} \leq \gamma \left(\left\|\overline{u}\right\|_{L^2(\Omega)} + \left\|h\right\|_{L^2(\Omega)} \right).$$

(ii) Recall that the equation is

$$\int_\Omega \left\langle \nabla \overline{u}; \nabla \varphi \right\rangle dx = \int_\Omega h\, \varphi\, dx, \quad \forall\, \varphi \in W_0^{1,2}(\Omega)$$

and we already have the estimate (2). Let $O \subset \overline{O} \subset \Lambda \subset \overline{\Lambda} \subset \Omega$ be open sets. The very same argument of Theorem 4.8 that leads to (2) gives

$$\left\|\overline{u}\right\|_{W^{2,2}(O)} \leq \gamma_1 \left(\left\|\overline{u}\right\|_{W^{1,2}(\Lambda)} + \left\|h\right\|_{L^2(\Lambda)} \right).$$

Apply (i) to get

$$\left\|\nabla \overline{u}\right\|_{L^2(\Lambda)} \leq \gamma_2 \left(\left\|\overline{u}\right\|_{L^2(\Omega)} + \left\|h\right\|_{L^2(\Omega)} \right).$$

Finally, combine the above two inequalities to find the result.

(iii) Choosing $\varphi = \overline{u} \in W_0^{1,2}(\Omega)$, we find for $\epsilon > 0$ fixed

$$\left\|\nabla \overline{u}\right\|_{L^2}^2 = \int_\Omega \left\langle \nabla \overline{u}; \nabla \overline{u} \right\rangle dx = \int_\Omega h\, \overline{u} \leq \epsilon \left\|\overline{u}\right\|_{W^{1,2}}^2 + \frac{1}{\epsilon} \left\|h\right\|_{L^2}^2 ,$$

where, unless stated otherwise, all L^2 norms are understood in $L^2(\Omega)$. Invoking Poincaré inequality, we have that there exists $\theta > 0$ such that

$$\theta \|\overline{u}\|_{W^{1,2}}^2 \leq \|\nabla \overline{u}\|_{L^2}^2 \ .$$

Combining the two inequalities we find, choosing ϵ small enough, that

$$\|\overline{u}\|_{W^{1,2}}^2 \leq \frac{1}{\epsilon(\theta - \epsilon)} \|h\|_{L^2}^2 \ . \tag{3}$$

This estimate, together with (2), leads to the existence of $\gamma > 0$ such that

$$\|\overline{u}\|_{W^{2,2}(O)} \leq \gamma \|h\|_{L^2(\Omega)} \ . \quad \blacksquare$$