ON MEAN FIELD GAMES

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I. INTRODUCTION

- New class of models for the average (Mean Field) behavior of “small” agents (Games) started in the early 2000’s by J-M. Lasry and P-L. Lions.

- Requires new mathematical theories.

- Numerous applications: economics, finance, social networks, crowd motions...


- A research community in expansion: mathematics, economics, finance.

- Some written references but most of the existing mathematical material to be found in the Collège de France videotapes (4 × 18h) that can be downloaded...!
- Combination of Mean Field theories (classical in Physics and Mechanics) and the notion of Nash equilibria in Games theory.

- Nash equilibria for continua of “small” players: a single heterogeneous group of players (adaptations to several groups . . .).

- Interpretation in particular cases (but already general enough!) like process control of McKean-Vlasov . . .

- Each generic player is “rational” i.e. tries to optimize (control) a criterion that depends on the others (the whole group) and the optimal decision affects the behavior of the group (however, this interpretation is limited to some particular situations . . .).

- Huge class of models: agents → particles, no dep. on the group are two extreme particular cases.
II. A REALLY SIMPLE EXAMPLE

- Simple example, not new but gives an idea of the general class of models (other “simple” exs later on).

- $E$ metric space, $N$ players ($1 \leq i \leq N$) choose a position $x_i \in E$ according to a criterion $F_i(X)$ where $X = (x_1, \ldots, x_N) \in E^N$.

- Nash equilibrium : $\bar{X} = (\bar{x}_1, \ldots, \bar{x}_N)$ if for all $1 \leq i \leq N \bar{x}_i \min$ over $E$ of $F_i(\tilde{x}_1, \ldots, \tilde{x}_{i-1}, x_i, \tilde{x}_{i+1}, \ldots \tilde{x}_N)$.

- Usual difficulties with the notion

- $N \to \infty$ ? simpler ?

- Indistinguishable players :

\[
F_i(X) = F(x_i, \{x_j\}_{j \neq i}), \text{ } F \text{ sym. in } (x_j)_{j \neq i}
\]
• Part of the mathematical theories is about $N \to \infty$:

$$F_i = F(x, m) \quad x \in E, \ m \in \mathcal{P}(E)$$

where $x = x_i$, $m = \frac{1}{N-1} \sum_{j \neq i} \delta_{x_j}$

• “Thm” : Nash equilibria converge, as $N \to \infty$, to solutions of

$$(\text{MFG}) \quad \forall x \in \text{Supp} \ m, F(x, m) = \inf_{y \in E} F(y, m)$$

• Facts : i) general existence and stability results

ii) uniqueness if $(m \to F(\bullet, m))$ monotone

iii) If $F = \Phi'(m)$, then $(\min \Phi) \in \mathcal{P}(E)$ yields one solution of MFG.
Example: $E = \mathbb{R}^d$, $F_i(X) = f(x_i) + g\left(\frac{\#\{j/|x_i - x_j| < \varepsilon\}}{(N - 1)|B_\varepsilon|}\right)$

$g \uparrow$ aversion crowds, $g \downarrow$ like crowds

$$F(x, m) = f(x) + g(m \ast 1_{B_\varepsilon}(x)(|B_\varepsilon|^{-1}))$$

$$\varepsilon \to 0 \quad F(x, m) = f(x) + g(m(x))$$

\[(\text{MFG}) \quad \text{supp} \ m \subset \text{Arg min} \left(f(x) + g(m(x))\right)\]

$- g \uparrow$ uniqueness, $g \downarrow$ non uniqueness

$$\min\left\{\int fm + \int G(m)/m \in \mathcal{P}(E)\right\}, \quad G = \int_0^z f(s)ds$$

$- \text{explicit solution if } g \uparrow: m = g^{-1}(\lambda - f), \lambda \in \mathbb{R} \text{ s.t. } \int m = 1$
III. GENERAL STRUCTURE

- Particular case: dynamical problem, horizon $T$, continuous time and space, Brownian noises (both indep. and common), no intertemporal preference rate, control on drifts (Hamiltonian $H$), criterion dep. only on $m$

- $U(x, m, t)$ ($x \in \mathbb{R}^d$, $m \in \mathcal{P}(\mathbb{R}^d)$ or $\mathcal{M}_+(\mathbb{R}^d)$, $t \in [0, T]$ and $H(x, p, m)$ (convex in $p \in \mathbb{R}^d$)

- MFG master equation

\[
\begin{align*}
\frac{\partial U}{\partial t} - (\nu + \alpha)\Delta x U + H(x, \nabla_x U, m) + \\
+ \langle \frac{\partial U}{\partial m}, -(\nu + \alpha)\Delta m + \text{div} \left( \frac{\partial H}{\partial p} m \right) \rangle + \\
- \alpha \frac{\partial U}{\partial m^2} (\nabla m, \nabla m) + 2\alpha \langle \frac{\partial}{\partial m} \nabla_x U, \nabla m \rangle = 0
\end{align*}
\]

and $U \mid_{t=0} = U_0(x, m)$ (final cost)

- $\nu$ amount of ind. rand., $\alpha$ amount of common rand.
• $\infty$ $d$ problem!

• If $\nu = 0$ (ind) : Nash $N$ special case

using $x = x_i$, $m = \frac{1}{N-1} \sum_{j \neq i} \delta_{x \tilde{N}j}$

• Aggregation/decentralization : IF $H(x, p, m) = H(x, p) + F'(m)$ and $U_0 = \Phi'(m)$, then $U = \frac{\partial \Phi}{\partial m}$ solves MFG if $\Phi$ solves $HJB$ on $\mathcal{P}(E)$ for the optimal control of a SPDE

• Particular case : many extensions and variants . . .
IV. TWO PARTICULAR CASES

• $\infty \ d$ problem in general but reductions to finite $d$ in two cases

1. Indep. noises ($\alpha = 0$)

int. along caract. in $m$ yields

\[
\begin{aligned}
\frac{\partial u}{\partial t} - \nu \Delta u + H(x, \nabla u, m) &= 0 \\
\frac{\partial m}{\partial t} + \nu \Delta m + \text{div} \left( \frac{\partial H}{\partial p} m \right) &= 0
\end{aligned}
\]

(MFGi) \quad u \mid_{t=0} = U_0(x, m(0)), m \mid_{t=T} = \bar{m}

where $\bar{m}$ is given

FORWARD — BACKWARD system!

contains as particular cases: $HJB$, heat, porous media, $FP$, $V$, $B$, Hartree, semilinear elliptic, barotropic Euler . . .
2. Finite state space \((i \leq i \leq k)\)

\[
(MFGf) \quad \frac{\partial U}{\partial t} + (F(x, U) \cdot \nabla) U = G(x, U), \quad U \mid_{t=0} = U_0
\]

(no common noise here to simplify . . .)

\[x \in \mathbb{R}^k, \; U \rightarrow \mathbb{R}^k, \; F \text{ and } G : \mathbb{R}^{2k} \rightarrow \mathbb{R}^k\]

non-conservative hyperbolic system

Example : If \(F = F(U) = H'(U), \; G \equiv 0\)

and if \(U_0 = \nabla \varphi_0 \) (\(\varphi_0 \rightarrow \mathbb{R}\)) then

- solve \(HJ\)

\[
\frac{\partial \varphi}{\partial t} + H(\nabla \varphi) = 0, \quad \varphi \mid_{t=0} = \varphi_0
\]

- take \(U = \nabla \varphi\), “\(U\) solves” \((MFGf)\) in this case
V. OVERVIEW AND PERSPECTIVES

Lots of questions, partial results exist, many open problems

- Existence/regularity:
  - (MFGi) "simple" if $H$ "smooth" in $m$ (or if $H$ almost linear . . . ), OK if monotone (Zoom 1)
  - (MFGf) OK if $(G, F)$ mon. on $\mathbb{R}^{2k}$ or small time (Zoom 2)

- Uniqueness: OK if "monotone" or $T$ small . . .

- Non existence, non uniqueness, non regularity (!)

- Qualitative properties, stationary states and stability, comparison, cycles . . .

- $N \rightarrow \infty$ (see above)

- Numerical methods (currently, 3 "general" methods and some particular cases)

- Variants: other noises, several populations . . .

- Random heterogeneity, partial info . . .

- Applications (MFG Labs . . .)
- intertemporal preference rates \((+\lambda \to \infty)\) effective models
- macroscopic limits
- ? Beyond MFG ? (fluctuations, LD, transitions)
- Two more S. examples :
  - at which time will the meeting start?
  - the (mexican) wave
\[ \begin{align*}
\text{(MFGi)} \quad \left\{ \begin{array}{ll}
\frac{\partial u}{\partial t} - \nu \Delta u + H(x, \nabla u) &= f(x, m) \\
u |_{t=0} &= U_0(x), \quad m |_{t=T} = \bar{m} \\
\frac{\partial m}{\partial t} + \nu \Delta m + \text{div} \left( \frac{\partial H}{\partial p} m \right) &= 0
\end{array} \right.
\end{align*} \]

- \( m \mapsto f(\bullet, m) \) smoothing operator
  - \( \exists \) regular solution
- uniqueness if operator monotone or if \( T \) small
- \( f(m(x)) \uparrow: \exists ! \) regular solution \( \nu > 0 \)
- \( f(m(x)) \uparrow: \) if \( \nu = 0 \) \( m = f^{-1}(\frac{\partial u}{\partial t} + H(x, \nabla u)) \)

Equation in \( m \) becomes quasilinear elliptic equation of second order \( (x \in Q, \, t \in [0, \, T]) \) with “elliptic” boundary conditions

\[ u |_{t=0} = U_0(x), \quad \frac{\partial u}{\partial t} + H(\nabla u) = f(\bar{m}) \text{ if } t = T \]
(MFGf) \[
\begin{cases}
\frac{\partial u}{\partial t} + (F(x, U) \cdot \nabla) U = G(x, U) \quad x \in \mathbb{R}^d \\
U \rightarrow \mathbb{R}^d, \quad U |_{t=0} = U_0(x)
\end{cases}
\]

- shocks (discontinuities of $U$) in finite time in general
- well-posed problem on $[0, T_{\text{max}})$ ($T_{\text{max}} \leq +\infty$)
- $\exists$ !regular solution monotone in $x$ if $U_0$ monotone and $(G, F)$ monotone of $\mathbb{R}^{2,k}$ in $\mathbb{R}^{2k}(+ \ldots )$
- + change of unknown functions :

ex. : $\frac{\partial U}{\partial t} + (F(U) \cdot \nabla) U = 0$
then $V = F(U)$ solves

$$\frac{\partial V}{\partial t} + (V \cdot \nabla) V = 0$$

max class of regularity

$$\forall \delta > 0, \inf_{x \in \mathbb{R}^d} \text{dist}(\text{Sp}(DV_0(x)), (-\infty, \delta]) > 0$$

($V_0 = F(U_0)$ gives the maximum class of regularity $\sim$ composed of 2 monotone applications)

Rem. : gives new results of regularity for Hamilton-Jacobi equations of the first order.